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The Electromagnetic Field in Accelerated Frames

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1. Introduction

Maxwell's electromagnetic theory is more than a century old. It is a well established and understood theory. Usually the theory is presented in standard textbooks as a field theory in flat space-time (1; 2). The establishment of the theory in curved space-time (3) requires the understanding of how exactly the Faraday tensor couples with the gravitational field, and presently this is an open issue. In the ordinary description of electrodynamics in flat space-time one almost always assumes that the sources and fields are established in an inertial reference frame. Very few investigations (4) attempt to extend the analysis to accelerated frames. Such extension is mandatory because most frames in nature are, in one or another way, accelerated.

Until recently the attempts to describe the electromagnetic field in an accelerated frame consisted in performing a coordinate transformation of the Faraday tensor defined in an inertial frame in flat space-time. For this purpose one considers a coordinate transformation from the flat space-time cartesian coordinates to coordinates that describe a hyperbola in Rindler space, in case of uniform acceleration. This procedure is not satisfactory for two reasons. First, a coordinate transformations is not a frame transformation. A coordinate transformation is carried out on vectors and tensors on a manifold, and they just express the fact that (i) a point on the manifold may be labelled by different coordinates in different charts, and that (ii) one can work with any set of coordinates. On the other hand, a frame transformation is a Lorentz tranformation, it satisfies the properties of the Lorentz group and is carried out in the tangent space of the manifold.

The second reason is that by considering an accelerated frame as a frame obtained by a coordinate transformation, one cannot provide satisfactory answers to situations that are eventually understood as paradoxes, because the inertial and "accelerated" fields are described in different coordinates. One of these paradoxes is the following: are the two situations, (i) an accelerated charge in an inertial frame, and (ii) a charge at rest in an inertial frame described from the perspective of an accelerated frame, physically equivalent?

The procedure to be considered here consists, first, in assuming that the Faraday tensor and Maxwell's equations are abstract tensor quantities in space-time. Then we make use of tetrad fields to project the electromagnetic field either on an inertial or on a non-inertial frame, in the same coordinate system, in flat space-time. Tetrad fields constitute a set of four orthonormal vectors, that are adapted to observers that follow arbitrary paths in space-time. They constitute the local frame of these observers. Since the fields in the inertial frame and in

the accelerated frame are defined in the same coordinate system, they can be compared with each other unambiguously.

Given any set of tetrad fields, we may construct the acceleration tensor, as we will show. This tensor determines the inertial (i.e., non-gravitational) accelerations that act on a given observer. For instance, a stationary observer in space-time undergoes inertial forces, otherwise it would follow a geodesic motion determined by the gravitational field. A given frame (or a given tetrad field) may be characterized by the inertial accelerations.

In this chapter we will obtain the general form of Maxwell's equations that hold in inertial or noninertial frames. The formalism ensures that the procedure for projecting electromagnetic fields in noninertial frames is mathematically and physically consistent, and allows the investigation of several paradoxes. It is possible to conclude, for instance, that the radiation of an accelerated charged particle in an inertial frame is different from the radiation of the same charged particle measured in a frame that is co-accelerated (equally accelerated) with the particle. Consequently, the accelerated motion in space-time is not relative, and the radiation of an accelerated charged particle is an absolute feature of the theory (5).

We will study in detail the description of plane and spherical electromagnetic waves in linearly accelerated frames in Minkowski space-time. We will show that (i) the amplitude, (ii) the frequency and the wave vector of the plane wave, and (iii) the Poynting vector in the accelerated frame vary (decrease) with time, while the light speed remains constant.

Notation:

1. Space-time indices $\mu, \nu, \dots$ and Lorentz (SO(3,1)) indices $a, b, \dots$ run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i, \quad a = (0), (i)$.

2. The space-time is flat, and therefore the metric tensor in cartesian coordinates is given by $g_{\mu\nu} = (-1, +1, +1, +1)$

3. The tetrad field is represented by $e^a{}_\mu$. The flat, tangent space Minkowski space-time metric tensor raises and lowers tetrad indices and is fixed by $\eta_{ab} = e_{a\mu}e_{b\nu}g^{\mu\nu} = (-1, +1, +1, +1)$.

4. The frame components are given by the inverse tetrads $e_a{}^\mu$, although we may as well refer to $\{e^a{}_\mu\}$ as the frame. The determinant of the tetrad field is represented by $e = \det(e^a{}_\mu)$

2. Reference frames in space-time

The electromagnetic field is described by the Faraday tensor $F^{\mu\nu}$. In the present analysis we will consider that $\{F^{\mu\nu}\}$ are just *tensor components* in the flat Minkowski space-time described by *arbitrary* coordinates x^μ. The projection of $F^{\mu\nu}$ on inertial or noninertial frames yields the electric and magnetic fields E_x, E_y, E_z, B_x, B_y and B_z, which are the *frame components* of $\{F^{\mu\nu}\}$. The projection is carried out with the help of tetrad fields $e^a{}_\mu$. For instance, $E_x = -cF^{(0)(1)}$, where c is the speed of light and $F^{(0)(1)} = e^{(0)}{}_\mu e^{(1)}{}_\nu F^{\mu\nu}$. The study of the kinematical properties of tetrad fields is mandatory for the characterization of reference frames.

