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Free Vibration Analysis of Centrifugally Stiffened Non Uniform Timoshenko Beams

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1. Introduction

Rotating beams – like structures are widely used in many engineering fields and are of great interest as they can be used to model blades of wind turbines, helicopter rotors, robotic manipulators, turbo-machinery and aircraft propellers. The governing differential equations of motion in free vibration of a non-uniform rotating Timoshenko beam, with general elastic restraints at the ends are solved using the differential quadrature method, (Bellman & Roth, 1986; Felix et al., 2008, 2009). The equations of motion are derived to include the effects of shear deformation, rotary inertia, hub radius, ends elastically restrained and non-uniform variation of the cross-sectional area of the beam. The presence of a centrifugal force due to the rotational motion is considered as Banerjee has developed, using Hamilton's principle to capture the centrifugal stiffening arising in fast rotating structures, (Banerjee, 2001). With the proposed model, a great number of different situations are admitted to be solved. Particular cases with classical restraints can be deduced for limiting values of the rigidities. Also step changes in cross-section are considered (Naguleswaran, 2004).

The natural vibration frequencies and mode shapes of rotating beams have been a topic of interest and have received considerable attention. A large number of researchers have studied the dynamic behavior of rotating uniform or tapered Euler-Bernoulli beams. (Yang el al., 2004; Özdemir & Kaya, 2006; Lin & Hsiao, 2001). Banerjee derived the dynamic stiffness matrix of a rotating Bernoulli-Euler beam using the Frobenius method of solution in power series and he includes the presence of an axial force at the outboard end of the beam in addition to the existence of the usual centrifugal (Banerjee, 2000).

Not so many studies have tackled the problem of rotating beams taking into account rotary inertia, shear deformation and their combined effects, hub radius and ends elastically restrained, (Bambill et al., 2010). In applications where the rotary inertia and the shear deformation effects are not significant, an analysis based on the Euler–Bernoulli beam theory can be used. However, Timoshenko theory allows describing the vibration of short beams, sandwich composite beams or high modes of a slender beam, (Rossi et al., 1991; Seon et al., 1999). (Banerjee et al., 2006) investigated the free bending vibration of rotating tapered Timoshenko beams by the dynamic stiffness method. (Ozgumus & Kaya, 2010) used the Differential Transform Method for free vibration analysis of a rotating, tapered Timoshenko beam.

The finite element method was used by (Hodges & Rutkowski, 1981). (Vinod et al., 2007) presented a study about spectral finite element formulation for a rotating beam subjected to small duration impact. (Gunda & Ganguli, 2008) developed a new beam finite element whose basis functions were obtained by the exact solution of the governing static homogenous differential equation of a stiff string, which resulted from an approximation in the rotating beam equation. (Singh et al., 2007) used the Genetic Programming to create an approximate model of rotating beams. (Gunda et al., 2007) introduced a low degree of freedom model for dynamic analysis of rotating tapered beams based on a numerically efficient superelement, developed using a combination of polynomials and Fourier series as shape functions. (Kumar & Ganguli, 2009) looked for rotating beams whose eigenpair, frequency and mode-shape, is the same as that of uniform non rotating beams for a particular mode. An interesting paper (Ganesh & Ganguli, 2011) presented physics based basis function for vibration analysis of high speed rotating beams using the finite element method. The basis function gave rise to shape functions which depend on position of the element in the beam, material, geometric properties and rotational speed of the beam.

The present study tries to provide not only solutions for practical engineering situations but they also may be useful as benchmark for comparing other numerical models. The proposed differential quadrature method, offers a useful and accurate procedure for the solution of linear and non linear partial differential equations. It was used by Bellman in the 1970's. He used this method to calculate the natural frequencies of transverse vibration of a rotating cantilever beam. (Bellman & Casti, 1971). Other authors have used the differential quadrature method and recognized it as an effective technique for solving this kind of problems, (Bert & Malik, 1996; Shu & Chen, 1999; Choi et al., 2000; Liu & Wu, 2001; Shu, 2000).

Numerical results are obtained for the natural frequencies of transverse vibration and the mode shapes of rotating beams considering the elastic restraints, with non uniform variation of the cross-sectional area. Some of those cases have also been solved using the finite element method, and the sets of results are in excellent agreement.

2. Theory

Figure 1 shows the rotating tapered beam considered in the present paper. The beam could have step jumps in cross section and rotates at speed $\overline{\eta}$. The $\overline{X}$-axis coincides with the centroidal axis of the beam, the $\overline{Y}$-axis is parallel with the axis of rotation and the $\overline{Z}$-axis lies in the plane of rotation. L is the length of the beam, L_k is the length of the segment k and L_d is the length of the last segment of the beam. The displacement in the $\overline{Y}$ direction is denoted as $\overline{w}$ and the section rotation is denoted as $\overline{\psi}$. Only displacements in the $\overline{X}-\overline{Y}$ plane are taken into account and the Coriolis effects are not considered.

The centrifugal force of a beam element at a distance $\overline{R}_k + \overline{x}_k$ from the axis of rotation can be expressed as

$$d\overline{F}_k = \overline{\eta}^2 (\overline{R}_k + \overline{x}_k)\, dm \tag{1}$$

where $dm = \rho\, A_k(\overline{x}_k)\, d\overline{x}_k$ is its mass, with ρ the mass density of material, and $A_k(\overline{x}_k)$, is the cross-sectional area at $\overline{x}_k$. Figure 2. The centrifugal force $\overline{N}_k(\overline{x}_k)$ generated by $\overline{\eta}$ is

$$d\overline{N}_k(\overline{x}_k) = \overline{\eta}^2 \rho\, (\overline{R}_k + \overline{x}_k)\, A_k(\overline{x}_k)\, d\overline{x}_k \tag{2}$$

The total axial force at the cross section located at $\bar{R}_k + \bar{x}_k$ is

$$\bar{N}_k(\bar{x}_k) = \bar{\eta}^2 \rho \int_{\bar{x}_k}^{L_k} (\bar{R}_k + \bar{x}_k) A_k(\bar{x}_k) d\bar{x}_k + \bar{F}_{k+1} = \bar{\eta}^2 \rho \left(\bar{R}_k \int_{\bar{x}_k}^{L_k} A_k(\bar{x}_k) d\bar{x}_k + \int_{\bar{x}_k}^{L_k} A_k(\bar{x}_k) \bar{x}_k \, d\bar{x}_k \right) + \bar{F}_{k+1} \quad (3)$$

$\bar{F}_{k+1}$ is the outboard force at the end of the segment k, due to the adjacent segments k+1 to d.

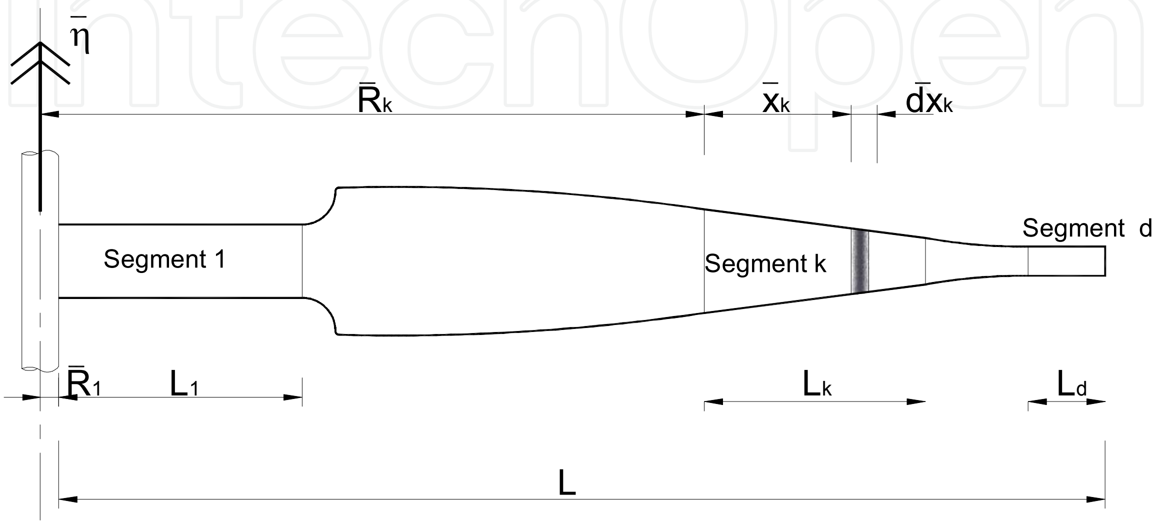


Fig. 1. Rotating beam model

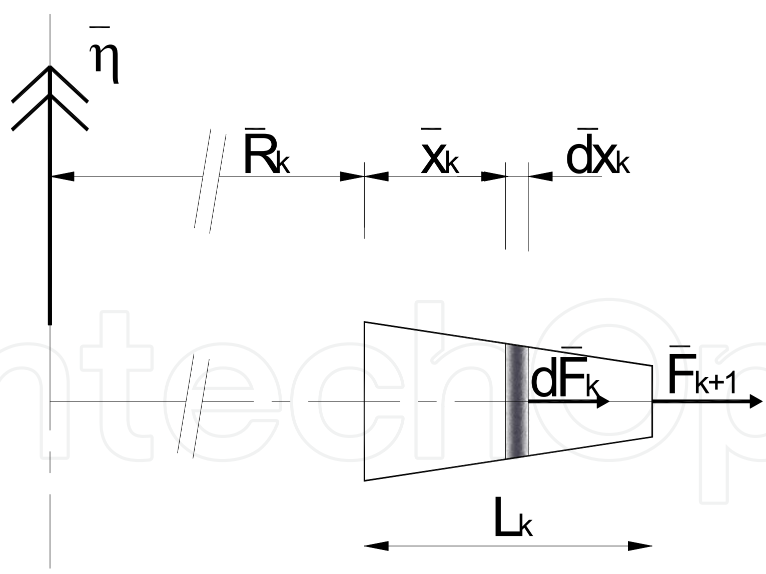


Fig. 2. Rotating beam segment k of length L_k

Finally, the tensile force can be written as

$$\bar{N}_k(\bar{x}_k) = \bar{\eta}^2 \rho \left(\bar{R}_k V_k(L_k) + \Phi_k(L_k) - \bar{R}_k V_k(\bar{x}_k) - \Phi_k(\bar{x}_k) \right) + \bar{F}_{k+1} \quad (4)$$

with

$$V_k(\bar{x}_k) = \int_0^{\bar{x}_k} A_k(\bar{x}_k)\, d\bar{x}_k \ ; \ \Phi_k(\bar{x}_k) = \int_0^{\bar{x}_k} A_k(\bar{x}_k)\bar{x}_k\, d\bar{x}_k \tag{5a,b}$$

