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# New Techniques for Optimizing the Norm of Robust Controllers of Polytopic Uncertain Linear Systems

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## 1. Introduction

The history of linear matrix inequalities (LMIs) in the analysis of dynamical systems dates from over 100 years. The story begins around 1890 when Lyapunov published his work introducing what is now called the Lyapunov's theory (Boyd et al., 1994). The researches and publications involving the Lyapunov's theory have grown up a lot in recent decades (Chen, 1999), opening a very wide range for various approaches such as robust stability analysis of linear systems (Montagner et al., 2009), LMI optimization approach (Wang et al., 2008),  $\mathcal{H}_2$  (Apkarian et al., 2001; Assunção et al., 2007a; Ma & Chen, 2006) or  $\mathcal{H}_\infty$  (Assunção et al., 2007b; Chilali & Gahinet, 1996; Lee et al., 2004) robust control, design of controllers for systems with state feedback (Montagner et al., 2005), and design of controllers for systems with state-derivative feedback (Cardim et al., 2009). The design of robust controllers can also be applied to nonlinear systems.

In addition to the various current controllers design techniques, the design of robust controllers (or controller design by quadratic stability) using LMI stands out for solving problems that previously had no known solution. These designs use specialized computer packages (Gahinet et al., 1995), which made the LMIs important tools in control theory.

Recent publications have found a certain conservatism inserted in the analysis of quadratic stability, which led to a search for solutions to eliminate this conservatism (de Oliveira et al., 1999). Finsler's lemma (Skelton et al., 1997) has been widely used in control theory for the stability analysis by LMIs (Montagner et al., 2009; Peaucelle et al., 2000), with better results than the quadratic stability of LMIs, but with extra matrices, which allows a certain relaxation in the stability analysis (here called extended stability), by obtaining a larger feasibility region. The advantage found in its application to design of state feedback is the fact that the synthesis of gain  $K$  becomes decoupled from Lyapunov's matrix  $P$  (Oliveira et al., 1999), leaving Lyapunov's matrix free as it is necessarily symmetric and positive defined to meet the initial restrictions.

The reciprocal projection lemma used in robust control literature  $\mathcal{H}_2$  (Apkarian et al., 2001), can also be used for the synthesis of robust controllers, eliminating in a way the existing conservatism, as it makes feasible dealing with multiple Lyapunov's matrices, as in the case of

extended stability point, allowing extra matrices through a relaxation in the case of extended stability, making feasible a relaxation in the stability analysis (here called projective stability) through extra matrices. The synthesis of the controller  $K$  is depending now on an auxiliary matrix  $V$ , not necessarily symmetrical, and in this situation it becomes completely decoupled from Lyapunov's matrix  $P$ , leaving it free.

Two critical points in the design of robust controllers are explored here. One of them is the magnitude of the designed controllers that are often high, affect their practical implementation and therefore require a minimization of the gains of the controller to facilitate its implementation (optimization of the norm of  $K$ ). The other one is the fact that the system settling time can be larger than the required specifications of the project, thus demanding restrictions on LMIs to limit the decay rate, formulated with the inclusion of the parameter  $\gamma$  in LMIs.

The main focus of this work is to propose new methods for optimizing the controller's norm, through a different approach from that found in (Chilali & Gahinet, 1996), and compare it with the optimization method presented in (Assunção et al., 2007c) considering the different criteria of stability, aiming at the advantages and disadvantages of each method, as well as the inclusion of a decay rate (Boyd et al., 1994) in LMIs formulation.

In (Šiljak & Stipanovic, 2000) an optimization of the controllers's norm was proposed for decentralized control, but without the decay rate, so no comparisons were made with this work due to the necessity to insert this parameter to improve the performance of the system response.

The LMIs of optimization that will be used for new design techniques, had to be reformulated because the matrix controller synthesis does not depend more on a symmetric matrix, a necessary condition for the formulation of the existing LMI optimization. Comparisons will be made through a practical implementation in the Quanser's 3-DOF helicopter (Quanser, 2002) and a general analysis involving 1000 randomly generated polytopic uncertain systems.

## 2. Quadratic stability of continuous time linear systems

Consider (1) an autonomous linear dynamic system without state feedback. Lyapunov proved that the system

$$\dot{x}(t) = Ax(t) \quad (1)$$

with  $x(t) \in \mathbb{R}^n$  e  $A \in \mathbb{R}^{n \times n}$  a known matrix, is asymptotically stable (i.e., all trajectories converge to zero) if and only if there exists a matrix  $P = P' \in \mathbb{R}^{n \times n}$  such that the LMIs (2) and (3) are met (Boyd et al., 1994).

$$A'P + PA < 0 \quad (2)$$

$$P > 0 \quad (3)$$

Consider in equation (2) that  $A$  is not precisely known, but belongs to a polytopic bounded uncertainty domain  $\mathcal{A}$ . In this case, the matrix  $A$  within the domain of uncertainty can be written as convex combination of vertexes  $A_j$ ,  $j = 1, \dots, N$ , of the convex bounded uncertainty domain (Boyd et al., 1994), i.e.  $A(\alpha) \in \mathcal{A}$  and  $\mathcal{A}$  shown in (4).

$$\mathcal{A} = \{A(\alpha) \in \mathbb{R}^{n \times n} : A(\alpha) = \sum_{j=1}^N \alpha_j A_j, \sum_{j=1}^N \alpha_j = 1, \alpha_j \geq 0, j = 1 \dots N\} \quad (4)$$

A sufficient condition for stability of the convex bounded uncertainty domain  $\mathcal{A}$  (now on called polytope) is given by the existence of a Lyapunov's matrix  $P = P' \in \mathbb{R}^{n \times n}$  such that the LMIs (5) and (6)

$$A(\alpha)'P + PA(\alpha) < 0 \quad (5)$$

$$P > 0 \quad (6)$$

are checked for every  $A(\alpha) \in \mathcal{A}$  (Boyd et al., 1994). This stability condition is known as quadratic stability and can be easily verified in practice thanks to the convexity of Lyapunov's inequality that turns the conditions (5) and (6) equivalent to checking the existence of  $P = P' \in \mathbb{R}^{n \times n}$  such that conditions (7) and (8) are met with  $j = 1, \dots, N$ .

$$A_j'P + PA_j < 0 \quad (7)$$

$$P > 0 \quad (8)$$

It can be observed that (5) can be obtained multiplying by  $\alpha_j \geq 0$  and adding in  $j$  of  $j = 1$  to  $j = N$ .

Due to being a sufficient condition for stability of the polytope  $\mathcal{A}$ , conservative results are generated, nevertheless this quadratic stability has been widely used for robust controllers's synthesis.

### 3. Decay rate restriction for closed-loop systems

Consider a linear time invariant controllable system described in (9)

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \quad (9)$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  the matrix of system input,  $x(t) \in \mathbb{R}^n$  the state vector and  $u(t) \in \mathbb{R}^m$  the input vector. Assuming that all state are available for feedback, the control law for the same feedback is given by (10)

$$u(t) = -Kx(t) \quad (10)$$

being  $K \in \mathbb{R}^{m \times n}$  a constant elements matrix. Often the norm of the controller  $K$  can be high, leading to saturation of amplifiers and making the implementation in analogic systems difficult. Thus it is necessary to reduce the norm of the controllers elements to facilitate its implementation.

Considering the controlled system (9) - (10), the decay rate (or largest Lyapunov's exponent) is defined as the largest positive constant  $\gamma$ , such that (11)

$$\lim_{t \rightarrow \infty} e^{\gamma t} \|x(t)\| = 0 \quad (11)$$

remains for all trajectories  $x(t)$ ,  $t > 0$ .

From the quadratic Lyapunov's function (12),

$$V(x(t)) = x(t)'Px(t) \quad (12)$$

to establish a lower limit on the decay rate of (9), with (13)

$$\dot{V}(x(t)) \leq -2\gamma V(x(t)) \quad (13)$$

for all trajectories (Boyd et al., 1994).

From (12) and (9), (14) can be found.

$$\begin{aligned}\dot{V}(x(t)) &= \dot{x}(t)'Px(t) + x(t)'P\dot{x}(t) \\ &= x(t)'(A - BK)'Px(t) + x(t)'P(A - BK)x(t)\end{aligned}\quad (14)$$

Adding the restriction on the decay rate (13) in the equation (14) and making the appropriate simplifications, (15) and (16) are met.

$$(A - BK)'P + P(A - BK) < -2\gamma P \quad (15)$$

$$P > 0 \quad (16)$$

As the inequality (15) became a bilinear matrix inequality (BMI) it is necessary to perform manipulations to fit them back into the condition of LMIs. Multiplying the inequalities (17) and (18) on the left and on the right by  $P^{-1}$ , making  $X = P^{-1}$  and  $G = KX$  results:

$$AX - BG + XA' - G'B' + 2\gamma X < 0 \quad (17)$$

$$X > 0 \quad (18)$$

If the LMIs (17) and (18) are feasible, a controller that stabilizes the closed-loop system can be given by  $K = GX^{-1}$ .

Consider the linear uncertain time-invariant system (19).

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)u(t) \quad (19)$$

This system can be described as convex combination of the polytope's vertexes shown in (20).

$$\dot{x}(t) = \sum_{j=1}^r \alpha_j A_j x(t) + \sum_{j=1}^r \alpha_j B_j u(t) \quad (20)$$

with A and B belonging to the uncertainty polytope (21)

$$(\mathcal{A}, \mathcal{B}) = \{(A, B)(\alpha) \in \mathbb{R}^{n \times n} : (A, B)(\alpha) = \sum_{j=1}^N \alpha_j (A, B)_j, \sum_{j=1}^N \alpha_j = 1, \alpha_j \geq 0, j = 1 \dots N\} \quad (21)$$

being  $r$  the number of vertexes (Boyd et al., 1994).

Knowing the existing theory for uncertain systems, Theorem 3.1 theorem can be enunciated (Boyd et al., 1994):

**Theorem 3.1.** *A sufficient condition which guarantees the stability of the uncertain system (20) subject to decay rate  $\gamma$  is the existence of matrices  $X = X' \in \mathbb{R}^{n \times n}$  and  $G \in \mathbb{R}^{m \times n}$ , such that (22) and (23) are met.*

$$A_j X - B_j G + X A_j' - G' B_j' + 2\gamma X < 0 \quad (22)$$

$$X > 0 \quad (23)$$

with  $j = 1, \dots, r$ .

When the LMIs (22) and (23) are feasible, a state feedback matrix which stabilizes the system can be given by (24).

$$K = GX^{-1} \quad (24)$$

*Proof.* The proof can be found at (Boyd et al., 1994). □

Thus, it can be feedback into the uncertain system shown in (19) being (22) and (23) sufficient conditions for the polytope asymptotic stability, now for a closed-loop system subject to decay rate.