Tetrad fields constitute a set of four orthonormal vectors in space-time, $\{e^{(0)}{}_\mu, e^{(1)}{}_\mu, e^{(2)}{}_\mu, e^{(3)}{}_\mu\}$, that establish the local reference frame of an observer that moves along a trajectory C, represented by functions $x^\mu(s)$ (6–8) (s is the proper time of the

observer). The tetrad field yields the space-time metric tensor $g_{\mu\nu}$ by means of the relation $e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu}$, and $e^{(0)}{}_\mu$ and $e^{(i)}{}_\mu$ are timelike and spacelike vectors, respectively.

We identify the $a = (0)$ component of $e_a{}^\mu$ with the observer's velocity u^μ along the trajectory C, i.e., $e_{(0)}{}^\mu = u^\mu/c = dx^\mu/(cd\tau)$. The observer's acceleration a^μ is given by the absolute derivative of u^μ along C,

$$a^\mu = \frac{Du^\mu}{d\tau} = c\,\frac{De_{(0)}{}^\mu}{d\tau}. \tag{1}$$

The absolute derivative is constructed with the help of the Christoffel symbols. Thus $e_{(0)}{}^\mu$ and its absolute derivative determine the velocity and acceleration along the worldline of an observer adapted to the frame. The set of tetrad fields for which $e_{(0)}{}^\mu$ describes a congruence of timelike curves is adapted to a class of observers characterized by the velocity field $u^\mu = c\,e_{(0)}{}^\mu$ and by the acceleration a^μ. If $e^a{}_\mu = \delta^a_\mu$ everywhere in space-time, then $e^a{}_\mu$ is adapted to static observers, and $a^\mu = 0$.

We may consider not only the acceleration of observers along trajectories whose tangent vectors are given by $e_{(0)}{}^\mu$, but the acceleration of the whole frame along C. The acceleration of the frame is determined by the absolute derivative of $e_a{}^\mu$ along the path $x^\mu(\tau)$. Thus, assuming that the observer carries a frame, the acceleration of the latter along the path is given by (4; 9),

$$\frac{De_a{}^\mu}{d\tau} = \phi_a{}^b e_b{}^\mu, \tag{2}$$

where ϕ_{ab} is the antisymmetric acceleration tensor of the frame ($\phi_{ab} = -\phi_{ba}$). According to Refs. (4; 9), in analogy with the Faraday tensor we can identify $\phi_{ab} \equiv (\vec{a}/c, \vec{\Omega})$, where $\vec{a}$ is the translational acceleration ($\phi_{(0)(i)} = a_{(i)}/c$) and $\vec{\Omega}$ is the frequency of rotation ($\phi_{(i)(j)} = \epsilon_{(i)(j)(k)}\Omega^{(k)}$) of the spatial frame with respect to a nonrotating (Fermi-Walker transported (6; 8)) frame. It follows from Eq. (2) that

$$\phi_a{}^b = e^b{}_\mu \frac{De_a{}^\mu}{d\tau}. \tag{3}$$

Therefore given any set of tetrad fields for an arbitrary space-time, its geometrical interpretation may be obtained by suitably interpreting the velocity field $u^\mu = e_{(0)}{}^\mu$ and the acceleration tensor ϕ_{ab}.

Using the definiton of the absolute derivative, we can write Eq. (3) as

$$\begin{aligned}
\phi_a{}^b &= e^b{}_\mu \left(\frac{de_a{}^\mu}{d\tau} + \Gamma^\mu{}_{\lambda\sigma} \frac{dx^\lambda}{d\tau} e_a{}^\sigma \right) \\
&= e^b{}_\mu \left(\frac{dx^\lambda}{d\tau} \frac{\partial e_a{}^\mu}{\partial x^\lambda} + \Gamma^\mu{}_{\lambda\sigma} \frac{dx^\lambda}{d\tau} e_a{}^\sigma \right) \\
&= e^b{}_\mu u^\lambda \left(\frac{\partial e_a{}^\mu}{\partial x^\lambda} + \Gamma^\mu{}_{\lambda\sigma} e_a{}^\sigma \right) = e^b{}_\mu u^\lambda \nabla_\lambda e_a{}^\mu.
\end{aligned} \tag{4}$$

Following Ref. (7), we take into account the orthogonality of the tetrads and write Eq. (4) as $\phi_a{}^b = -u^\lambda e_a{}^\mu \nabla_\lambda e^b{}_\mu$, where $\nabla_\lambda e^b{}_\mu = \partial_\lambda e^b{}_\mu - \Gamma^\sigma{}_{\lambda\mu} e^b{}_\sigma$. Next we consider the identity

$\partial_\lambda e^b{}_\mu - \Gamma^\sigma{}_{\lambda\mu} e^b{}_\sigma + {}^0\omega_\lambda{}^b{}_c e^c{}_\mu = 0$, where ${}^0\omega_\lambda{}^b{}_c$ is the Levi-Civita spin connection given by Eq. (21) below, and express $\phi_a{}^b$ according to

$$\phi_a{}^b = u^\lambda e_a{}^\mu \left({}^0\omega_\lambda{}^b{}_c e^c{}_\mu\right) = c\, e_{(0)}{}^\mu\left({}^0\omega_\mu{}^b{}_a\right). \tag{5}$$