The expressions for shear force and bending moment at an instant t in the rotating beam are

$$\bar{Q}_k^*(\bar{x}_k,t) = \bar{N}_k(\bar{x}_k)\frac{\partial \bar{w}_k(\bar{x}_k,t)}{\partial \bar{x}_k} + \kappa\, GA_k(\bar{x}_k)\left(\frac{\partial \bar{w}_k(\bar{x}_k,t)}{\partial \bar{x}_k} - \bar{\psi}_k(\bar{x}_k,t)\right) \tag{6}$$

$$\bar{M}_k^*(\bar{x}_k,t) = EI_k(\bar{x}_k)\frac{\partial \bar{\psi}_k(\bar{x}_k,t)}{\partial \bar{x}_k} \tag{7}$$

where $I_k(\bar{x}_k)$ is the second moment of area of the beam cross-section; t the time; $\bar{w}_k(\bar{x},t)$ the transverse displacement; $\bar{\psi}_k(\bar{x},t)$ the section rotation; E the Young's modulus; ν the Poisson's ratio; $G = E/2(1+\nu)$ the shear modulus and κ is the shear factor.

The governing differential equations of motion of a rotating Timoshenko beams (Banerjee, 2001) are:

$$\frac{\partial \bar{Q}_k^*(\bar{x}_k,t)}{\partial \bar{x}_k} = \rho A_k(\bar{x}_k)\frac{\partial^2 \bar{w}_k(\bar{x}_k,t)}{\partial t^2}$$
$$\bar{Q}_k^*(\bar{x}_k,t) - N_k(\bar{x}_k)\frac{\partial \bar{w}_k(\bar{x}_k,t)}{\partial \bar{x}_k} + \frac{\partial \bar{M}_k^*(\bar{x}_k,t)}{\partial \bar{x}_k} + \rho I_k(\bar{x}_k)\bar{\eta}^2\bar{\psi}_k(\bar{x}_k,t) = \rho I_k(\bar{x}_k)\frac{\partial^2 \bar{\psi}_k(\bar{x}_k,t)}{\partial t^2} \tag{8a,b}$$

Assuming simple harmonic oscillation

$$\bar{w}_k(\bar{x}_k,t) = \bar{W}_k(\bar{x}_k)e^{i\omega t} \quad ; \quad \bar{\psi}_k(\bar{x}_k,t) = \bar{\Psi}_k(\bar{x}_k)e^{i\omega t} \tag{9a,b}$$

where ω is the circular frequency in radian per second.
The bending moment and the shear force are expressed as

$$\bar{Q}_k^*(\bar{x}_k,t) = \bar{Q}_k(\bar{x}_k)e^{i\omega t}; \ \ \bar{M}_k^*(\bar{x}_k,t) = \bar{M}_k(\bar{x}_k)e^{i\omega t} \tag{10a,b}$$

where

$$\bar{Q}_k(\bar{x}_k) = \left(\bar{N}_k(\bar{x}_k) + \kappa\, GA_k(\bar{x}_k)\right)\frac{d\bar{W}_k(\bar{x}_k)}{d\bar{x}_k} - \kappa\, GA_k(\bar{x}_k)\bar{\Psi}_k(\bar{x}_k) \ ; \bar{M}_k(\bar{x}_k) = EI_k(\bar{x}_k)\frac{d\bar{\Psi}_k(\bar{x}_k)}{d\bar{x}_k} \tag{11a,b}$$

Substituting equations (9-10) into equations (8), the equations of motion for the free vibration of the segment k of the rotating beam result in:

$$-\frac{d\bar{Q}_k(\bar{x}_k)}{d\bar{x}_k} = \rho A_k(\bar{x}_k)\,\omega^2 \bar{W}_k(\bar{x}_k)$$
$$-\bar{Q}_k(\bar{x}_k) + \bar{N}_k(\bar{x}_k)\frac{d\bar{W}_k(\bar{x}_k)}{d\bar{x}_k} - \frac{d\bar{M}_k(\bar{x}_k)}{d\bar{x}_k} - \rho I_k(\bar{x}_k)\bar{\eta}^2\,\bar{\Psi}_k(\bar{x}_k) = \rho I_k(\bar{x}_k)\,\omega^2\,\bar{\Psi}_k(\bar{x}_k) \tag{12a,b}$$

Replacing equations (11) into equations (12), the differential equations of motion become:

$$-\frac{d\bar{N}_k(\bar{x}_k)}{d\bar{x}_k}\frac{d\bar{W}_k(\bar{x}_k)}{d\bar{x}_k}-\bar{N}_k(\bar{x}_k)\frac{d^2\bar{W}_k(\bar{x}_k)}{d\bar{x}_k^2}-\kappa G A_k(\bar{x}_k)\left(\frac{d^2\bar{W}_k(\bar{x}_k)}{d\bar{x}_k^2}-\frac{d\bar{\Psi}_k(\bar{x}_k)}{d\bar{x}}\right)-$$
$$\kappa G\frac{dA_k(\bar{x}_k)}{d\bar{x}_k}\left(\frac{d\bar{W}_k(\bar{x}_k)}{d\bar{x}_k}-\bar{\Psi}_k(\bar{x}_k)\right)=\rho A_k(\bar{x}_k)\,\omega^2\bar{W}_k(\bar{x}_k)$$
$$-\kappa G A_k(\bar{x}_k)\left(\frac{d\bar{W}_k(\bar{x}_k)}{d\bar{x}_k}-\bar{\Psi}_k(\bar{x}_k)\right)-EI_k(\bar{x}_k)\frac{d^2\bar{\Psi}_k(\bar{x}_k)}{d\bar{x}_k^2}-$$
$$E\frac{dI_k(\bar{x}_k)}{d\bar{x}_k}\frac{d\bar{\Psi}_k(\bar{x}_k)}{d\bar{x}_k}-\rho I_k(\bar{x}_k)\bar{\eta}^2\,\bar{\Psi}_k(\bar{x}_k)=\rho I_k(\bar{x}_k)\,\omega^2\,\bar{\Psi}_k(\bar{x}_k) \quad (13a,b)$$

The term $\rho I_k(\bar{x}_k)\bar{\eta}^2\,\bar{\Psi}_k(\bar{x}_k)$ included in equation (13.b) was introduced by Banerjee, 2001. This term generates more realistic results especially for high rotational speeds, $\bar{\eta}^2$.

The conditions for displacements and forces between adjacent segments, *k* and *k*+1, are:

$$\bar{W}_k(L_k)-\bar{W}_{k+1}(0)=0\,;\ \ \bar{\Psi}_k(L_k)-\bar{\Psi}_{k+1}(0)=0 \quad (14a,b)$$

$$\bar{Q}_k(L_k)-\bar{Q}_{k+1}(0)=0\,;\ \ \bar{M}_k(L_k)-\bar{M}_{k+1}(0)=0 \quad (15a,b)$$

Figure 3 shows the beam elastically restrained at both ends.

The boundary conditions of the beam at its ends are, for the first segment *k*=1, at $\overline{x_1}=0$:

$$\bar{Q}_1(0)-\bar{K}_{W1}\ \ \bar{W}_1(0)=0\,;\ \bar{M}_1(0)-\bar{K}_{\Psi 1}\ \bar{\Psi}_1(0)=0 \quad (16a,b)$$

and for the last segment *k*=*d*, at $\overline{x_d}=L_d$:

$$\bar{Q}_d(L_d)-\bar{K}_{Wd}\ \ \bar{W}_d(0)=0\,;\ \bar{M}_d(L_d)-\bar{K}_{\Psi d}\ \bar{\Psi}_d(0)=0 \quad (17a,b)$$

The four spring constants are denoted as: $\bar{K}_{W1}, \bar{K}_{Wd}, \bar{K}_{\Psi 1}, \bar{K}_{\Psi d}$.

The expressions and parameters in dimensionless form are defined as follows:

$$\Omega^2=\frac{\rho A_1(0)}{EI_1(0)}L^4\omega^2;\ \ \eta^2=\frac{\rho A_1(0)}{EI_1(0)}L^4\bar{\eta}^2;$$

$$x=\frac{\bar{x}_k}{L_k};\ l_k=\frac{L_k}{L};\ r_k^2=\frac{I_k(0)}{A_k(0)};\ s_k=\frac{L}{r_k};\ R_k=\frac{\bar{R}_k}{L};\ W_k(x)=\frac{\bar{W}_k(\bar{x}_k)}{L_k};\ \Psi_k(x)=\bar{\Psi}_k(\bar{x}_k);$$

$$a_k(x)=\frac{A_k(\bar{x}_k)}{A_k(0)}\,;\ b_k(x)=\frac{I_k(\bar{x}_k)}{I_k(0)}\,;\ v_k(x)=\frac{V_k(\bar{x}_k)}{l_k\,A_k(0)}\,;\ \phi_k(x)=\frac{\Phi_k(\bar{x}_k)}{l_k^2\,A_k(0)}\,;$$

$$N_{k+1}=\frac{\bar{F}_{k+1}}{EA_k(0)};\ \ N_k(x)=\frac{\bar{N}_k(\bar{x}_k)}{EA_k(0)};\ \ Q_k(x)=\frac{\bar{Q}_k(\bar{x}_k)}{EA_k(0)};\ \ M_k(x)=\frac{L_k}{EI_k(0)}\bar{M}_k(\bar{x}_k)\,;$$

and $K_{Wj} = \overline{K}_{Wj} \dfrac{L}{EA_1(0)}$; $K_{\Psi j} = \overline{K}_{\Psi j} \dfrac{L}{EI_1(0)}$; with j =1 or j=d.