#### 4. Optimization of the $K$ matrix norm of the closed-loop system

In many situations the norm of the state feedback matrix is high, precluding its practical implementation. Thus Theorem 4.1 was proposed in order to limit the norm of  $K$  (Assunção et al., 2007c; Faria et al., 2010).

**Theorem 4.1.** *Given an fixed constant  $\mu_0 > 0$ , that enables to find feasible results, it can be obtained a constraint for the  $K \in \mathbb{R}^{m \times n}$  matrix norm from the state feedback, with  $K = GX^{-1}$ ,  $X = X' > 0 \in \mathbb{R}^{n \times n}$  and  $G \in \mathbb{R}^{m \times n}$  finding the minimum value  $\beta$ ,  $\beta > 0$  such that  $KK' < \frac{\beta}{\mu_0^2} I_m$ . The optimum value for  $\beta$  can be found solving the optimization problem with the LMIs (25), (26) and (27).*

$$\begin{aligned} & \min \beta \\ & \text{s.t.} \begin{bmatrix} \beta I_m & G \\ G' & I_n \end{bmatrix} > 0 \end{aligned} \quad (25)$$

$$X > \mu_0 I_n \quad (26)$$

$$A_j X - B_j G + X A_j' - G' B_j' + 2\gamma X < 0 \quad (27)$$

where  $I_m$  and  $I_n$  are the identity matrices of  $m$  and  $n$  order respectively.

*Proof.* The proof can be found at (Assunção et al., 2007c). □

#### 5. New optimization of the $K$ matrix norm of the closed-loop system

It can be verified that the LMIs given in Theorem 4.1 can produce conservative results, so in order to find better results, new methodologies are proposed.

Using the theory presented in (Assunção et al., 2007c) for the optimization of the norm of robust controllers subject to failures, it is proposed an alternative approach for the same problem grounded in Lemma (5.1).

The approach of the optimum norm used was modified to fit to the new structures of LMIs that will be given in sequence. At first, this new approach has produced better results comparing to the existing ones for the optimization stated in Theorem 4.1 using the set of LMIs (22) and (23).

**Lemma 5.1.** *Consider  $L \in \mathbb{R}^{n \times m}$  a given matrix and  $\beta \in \mathbb{R}$ ,  $\beta > 0$ . The conditions*

1.  $L'L \leq \beta I_m$
2.  $LL' \leq \beta I_n$

*are equivalent.*

*Proof.* Note that if  $L = 0$  the lemma conditions are verified. Then consider the case where  $L \neq 0$ .

Note that in the first statement of the lemma, (28) is met

$$L'L \leq \beta I_m \Leftrightarrow x'(L'L)x \leq \beta x'x \quad (28)$$

for all  $x \in \mathbb{R}^m$ .

Knowing that (29) is true

$$x'(L'L)x \leq \lambda_{\max}(L'L)x'x \quad (29)$$

and  $\lambda_{\max}(L'L)$  the maximum eigenvalue of  $L'L$ , which is real (every symmetric matrix has only real eigenvalues). Besides, when  $x$  is equal to the eigenvector of  $L'L$  associated to the eigenvalue  $\lambda_{\max}(L'L)$ , and  $x'(L'L)x = \lambda_{\max}(L'L)x'x$ . Thus, from (28) and (29),  $\beta \geq \lambda_{\max}(L'L)$ .

Similarly, for every  $z \in \mathbb{R}^n$ , the second assertion of the lemma results in (30).

$$LL' \leq \beta I_n \Leftrightarrow z'(LL')z \leq \lambda_{\max}(LL')z'z \leq \beta z'z \quad (30)$$

and then,  $\beta \geq \lambda_{\max}(LL')$ .

Now, note that the condition (31) is true (Chen, 1999).

$$\lambda^m \det(\lambda I_n - L'L) = \lambda^n \det(\lambda I_m - LL') \quad (31)$$

Consequently, every non-zero eigenvalue of  $L'L$  is also an eigenvalue of  $LL'$ . Therefore,  $\lambda_{\max}(L'L) = \lambda_{\max}(LL')$ , and from (29) and (30) the lemma is proved.  $\square$

Knowing that  $P = X^{-1}$  is the matrix used to define Lyapunov's quadratic function, Theorem 5.1 is proposed.

**Theorem 5.1.** Given a constant  $\mu_0 > 0$ , a constraint for the state feedback  $K \in \mathbb{R}^{m \times n}$  matrix norm is obtained, with  $K = GX^{-1}$ ,  $X = X' > 0$ ,  $X \in \mathbb{R}^{n \times n}$  and  $G \in \mathbb{R}^{m \times n}$  by finding the minimum of  $\beta$ ,  $\beta > 0$  such that  $K'K < \frac{\beta}{\mu_0} I_n$ . You can get the minimum  $\beta$  solving the optimization problem with the LMIs (32), (33) and (34).

$$\begin{aligned} & \min \beta \\ & \text{s.t.} \begin{bmatrix} X & G' \\ G & \beta I_m \end{bmatrix} > 0 \end{aligned} \quad (32)$$

$$X > \mu_0 I_n \quad (33)$$

$$A_j X - B_j G + X A_j' - G' B_j' + 2\gamma X < 0 \quad (34)$$

where  $I_m$  and  $I_n$  are the identity matrices of  $m$  and  $n$  order respectively.

*Proof.* Applying the Schur complement for the first inequality of (32) results in (35).

$$\beta I_m > 0 \text{ e } X - G'(\beta I_m)^{-1}G > 0 \quad (35)$$



Thus, from (35), (36) is found.

$$X > \frac{1}{\beta} G'G \Rightarrow G'G < \beta X \quad (36)$$

Replacing  $G = KX$  in (36) results in (37)

$$XK'KX < \beta X \Rightarrow K'K < \beta X^{-1} \quad (37)$$

So from (33), (37) and (33), (38) is met.

$$K'K < \frac{\beta}{\mu_0} I_n \quad (38)$$

on which  $K$  is the optimal controller associated with (22).  $\square$

It follows that minimizing the norm of a matrix is equivalent to the minimization of a  $\beta > 0$  variable such that  $K'K < \frac{\beta}{\mu_0} I_n$ , with  $\mu_0 > 0$ . Note that the position of the transposed matrix was replaced in this condition, comparing to that used in Theorem 4.1.

A comparison will be shown between the optimization methods, using the robust LMIs with decay rate (22) and (23) in the results section. Since the new method may suit the relaxed LMIs listed below, it was used in the comparative analysis for the control design for extended stability and projective stability.

Finsler's lemma shown in Lemma (5.2) can be used to express stability conditions referring to matrix inequalities, with advantages over existing Lyapunov's theory (Boyd et al., 1994), because it introduces new variables and generate new degrees of freedom in the analysis of uncertain systems with the possibility of nonlinearities elimination.

**Lemma 5.2** (Finsler). Consider  $w \in \mathbb{R}^{n_x}$ ,  $\mathcal{L} \in \mathbb{R}^{n_x \times n_x}$  and  $\mathcal{B} \in \mathbb{R}^{m_x \times n_x}$  with  $\text{rank}(\mathcal{B}) < n_x$  e  $\mathcal{B}^\perp$  a basis for the null space of  $\mathcal{B}$  (i.e.,  $\mathcal{B}\mathcal{B}^\perp = 0$ ). Then the following conditions are equivalent:

1.  $w' \mathcal{L} w < 0, \forall w \neq 0 : \mathcal{B} w = 0$
2.  $\mathcal{B}^{\perp'} \mathcal{L} \mathcal{B}^\perp < 0$
3.  $\exists \mu \in \mathbb{R} : \mathcal{L} - \mu \mathcal{B}' \mathcal{B} < 0$
4.  $\exists \mathcal{X} \in \mathbb{R}^{n_x \times m_x} : \mathcal{L} + \mathcal{X} \mathcal{B} + \mathcal{B}' \mathcal{X}' < 0$

*Proof.* Finsler's lemma proof can be found at (Oliveira & Skelton, 2001; Skelton et al., 1997).  $\square$

### 5.1 Stability of systems using Finsler's lemma restricted by the decay rate

Consider the closed-loop system (9). Defining  $w = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ ,  $\mathcal{B} = [(A - BK) \ -I]$ ,  $\mathcal{B}^\perp = \begin{bmatrix} I \\ (A - BK) \end{bmatrix}$  and  $\mathcal{L} = \begin{bmatrix} 2\gamma P & P \\ P & 0 \end{bmatrix}$ . Note that  $\mathcal{B} w = 0$  corresponds to (9) and  $w' \mathcal{L} w < 0$  corresponds to stability constraint with decay rate given by (12) and (13). In this case the dimensions of the lemma's variables (5.2) are:  $n_x = 2n$  and  $m_x = n$ . Considering that  $P$  is the matrix used to define the quadratic Lyapunov's function (12), the properties 1 and 2 of Finsler's lemma can be written as:



1.  $\exists P = P' > 0$  such that

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix}' \begin{bmatrix} 2\gamma P & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} < 0 \quad \forall x, \dot{x} \neq 0 : [(A - BK) - I] \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = 0$$

2.  $\exists P = P' > 0$  such that

$$\begin{bmatrix} I \\ (A - BK) \end{bmatrix}' \begin{bmatrix} 2\gamma P & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I \\ (A - BK) \end{bmatrix} < 0$$

which results in the equations of stability, according to Lyapunov, including decay rate:

1.  $x(t)'P\dot{x}(t) + \dot{x}(t)'Px(t) + 2\gamma x(t)'Px(t) < 0 \quad \forall x, \dot{x} \neq 0 : \dot{x}(t) = (A - BK)x(t)$
2.  $P(A - BK) + (A - BK)'P + 2\gamma P < 0$

Thus, it is possible to characterize stability through Lyapunov's quadratic function ( $V(x(t)) = x(t)'Px(t)$ ), generating new degrees of freedom for the synthesis of controllers.

From Finsler's lemma proof follows that if the properties 1 and 2 are true, then properties 3 and 4 will also be true. Thus, the fourth propriety can be written as (39).

4.  $\exists \mathcal{X} \in \mathbb{R}^{2n \times n}, P = P' > 0$  such that

$$\begin{bmatrix} 2\gamma P & P \\ P & 0 \end{bmatrix} + \mathcal{X} [(A - BK) - I] + \begin{bmatrix} (A - BK)' \\ -I \end{bmatrix} \mathcal{X}' < 0. \quad (39)$$

Choosing conveniently the matrix of variables  $\mathcal{X} = \begin{bmatrix} Z \\ aZ \end{bmatrix}$ , with  $Z \in \mathbb{R}^{n \times n}$  invertible and not necessarily symmetric and  $a > 0$  a fixed relaxation constant of the LMI (Pipeleers et al., 2009).