Finally we make use of the identity ${}^0\omega_\mu{}^a{}_b = -K_\mu{}^a{}_b$, where $K_\mu{}^a{}_b$ is the contortion tensor defined by

$$K_{\mu ab} = \frac{1}{2} e_a{}^\lambda e_b{}^\nu (T_{\lambda\mu\nu} + T_{\nu\lambda\mu} + T_{\mu\lambda\nu}), \tag{6}$$

where

$$T^\lambda{}_{\mu\nu} = e_a{}^\lambda T^a{}_{\mu\nu} = e_a{}^\lambda \left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu\right), \tag{7}$$

is the object of anholonomy. Note that $T^\lambda{}_{\mu\nu}$ is also the torsion tensor of the Weitzenböck space-time. After simple manipulations we arrive at

$$\phi_{ab} = \frac{c}{2}\left[T_{(0)ab} + T_{a(0)b} - T_{b(0)a}\right], \tag{8}$$

where $T_{abc} = e_b{}^\mu e_c{}^\nu T_{a\mu\nu}$. The expression above is not covariant under local Lorentz (SO(3,1) or frame) transformations, but is invariant under coordinate transformations. The noncovariance under local Lorentz transformations allows us to take the values of ϕ_{ab} to characterize the frame.

In order to measure field quantities with magnitude and direction (velocity, acceleration, etc.), an observer must project these quantities on the frame carried by the observer. The projection of a vector V^μ on a particular frame is determined by

$$V^a(x) = e^a{}_\mu(x)\, V^\mu(x), \tag{9}$$

and the projection of a tensor $T^{\mu\nu}$ is

$$T^{ab}(x) = e^a{}_\mu(x)\, e^b{}_\nu(x)\, T^{\mu\nu}(x). \tag{10}$$

Note that the projections are carried out in the same coordinate system.

We consider now an accelerated observer that follows a worldline $\bar{x}^\mu(\tau)$ in Minkowski space-time and carries a tetrad $e^a{}_\mu$, such that $e_{(0)}{}^\mu = u^\mu/c$ and $De_a{}^\mu/d\tau = \phi_a{}^b e_b{}^\mu$. At each instant τ of proper time along the worldline there are spacelike geodesics orthogonal to the worldline that form a local spacelike hypersurface. The observer can assign local coordinates $x^a = \{x^{(0)}, x^{(i)}\} = \{c\tau, \vec{x}'\}$ to an event, which is also described by Cartesian coordinates $x^\mu = \{ct, \vec{x}\}$ belonging to this hypersurface, where

$$x^{(0)} = c\tau, \qquad x^{(i)} = [x^\mu - \bar{x}^\mu]e^{(i)}{}_\mu. \tag{11}$$

The inverse transformation reads

$$x^\mu = \bar{x}^\mu + e_{(i)}{}^\mu x^{(i)}. \tag{12}$$

If we differentiate both sides of this equation over the worldline, we find

$$dx^\mu = \left(\frac{1}{c} \frac{d\bar{x}^\mu}{d\tau} + \frac{1}{c} \frac{de_{(i)}{}^\mu}{d\tau} x^{(i)} \right) dx^{(0)} + e_{(i)}{}^\mu dx^{(i)}$$

$$= \left(e_{(0)}{}^\mu + \frac{1}{c} \phi_{(i)}{}^a e_a{}^\mu x^{(i)} \right) dx^{(0)} + e_{(i)}{}^\mu dx^{(i)}. \tag{13}$$

Substituting Eq. (13) into the line element $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, we obtain the metric in the local coordinate system of an accelerated observer,

$$ds^2 = \left[-\left(1 + \frac{\vec{a} \cdot \vec{x'}}{c^2} \right)^2 + \frac{1}{c^2} \left(\vec{\Omega} \times \vec{x'} \right)^2 \right] (dx^{(0)})^2 + \left(\frac{2}{c} \vec{\Omega} \times \vec{x'} \right) dx^{(0)} dx^{(i)}$$

$$+ \eta_{(i)(j)} dx^{(i)} dx^{(j)}, \tag{14}$$

where we used $\phi_{(i)}{}^{(0)} x^{(i)} = (\vec{a} \cdot \vec{x'})/c$ and $\phi_{(j)}{}^{(i)} x^{(j)} = \left(\vec{\Omega} \times \vec{x'} \right)^{(i)}$.

We see from Eq. (14) that $\eta_{(0)(0)} \cong -1$ only in the regions of space-time where

$$|\vec{x'}| \ll \frac{c^2}{|\vec{a}|}, \qquad \text{and} \qquad |\vec{x'}| \ll \frac{c}{|\vec{\Omega}|}. \tag{15}$$

Furthermore, some $c\tau =$ constant surfaces will intersect each other if we extend the spatial local coordinates far away from the observer's worldline, which is not an admissible situation. Since we cannot assign two sets of coordinates for the same event, the local spatial coordinates have a limit of validity. In fact, the local coordinate system of Eq. (11) is valid only in those regions in the neighborhood of the observer's wordline in which Eqs. (15) hold. We call $c^2/|\vec{a}|$ the translational acceleration length and $c/|\vec{\Omega}|$ the rotational acceleration length. On the Earth's surface, for example, we have ($|\vec{a}| = 9,8\,\text{m/s}^2$, $|\vec{\Omega}| = \Omega_\oplus$)

$$\frac{c^2}{|\vec{a}|} = 9.46 \cdot 10^{15}\,\text{m} \approx 1\,\text{ly} \qquad \text{and} \qquad \frac{c}{|\vec{\Omega}|} = 4.125 \cdot 10^{12}\,\text{m} \approx 27.5\,\text{AU}. \tag{16}$$

Hence we can use the local coordinates x^a with confidence in most experimental situations in a laboratory on the Earth, where $|\vec{x'}|$ is negligible comparing to the acceleration lengths.