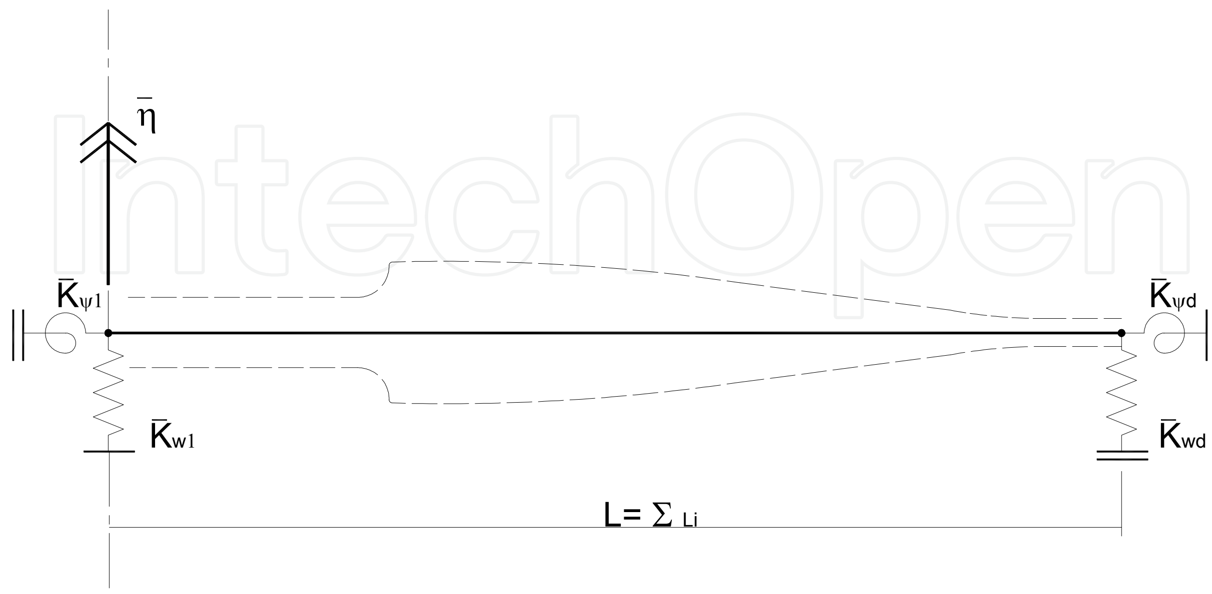


Fig. 3. Elastic restraints of the rotating beam

In each segment k of the beam, x varies between 0 and 1.

The axial force, the shear force and the bending moment in the adimensional form become:

$$N_k(x) = \eta^2 \frac{l_k^2}{s_1^2}\left(R_k v_k(1) + \phi_k(1) - R_k v_k(x) - \phi_k(x)\right) + N_{k+1}; \text{ with } s_1 = \frac{L}{r_1} \;;\; r_1^2 = \frac{I_1(0)}{A_1(0)} \quad (18)$$

$$Q_k(x) = \left(N_k(x) + \frac{\kappa}{2(1+\nu)} a_k(x)\right)\frac{dW_k(x)}{dx} - \frac{\kappa}{2(1+\nu)} a_k(x)\Psi_k(x) \quad (19)$$

$$M_k(x) = b_k(x)\frac{d\Psi_k(x)}{dx} \quad (20)$$

And the equations of motion in dimensionless form are:

$$\begin{aligned} &\eta^2 a_k(x)\left(R_k + x\right)\frac{dW_k(x)}{dx} - \frac{s_1^2}{l_k^2} N_k(x)\frac{d^2W_k(x)}{dx^2} - \frac{\kappa}{2(1+\nu)}\frac{s_1^2}{l_k^2} a_k(x)\left(\frac{d^2W_k(x)}{dx^2} - \frac{d\Psi_k(x)}{dx}\right) - \\ &- \frac{\kappa}{2(1+\nu)}\frac{s_1^2}{l_k^2}\frac{da_k(x)}{dx}\left(\frac{dW_k(x)}{dx} - \Psi_k(x)\right) = \Omega^2 a_k(x) W_k(x) \end{aligned} \quad (21)$$

$$\begin{aligned} &-s_1^2 \frac{\kappa}{2(1+\nu)} s_k^2 a_k(x)\left(\frac{dW_k(x)}{dx} - \Psi_k(x)\right) - \frac{s_1^2}{l_k^2} b_k(x)\frac{d^2\Psi_k(x)}{dx^2} - \frac{s_1^2}{l_k^2}\frac{db_k(x)}{dx}\frac{d\Psi_k(x)}{dx} - \\ &- \eta^2 b_k(x)\Psi_k(x) = \Omega^2 b_k(x)\Psi_k(x) \end{aligned} \quad (22)$$

The equations (14), which satisfy continuity of displacement and rotation, can be expressed in dimensionless form as follows:

$$l_k W_k(1) - l_{k+1} W_{k+1}(0) = 0\ ;\ \Psi_k(1) - \Psi_{k+1}(0) = 0 \tag{23a,b}$$

and the equations (15) of compatibility of the bending moment and the shear force, result in the following adimensional equations:

$$\alpha_k Q_k(1) - \alpha_{k+1} Q_{k+1}(0) = 0\ ;\ \frac{\beta_k}{l_k} M_k(1) - \frac{\beta_{k+1}}{l_{k+1}} M_{k+1}(0) = 0$$

or

$$\alpha_k \left[\left(N_k(x) + \frac{\kappa}{2(1+\nu)} a_k(x) \right) \frac{dW_k(x)}{dx} - \frac{\kappa}{2(1+\nu)} a_k(x) \Psi_k(x) \right] \Bigg|_{x=1} -$$

$$-\alpha_{k+1} \left[\left(N_{k+1}(x) + \frac{\kappa}{2(1+\nu)} a_{k+1}(x) \right) \frac{dW_{k+1}(x)}{dx} - \frac{\kappa}{2(1+\nu)} a_{k+1}(x) \Psi_{k+1}(x) \right] \Bigg|_{x=0} = 0; \tag{24a,b}$$

$$\frac{\beta_k}{l_k} b_k(x) \frac{d\Psi_k(x)}{dx} \Bigg|_{x=1} - \frac{\beta_{k+1}}{l_{k+1}} b_{k+1}(x) \frac{d\Psi_{k+1}(x)}{dx} \Bigg|_{x=0} = 0$$

where $\alpha_k = \dfrac{A_k(0)}{A_1(0)}$; $\beta_k = \dfrac{I_k(0)}{I_1(0)}$.

The boundary conditions at the end closest to the axis of rotation, segment 1, x=0, are:

$$Q_1(0) - K_{W1} l_1 W_1(0) = 0\ ;\ \left(N_1(0) + \frac{\kappa a_1(0)}{2(1+\nu)} \right) \frac{dW_1(x)}{dx} \Bigg|_{x=0} - \frac{\kappa a_1(0) \Psi_1(0)}{2(1+\nu)} - K_{W1} l_1 W_1(0) = 0$$

$$M_1(0) - K_{\Psi 1} l_1 \Psi_1(0) = 0\ ;\ b_1(0) \frac{d\Psi_1(x)}{dx} \Bigg|_{x=0} - K_{\Psi 1} l_1 \Psi_1(0) = 0 \tag{25a,b}$$

and at the other end of the rotating beam, segment d, x=1, they are:

$$Q_d(1) - K_{Wd} \frac{l_d}{\alpha_d} W_d(1) = 0;\ \left(N_d(1) + \frac{\kappa\, a_d(1)}{2(1+\nu)} \right) \frac{dW_d(x)}{dx} \Bigg|_{x=1} - \frac{\kappa\, a_d(1)}{2(1+\nu)} \Psi_d(1) - \frac{K_{Wd} l_d}{\alpha_d} W_d(1) = 0$$

$$M_d(1) - \frac{K_{\Psi d} l_d}{\beta_d} \Psi_d(1) = 0\ ;\ b_d(1) \frac{d\Psi_d(x)}{dx} \Bigg|_{x=1} - \frac{K_{\Psi d}\, l_d}{\beta_d} \Psi_d(1) = 0 \tag{26a,b}$$

where $N_d(1)$ is an outboard force at the end of the beam, farthest from the axis of rotation, that is equal to zero in the present study.

3. Differential Quadrature Method, DQM

In order to obtain the DQM analog equations from the governing equations of the rotating beam, the beam segment domain is discretized in a grid of i points, using the Chebyshev – Gauss - Lobato expression, (Shu, 2000). (See Fig. A.1 in Appendix A)