Developing the equation (39) and applying the congruence transformation  $\begin{bmatrix} Z^{-1} & 0 \\ 0 & Z^{-1} \end{bmatrix}$  on the left and  $\begin{bmatrix} Z^{-1} & 0 \\ 0 & Z^{-1} \end{bmatrix}'$  on the right, is found (40).

$$\begin{bmatrix} AZ'^{-1} + Z^{-1}A' - BKZ'^{-1} - Z^{-1}K'B' + 2\gamma Z^{-1}PZ'^{-1} & Z^{-1}PZ'^{-1} + aZ^{-1}A' - aZ^{-1}K'B' - Z'^{-1} \\ Z^{-1}PZ'^{-1} + aAZ'^{-1} - aBKZ'^{-1} - Z^{-1} & -aZ'^{-1} - aZ^{-1} \end{bmatrix} < 0$$

Making  $Y = Z'^{-1}$ ;  $G = KY$  and  $Q = Y'PY$ , there were found LMIs (40) and (41) subject to decay rate  $\gamma$ .

$$\begin{bmatrix} AY + Y'A' - BG - G'B' + 2\gamma Q & Q + aY'A' - aG'B' - Y \\ Q + aAY - aBG - Y' & -aY - aY' \end{bmatrix} < 0, \quad (40)$$

$$Q > 0 \quad (41)$$

with  $Y \in \mathbb{R}^{n \times n}, Y \neq Y', G \in \mathbb{R}^{m \times n}$  and  $Q \in \mathbb{R}^{n \times n}, Q = Q' > 0$ , for some  $a > 0$ .

These LMIs meet the restrictions for the asymptotic stability (Feron et al., 1996) of the system described in (9) with state feedback given by (10). It can be checked that the first principal minor of the LMI (40) has the structure of the result found in the theorem of quadratic stability with decay rate (Faria et al., 2009). Nevertheless, there is also, as stated in the Finsler's lemma, a greater degree of freedom because the matrix of variables  $Y$ , responsible for the synthesis of the controller, doesn't need to be symmetric and the Lyapunov's matrix now

turned into  $Q$ , which remains restricted to positive definite, is partially detached from the controller synthesis, since that  $Q = Y'PY$ .

The stability of the LMIs derived from Finsler's lemma stability is commonly called extended stability and it will be designated this way now on.

## 5.2 Robust stability of systems using Finsler's lemma restricted by the decay rate

As discussed for the condition of quadratic stability, the stability analysis can be performed for a robust stability condition considering the continuous time linear system as a convex combination of  $r$  vertexes of the polytope described in (20). The advantage of using the Finsler's lemma for robust stability analysis is the freedom of Lyapunov's function, now defined as  $Q(\alpha) = \sum_{j=1}^r \alpha_j Q_j$ ,  $\sum_{j=1}^r \alpha_j = 1$ ,  $\alpha_j \geq 0$  e  $j = 1 \dots r$ , i.e., it can be defined a Lyapunov's function  $Q_j$  for each vertex  $j$ . As  $Q(\alpha)$  depends on  $\alpha$ , the Lyapunov matrix use fits to time-invariant polytopic uncertainties, being permitted rate of variation sufficiently small. To verify this, Theorem 5.2 is proposed.

**Theorem 5.2.** *A sufficient condition which guarantees the stability of the uncertain system (20) is the existence of matrices  $Y \in \mathbb{R}^{n \times n}$ ,  $Q \in \mathbb{R}^{n \times n}$ ,  $Q_j = Q_j' > 0$  e  $G \in \mathbb{R}^{m \times n}$ , decay rate greater than  $\gamma$  and a fixed constant  $a > 0$  such that the LMIs (42) and (43) are met.*

$$\begin{bmatrix} A_j Y + Y' A_j' - B_j G - G' B_j' + 2\gamma Q_j & Q_j + a Y' A_j' - a G' B_j' - Y \\ Q_j + a A_j Y - a B_j G - Y' & -a Y - a Y' \end{bmatrix} < 0 \quad (42)$$

$$Q_j > 0 \quad (43)$$

with  $j = 1, \dots, r$ . When the LMIs (42) and (43) are feasible, a state feedback matrix which stabilizes the system can be given by (44).

$$K = GY^{-1} \quad (44)$$

*Proof.* Multiplying (42) and (43) by  $\alpha_j \geq 0$ , and adding in  $j$ , for  $j = 1$  to  $j = N$ , LMIs (45) and (46) are found.

$$\begin{bmatrix} (\sum_{j=1}^r \alpha_j A_j) Y + Y' (\sum_{j=1}^r \alpha_j A_j)' - (\sum_{j=1}^r \alpha_j B_j) G - G' (\sum_{j=1}^r \alpha_j B_j)' + 2\gamma (\sum_{j=1}^r \alpha_j Q_j) \\ (\sum_{j=1}^r \alpha_j Q_j) + a (\sum_{j=1}^r \alpha_j A_j) Y - a (\sum_{j=1}^r \alpha_j B_j) G - Y' \\ (\sum_{j=1}^r \alpha_j Q_j) + a Y' (\sum_{j=1}^r \alpha_j A_j)' - a G' (\sum_{j=1}^r \alpha_j B_j)' - Y \\ -a Y - a Y' \end{bmatrix} < 0$$

$$\sum_{j=1}^r \alpha_j Q_j > 0$$

$$\begin{bmatrix} A(\alpha) Y + Y' A(\alpha)' - B(\alpha) G - G' B(\alpha)' + 2\gamma Q(\alpha) & Q(\alpha) + a Y' A(\alpha)' - a G' B(\alpha)' - Y \\ Q(\alpha) + a A(\alpha) Y - a B(\alpha) G - Y' & -a Y - a Y' \end{bmatrix} < 0 \quad (45)$$

$$Q(\alpha) > 0 \quad (46)$$

with  $Q(\alpha) = \sum_{j=1}^r \alpha_j Q_j$ ,  $\sum_{j=1}^r \alpha_j = 1$ ,  $\alpha_j \geq 0$  and  $j = 1 \dots r$ .  $\square$

Thus, the uncertain system shown can be fed back in (19) with (45) and (46) sufficient conditions for asymptotic stability of the polytope.

**Observation 1.** In the LMIs (42) and (43), the constant "a" has to be fixed for all vertexes and to satisfy the LMIs and it can be found through a one-dimensional search.

### 5.3 Optimization of the $K$ matrix norm using Finsler's lemma

The motivation for the study of an alternative optimization of the  $K$  matrix norm of state feedback control was due to less conservative results obtained with Finsler's lemma. This way expecting to find, for some situations, controllers with lower gains, thus being easier to implement than those designed using the existing quadratic stability theory (Faria et al., 2010), avoiding the signal control saturations.

Some difficulty in applying the existing theorem (Faria et al., 2010) was found to the new structure of LMIs, as the controller synthesis matrix  $Y$  is not symmetric, a condition that was necessary for the development of Theorem (4.1) when the controller synthesis matrix was  $X = P^{-1}$ . Thus, Theorem 5.3 is proposed.

**Theorem 5.3.** A constraint for the  $K \in \mathbb{R}^{m \times n}$  matrix norm of state feedback can be obtained, with  $K = GY^{-1}$  and  $Q_j = Y'P_jY$ , being  $Y \in \mathbb{R}^{n \times n}$ ,  $G \in \mathbb{R}^{m \times n}$  and  $P \in \mathbb{R}^{n \times n}$ ,  $P_j = P_j' > 0$  finding the minimum  $\beta$ ,  $\beta > 0$ , such that  $K'K < \beta P_j$ ,  $j = 1 \dots N$ . You can get the optimal value of  $\beta$  solving the optimization problem with the LMIs (47) and (48).

$$\begin{aligned} & \min \beta \\ & \text{s.t.} \begin{bmatrix} Q_j & G' \\ G & \beta I_m \end{bmatrix} > 0 \end{aligned} \quad (47)$$

$$\begin{bmatrix} A_j Y + Y' A_j' - B_j G - G' B_j' + 2\gamma Q_j & Q_j + a Y' A_j' - a G' B_j' - Y \\ Q_j + a A_j Y - a B_j G - Y' & -a Y - a Y' \end{bmatrix} < 0 \quad (48)$$

where  $I_m$  denotes the identity matrix of  $m$  order.

*Proof.* Applying the Schur complement for (47) results in (49).

$$\beta I_m > 0 \text{ and } Q_j - G'(\beta I_m)^{-1}G > 0 \quad (49)$$

Thus, from (49), (50) is found.

$$Q_j > \frac{1}{\beta} G'G \Rightarrow G'G < \beta Q_j \quad (50)$$

Replacing  $G = KY$  and  $Q_j = Y'P_jY$  in (50), (51) is met.

$$Y'K'KY < \beta Y'P_jY \Rightarrow K'K < \beta P_j \quad (51)$$

on which  $K$  is the optimal controller associated with (42) and (43).  $\square$

Thus it was possible the adequacy of the proposed optimization method with the minimization of a scalar  $\beta$ , using the inequality of minimization  $K'K < \beta P_j$  with  $P_j$  the Lyapunov's matrix, to the new relaxed parameters.

#### 5.4 Stability of systems using reciprocal projection lemma restricted by the decay rate

Another tool that can be used for stability analysis using LMIs is the reciprocal projection lemma (Apkarian et al., 2001) set out in Lemma 5.3.

**Lemma 5.3** (reciprocal projection lemma). *Consider  $Y = Y' > 0$  a given matrix. The following statements are equivalent*

1.  $\psi + S + S' < 0$

2. The following LMI is feasible for  $W$

$$\begin{bmatrix} \psi + Y - (W + W') & S' + W' \\ S + W & -Y \end{bmatrix} < 0$$

*Proof.* Reciprocal projection lemma proof can be found at (Apkarian et al., 2001). □

Consider the Lyapunov's inequality subject to a decay rate given by (15) and (16), which can be rewritten as (52) and (53).

$$(A - BK)X + X(A - BK)' + 2\gamma X < 0 \tag{52}$$

$$X > 0 \tag{53}$$

where  $X \triangleq P^{-1}$  e  $P$  is the Lyapunov's matrix. The original Lyapunov's inequality (15) can be recovered by multiplying the inequality (52) on the left and on the right by  $P$ .