3. The formulation of Maxwell's theory in moving frames

The vector potential A^μ, the Faraday tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the four-vector current J^μ are vector and tensor components in space-time. Space-time indices are raised and lowered by means of the flat space-time metric tensor $g_{\mu\nu} = (-1, +1, +1, +1)$. On a particular frame the electromagnetic quantities are projected and measured according to $A^a(x) = e^a{}_\mu(x) A^\mu(x)$ and $F^{ab}(x) = e^a{}_\mu(x) e^a{}_\nu(x) F^{\mu\nu}(x)$.

An inertial frame is characterized by the vanishing of the acceleration tensor ϕ_{ab}. A realization of an inertial frame in Minkowski space-time is given by $e^a{}_\mu(t, x, y, z) = \delta^a_\mu$. It is easy to verify

that this frame satisfies $\phi_{ab} = 0$. More generally, all tetrad fields that are function of space-time *independent* parameters (boost and rotation parameters) determine inertial frames. Suppose that A^a are componentes of the vector potential in an inertial frame, i.e., $A^a = (e^a{}_\mu)_{in} A^\mu = \delta^a_\mu A^\mu$. The components of A^a in a noninertial frame are obtained by means of a local Lorentz transformation,

$$\tilde{A}^a(x) = \Lambda^a{}_b(x) A^b(x), \tag{17}$$

where $\Lambda^a{}_b(x)$ are space-time dependent matrices that satisfy

$$\Lambda^a{}_c(x) \Lambda^b{}_d(x) \eta_{ab} = \eta_{cd}. \tag{18}$$

In terms of covariant indices we have $\tilde{A}_a(x) = \Lambda_a{}^b(x) A_b(x)$. An alternative but completely equivalent way of obtaining the field components $\tilde{A}_a(x)$ consists in performing a frame transformation by means of a suitable noninertial frame $e^a{}_\mu$, namely, in projecting A^μ on the noninertial frame,

$$\tilde{A}^a(x) = e^a{}_\mu(x) A^\mu(x). \tag{19}$$

Of course we have $\Lambda^a{}_b \delta^b_\mu = \Lambda^a{}_b (e^b{}_\mu)_{in} = e^a{}_\mu$.

The covariant derivative of A_a is defined by

$$D_a A_b = e_a{}^\mu D_\mu A_b$$

$$= e_a{}^\mu (\partial_\mu A_b - {}^0\omega_\mu{}^c{}_b A_c), \tag{20}$$

where

$${}^0\omega_{\mu a b} = -\frac{1}{2} e^c{}_\mu (\Omega_{abc} - \Omega_{bac} - \Omega_{cab}),$$

$$\Omega_{abc} = e_{a\nu}(e_b{}^\mu \partial_\mu e_c{}^\nu - e_c{}^\mu \partial_\mu e_b{}^\nu), \tag{21}$$

is the metric-compatible Levi-Civita connection considered in Eq. (5). Note that we are considering the flat space-time, and yet this connection may be nonvanishing. In particular, for noninertial frames it is nonvanishing. The Weitzenböck torsion tensor $T^a{}_{\mu\nu}$ is also nonvanishing. However, the curvature tensor constructed out of ${}^0\omega_{\mu a b}$ vanishes identically: $R^a{}_{b\mu\nu}({}^0\omega) \equiv 0$.

Under a local Lorentz transformation the spin connection transforms as

$$\widetilde{{}^0\omega}_\mu{}^a{}_b = \Lambda^a{}_c({}^0\omega_\mu{}^c{}_d)\Lambda_b{}^d + \Lambda^a{}_c\partial_\mu\Lambda_b{}^c. \tag{22}$$

It follows from eqs. (17), (21) and (22) that under a local Lorentz transformation we have

$$\tilde{D}_a\tilde{A}_b = \Lambda_a{}^c(x)\Lambda_b{}^d(x)\, D_c A_d. \tag{23}$$

The natural definition of the Faraday tensor in a noninertial frame is

$$F_{ab} = D_a A_b - D_b A_a. \tag{24}$$

In view of eq. (24) we find that the tensors F_{ab} and $\tilde{F}_{ab}$ in two arbitrary frames are related by

$$\tilde{F}_{ab} = \Lambda_a{}^c(x)\Lambda_b{}^d(x)F_{cd}. \tag{25}$$

The Faraday tensor defined by eq. (24) is related to the standard expression defined in inertial frames. By substituting (20) in (24) we find

$$F_{ab} = e_a{}^\mu(\partial_\mu A_b - {}^0\omega_\mu{}^m{}_b A_m) - e_b{}^\mu(\partial_\mu A_a - {}^0\omega_\mu{}^m{}_a A_m)$$

$$= e_a{}^\mu(\partial_\mu A_b) - e_b{}^\mu(\partial_\mu A_a) + ({}^0\omega_{abm} - {}^0\omega_{bam})A^m. \tag{26}$$

We make use of the *identity*

$${}^0\omega_{abm} - {}^0\omega_{bam} = T_{mab}, \tag{27}$$

where T_{mab} is given by eq. (7), and write

$$F_{ab} = e_a{}^\mu e_b{}^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) + T^m{}_{ab}A_m$$

$$+ e_a{}^\mu(\partial_\mu e_b{}^\nu)A_\nu - e_b{}^\mu(\partial_\mu e_a{}^\nu)A_\nu. \tag{28}$$