Equations (18, 19, 20) assumed the form:

$$N_k(x_i) = \eta^2 \frac{l_k^2}{s_1^2}\left(R_k\, v_k(1) + \phi_k(1) - R_k\, v_k(x_i) - \phi_k(x_i)\right) + N_{k+1} \tag{27}$$

$$Q_k(x_i) = \left(N_k(x_i) + \frac{\kappa}{2(1+\nu)} a_k(x_i)\right)\sum_{j=1}^{n} A_{ij}^{(1)}\, W_{kj} - \frac{\kappa}{2(1+\nu)} a_k(x_i)\Psi_{ki} \tag{28}$$

$$M_k(x_i) = b_k(x_i)\sum_{j=1}^{n} A_{ij}^{(1)}\,\Psi_{kj} \tag{29}$$

The equations of motion (21) and (22) become:

$$\begin{aligned}&\left(\eta^2 a_k(x_i)\left(R_k + x_i\right) - \frac{\kappa}{2(1+\nu)}\frac{s_1^2}{l_k^2}\frac{da_k(x_i)}{dx}\right)\sum_{j=1}^{n}\left(A_{ij}^{(1)}\right) W_{kj} - \\ &-\left(\frac{s_1^2}{l_k^2}N_k(x_i) + \frac{\kappa}{2(1+\nu)}\frac{s_1^2}{l_k^2}a_k(x_i)\right)\sum_{j=1}^{n}\left(A_{ij}^{(2)}\right) W_{kj} + \frac{\kappa}{2(1+\nu)}\frac{s_1^2}{l_k^2}a_k(x_i)\sum_{j=1}^{n} A_{ij}^{(1)}\,\Psi_{kj} + \\ &+\frac{\kappa}{2(1+\nu)}\frac{s_1^2}{l_k^2}\frac{da_k(x_i)}{dx}\Psi_{ki} = \Omega^2 a_k(x_i)\, W_{ki}\end{aligned} \tag{30}$$

$$\begin{aligned}&-\frac{\kappa}{2(1+\upsilon)}s_1^2 s_k^2 a_k(x_i)\sum_{j=1}^{n} A_{ij}^{(1)}\, W_{kj} - \frac{s_1^2}{l_k^2} b_k(x_i)\sum_{j=1}^{n} A_{ij}^{(2)}\,\Psi_{kj} + \\ &+\left(\frac{\kappa}{2(1+\nu)}s_1^2 s_k^2 a_k(x_i) - \eta^2 b_k(x_i)\right)\Psi_{ki} - \frac{s_1^2}{l_k^2}\frac{db_k(x_i)}{dx}\sum_{j=1}^{n} A_{ij}^{(1)}\,\Psi_{kj} = \Omega^2 b_k(x_i)\Psi_{ki}\end{aligned} \tag{31}$$

where the $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$ are the weighting coefficients of linear algebraic equations. (See Appendix A.1 for more details).

Finally, the conditions (23) and (24) are replaced by:

$$l_k\, W_{kn} - l_{k+1}\, W_{(k+1)\,1} = 0\,;\;\; \Psi_{kn} - \Psi_{(k+1)\,1} = 0\,; \tag{32a,b}$$

$$\begin{aligned}&\alpha_k\left(\left(N_k(1) + \frac{\kappa}{2(1+\nu)}a_k(1)\right)\sum_{j=1}^{n} A_{nj}^{(1)} W_{kj} - \frac{\kappa}{2(1+\nu)}a_k(1)\Psi_{kn}\right) \\ &-\alpha_{k+1}\left(\left(N_{k+1}(0) + \frac{\kappa}{2(1+\nu)}a_{k+1}(0)\right)\sum_{j=1}^{n} A_{1j}^{(1)} W_{(k+1)j} - \frac{\kappa}{2(1+\nu)}a_{k+1}(0)\Psi_{k1}\right) = 0; \\ &\frac{\alpha_k}{l_k} b_k(1)\sum_{j=1}^{n} A_{nj}^{(1)}\,\Psi_{kj} - \frac{\alpha_{k+1}}{l_{k+1}} b_{k+1}(0)\sum_{j=1}^{n} A_{1j}^{(1)}\,\Psi_{(k+1)j} = 0\end{aligned} \tag{33a,b}$$

and the boundary conditions (25) and (26) replaced by:

$$\left(N_1(0)+\frac{\kappa}{2(1+\nu)}a_1(0)\right)\sum_{j=1}^{n}A_{1j}^{(1)}W_{1j}-\frac{\kappa}{2(1+\nu)}a_1(0)\Psi_{11}-l_1K_{W1}W_{11}=0\ ;$$

$$K_{\Psi 1}\Psi_{11}-\frac{b_1(0)}{l_1}\sum_{j=1}^{n}A_{1j}^{(1)}\Psi_{1j}=0 \qquad (34a,b)$$

$$\left(N_d(1)+\frac{\kappa}{2(1+\nu)}a_d(1)\right)\sum_{j=1}^{n}A_{nj}^{(1)}W_{dj}-\frac{\kappa}{2(1+\nu)}a_d(1)\Psi_{dn}-l_dK_{Wd}W_{dn}=0;$$

$$K_{\Psi n}\Psi_{dn}-\frac{b_d(1)}{l_d}\sum_{j=1}^{n}A_{nj}^{(1)}\Psi_{dj}=0 \qquad (35a,b)$$

The DQM linear equation system is used to determine the natural frequencies and mode shapes of the rotating beam.
The number of terms taken in the summations had been studied for many situations and the system has acceptable convergence by n= 21 terms. (See Table 1)

4. Finite element method, MEF

An independent set of results for the natural frequencies, was also obtained by a finite element code. (Bambill et al., 2010). The finite element model employed in the analysis has 3000 beam elements of two nodes in the longitudinal direction (Rossi, 2007). See Table 2. This number of elements was proved to be enough with a convergence analysis.

The beam model also takes into account the shear deformation (Timoshenko beam's theory) and the increase in bending stiffness induced by the centrifugal force.
The term $\rho I_k(\overline{x}_k)\overline{\eta}^2\,\overline{\Psi}_k(\overline{x}_k)$ of equation (13.b) was not included in the finite element formulation. Probably for this reason some small differences between both sets of numerical results (DQM and FEM) begin to appear when the rotational speed η increases.

5. Numerical results

In the following examples some calculations were performed over elliptical cross sections. ($\kappa = 0.886364$). Without loss of generality, one may choose to keep constant width e_k=e and vary the height $h_k(x)$ in each segment of the beam. The area and the second moment of area of the cross section of the beam will be $A_k(x)=\frac{\pi\, e h_k(x)}{4}$, $I_k(x)=\frac{\pi\, e h_k^3(x)}{64}$, and for this particular situation there are:

$$a_k(x)=\frac{h_k(x)}{h_k(0)}\ ;\ b_k(x)=\left(\frac{h_k(x)}{h_k(0)}\right)^3$$

The following formula is proposed to a quadratic variation of the height in each segment of beam:

$$h_k(x)=c_{0k}+c_{1k}\,x+c_{2k}\,x^2$$

And the slope is the derivative of this function

$$h'_k(x) = \frac{dh_k(x)}{dx} = c_{1k} + 2c_{2k}\,x$$

where c_{0k}, c_{1k} and c_{2k} are constants, which are defined by the heights and slopes at both ends of each segment *k*. The heights and slopes at each end are identified with the subscript *A* for *x*=0: h_{Ak}; h'_{Ak} and with the subscript *B* for *x*=1: h_{Bk}; h'_{Bk}.

If the segment of the beam shows a linear variation of height , $c_{2k} = 0$ and

$$h_{Ak} = c_{0k}\,;\; h_{Bk} = c_{0k} + c_{1k}\,; h'_{Ak} = h'_{Bk} = c_{1k}$$

As it can be seen in Table 1, the frequency coefficients calculated by the Differential Quadrature Method, DQM, using a summation with $n \geq 19$ (*i*= 1, 2, 3, …, *n*) points, show none significant improvement.

n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	15.6861	29.2939	49.1602	63.9792	112.610
7	15.1981	28.9907	46.9070	64.9219	88.8670
9	14.9057	29.5079	47.4960	64.7054	87.4079
11	14.8340	29.6332	47.6579	64.7247	87.6724
13	14.8281	29.6467	47.6811	64.7310	87.7047
15	14.8291	29.6464	47.6820	64.7319	87.7079
17	14.8295	29.6460	47.6816	64.7320	87.7080
19	14.8296	29.6459	47.6815	64.7320	87.7080
21	14.8296	29.6459	47.6815	64.7320	87.7080

Table 1. Convergence analysis of the DQM, for a two-span rotating Timoshenko beam elastically restrained al both ends, with a quadratic variation of height.

The frequency coefficients in Table 1, correspond to a beam of two segments, rotating at speed $\eta = 10$, whose characteristics are: elliptical cross section; $\nu = 0.3$; $\kappa = 0.886364$; R_1=0; $l_1 / L = l_2 / L = 1/2$; $s_1 = \sqrt{300}$; $h_{B1} / h_{A1} = 1/2$; $h'_{B1} = 0$; $h_{A2} / h_{B1} = 1/2$; $h_{B2} / h_{A2} = 1/2$; $h'_{A2} = 0$; $K_{W1} = 10$; $K_{\psi 1} = 5$; $K_{Wd} = 0.1$; $K_{\psi d} = 1$.

In Table 2 the values obtained for the natural frequency coefficients using the finite element method are presented for $\eta = \sqrt{\rho A_0 / EI_0}\, L^2\, \overline{\eta} = 0$ and $\eta = 10$. The number of elements is increased from 10 to 3000.

The model of the rotating beam of Table 2 has the following characteristics: one segment; rectangular cross section; $\nu = 0.3$; $\kappa = 10(1+\nu)/(12+11\nu)$ =0.849673; R_1=0; $s_1 = \sqrt{300}$; $h_B / h_A = 1/4$; h'_B=0; $K_{W1} \to \infty$; $K_{\psi 1} \to \infty$; $K_{Wd} = 0$; $K_{\psi d} = 0$.

In the first examples it is assumed a perfect clamped condition at the axis of rotation, given by: $K_{W1} \to \infty$ and $K_{\psi 1} \to \infty$. (Tables 3, 4 and 5).

Table 3 presents the effect of the rotational speed parameter η on the natural frequency coefficients of a rotating cantilever beam of one segment, ($K_{W1} \to \infty ; K_{\psi 1} \to \infty$; $K_{Wd} = 0$; $K_{\psi d} = 0$). The results correspond to a linear variation of height and a comparison is made with

(Barnejee, 2006) when Banerjee´s parameter is n=1. As it can be observed the agreement is excellent.

$\eta = 0$					
Number of elements	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
10	3.38628165	11.7689336	26.5951854	46.6658427	71.0448001
100	3.37398143	11.7248502	26.4438604	46.1408176	69.5136708
1000	3.37385398	11.7243988	26.4423706	46.1357196	69.4986357
2000	3.37385302	11.7243954	26.4423593	46.1356810	69.4985219
3000	3.37385284	11.7243946	26.4423572	46.1356739	69.4985008
$\eta = 10$					
10	11.6074237	25.8805102	44.0407905	66.3753084	92.6859627
100	11.6098042	25.7094320	43.5638284	65.4674874	90.8491237
1000	11.6098077	25.7074626	43.5585908	65.4579769	90.8301746
2000	11.6098078	25.7074476	43.5585511	65.4579049	90.8300310
3000	11.6098078	25.7074448	43.5585437	65.4578915	90.8300044

Table 2. Convergence analysis of the frequency coefficients $\Omega_i = \sqrt{\rho A_0 / EI_0}\, L^2\, \omega_i$ using MEF.