Assuming  $\psi \triangleq 0$  e  $S' = (A - BK)X + \gamma X$ , it will be verified that the first claim of the reciprocal projection lemma will be exactly Lyapunov's inequality subject to the decay rate described in (52):

$$\psi + S + S' = (A - BK)X + X(A - BK)' + 2\gamma X < 0$$

From the reciprocal projection lemma, if the first statement is true, then the second one will also be true as (54) shows.

$$\begin{bmatrix} Y - (W + W') & (A - BK)X + \gamma X + W' \\ X(A - BK)' + \gamma X + W & -Y \end{bmatrix} < 0 \tag{54}$$

Multiplying (54) on the left and on the right by  $\begin{bmatrix} I & 0 \\ 0 & X^{-1} \end{bmatrix}$  with  $P \triangleq X^{-1}$  results in (55).

$$\begin{bmatrix} Y - (W + W') & (A - BK) + \gamma I + W'P \\ (A - BK)' + \gamma I + PW & -PYP \end{bmatrix} < 0 \tag{55}$$

Multiplying (55) on the left and on the right by  $\begin{bmatrix} W'^{-1} & 0 \\ 0 & I \end{bmatrix}$  and  $\begin{bmatrix} W^{-1} & 0 \\ 0 & I \end{bmatrix}$  respectively with  $V \triangleq W^{-1}$ , (56) is found.

$$\begin{bmatrix} V'YV - (V + V') & V'(A - BK) + \gamma V' + P \\ (A - BK)'V + \gamma V + P & -PYP \end{bmatrix} < 0 \tag{56}$$

Applying the Schur complement in  $V'YV$ , (57) is found.

$$\begin{bmatrix} -(V + V') & V'(A - BK) + \gamma V' + P & V' \\ (A - BK)'V + \gamma V + P & -PYP & 0 \\ V & 0 & -Y^{-1} \end{bmatrix} < 0 \quad (57)$$

performing the linearizing variable change  $Y \triangleq P^{-1}$  results in (58).

$$\begin{bmatrix} -(V + V') & V'(A - BK) + \gamma V' + P & V' \\ (A - BK)'V + \gamma V + P & -P & 0 \\ V & 0 & -P \end{bmatrix} < 0 \quad (58)$$

In literature it can be found a formulation close to the insertion of the decay rate but with different positioning of the parameter of decay rate (Shen et al., 2006). It is easy to verify that some conservatism was introduced with the choice of  $Y \triangleq P^{-1}$ , but the state feedback matrix is unrelated to the Lyapunov's matrix  $P$ , which results in relaxation of Lyapunov's LMI. Using the dual form  $(A - BK) \rightarrow (A - BK)'$  (Apkarian et al., 2001) results in inequality (59).

$$\begin{bmatrix} -(V + V') & V'(A - BK)' + \gamma V' + P & V' \\ (A - BK)V + \gamma V + P & -P & 0 \\ V & 0 & -P \end{bmatrix} < 0 \quad (59)$$

Performing the change of variable  $Z \triangleq KV$  and inserting the constraint  $P > 0$ , the LMIs (60) and (61) that guarantee system stability can be found.

$$\begin{bmatrix} -(V + V') & V'A' - Z'B' + \gamma V' + P & V' \\ AV - BZ + \gamma V + P & -P & 0 \\ V & 0 & -P \end{bmatrix} < 0 \quad (60)$$

$$P > 0 \quad (61)$$

The inequalities (60) and (61) are LMIs, and being feasible, it is deduced a state feedback matrix that can stabilize the system (9) - (10) given by (62).

$$K = ZV^{-1} \quad (62)$$

The result of relaxation of LMIs is interesting in the design of robust controllers, proposed below.

### 5.5 Robust stability of systems using reciprocal projection lemma restricted by the decay rate

A stability analysis for a robust stability condition can be performed considering the continuous time linear system an convex combination of  $r$  vertexes of the polytope described in (20). As in the extended stability case, the advantage of using the reciprocal projection lemma for robust stability analysis is the Lyapunov's function degree of freedom, now defined as  $P(\alpha) = \sum_{j=1}^r \alpha_j P_j$ ,  $\sum_{j=1}^r \alpha_j = 1$ ,  $\alpha_j \geq 0$  e  $j = 1 \dots r$ , i.e., it is defined a Lyapunov's function  $P_j$  for

each vertex  $j$ . As described before Theorem 5.2, the use of  $P(\alpha)$  fits to time-invariant polytopic uncertainties, being permitted rate of variation sufficiently small. To verify this, Theorem 5.4 is proposed.

**Theorem 5.4.** *A sufficient condition which guarantees the stability of the uncertain system (20) is the existence of matrices  $V \in \mathbb{R}^{n \times n}$ ,  $P_j = P_j' \in \mathbb{R}^{n \times n}$  and  $Z \in \mathbb{R}^{m \times n}$ , such that LMIs (63) and (64) are met.*

$$\begin{bmatrix} -(V + V') & V'A'_j - Z'B'_j + \gamma V' + P_j & V' \\ A_j V - B_j Z + \gamma V + P_j & -P_j & 0 \\ V & 0 & -P_j \end{bmatrix} < 0 \quad (63)$$

$$P_j > 0 \quad (64)$$

with  $j = 1, \dots, r$ .

When the LMIs (63) and (64) are feasible, a state feedback matrix which stabilizes the system can be given by (65).

$$K = ZV^{-1} \quad (65)$$

*Proof.* Multiplying (63) and (64) by  $\alpha_j \geq 0$ , and adding in  $j$ , for  $j = 1$  to  $j = N$ , (65) and (66) are found.

$$\begin{bmatrix} -(V + V') \\ (\sum_{j=1}^r \alpha_j A_j)V - (\sum_{j=1}^r \alpha_j B_j)Z + \gamma V + (\sum_{j=1}^r \alpha_j P_j) \\ V \\ V'(\sum_{j=1}^r \alpha_j A'_j) - Z'(\sum_{j=1}^r \alpha_j B'_j) + \gamma V' + (\sum_{j=1}^r \alpha_j P_j) & V' \\ -(\sum_{j=1}^r \alpha_j P_j) & 0 \\ 0 & -(\sum_{j=1}^r \alpha_j P_j) \end{bmatrix} < 0$$

$$(\sum_{j=1}^r \alpha_j P_j) > 0$$

$$\begin{bmatrix} -(V + V') & V'A'(\alpha) - Z'B'(\alpha) + \gamma V' + P(\alpha) & V' \\ A(\alpha)V - B(\alpha)Z + \gamma V + P(\alpha) & -P(\alpha) & 0 \\ V & 0 & -P(\alpha) \end{bmatrix} < 0 \quad (65)$$

$$P(\alpha) > 0 \quad (66)$$

with  $P(\alpha) = \sum_{j=1}^r \alpha_j P_j$ ,  $\sum_{j=1}^r \alpha_j = 1$ ,  $\alpha_j \geq 0$  and  $j = 1 \dots r$ . □

It appears that  $K$  is unique and there are  $r$  Lyapunov's matrices  $P_j$ , generating a relaxation in the LMIs. The same trend was observed in the formulation via Finsler's lemma in which variables were the Lyapunov's matrices  $Q_j$ , but in (65) and (66) there is a greater degree of freedom with the inclusion of  $V$  in the design of the control matrix  $K$ ,  $V$  being totally disconnected from  $P_j$ ,  $j = 1, \dots, n$ .

### 5.6 Optimization of the $K$ matrix norm using reciprocal projection lemma

A study was carried out to fit the LMIs to the new relaxed parameters once the state feedback matrix  $K$  is completely detached from the Lyapunov's matrix  $P(\alpha)$ . Therefore, relevant changes took place in the optimization proposed in this study to suit the reciprocal projection lemma. This optimization has provided interesting results in practice.

Due to the lack of relations to assemble LMI able to optimize the module of  $K$  it was proposed a minimization procedure similar to the optimization procedure for redesign presented in (Chang et al., 2002) inserting an extra restriction to the LMIs (63) and (64).

Thus Theorem 5.5 was proposed.

**Theorem 5.5.** A constraint for the  $K \in \mathbb{R}^{m \times n}$  matrix norm of state feedback is obtained, with  $K = ZV^{-1}$ ,  $V \in \mathbb{R}^{n \times n}$  and  $Z \in \mathbb{R}^{m \times n}$  finding the minimum  $\beta$ ,  $\beta > 0$ , such that  $K'K < \beta M$ , being  $M = V'^{-1}V^{-1}$  and therefore  $M = M' > 0$ . You can get the optimal value of  $\beta$  solving the optimization problem with the LMIs (67) and (68).

$$\begin{aligned} & \min \beta \\ & \text{s.t. } \begin{bmatrix} I_n & Z' \\ Z & \beta I_m \end{bmatrix} > 0 \end{aligned} \quad (67)$$

$$\text{(Set of LMIs (63) and (64))} \quad (68)$$

which  $I_m$  and  $I_n$  denote the identity matrices of  $m$  and  $n$  order respectively.

*Proof.* Applying the Schur complement in (67) results in (69).

$$\beta I_m > 0 \text{ e } I_n - Z'(\beta I_m)^{-1}Z > 0 \quad (69)$$

Thus, from (69), (70) is found.

$$I_n > \frac{1}{\beta} Z'Z \Rightarrow Z'Z < \beta I_n \quad (70)$$

Replacing  $Z = KV$  in (70) results in (71).

$$V'K'KV < \beta I_n \quad (71)$$

Multiplying on the left and on the right (71) for  $V'^{-1}$  e  $V^{-1}$  respectively and naming  $V'^{-1}V^{-1} = M$  (72) is met.

$$V'K'KV < \beta I_n \Rightarrow K'K < \beta M \quad (72)$$

where  $K$  is the optimal controller associated with (63) and (64).  $\square$

Due to  $M$  being defined as  $M = V'^{-1}V^{-1}$  and so  $M = M' > 0$ , it is possible to find a relationship that optimizes the matrix  $K$  minimizing a scalar  $\beta$ , with the relation of minimizing  $K'K < \beta M$ .



## 6. Practical application in the 3-DOF helicopter

Consider the schematic model in Figure (2) of the 3-DOF helicopter (Quanser, 2002) shown in Figure (1). Two DC motors are mounted at the two ends of a rectangular frame and drive two propellers. The motors axis are parallel and the thrust vector is normal to the frame. The helicopter frame is suspended from the instrumented joint mounted at the end of a long arm and is free to pitch about its center (Quanser, 2002).