In view of the orthogonality of the tetrad fields we have

$$\partial_\mu e_b{}^\nu = -e_b{}^\lambda(\partial_\mu e^c{}_\lambda)e_c{}^\nu. \tag{29}$$

With the help of the equation above we find that the last two terms of eq. (28) may be rewritten as

$$e_a{}^\mu(\partial_\mu e_b{}^\nu)A_\nu - e_b{}^\mu(\partial_\mu e_a{}^\nu)A_\nu = -T^m{}_{ab}A_m. \tag{30}$$

Therefore the last three terms of (28) cancel each other and finally we have

$$F_{ab} = e_a{}^\mu e_b{}^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{31}$$

The equation above shows that given the abstract, tensorial expression of the Faraday tensor we can simply project it on any moving frame in Minkowski space-time. This is exactly the procedure adopted by Mashhoon (4) in the investigation of electrodynamics in accelerated frames. Mashhoon is interested in developing the non-local formulation of electrodynamics. However, if we restrict attention to the evaluation of total quantities, such as the integration of the Poynting vector and the total radiated power (and not to pointwise measurements), then the standard formulation suffices to arrive at qualitative conclusions.

We may obtain Maxwell's equations with sources from an action integral determined by the Lagrangian density

$$L = -\frac{1}{4}e\,F^{ab}F_{ab} - \mu_0\,e\,A_b J^b, \tag{32}$$

where $e = \det(e^a{}_\mu)$, $J^b = e^b{}_\mu J^\mu$ and μ_0 is the magnetic permeability constant. Although in flat space-time we have $e = 1$, we keep e in the expressions below because it allows a straightforward inclusion of the gravitational field. Note that in view of eq. (31) we have

$$F^{ab}F_{ab} = F^{\mu\nu}F_{\mu\nu}. \tag{33}$$

Therefore L is frame independent, besides being invariant under coordinate transformations. The field equations derived from L are

$$\partial_\mu(e\,F^{\mu b}) + e\,F^{\mu c}\,(^0\omega_\mu{}^b{}_c) = \mu_0\,e\,J^b\,, \tag{34}$$

or

$$e_b{}^\nu[\partial_\mu(e\,F^{\mu b}) + e\,F^{\mu c}\,(^0\omega_\mu{}^b{}_c)] = \mu_0\,e\,J^\nu\,, \tag{35}$$

where $F^{\mu c} = e_b{}^\mu F^{bc}$. In view of eq. (33) it is clear that the equations above are equivalent to the standard form of Maxwell's equations in flat space-time.

The second set of Maxwell's equations is obtained by working out the quantity $D_a F_{bc} + D_b F_{ca} + D_c F_{ab}$, where the covariant derivative of $D_a F_{bc}$ is defined by

$$\begin{aligned}
D_a F_{bc} &= e_a{}^\mu D_\mu F_{bc} \\
&= e_a{}^\mu(\partial_\mu F_{bc} - {}^0\omega_\mu{}^m{}_b F_{mc} - {}^0\omega_\mu{}^m{}_c F_{bm})\,.
\end{aligned} \tag{36}$$

Taking into account relations (27) and (29) we find that the source free Maxwell's equations in an arbitrary moving frame are given by

$$D_a F_{bc} + D_b F_{ca} + D_c F_{ab} = e_a{}^\mu e_b{}^\nu e_c{}^\lambda(\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu}) = 0\,, \tag{37}$$

in agreement with the standard description.

We refer the reader to Ref. (5), where we consider an accelerated frame with velocity $v(t)$ with respect to an inertial frame, and describe Gauss law in the accelerated frame for the situations (i) in which the source is at rest in the inertial frame, and (ii) in which the source is at rest in the accelerated frame.

4. Plane electromagnetic waves in a linearly accelerated frame

In this section we consider an observer in Minkowski space-time that is uniformly accelerated in the positive x direction. The wordline and velocity of the observer in terms of its proper time τ are

$$\bar{x}^\mu = \left\{\frac{c^2}{a}\sinh\left(\frac{a\tau}{c}\right), \frac{c^2}{a}\left[\cosh\left(\frac{a\tau}{c}\right) - 1\right], 0, 0\right\}, \tag{38}$$

and

$$u^\mu = \frac{d\bar{x}^\mu}{d\tau} = \left\{c\cosh\left(\frac{a\tau}{c}\right), c\sinh\left(\frac{a\tau}{c}\right), 0, 0\right\}, \tag{39}$$

respectively.