η		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	DQM	3.82377	18.3171	47.2638	90.4468	147.992
	(Barnejee,2006)	3.82379	18.3173	47.2648	90.4505	148.002
2	DQM	4.43680	18.9365	47.8706	91.0589	148.609
	(Barnejee,2006)	4.43680	18.9366	47.8717	91.0625	148.619
4	DQM	5.87874	20.6850	49.6446	92.8693	150.444
	(Barnejee,2006)	5.87877	20.6851	49.6456	92.8730	150.454
6	DQM	7.65512	23.3091	52.4622	95.8054	153.450
	(Barnejee,2006)	7.65514	23.3093	52.4632	95.8090	153.460
8	DQM	9.55392	26.5435	56.1584	99.7601	157.555
	(Barnejee,2006)	9.55396	26.5437	56.1595	99.7638	157.564
10	DQM	11.5015	30.1825	60.5628	104.608	162.668
	(Barnejee,2006)	11.5015	30.1827	60.5639	104.612	162.677

Table 3. Frequency coefficients $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2\, \omega_i$ for a one-span beam, $l_1 / L = 1$; $s_1 = \sqrt{1000}$; $h_B / h_A = 1/2$; $K_{W1} \to \infty$; $K_{\psi 1} \to \infty$; $K_{Wd} = 0$; $K_{\psi d} = 0$.

All the calculations performed for the following Tables and Graphics used $R_1 = 0$; and $\nu = 0.30$; $\kappa = 0.886364$ (elliptical cross section).

The DQM results are determined using n = 21 in each segment of the beam, and the MEF results were obtained with 3000 elements.

The beam considered in Table 4 has one segment and is elastically restrained at its outer end. The parameter of rotation speed η is taken equal to 10. The Table presents the frequency coefficients for the first five mode shapes which correspond to different sets of elastically boundary conditions given by the spring constant parameters K_{Wd} and $K_{\psi d}$. The other details of the beam are specified in the legend of the table.

The beam model considered in Table 5 has two segments of equal length and similar conditions and parameters as Table 4.

$K_{\psi d}$	K_{Wd}	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	DQM	11.2148	27.6174	50.0089	77.5866	108.472
		FEM	11.2375	27.6743	50.0711	77.6432	108.523
	0.1	DQM	15.4254	32.5178	52.8516	79.0733	109.357
		FEM	15.4438	32.5548	52.9087	79.1298	109.408
	1	DQM	18.0157	40.3494	65.6538	92.2848	119.836
		FEM	18.0465	40.3841	65.6882	92.3208	119.877
	10	DQM	18.3978	41.6361	69.0216	99.4111	131.859
		FEM	18.4315	41.6757	69.0611	99.4484	131.893
	$\to\infty$	DQM	18.4417	41.7750	69.3474	100.033	132.894
		FEM	18.4757	41.8151	69.3878	100.071	132.929
1	0	DQM	11.3941	29.3678	53.0174	81.0192	112.024
		FEM	11.4148	29.4104	53.0660	81.0662	112.068
	0.1	DQM	15.6233	32.8965	55.0307	82.2247	112.825
		FEM	15.6400	32.9308	55.0763	82.2710	112.868
	1	DQM	19.2962	41.3980	65.9339	92.3365	120.822
		FEM	19.3219	41.4295	65.9674	92.3723	120.859
	10	DQM	19.9179	43.3987	70.5662	100.622	132.723
		FEM	19.9463	43.4345	70.6034	100.658	132.756
	$\to\infty$	DQM	19.9899	43.6199	71.0558	101.509	134.136
		FEM	20.0187	43.6562	71.0937	101.546	134.170
10	0	DQM	11.4913	30.3954	55.1815	84.0260	115.628
		FEM	11.5115	30.4328	55.2229	84.0663	115.665
	0.1	DQM	15.7621	33.1503	56.5688	84.8630	116.210
		FEM	15.7780	33.1835	56.6092	84.9031	116.247
	1	DQM	20.4765	42.5961	66.2635	92.3899	121.730
		FEM	20.4994	42.6248	66.2963	92.4255	121.765
	10	DQM	21.3539	45.5548	72.8197	102.560	134.141
		FEM	21.3795	45.5875	72.8543	102.594	134.174
	$\to\infty$	DQM	21.4553	45.8807	73.5609	103.912	136.261
		FEM	21.4813	45.9139	73.5962	103.947	136.294
$\to\infty$	0	DQM	11.5091	30.5860	55.6105	84.6706	116.454
		FEM	11.5291	30.6228	55.6510	84.7101	116.491
	0.1	DQM	15.7905	33.2010	56.8768	85.4233	116.975
		FEM	15.8064	33.2340	56.9165	85.4625	117.012
	1	DQM	20.7557	42.9193	66.3549	92.4035	121.943
		FEM	20.7782	42.9475	66.3875	92.4392	121.978
	10	DQM	21.6961	46.1510	73.5285	103.223	134.643
		FEM	21.7214	46.1832	73.5626	103.257	134.675
	$\to\infty$	DQM	21.8045	46.5039	74.3452	104.735	137.026
		FEM	21.8302	46.5368	74.3801	104.769	137.059

Table 4. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a one-span rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $\nu = 0.3$; $s_1 = \sqrt{300}$; $h_B / h_A = 1/2$; $h'_B = 0$; $K_{W1} \to \infty$; $K_{\psi 1} \to \infty$; $\eta = 10$.

$K_{\psi d}$	K_{Wd}	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	DQM	11.8651	24.5717	40.8347	59.8775	81.1573
		FEM	11.8796	24.5914	40.8559	59.9110	81.1826
	0.1	DQM	15.2667	30.3903	49.8409	67.9228	90.5546
		FEM	15.2858	30.4064	49.8638	67.9498	90.5743
	1	DQM	15.6938	31.4585	52.2093	72.0062	98.7038
		FEM	15.7140	31.4754	52.2362	72.0330	98.7265
	10	DQM	15.7412	31.5756	52.4458	72.4214	99.4803
		FEM	15.7616	31.5927	52.4732	72.4482	99.5038
	$\to\infty$	DQM	15.7466	31.5887	52.4718	72.4669	99.5627
		FEM	15.7669	31.6059	52.4993	72.4937	99.5862
1	0	DQM	11.9142	25.1342	42.8878	62.4877	85.6040
		FEM	11.9288	25.1532	42.9079	62.5196	85.6258
	0.1	DQM	16.2121	31.6526	50.5459	67.9979	90.8436
		FEM	16.2314	31.6672	50.5682	68.0245	90.8635
	1	DQM	16.9952	33.8476	55.0090	75.3283	102.166
		FEM	17.0160	33.8634	55.0372	75.3520	102.190
	10	DQM	17.0842	34.0961	55.4704	76.2542	103.723
		FEM	17.1052	34.1120	55.4993	76.2779	103.748
	$\to\infty$	DQM	17.0942	34.1238	55.5205	76.3541	103.882
		FEM	17.1152	34.1398	55.5496	76.3778	103.907
10	0	DQM	11.9157	25.1505	42.9498	62.5733	85.7690
		FEM	11.9302	25.1695	42.9699	62.6051	85.7907
	0.1	DQM	16.2528	31.7152	50.5831	68.0018	90.8571
		FEM	16.2721	31.7297	50.6053	68.0283	90.8770
	1	DQM	17.0528	33.9729	55.1728	75.5622	102.430
		FEM	17.0737	33.9886	55.2011	75.5857	102.453
	10	DQM	17.1437	34.2281	55.6450	76.5200	104.034
		FEM	17.1648	34.2440	55.6741	76.5435	104.059
	$\to\infty$	DQM	17.1539	34.2566	55.6962	76.6231	104.197
		FEM	17.1750	34.2725	55.7255	76.6466	104.222
$\to\infty$	0	DQM	11.9158	25.1524	42.9569	62.5831	85.7880
		FEM	11.9304	25.1713	42.9770	62.6149	85.8097
	0.1	DQM	16.2575	31.7225	50.5875	68.0023	90.8587
		FEM	16.2768	31.7370	50.6097	68.0288	90.8786
	1	DQM	17.0595	33.9876	55.1921	75.5901	102.461
		FEM	17.0804	34.0033	55.2205	75.6136	102.485
	10	DQM	17.1506	34.2436	55.6656	76.5517	104.071
		FEM	17.1717	34.2595	55.6947	76.5752	104.096
	$\to\infty$	DQM	17.1609	34.2722	55.7169	76.6551	104.234
		FEM	17.1820	34.2881	55.7461	76.6786	104.259

Table 5. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a two-span elastically restrained rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $l_1 / L = 1/2$ $l_2 / L = 1/2$, $h_{B1} / h_{A1} = 1/2$, $h'_{B1} = 0$, $h_{A2} / h_{B1} = 1/2$, $h_{B2} / h_{A2} = 1/2$, $h'_{A2} = 0$, $K_{W1} \to \infty$ $K_{\psi 1} \to \infty$, $\eta = 10$.

Next Tables, 6 to 10, correspond to beams of two segments, elastically restrained at both ends and any particular details are expressed in each legend.