The arm is gimbaled on a 2-DOF instrumented joint and is free to pitch and yaw. The other end of the arm carries a counterweight such that the effective mass of the helicopter is light enough for it to be lifted using the thrust from the motors. A positive voltage applied to the front motor causes a positive pitch while a positive voltage applied to the back motor causes a negative pitch (angle *pitch* ( $\rho$ )). A positive voltage to either motor also causes an elevation of the body (i.e., pitch of the arm). If the body pitches, the thrust vectors result in a travel of the body (i.e., yaw ( $\epsilon$ ) of the arm) as well. If the body pitches, the impulsion vector results in the displacement of the system (i.e., travel ( $\lambda$ ) of the system).

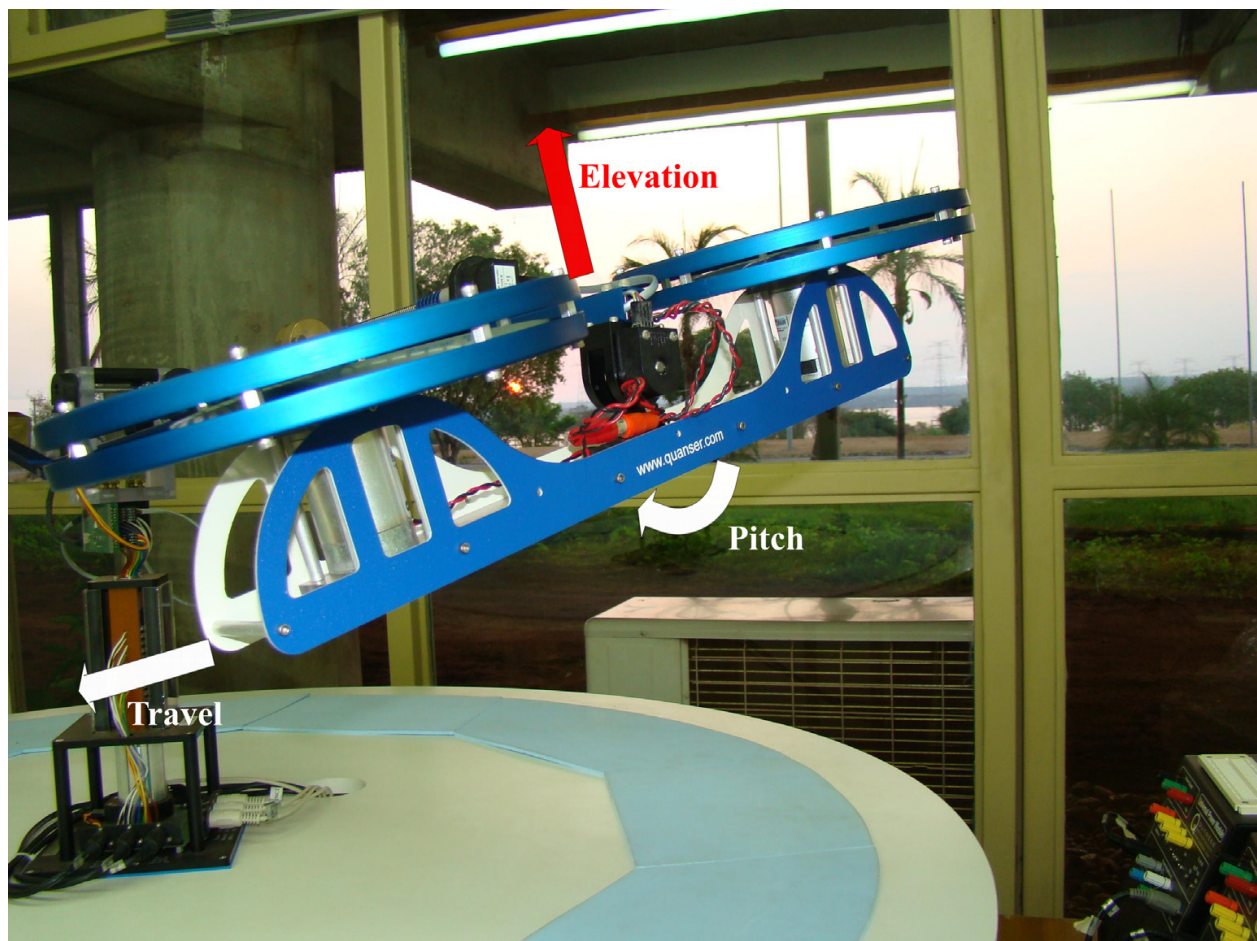


Fig. 1. Quanser's 3-DOF helicopter of UNESP - Campus Ilha Solteira.

The objective of this experiment is to design a control system to track and regulate the elevation and travel of the 3-DOF Helicopter.

The 3-DOF Helicopter can also be fitted with an active mass disturbance system that will not be used in this work.

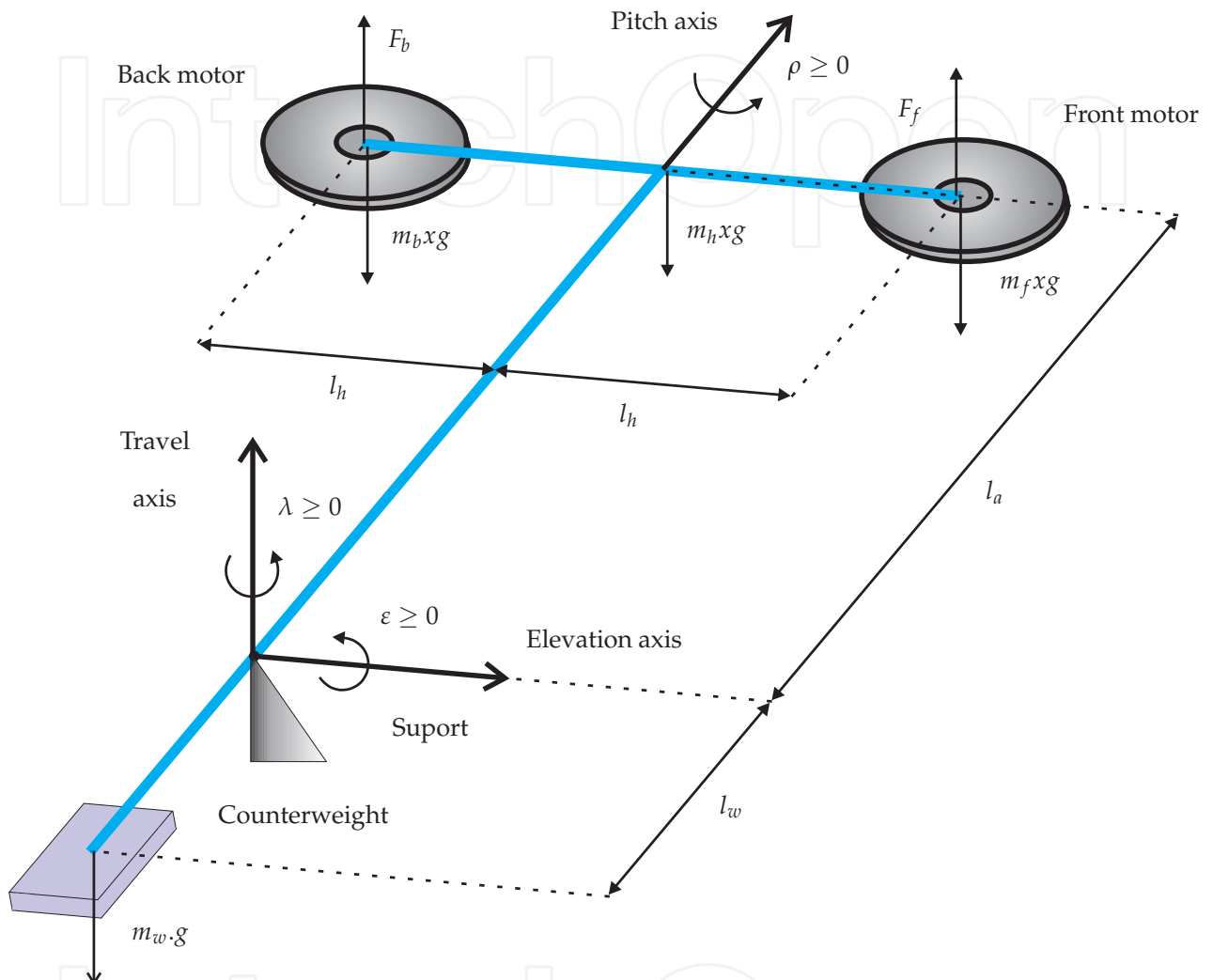


Fig. 2. Schematic drawing of 3-DOF Helicopter

The state space model that describes the helicopter is (Quanser, 2002) shown in (73).

$$\begin{bmatrix} \dot{\varepsilon} \\ \dot{\rho} \\ \dot{\lambda} \\ \ddot{\varepsilon} \\ \ddot{\rho} \\ \ddot{\lambda} \\ \dot{\xi} \\ \dot{\gamma} \end{bmatrix} = A \begin{bmatrix} \varepsilon \\ \rho \\ \lambda \\ \dot{\varepsilon} \\ \dot{\rho} \\ \dot{\lambda} \\ \xi \\ \gamma \end{bmatrix} + B \begin{bmatrix} V_f \\ V_b \end{bmatrix} \quad (73)$$

The variables  $\zeta$  and  $\gamma$  represent the integrals of the angles  $\varepsilon$  of yaw and  $\lambda$  of travel, respectively. The matrices A and B are presented in sequence.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2m_f l_a - m_w l_w g}{2m_f l_a^2 + 2m_f l_h^2 + m_w l_w^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{l_a k_f}{m_w l_w^2 + 2m_f l_a^2} & \frac{l_a k_f}{m_w l_w^2 + 2m_f l_a^2} \\ \frac{1}{2} \frac{k_f}{m_f l_h} & -\frac{1}{2} \frac{k_f}{m_f l_h} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The values used in the project were those that appear in the MATLAB programs for implementing the original design manufacturer, to maintain fidelity to the parameters. The constants used are described in Table (1).

Power constant of the propeller (found experimentally)	$k_f$	0.1188
Mass of the helicopter body (kg)	$m_h$	1.15
Mass of counterweight (kg)	$m_w$	1.87
Mass of the whole front of the propeller (kg)	$m_f$	$m_h/2$
Mass of the whole back of the propeller (kg)	$m_b$	$m_h/2$
Distance between each axis of pitch and motor (m)	$l_h$	0.1778
Distance between the lift axis and the body of the helicopter (m)	$l_a$	0.6604
Distance between the axis of elevation and the counterweight (m)	$l_w$	0.4699
Gravitational constant ( $m/s^2$ )	$g$	9.81

Table 1. Helicopter parameters

Practical implementations of the controllers were carried out in order to view the controller acting in real physical systems subject to failures.