A simple form of tetrad fields adapted to the observer with velocity u^μ, i.e., for which $e_{(0)}{}^\mu = u^\mu/c$ and $e^a{}_\mu e_{a\nu} = \eta_{\mu\nu}$, is given by

$$e^a{}_\mu = \begin{pmatrix} \cosh(a\tau/c) & -\sinh(a\tau/c) & 0 & 0 \\ -\sinh(a\tau/c) & \cosh(a\tau/c) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{40}$$

If we substitute the tetrad fields and the inverses into Eq. (4), we see that the only nonvanishing component of ϕ_{ab} is

$$\phi_{(0)}{}^{(1)} = \frac{a}{c}. \tag{41}$$

The frame described by Eq. (40) is moving with uniform acceleration a in the positive x direction, and its axes are oriented along the global Cartesian frame. In view of Eqs. (12), (38) and (40), it follows that

$$t = \frac{c}{a} \left(1 + \frac{ax'}{c^2}\right) \sinh\left(\frac{a\tau}{c}\right),$$

$$x = \frac{c^2}{a} \left(1 + \frac{ax'}{c^2}\right) \cosh\left(\frac{a\tau}{c}\right) - \frac{c^2}{a},$$

$$y = y',$$

$$z = z'. \tag{42}$$

We note that Eq. (39) can be given alternatively in terms of the time coordinate t of the inertial frame by

$$u^\mu(t) = \{c\gamma(t), c\beta(t)\gamma(t), 0, 0\}, \tag{43}$$

where

$$\gamma(t) = \sqrt{1 + a^2 t^2/c^2}, \qquad \text{and} \qquad \beta(t)\gamma(t) = at/c.$$

In terms of the coordinates (t, x, y, z) adapted to the inertial frame, the Faraday tensor for a plane electromagnetic wave that propagates in the positive x direction reads

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & -E_y/c & 0 \\ 0 & 0 & -B_z & 0 \\ E_y/c & B_z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{44}$$

where

$$E_y(t, \vec{x}) = E_0 \cos(kx - \omega t), \tag{45}$$

$$B_z(t, \vec{x}) = \frac{E_0}{c} \cos(kx - \omega t). \tag{46}$$

In these expressions k is the wave number and ω is the frequency of the wave, which are related by $k = |\vec{k}| = \omega/c$. The speed of propagation of the electromagnetic wave is

$$v_p = \frac{\omega}{|\vec{k}|} = c. \tag{47}$$

The expression of the electromagnetic field in the inertial frame is formally obtained out of Eqs. (45) and (46) by means of the tetrad field $e^a{}_\mu = \delta^a_\mu$. However we will consider that (45) and (46) do represent the fields in the inertial frame.

In view of the expressions above we see that the only nonzero component of the Poynting vector

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}, \tag{48}$$

is given by

$$S_x = \frac{(E_0)^2}{\mu_0 c} \cos^2 (kx - \omega t), \tag{49}$$

where μ_0 is the magnetic permeability constant. Thus the energy flux of the electromagnetic wave points in the same direction of the wave propagation.

In order to obtain the electric and magnetic field components of the electromagnetic wave in the uniformly accelerated frame, we insert Eqs. (44) and (40) into $F^{ab} = e^a{}_\mu e^b{}_\nu F^{\mu\nu}$. We obtain

$$E_{(y)} = \cosh\left(\frac{a\tau}{c}\right) E_y - c \sinh\left(\frac{a\tau}{c}\right) B_z, \tag{50}$$

$$B_{(z)} = -\frac{1}{c} \sinh\left(\frac{a\tau}{c}\right) E_y + \cosh\left(\frac{a\tau}{c}\right) B_z, \tag{51}$$

where E_y and B_z are given by (45) and (46), respectively.

In Eqs. (50) and (51) E_y and B_z are expressed in terms of the coordinates (t, x). In order to present the electric and magnetic fields in terms of the coordinates $(\tau, \vec{x'})$ of the accelerated frame we make use of Eq. (42). We arrive at

$$E_{(y)}(\tau, \vec{x'}) = E_0 \, e^{-a\tau/c} \cos\left[k\left(e^{-a\tau/c}\right) x' - \frac{\omega c}{a}\left(1 - e^{-a\tau/c}\right)\right], \tag{52}$$

$$B_{(z)}(\tau, \vec{x'}) = \frac{E_0}{c} \, e^{-a\tau/c} \cos\left[k\left(e^{-a\tau/c}\right) x' - \frac{\omega c}{a}\left(1 - e^{-a\tau/c}\right)\right], \tag{53}$$

where we used

$$e^{-a\tau/c} = \cosh(a\tau/c) - \sinh(a\tau/c).$$

The only nonzero component of the Poynting vector is

$$S_{(x)} = \frac{(E_0)^2}{\mu_0 c} \, e^{-2a\tau/c} \cos^2\left[k\left(e^{-a\tau/c}\right) x' - \frac{\omega c}{a}\left(1 - e^{-a\tau/c}\right)\right], \tag{54}$$

We see that the density of energy flux decreases in time by a factor $e^{-2a\tau/c}$ in a frame that is uniformly accelerated in the same direction of the propagation of the electromagnetic wave.

The amplitudes in Eqs. (52) and (53) may be written as

$$E_{(0)} = E_0 \, e^{-a\tau/c}, \tag{55}$$

$$B_{(0)} = \frac{E_0}{c} \, e^{-a\tau/c} = \frac{E_{(0)}}{c}. \tag{56}$$

The identification of the wave number and of the frequency of the wave in the accelerated frame is made by means of a projection of the wave vector $k_\mu = (-\omega/c, k, 0, 0)$ from the inertial to the accelerated frame, according to $k^a = e^a{}_\mu k^\mu$. We recall that this procedure is equivalent to performing a local Lorentz transformation where the coefficients $\Lambda^a{}_b$ of the transformation satisfy $e^a{}_\mu = \Lambda^a{}_b \, (e^b{}_\mu)_{in} = \Lambda^a{}_b \, \delta^b_\mu$. Thus we have

$$k' = k \, e^{-a\tau/c}, \tag{57}$$

$$\omega' = \omega\, e^{-a\tau/c}. \tag{58}$$

We conclude that the amplitude, wave number and frequency of the electromagnetic wave decrease in proper time by a factor $e^{-a\tau/c}$ in a frame that is uniformly accelerated in the same direction of the wave propagation. We note that the observer will never reach the speed of light. Considering Eqs. (57) and (58) we see that the speed of propagation of the electromagnetic wave in the uniformly accelerated frame is

$$v'_p = \frac{\omega'}{k'} = \frac{\omega\, e^{-a\tau/c}}{k\, e^{-a\tau/c}} = c\,. \tag{59}$$

Therefore the speed of the electromagnetic wave is independent of the observer's acceleration.