$K_{\psi d}$	K_{Wd}	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	DQM	9.98841	21.2706	37.3110	54.9224	77.7336
		FEM	10.0246	21.3074	37.3506	54.9699	77.7658
	0.1	DQM	12.4181	26.7466	45.4389	63.7717	87.4947
		FEM	12.4641	26.7810	45.4827	63.8054	87.5227
	1	DQM	12.7051	27.6834	47.3154	67.8701	94.9448
		FEM	12.7526	27.7193	47.3634	67.9035	94.9788
	10	DQM	12.7370	27.7869	47.5065	68.2901	95.6398
		FEM	12.7847	27.8230	47.5548	68.3236	95.6747
	$\to\infty$	DQM	12.7406	27.7985	47.5276	68.3362	95.7137
		FEM	12.7883	27.8347	47.5760	68.3697	95.7487
1	0	DQM	10.0086	21.6505	39.0007	57.4853	82.1930
		FEM	10.0451	21.6876	39.0412	57.5291	82.2227
	0.1	DQM	13.1047	28.0775	46.2626	63.9615	87.6768
		FEM	13.1521	28.1101	46.3049	63.9941	87.7052
	1	DQM	13.6220	29.9765	49.8847	71.5084	98.2767
		FEM	13.6718	30.0120	49.9325	71.5384	98.3117
	10	DQM	13.6812	30.1918	50.2664	72.4302	99.6646
		FEM	13.7312	30.2278	50.3147	72.4604	99.7012
	$\to\infty$	DQM	13.6879	30.2159	50.3082	72.5299	99.8074
		FEM	13.7379	30.2519	50.3566	72.5601	99.8441
10	0	DQM	10.0092	21.6615	39.0505	57.5689	82.3537
		FEM	10.0457	21.6987	39.0910	57.6127	82.3835
	0.1	DQM	13.1336	28.1416	46.3059	63.9714	87.6854
		FEM	13.1811	28.1741	46.3481	64.0039	87.7138
	1	DQM	13.6618	30.0922	50.0321	71.7561	98.5256
		FEM	13.7116	30.1278	50.0799	71.7860	98.5607
	10	DQM	13.7222	30.3131	50.4233	72.7081	99.9556
		FEM	13.7723	30.3491	50.4716	72.7381	99.9923
	$\to\infty$	DQM	13.7290	30.3379	50.4661	72.8107	100.102
		FEM	13.7792	30.3739	50.5145	72.8408	100.139
$\to\infty$	0	DQM	10.0093	21.6628	39.0562	57.5786	82.3722
		FEM	10.0458	21.6999	39.0968	57.6224	82.4020
	0.1	DQM	13.1369	28.1491	46.3110	63.9726	87.6864
		FEM	13.1844	28.1817	46.3532	64.0051	87.7149
	1	DQM	13.6664	30.1057	50.0495	71.7856	98.5555
		FEM	13.7163	30.1414	50.0973	71.8155	98.5906
	10	DQM	13.7270	30.3273	50.4418	72.7411	99.9903
		FEM	13.7771	30.3634	50.4901	72.7711	100.027
	$\to\infty$	DQM	13.7338	30.3521	50.4847	72.8441	100.137
		FEM	13.7840	30.3882	50.5331	72.8741	100.174

Table 6. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a two-span elastically restrained rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $l_1 / L = 1/2$ $l_2 / L = 1/2$, $h_{B1} / h_{A1} = 1/2$, $h'_{B1} = 0$, $h_{A2} / h_{B1} = 1/2$, $h_{B2} / h_{A2} = 1/2$, $h'_{A2} = 0$, $K_{W1} \to \infty$, $K_{\psi 1} = 0.1$, $\eta = 10$.

$K_{\psi d}$	K_{Wd}	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	DQM	11.3734	23.4059	39.2570	56.9363	78.4787
		FEM	11.3904	23.4273	39.2820	56.9721	78.5041
	0.1	DQM	14.4551	28.9705	47.6260	65.1400	87.8678
		FEM	14.4773	28.9890	47.6542	65.1664	87.8894
	1	DQM	14.8320	29.9641	49.6507	69.1005	95.0494
		FEM	14.8553	29.9837	49.6827	69.1264	95.0757
	10	DQM	14.8738	30.0733	49.8536	69.5057	95.7100
		FEM	14.8972	30.0931	49.8861	69.5316	95.7371
	$\to\infty$	DQM	14.8785	30.0856	49.8761	69.5501	95.7801
		FEM	14.9019	30.1054	49.9086	69.5761	95.8073
1	0	DQM	11.4126	23.8883	41.1048	59.4030	82.7965
		FEM	11.4297	23.9092	41.1294	59.4361	82.8192
	0.1	DQM	15.3087	30.2367	48.3683	65.2804	88.0547
		FEM	15.3312	30.2539	48.3957	65.3061	88.0766
	1	DQM	15.9925	32.2677	52.2109	72.5363	98.2033
		FEM	16.0166	32.2865	52.2436	72.5592	98.2306
	10	DQM	16.0702	32.4974	52.6080	73.4315	99.5067
		FEM	16.0945	32.5165	52.6413	73.4545	99.5353
	$\to\infty$	DQM	16.0790	32.5231	52.6513	73.5282	99.6404
		FEM	16.1032	32.5422	52.6847	73.5513	99.6691
10	0	DQM	11.4138	23.9023	41.1598	59.4837	82.9526
		FEM	11.4309	23.9232	41.1844	59.5168	82.9753
	0.1	DQM	15.3450	30.2988	48.4074	65.2877	88.0636
		FEM	15.3676	30.3159	48.4348	65.3134	88.0854
	1	DQM	16.0434	32.3866	52.3585	72.7730	98.4379
		FEM	16.0676	32.4055	52.3914	72.7958	98.4653
	10	DQM	16.1228	32.6224	52.7651	73.6979	99.7796
		FEM	16.1471	32.6415	52.7985	73.7208	99.8083
	$\to\infty$	DQM	16.1317	32.6487	52.8094	73.7975	99.9164
		FEM	16.1560	32.6679	52.8428	73.8204	99.9453
$\to\infty$	0	DQM	11.4139	23.9039	41.1661	59.4929	82.9706
		FEM	11.4310	23.9248	41.1908	59.5260	82.9933
	0.1	DQM	15.3493	30.3061	48.4120	65.2886	88.0646
		FEM	15.3718	30.3232	48.4394	65.3142	88.0865
	1	DQM	16.0493	32.4005	52.3759	72.8012	98.4661
		FEM	16.0735	32.4194	52.4088	72.8240	98.4934
	10	DQM	16.1289	32.6371	52.7836	73.7295	99.8122
		FEM	16.1532	32.6562	52.8170	73.7524	99.8409
	$\to\infty$	DQM	16.1378	32.6635	52.8280	73.8295	99.9493
		FEM	16.1622	32.6827	52.8615	73.8524	99.9782

Table 7. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a two-span elastically restrained rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $l_1 / L = 1/2$ $l_2 / L = 1/2$, $h_{B1} / h_{A1} = 1/2$, $h'_{B1} = 0$, $h_{A2} / h_{B1} = 1/2$, $h_{B2} / h_{A2} = 1/2$, $h'_{A2} = 0$, $K_{W1} = 10$, $K_{\psi 1} = 10$, $\eta = 10$.

$K_{\psi d}$	K_{Wd}	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	DQM	11.0954	22.8658	38.6771	56.1987	78.0667
		FEM	11.1149	22.8898	38.7052	56.2367	78.0931
	0.1	DQM	14.0189	28.3688	46.9190	64.5736	87.5306
		FEM	14.0444	28.3902	46.9506	64.6011	87.5532
	1	DQM	14.3726	29.3408	48.8766	68.5581	94.6820
		FEM	14.3992	29.3634	48.9120	68.5850	94.7094
	10	DQM	14.4118	29.4478	49.0737	68.9659	95.3401
		FEM	14.4386	29.4706	49.1095	68.9930	95.3684
	$\to\infty$	DQM	14.4162	29.4598	49.0954	69.0107	95.4100
		FEM	14.4430	29.4826	49.1314	69.0377	95.4383
1	0	DQM	11.1299	23.3178	40.4680	58.6771	82.4016
		FEM	11.1495	23.3414	40.4960	58.7122	82.4254
	0.1	DQM	14.8296	29.6459	47.6815	64.7320	87.7080
		FEM	14.8556	29.6659	47.7122	64.7587	87.7309
	1	DQM	15.4691	31.6287	51.4150	72.0511	97.8483
		FEM	15.4967	31.6508	51.4509	72.0751	97.8766
	10	DQM	15.5418	31.8531	51.8028	72.9497	99.1495
		FEM	15.5696	31.8754	51.8392	72.9737	99.1791
	$\to\infty$	DQM	15.5499	31.8782	51.8452	73.0468	99.2832
		FEM	15.5778	31.9005	51.8817	73.0708	99.3129
10	0	DQM	11.1309	23.3309	40.5211	58.7582	82.5578
		FEM	11.1505	23.3545	40.5491	58.7932	82.5816
	0.1	DQM	14.8640	29.7082	47.7217	64.7403	87.7163
		FEM	14.8900	29.7282	47.7523	64.7669	87.7393
	1	DQM	15.5170	31.7461	51.5610	72.2905	98.0837
		FEM	15.5447	31.7682	51.5970	72.3143	98.1121
	10	DQM	15.5912	31.9764	51.9582	73.2187	99.4233
		FEM	15.6191	31.9987	51.9947	73.2426	99.4531
	$\to\infty$	DQM	15.5996	32.0021	52.0016	73.3186	99.5601
		FEM	15.6275	32.0245	52.0381	73.3426	99.5900
$\to\infty$	0	DQM	11.1310	23.3324	40.5272	58.7675	82.5758
		FEM	11.1506	23.3560	40.5552	58.8025	82.5996
	0.1	DQM	14.8680	29.7155	47.7264	64.7412	87.7173
		FEM	14.8940	29.7355	47.7570	64.7679	87.7402
	1	DQM	15.5225	31.7598	51.5783	72.3191	98.1119
		FEM	15.5503	31.7819	51.6143	72.3428	98.1403
	10	DQM	15.5970	31.9908	51.9765	73.2506	99.4560
		FEM	15.6249	32.0132	52.0131	73.2745	99.4857
	$\to\infty$	DQM	15.6053	32.0166	52.0200	73.3510	99.5931
		FEM	15.6333	32.0391	52.0566	73.3749	99.6230