The trajectory of the helicopter was divided into three stages. The first stage is to elevate the helicopter  $27.5^\circ$  reaching the yaw angle  $\varepsilon = 0^\circ$ . In the second stage the helicopter travels  $120^\circ$ , keeping the same elevation i.e., the helicopter reaches  $\lambda = 120^\circ$  with reference to the launch point. In the third stage the helicopter performs the landing recovering the initial angle  $\varepsilon = -27.5^\circ$ .

During the landing stage, more precisely in the instant 22 s, the helicopter loses 30% of the power back motor. The robust controller should maintain the stability of the helicopter and have small oscillation in the occurrence of this failure.

To add robustness to the system without any physical change, a 30% drop in power of the back motor is forced by inserting a timer switch connected to an amplifier with a gain of 0.7 in tension acting directly on engine, and thus being constituted a polytope of two vertexes

with an uncertainty in the input matrix of the system acting on the helicopter voltage between  $0,7V_b$  and  $V_b$ . The polytope described as follows.

Vertex 1 (100% of  $V_b$ ):

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.2304 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.0858 & 0.0858 \\ 0.5810 & -0.5810 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Vertex 2 (70% of  $V_b$ ):

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.2304 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.0858 & 0.0601 \\ 0.5810 & -0.4067 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Fixing the decay rate equal to 0.8, there were designed: a controller with quadratic stability using the existing optimization (Assunção et al., 2007c), a controller with quadratic stability with the proposed optimization and controllers with extended stability and projective stability also with the proposed optimization to perform the practical implementation.

The controller designed by quadratic stability with existing optimization (Theorem 4.1) is shown in (74) (Assunção et al., 2007c).

$$K = \begin{bmatrix} -46.4092 & -15.6262 & 21.3173 & -24.7541 & -3.9269 & 23.5800 & -27.4973 & 7.4713 \\ -70.3091 & 13.3795 & -10.1982 & -37.5960 & 4.3357 & -15.1521 & -41.5328 & -2.7935 \end{bmatrix} \quad (74)$$

where  $\|K\| = 107.83$ .

This controller was implemented in helicopter and the results are shown in Figure 3.

In (75) follows the quadratic stability controller design with the proposed optimization follows (Theorem 5.1).

$$K = \begin{bmatrix} -18.8245 & -12.2370 & 10.9243 & -13.9612 & -4.4480 & 14.6213 & -9.1334 & 3.2483 \\ -27.9219 & 10.6586 & -7.6096 & -20.1096 & 4.5602 & -11.0774 & -13.7202 & -2.2629 \end{bmatrix} \quad (75)$$

where  $\|K\| = 44.88$ .

This controller was implemented in the helicopter and the results are shown in Figure 4.

In (76) follows the extended stability controller design with the proposed optimization follows (Theorem 5.3). For this LMIs an  $a = 10^{-6}$  solves the problem. Though the Theorem 5.3

and Theorem 5.5 hypothesis establishes a sufficiently low time variation of  $\alpha$ . However, for comparison purposes of Theorem 5.3 and Theorem 5.5 with Theorem 5.1, the same abrupt loss of power test was done with controllers (76) and (77).

$$K = \begin{bmatrix} -23.7152 & -12.9483 & 9.8587 & -18.7322 & -4.9737 & 14.3283 & -10.7730 & 2.6780 \\ -33.8862 & 15.2923 & -11.6132 & -25.4922 & 6.0776 & -16.5503 & -15.8350 & -3.4475 \end{bmatrix} \quad (76)$$

where  $\|K\| = 56.47$ .

This controller was implemented in the helicopter and the results are shown in Figure (5).

In (77) follows the projective stability controller design with the proposed optimization follows (Theorem 5.5).

$$K = \begin{bmatrix} -50.7121 & -28.7596 & 35.1829 & -29.8247 & -7.9563 & 41.0906 & -28.8974 & 11.7405 \\ -66.5405 & 31.9853 & -34.7642 & -38.3173 & 9.9376 & -42.0298 & -38.3418 & -11.8207 \end{bmatrix} \quad (77)$$

where  $\|K\| = 110.46$ .

This controller was implemented in the helicopter and the results are shown in Figure 6.

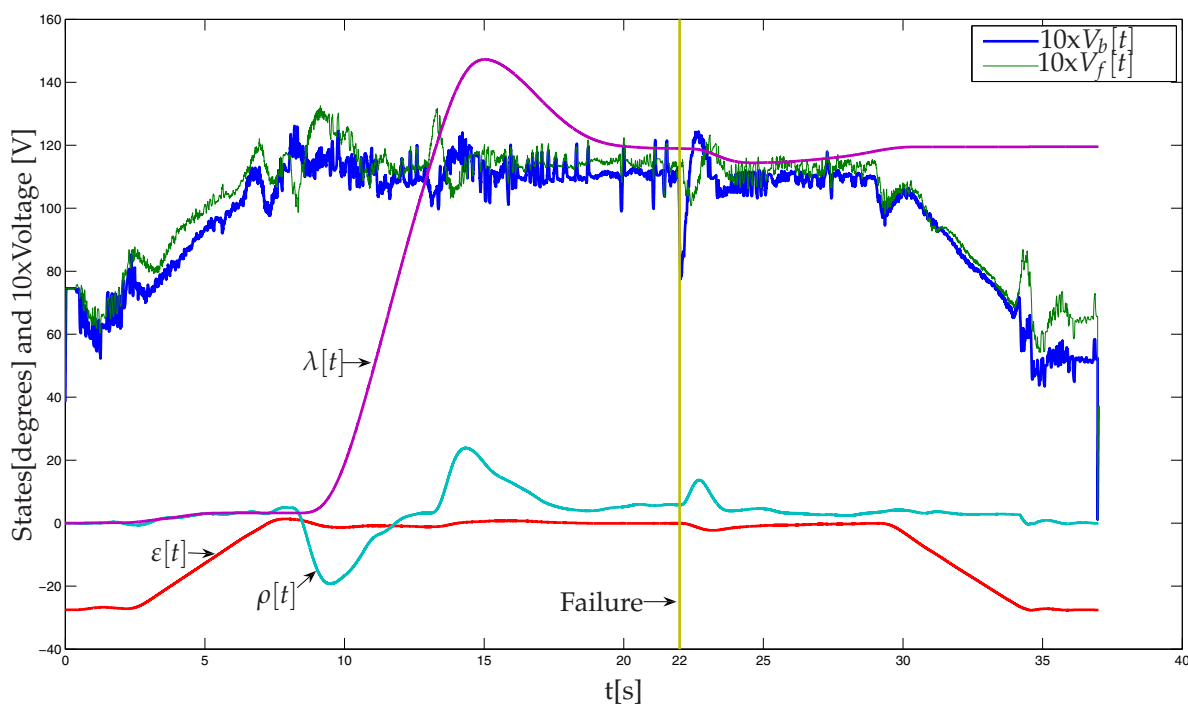


Fig. 3. Practical implementation of the designed K by quadratic stability with the optimization method presented in (Assunção et al., 2007c).

The graphics of Figures 3, 4, 5 and 6, refer to the actual data of the angles and voltages on the front motor ( $V_f$ ) and back motor ( $V_b$ ) measured with the designed controllers acting on the plant during the trajectory described as a failure in the instant 22 s. Tensions ( $V_f$ ) and ( $V_b$ ) on the motors were multiplied by 10 to match the scales of the two graphics.

Note that the variations of the amplitudes of ( $V_f$ ) and ( $V_b$ ) using optimized controllers proposed (75) and (76) in Figures 4 and 5 are smaller than those obtained with the existing

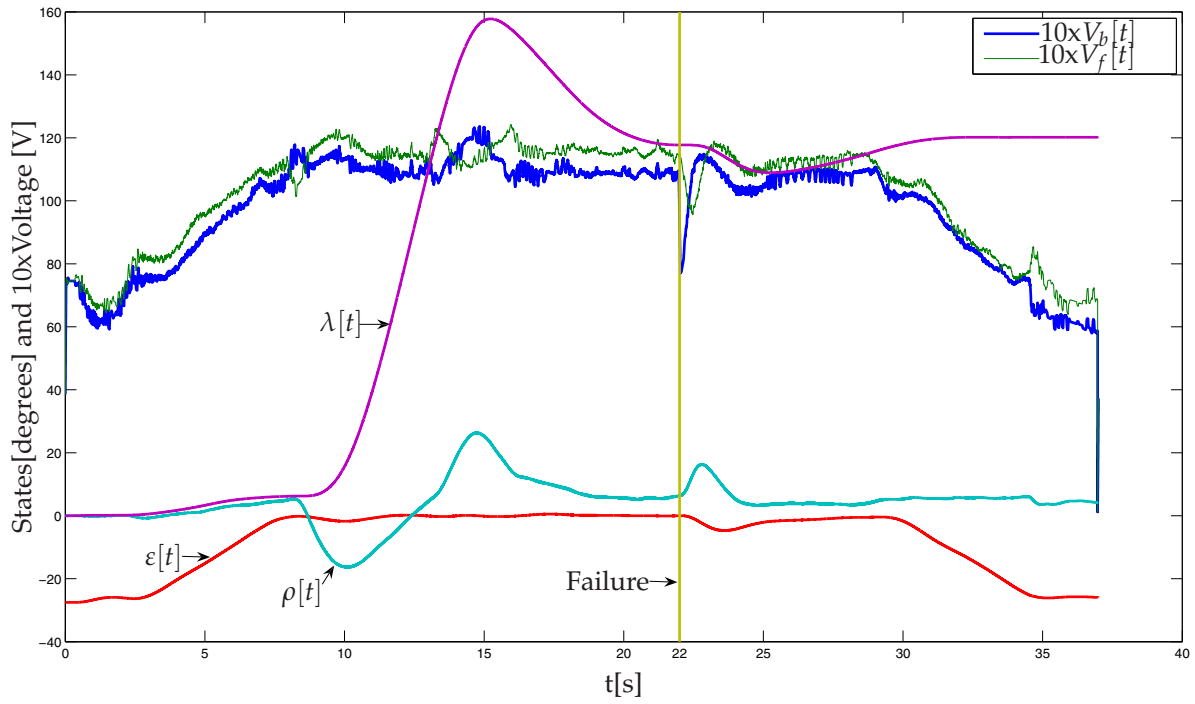


Fig. 4. Practical implementation of the K designed by quadratic stability with the proposed optimization method.