5. Spherical waves in a radially accelerated frame

We will repeat the analysis carried out in the previous section and consider the measurement of spherical electromagnetic waves, produced in an inertial frame, in a radially accelerated frame. A spherical wave in an inertial frame may be characterized by the following expressions for the electric and magnetic fields,

$$\vec{E}(t,r,\theta,\phi) = E_0 \frac{\sin\theta}{r} \left[\cos\left(kr - \omega t\right) - \frac{1}{kr}\sin\left(kr - \omega t\right) \right] \hat{\phi}, \tag{60}$$

$$\vec{B}(t,r,\theta,\phi) = -\frac{E_0}{c} \frac{\sin\theta}{r} \left[\cos\left(kr - \omega t\right) - \frac{1}{kr}\sin\left(kr - \omega t\right) \right] \hat{\theta}, \tag{61}$$

where the unit vectors $\hat{\phi}$ and $\hat{\theta}$ are defined in terms of the cartesian unit vectors by

$$\hat{\phi} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}},$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}. \tag{62}$$

A set of tetrad fields in spherical coordinates, adapted to an observer that undergoes uniform acceleration in the radial direction, is given by

$$e^a{}_\mu(t,r,\theta,\phi) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}, \tag{63}$$

where

$$\gamma = \sqrt{1 + \frac{a^2 t^2}{c^2}}, \qquad \gamma\beta = \frac{at}{c}. \tag{64}$$

The inverse components of Eq. (63) are such that $e_{(0)}{}^\mu(t,r,\theta,\phi) = (\gamma, \beta\gamma\, 0, 0)$. Therefore the frame is accelerated along the radial direction.

We start with the Faraday tensor in cartesian coordinates,

$$F^{\mu\nu}(t,x,y,z) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{65}$$

The electric and magnetic field components in the expression above are obtained out of Eqs. (60), (61) and (62). We must consider the expression above in spherical coordinates. So we perform the coordinate transformation

$$F'^{\alpha\beta}(t,r,\theta,\phi) = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} F^{\mu\nu}(t,x,y,z). \tag{66}$$

After some algebra we obtain

$$
\begin{aligned}
F'^{01} &= 0, \\
F'^{02} &= 0, \\
F'^{03} &= -\frac{E_0}{c}\frac{1}{r^2}\left[\cos(kr - \omega t) - \frac{1}{kr}\sin(kr - \omega t)\right], \\
F'^{12} &= 0, \\
F'^{23} &= 0, \\
F'^{31} &= \frac{E_0}{c}\frac{1}{r^2}\left[\cos(kr - \omega t) - \frac{1}{kr}\sin(kr - \omega t)\right].
\end{aligned}
\tag{67}
$$

The quantities in Eq. (67) represent both the abstract tensor components of the Faraday tensor in spherical coordinates, and the components of the Faraday tensor in an inertial frame. Next we project these tensor components on the accelerated frame defined by Eq. (63). We arrive at

$$
\begin{aligned}
F'^{(0)(1)} &= 0, \\
F'^{(0)(2)} &= 0, \\
F'^{(0)(3)} &= -\frac{E_0}{c}(\gamma - \gamma\beta)\frac{\sin\theta}{r}\left[\cos(kr - \omega t) - \frac{1}{kr}\sin(kr - \omega t)\right], \\
F'^{(1)(2)} &= 0, \\
F'^{(2)(3)} &= 0, \\
F'^{(3)(1)} &= \frac{E_0}{c}(\gamma - \gamma\beta)\frac{\sin\theta}{r}\left[\cos(kr - \omega t) - \frac{1}{kr}\sin(kr - \omega t)\right].
\end{aligned}
\tag{68}
$$

Note that the factor $(\gamma - \gamma\beta)$ may be rewritten as

$$\gamma - \gamma\beta = \sqrt{\frac{1-\beta}{1+\beta}}. \tag{69}$$

In order to verify how Eqs. (60) and (61) are modified in the accelerated frame we just compare the structure of Eqs. (67) and (68), and indentify (60) and (61) in the latter expression. We

obtain

$$\vec{E}(t,r,\theta,\phi) = E_0(\gamma - \beta\gamma)\frac{\sin\theta}{r}\left[\cos(kr - \omega t) - \frac{1}{kr}\sin(kr - \omega t)\right]\hat{\phi}, \tag{70}$$

$$\vec{B}(t,r,\theta,\phi) = -\frac{E_0}{c}(\gamma - \beta\gamma)\frac{\sin\theta}{r}\left[\cos(kr - \omega t) - \frac{1}{kr}\sin(kr - \omega t)\right]\hat{\theta}, \tag{71}$$

in the inertial frame coordinates.