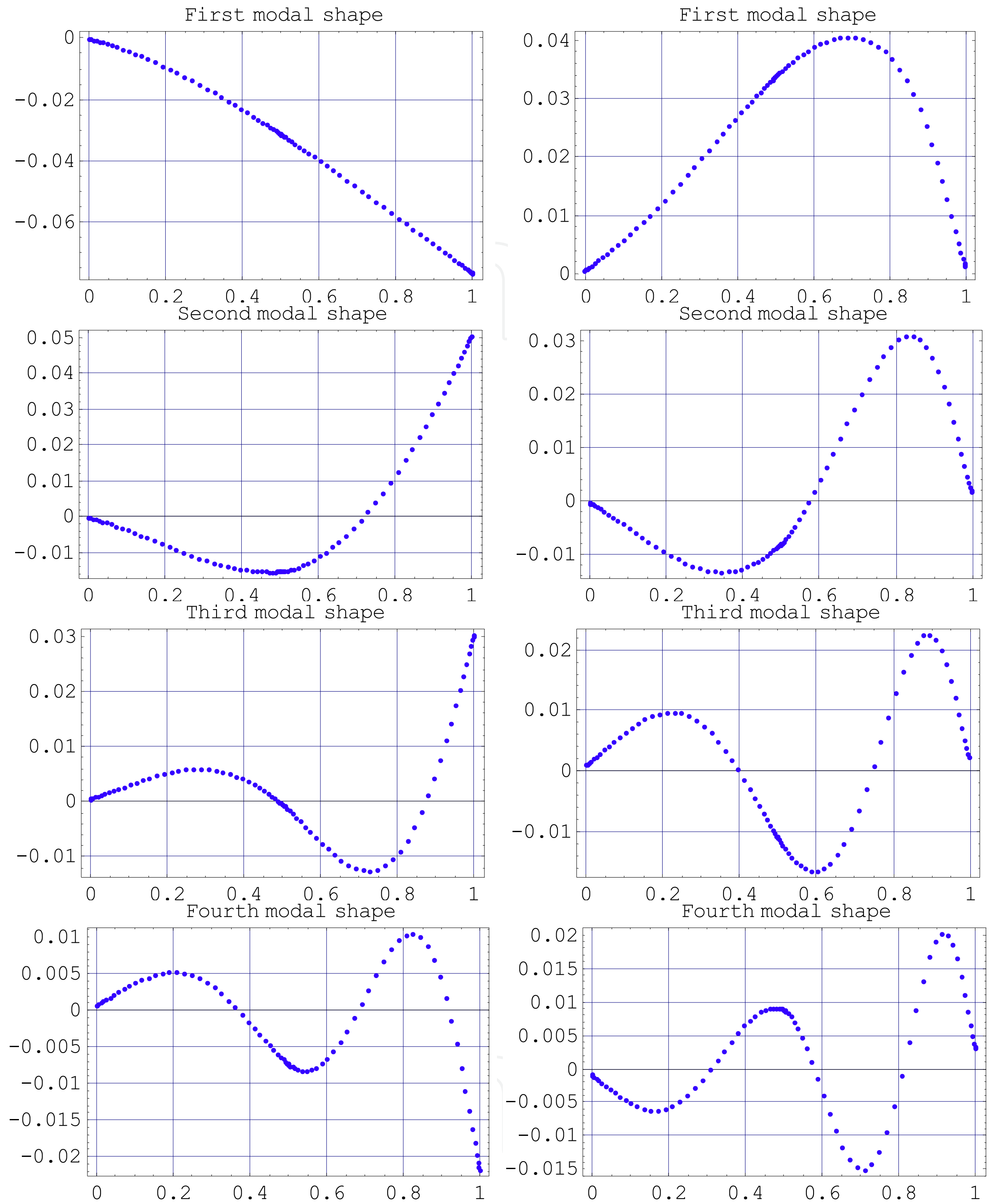
Table 8. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a two-span elastically restrained rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $l_1 / L = 1/2$ $l_2 / L = 1/2$, $h_{B1} / h_{A1} = 1/2$, $h'_{B1} = 0$, $h_{A2} / h_{B1} = 1/2$, $h_{B2} / h_{A2} = 1/2$, $h'_{A2} = 0$, $K_{W1} = 10$, $K_{\psi 1} = 5$, $\eta = 10$.

$K_{\psi d}$	K_{Wd}	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	DQM	10.3650	21.7083	37.5771	54.9536	77.3897
		FEM	10.3937	21.7398	37.6121	54.9961	77.4183
	0.1	DQM	12.9366	27.1514	45.6367	63.6520	86.9652
		FEM	12.9736	27.1805	45.6754	63.6816	86.9900
	1	DQM	13.2417	28.0899	47.4960	67.6756	94.0576
		FEM	13.2800	28.1204	47.5385	67.7048	94.0875
	10	DQM	13.2755	28.1934	47.6848	68.0876	94.7110
		FEM	13.3141	28.2242	47.7276	68.1169	94.7416
	$\to\infty$	DQM	13.2794	28.2050	47.7057	68.1329	94.7804
		FEM	13.3179	28.2358	47.7485	68.1621	94.8111
1	0	DQM	10.3892	22.1044	39.2705	57.4666	81.7463
		FEM	10.4182	22.1360	39.3061	57.5054	81.7724
	0.1	DQM	13.6565	28.4599	46.4416	63.8427	87.1268
		FEM	13.6946	28.4875	46.4789	63.8714	87.1519
	1	DQM	14.2060	30.3646	50.0205	71.2606	97.2421
		FEM	14.2462	30.3947	50.0627	71.2866	97.2727
	10	DQM	14.2687	30.5803	50.3961	72.1634	98.5382
		FEM	14.3091	30.6108	50.4388	72.1896	98.5700
	$\to\infty$	DQM	14.2757	30.6044	50.4372	72.2610	98.6715
		FEM	14.3161	30.6350	50.4800	72.2872	98.7035
10	0	DQM	10.3900	22.1160	39.3204	57.5486	81.9025
		FEM	10.4190	22.1475	39.3560	57.5874	81.9286
	0.1	DQM	13.6869	28.5231	46.4839	63.8527	87.1344
		FEM	13.7250	28.5506	46.5212	63.8813	87.1595
	1	DQM	14.2479	30.4798	50.1652	71.5043	97.4786
		FEM	14.2881	30.5099	50.2075	71.5301	97.5092
	10	DQM	14.3119	30.7011	50.5501	72.4364	98.8132
		FEM	14.3524	30.7317	50.5928	72.4624	98.8452
	$\to\infty$	DQM	14.3191	30.7259	50.5922	72.5368	98.9498
		FEM	14.3596	30.7565	50.6350	72.5629	98.9819
$\to\infty$	0	DQM	10.3901	22.1173	39.3262	57.5580	81.9205
		FEM	10.4190	22.1488	39.3618	57.5968	81.9466
	0.1	DQM	13.6904	28.5305	46.4889	63.8539	87.1353
		FEM	13.7286	28.5580	46.5261	63.8825	87.1604
	1	DQM	14.2527	30.4933	50.1823	71.5333	97.5069
		FEM	14.2930	30.5235	50.2245	71.5591	97.5375
	10	DQM	14.3169	30.7153	50.5682	72.4688	98.8460
		FEM	14.3574	30.7458	50.6110	72.4948	98.8780
	$\to\infty$	DQM	14.3241	30.7401	50.6105	72.5696	98.9829
		FEM	14.3646	30.7707	50.6532	72.5957	99.0150

Table 9. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a two-span elastically restrained rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $l_1 / L = 1/2$ $l_2 / L = 1/2$, $h_{B1} / h_{A1} = 1/2$, $h'_{B1} = 0$, $h_{A2} / h_{B1} = 1/2$, $h_{B2} / h_{A2} = 1/2$, $h'_{A2} = 0$, $K_{W1} = 10$, $K_{\psi 1} = 1$, $\eta = 10$.

$K_{\psi d}$	K_{Wd}	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	DQM	9.94295	21.1789	37.1287	54.4936	77.1422
		FEM	9.97877	21.2145	37.1668	54.5377	77.1717
	0.1	DQM	12.3497	26.6244	45.1340	63.3202	86.7544
		FEM	12.3950	26.6574	45.1757	63.3506	86.7801
	1	DQM	12.6335	27.5522	46.9619	67.3571	93.8220
		FEM	12.6802	27.5867	47.0072	67.3872	93.8529
	10	DQM	12.6651	27.6547	47.1481	67.7705	94.4732
		FEM	12.7119	27.6894	47.1937	67.8007	94.5049
	$\to\infty$	DQM	12.6686	27.6662	47.1687	67.8159	94.5425
		FEM	12.7155	27.7009	47.2144	67.8461	94.5743
1	0	DQM	9.96265	21.5532	38.7860	57.0231	81.5044
		FEM	9.99874	21.5890	38.8248	57.0634	81.5315
	0.1	DQM	13.0297	27.9504	45.9575	63.5235	86.9102
		FEM	13.0763	27.9817	45.9976	63.5530	86.9362
	1	DQM	13.5409	29.8282	49.4887	70.9743	97.0122
		FEM	13.5897	29.8623	49.5335	71.0012	97.0437
	10	DQM	13.5994	30.0410	49.8608	71.8782	98.3059
		FEM	13.6484	30.0754	49.9060	71.9052	98.3387
	$\to\infty$	DQM	13.6060	30.0648	49.9016	71.9759	98.4391
		FEM	13.6550	30.0993	49.9469	72.0030	98.4720
10	0	DQM	9.96324	21.5641	38.8347	57.1056	81.6606
		FEM	9.99933	21.5999	38.8736	57.1459	81.6876
	0.1	DQM	13.0583	28.0142	46.0008	63.5342	86.9175
		FEM	13.1049	28.0454	46.0408	63.5635	86.9436
	1	DQM	13.5802	29.9428	49.6333	71.2194	97.2490
		FEM	13.6291	29.9769	49.6781	71.2462	97.2806
	10	DQM	13.6399	30.1611	50.0148	72.1525	98.5813
		FEM	13.6890	30.1956	50.0600	72.1794	98.6142
	$\to\infty$	DQM	13.6467	30.1856	50.0566	72.2531	98.7177
		FEM	13.6958	30.2201	50.1018	72.2800	98.7508
$\to\infty$	0	DQM	9.96331	21.5653	38.8403	57.1151	81.6785
		FEM	9.99940	21.6011	38.8792	57.1553	81.7056
	0.1	DQM	13.0616	28.0216	46.0058	63.5354	86.9184
		FEM	13.1082	28.0529	46.0459	63.5648	86.9444
	1	DQM	13.5848	29.9563	49.6504	71.2486	97.2773
		FEM	13.6337	29.9904	49.6952	71.2753	97.3090
	10	DQM	13.6447	30.1752	50.0329	72.1851	98.6141
		FEM	13.6938	30.2097	50.0781	72.2120	98.6471
	$\to\infty$	DQM	13.6514	30.1997	50.0748	72.2860	98.7509
		FEM	13.7006	30.2342	50.1201	72.3129	98.7840