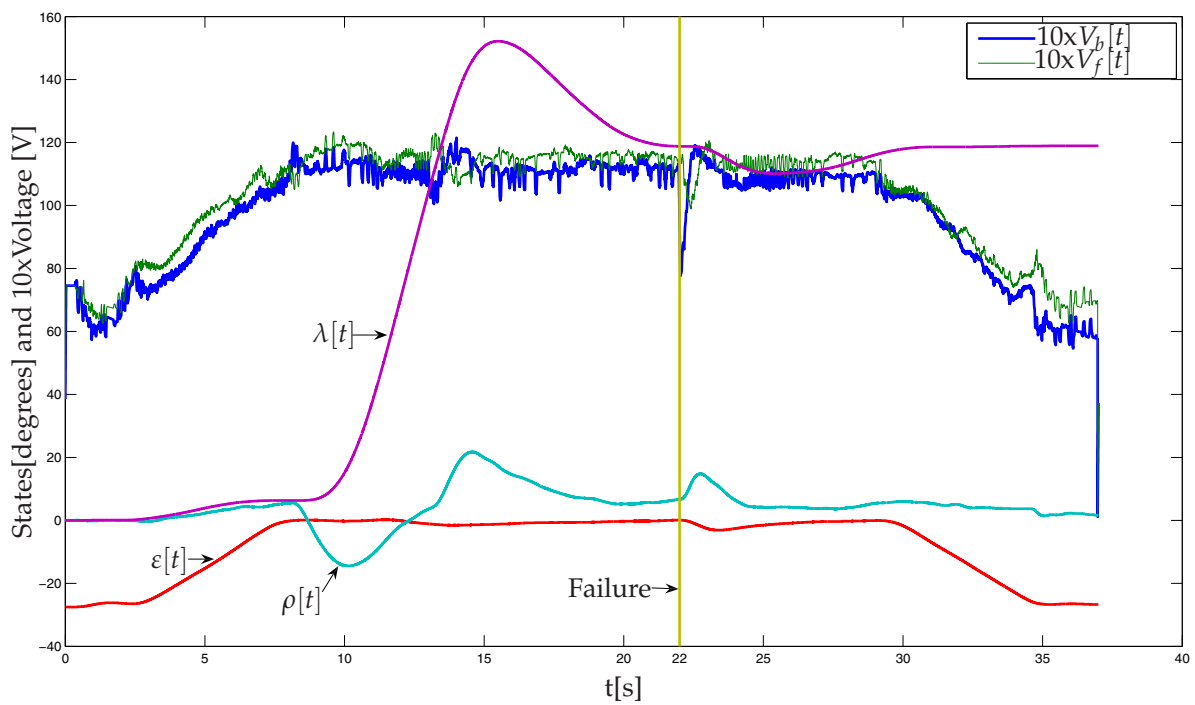


Fig. 5. Practical implementation of the K designed by extended stability with the proposed optimization method.



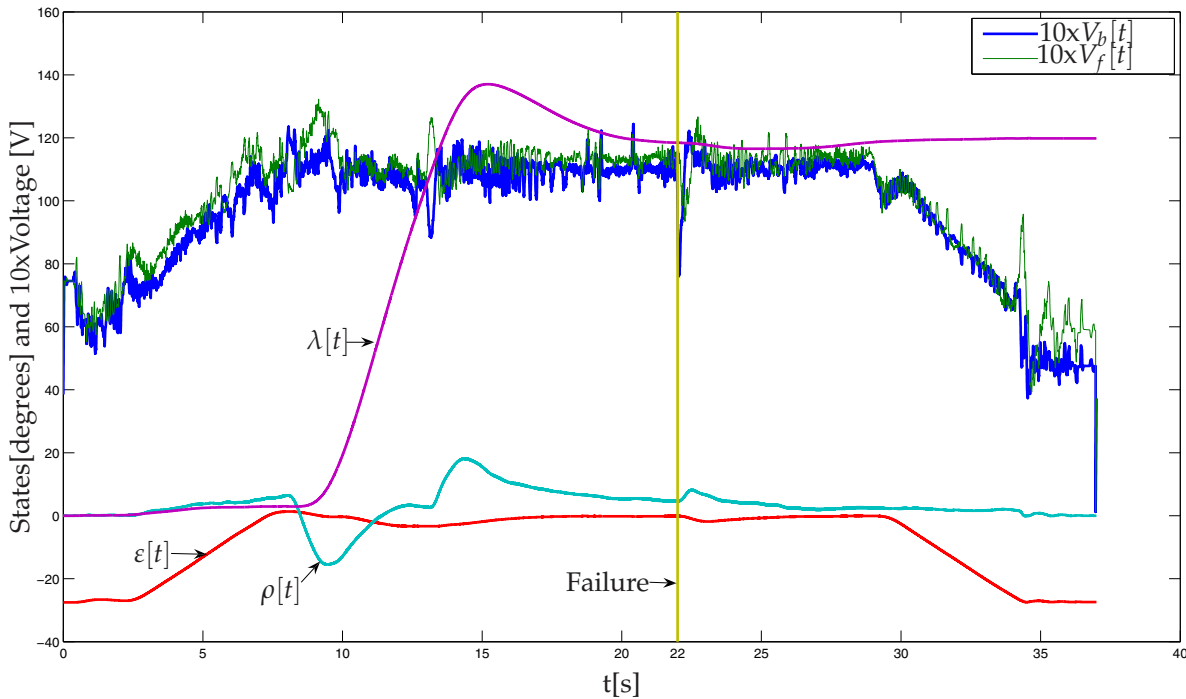


Fig. 6. Practical implementation of the  $K$  designed by projective stability with the proposed optimization method.

controller in the literature (74) shown in Figure 3. This is due to the fact that our proposed controllers (75) and (76) have lower gains than (74). For this implementation the projective stability designed controllers with proposed optimization (77) obtained the worst results as Figure 6.

It was checked that the  $\gamma$  used in the implementation of robust controllers, if higher, forces the system to have a quick and efficient recovery, with small fluctuations.

## 7. General comparison of the two optimization methods

In order to obtain more satisfactory results on which would be the best way to optimize the norm of  $K$ , a more general comparison has been made between the two methods as Theorems 4.1 and 5.1.

There were randomly generated 1000 uncertain polytopes of second order systems, with only one uncertain parameter (two vertexes) and after that, 1000 uncertain polytopes of fourth order uncertain systems, with two uncertain parameter (four vertexes). The 1000 uncertain polytopes were generated feasible in at least one case of optimization for  $\gamma = 0.5$ , and the consequences of  $\gamma$  increase were analyzed and plotted in a bar charts showing the number of controllers with lower norm due to the increase of  $\gamma$ , shown in Figure 7 for second-order systems and in Figure 8 to fourth-order systems.

The controllers designed with elevated values of  $\gamma$  do not have much practical application due to the fact that the increase of  $\gamma$  affect the increasing of the norm and make higher peaks of the transient oscillation, used here only for the purpose of analyzing feasibility and better



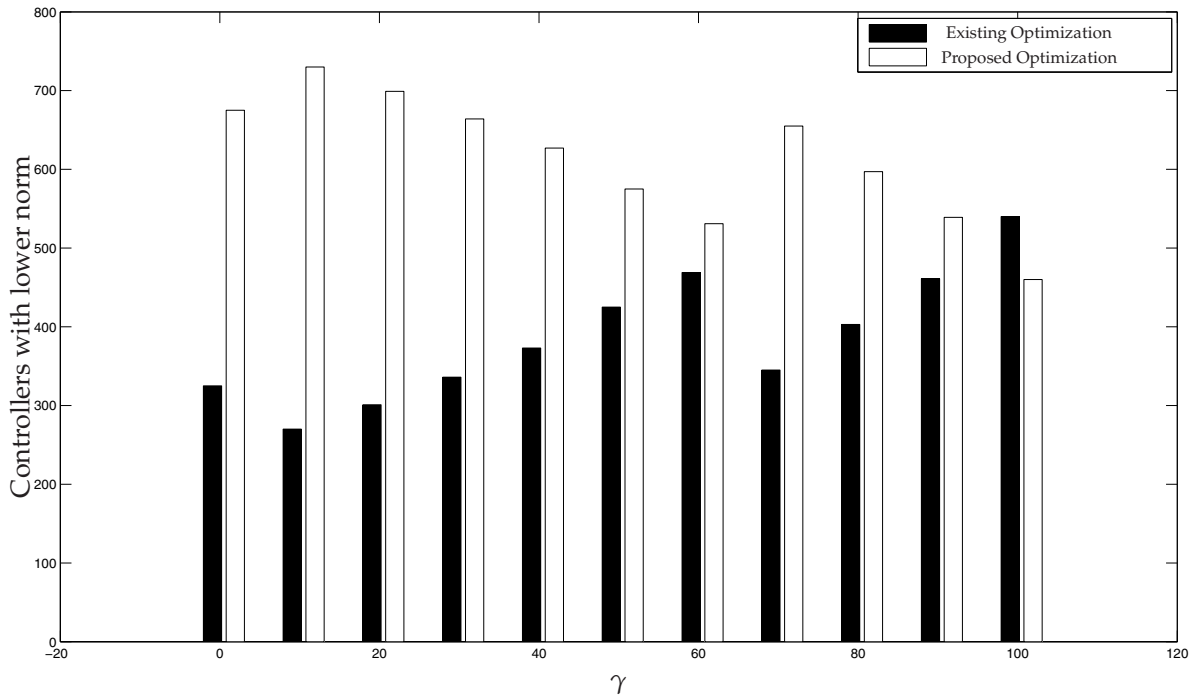


Fig. 7. Number of controllers with lower norm for 1000 uncertain polytopic systems of second-order randomly generated.

results for the norm of  $K$ , so comparisons were closed in  $\gamma = 100.5$ , because this  $\gamma$  is already considered high.

In Figure 8 can be seen that the proposed optimization method produces better results for all cases analyzed. Due to the complexity of the polytopes used in this case (fourth-order uncertain systems with two uncertainties (four vertexes)), is natural a loss of feasibility with the increase of  $\gamma$ , and yet the proposed method shows very good results.

## 8. General comparison of the new design and optimization methods

A generic comparison between the three methods of design and optimization of  $K$  was also carried out: design by quadratic stability with proposed optimization shown in Theorem 5.1, design and proposed optimization with extended stability shown in Theorem 5.3 (using the parameter  $a = 10^{-6}$  in the LMIs) and projective stability design with proposed optimization shown in Theorem 5.5.

Initially 1000 polytopes of second order uncertain systems were randomly generated, with only one uncertain parameter (two vertexes) and after that, fourth order uncertain systems, with two uncertain parameter (four vertexes). The 1000 polytopes were generated feasible in at least one case of optimization for  $\gamma = 0.5$  and the consequences of  $\gamma$  increase were analyzed. In fourth-order uncertain systems, the 1000 polytopes were generated feasible in at least one case of optimization for  $\gamma = 0.2$  and then, the consequences of  $\gamma$  of 0.2 in 0.2 increase were analyzed. This comparison was carried out with the intention of examining feasibility and better results for the norm of  $K$ . So, a bar graphics showing the number of controllers with lower norm with the increase of  $\gamma$  was plotted, and is shown in Figures 9 and 10.