By comparing Eqs. (70) and (71) with (60) and (61) we see that the major qualitative difference between these expressions is the emergence, in the former pair of equations, of the time dependent Doppler factor $(\gamma - \beta\gamma)$ given by Eq. (69). If the accelerated frame is at the radial position (r, θ) at the instant t, then the measured amplitude of the wave in the accelerated frame will be smaller by a factor $(\gamma - \beta\gamma)$ than if the frame were at rest at the same position. Thus the amplitude of the spherical wave in the accelerated frame varies with time, and approaches zero in the limit $t \to \infty$, since in this limit $\beta \to 1$.

6. Final comments

The tetrad field and its interpretation as a frame adapted to arbitrary observers in space-time allow the formulation of electrodynamics in accelerated frames. The idea is to project the electromagnetic vectorial and tensorial quantities in any moving frame by means of the tetrad field. Specific issues regarding electromagnetic radiation were discussed in ref. (5).

Of course all the results derived from the procedure adopted in this chapter are valid as long as the very concept of tetrad field and its interpretation are also valid. The justification behind the usage of tetrad fields for this purpose is given by principle of locality (10). The idea is the following. A physical measurement is considered to be reliable if it is performed in an inertial reference frame. Normally it is admitted that the observer is standing in an inertial frame. Measurements in accelerated frame are, in general, not easily performed. When an electromagnetic field quantity is projected in a frame by means of the tetrad field, it is assumed that this tetrad field is, at each instant of time, physically equivalent (identical) to another frame that is inertial and momentarily co-moving with the accelerated frame. The worldline of the two frames, the accelerated and the inertial, coincide at that instant of time. To a certain extent, the hypothesis of locality, together with the concept of tetrad field, extends the principle of relativity, since it relates inertial and non-inertial frames.

An interesting consequence of the present analysis is the following. Let us suppose that an accelerated observer in the context of section 4 measures the frequencies ω_1' and ω_2' at the instants of proper time τ_1 and τ_2, respectively,

$$\omega_1' = \omega\, e^{-a\tau_1/c}, \qquad \omega_2' = \omega\, e^{-a\tau_2/c}, \tag{72}$$

according to eq. (58). By dividing the two frequencies of the electromagnetic waves we obtain

$$a = \frac{c}{\Delta\tau}\ln\left(\frac{\omega_1'}{\omega_2'}\right), \tag{73}$$

where $\Delta\tau = \tau_2 - \tau_1$. Therefore the accelerated observer may determine the value of its own acceleration provided the luminosity of the source is constant and the acceleration is

uniform. This formula may be useful in the evaluation of the acceleration of the solar system, for instance, with respect to the distant supernovas, provided it is verified that in the interval $\Delta\tau$ the luminosity of the supernova is not substantially changed. Of course the resulting value will provide just the order of magnitude of the acceleration of the expansion of the universe.

The final expressions of the electric and magnetic fields in the accelerated frames, Eqs. (52-53), and (70-71), for plane and spherical waves, respectively, are related to the expressions in the inertial frame by means of simple time dependent functions. The simplicity of the final expressions ensures that the present tehcnique is correct, and suggests that all manifestations of electrodynamics may be investigated in any moving frame.

7. References

[1] *The Classical Theory of Fields*, L. D. Landau and E. M. Lifshitz (Pergamon, Oxford, 1908).

[2] *Classical Electrodynamics*, J. D. Jackson (Wiley, New York, 1999).

[3] *Foundations of Classical Electrodynamics*, F. W. Hehl and Y. N. Obukhov (Birkhäuser, Boston, 2003).

[4] B. Mashhoon, Ann. Phys. (Berlin) 12, 586 (2003) [hep-th/0309124]; Phys. Rev. A 70 (2004) 062103 [hep-th/0407278]; Phys. Rev. A 79 (2009) 062111 [arXiv:0903.1315].

[5] J. W. Maluf and S. C. Ulhoa, Ann. Phys. (Berlin) 522, 766 (2010) [arXiv:1009.3968].

[6] F. H. Hehl, J. Lemke and E. W. Mielke, "Two Lectures on Fermions and Gravity", in *Geometry and Theoretical Physics*, edited by J. Debrus and A. C. Hirshfeld (Springer, Berlin Heidelberg, 1991).

[7] J. W. Maluf, F. F. Faria and S. C. Ulhoa, Class. Quantum Grav. 24 (2007) 2743 [arXiv:0704.0986].

[8] J. W. Maluf and F. F. Faria, Ann. Phys. (Berlin) 17 (2008) 326 [arXiv:0804.2502].

[9] B. Mashhoon, Ann. Phys. (Berlin) 11, (2002) 532 [gr-qc/0206082].

[10] B. Mashhoon, Phys. Lett. A 143, 176 (1990); 145, 147 (1990).
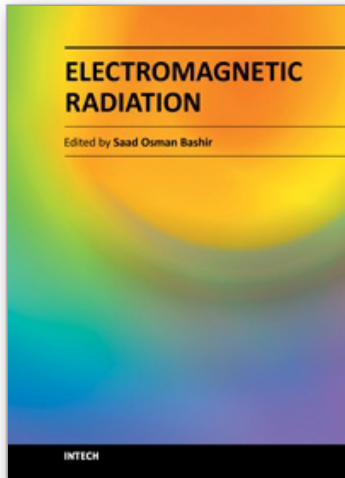
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