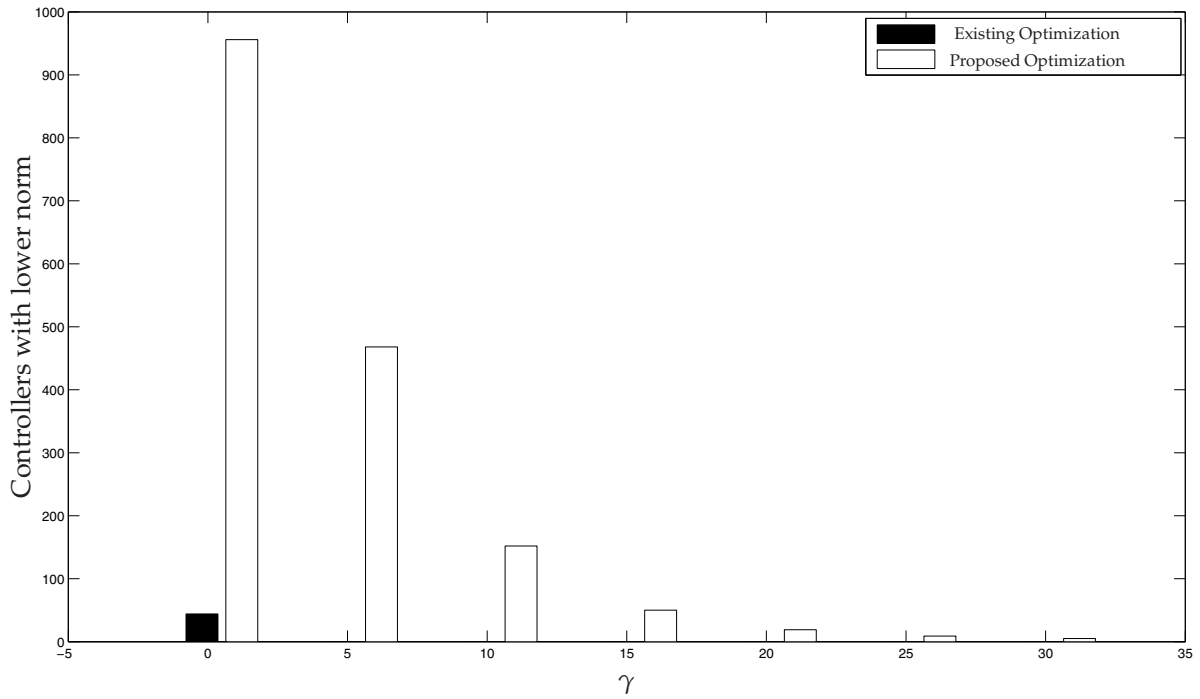


Fig. 8. Number of controllers with lower norm for 1000 uncertain polytopic systems of fourth-order randomly generated.

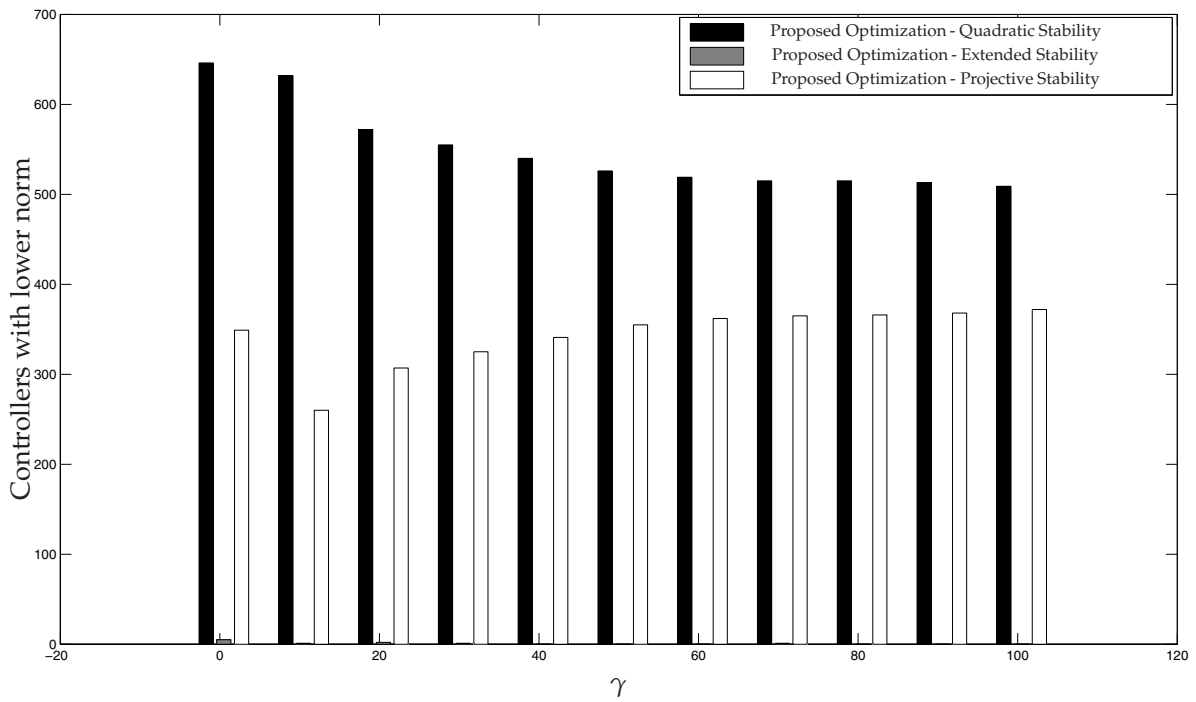


Fig. 9. Number of controllers with lower norm for 1000 uncertain polytopic systems of second-order randomly generated. All these methods are proposed in this work.

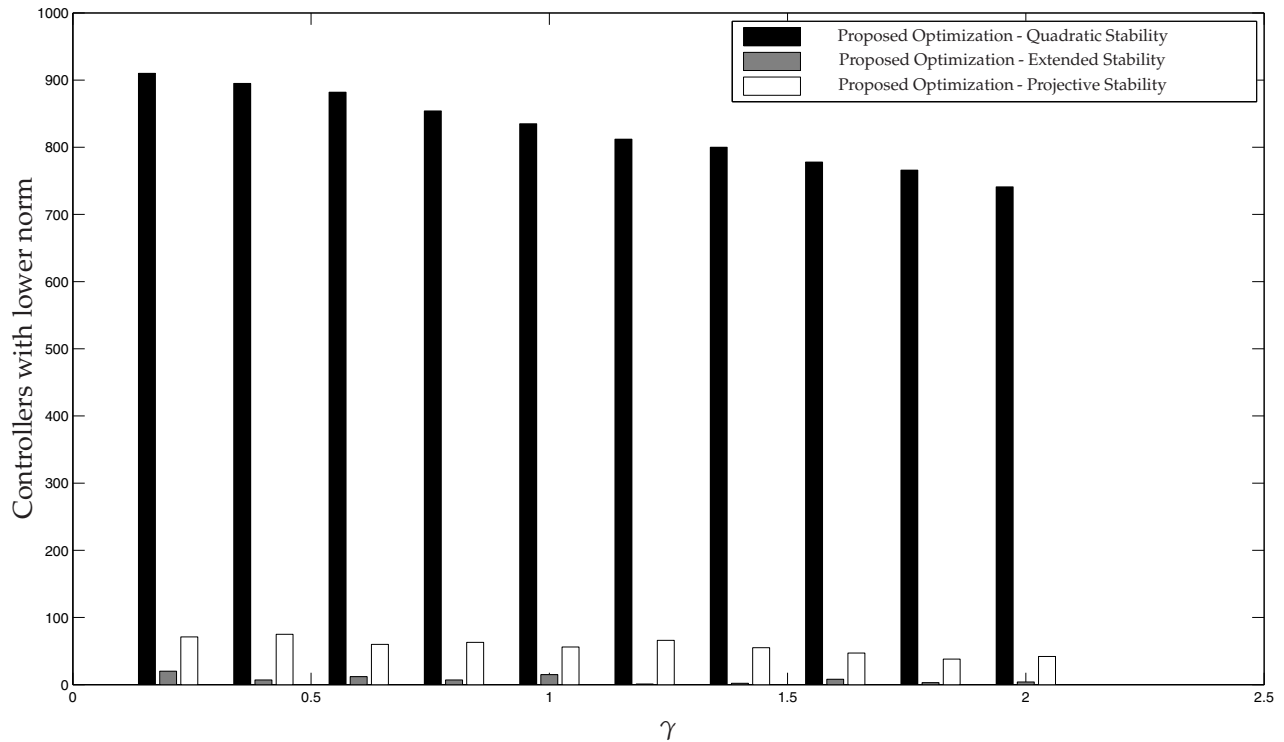


Fig. 10. Number of controllers with lower norm for 1000 uncertain polytopic systems of fourth-order randomly generated. All these methods are proposed in this work.

Both figures 9 and 10 show that the proposed optimization method using quadratic stability showed better results for the controller norm with the increase of  $\gamma$ , due to optimization this method no longer depend on the matrices that guarantee system stability as it can be seen in equation (22). In contrast, using the proposed optimizations with extended stability and projective stability, they still depend on the matrices that guarantee system stability as seen in equations (51) and (72) and this is the obstacle to finding better results for these methods.

## 9. Conclusions

At the 3-DOF helicopter practical application, the controllers designed with the proposed optimization showed lower values of the controller's norm designed by the existing optimization with quadratic stability, except the design for projective stability which had the worst value of the norm for this case, thus showing the advantage of the proposed method regarding implementation cost and required effort on the motors. These characteristics of optimality and robustness make our design methodology attractive from the standpoint of practical applications for systems subject to structural failure, guaranteeing robust stability and small oscillations in the occurrence of faults.

It is clear that the design of  $K$  via the optimization proposed here achieved better results than the existing optimizing  $K$  (Assunção et al., 2007c), using the LMI quadratic stability for second order polytopes with one uncertainty. The proposed optimization project continued to show better results even when the existing optimization has become totally infeasible for fourth order polytopes with two uncertainties.

By comparing the three optimal design methods proposed here (quadratic stability, extended stability, and projective stability) it can be concluded that the design using quadratic stability had a better performance for both analysis: 1000 second order polytopes with one uncertainty and for the 1000 fourth order polytopes with two uncertainties, showing so that the proposed optimization ensures best results when used with the quadratic stability.

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