Table 10. First natural frequencies $\Omega_i = \sqrt{\rho A_1(0) / EI_1(0)}\, L^2 \omega_i$ for a two-span elastically restrained rotating Timoshenko beam, with elliptical cross section and quadratic height variation along the axis. $l_1 / L = 1/2$ $l_2 / L = 1/2$, $h_{B1} / h_{A1} = 1/2$, $h'_{B1} = 0$, $h_{A2} / h_{B1} = 1/2$, $h_{B2} / h_{A2} = 1/2$, $h'_{A2} = 0$, $K_{W1} = 10$, $K_{\psi 1} = 0.1$, $\eta = 10$.



a) $K_{W1}=10$, $K_{\psi 1}=5$, $K_{Wd}=0$, $K_{\psi d}=0$ b) $K_{W1}=10$, $K_{\psi 1}=5$, $K_{Wd}=0.1$, $K_{\psi d}=1$

Fig. 4. Natural frequencies mode shapes for a two-span elastically restrained rotating Timoshenko beams, with elliptical cross section and quadratic height variation along the axis. $l_1/L=l_2/L=1/2$; $h_{B1}/h_{A1}=1/2$; $h'_{B1}=0$; $h_{A2}/h_{B1}=1/2$; $h_{B2}/h_{A2}=1/2$; $h'_{A2}=0$; $\eta=10$

Figure 4 shows the first four natural frequency mode shapes for beams, with two different kinds of boundary conditions: a) corresponds to $K_{W1}=10$, $K_{\psi 1}=5$, $K_{Wd}=0$, $K_{\psi d}=0$, while b) corresponds to $K_{W1}=10$, $K_{\psi 1}=5$, $K_{Wd}=0.1$, $K_{\psi d}=1$.

The next Figures, 5 and 6, present the variation of the fundament frequency parameter Ω_1 with the variation of the non-dimensional rotational speed η and the spring constant $K\psi_1$.

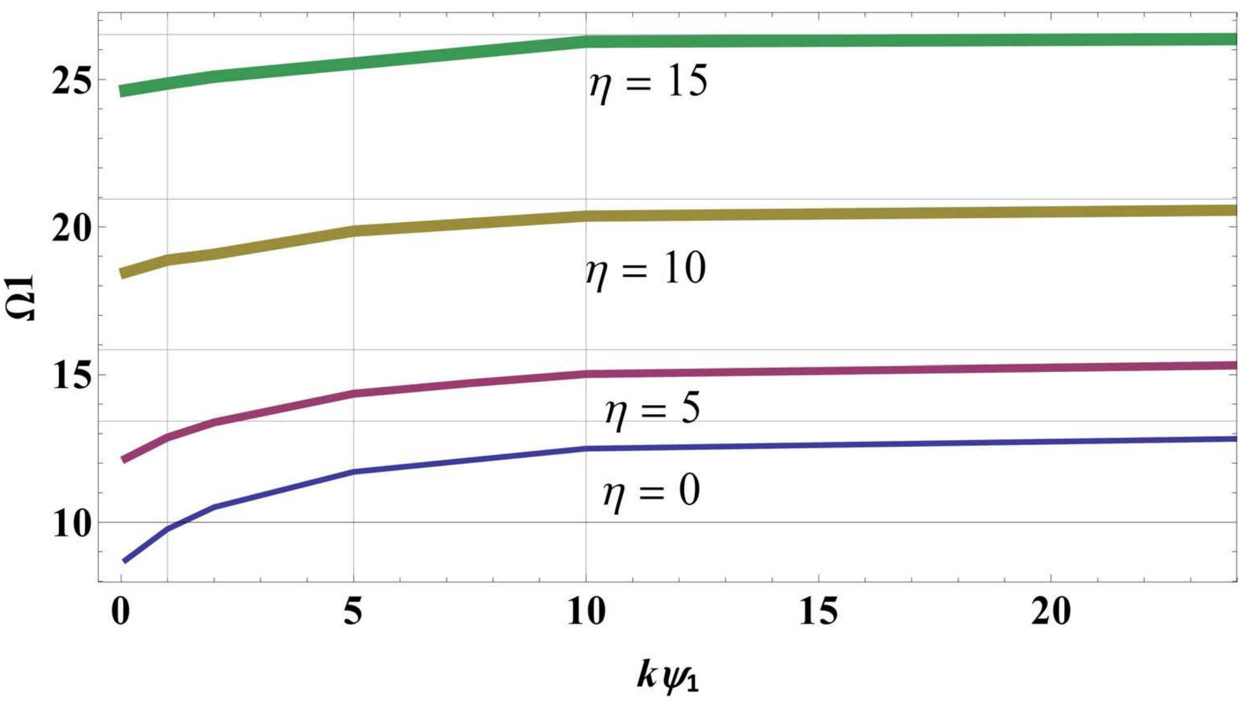


Fig. 5. The fundamental frequency coefficient Ω_1 of a one-span elastically restrained rotating Timoshenko beam versus the spring constant parameter of the rotational spring $K_{\psi 1}$, for different rotational speed parameters η. K_{w1}= 10; K_{wd}=1; $K_{\psi d}$=10

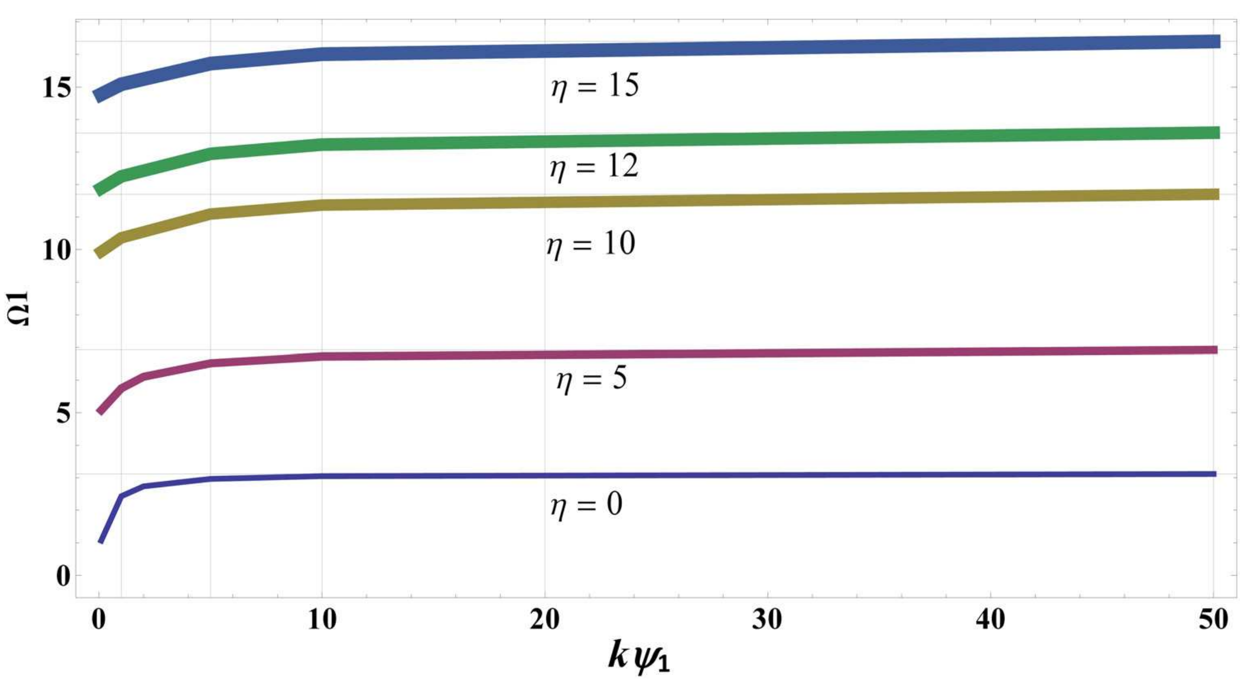


Fig. 6. The fundamental frequency coefficient Ω_1 of a two-span elastically restrained rotating Timoshenko beam versus the spring constant parameter of the rotational spring $K_{\psi 1}$, for different rotational speed parameters η. K_{w1}= 10; K_{wd}=0; $K_{\psi d}$=0

6. Conclusion

The differential quadrature method proves to be very efficient to obtain frequencies and mode shapes of natural vibration, for the rotating Timoshenko beam model.

The versatility of the proposed beam model (variable cross section, step change in cross section, elastic restraints at both ends) allows to solve a large number of individual cases.

Something interesting to point out is that because the method directly solves two ordinary differential equations, additional restrictions are not generated. This does not happen in other methodologies, such as the dynamic stiffness method (Banerjee, 2000, 2001).

As a matter of fact, the differential quadrature method has the same advantage as the finite element method and it needs less computer memory requirements than the FEM.

In particular the present results show that the frequency coefficients vary more significantly when the translational spring stiffness changes at the end of the beam farthest from the axis of rotation $K_{\psi d}$.

7. Appendix A

As Shu presents in his book (Shu, 2000), the differential quadrature method, DQM, is a numerical technique for solving differential equations.

In order to obtain the DQM analog equations to the governing equations of the rotating beam and its boundary conditions, the beam domain is discretized in a grid of points using the Chebyshev – Gauss - Lobato expression, (Shu & Chen, 1999):

$$x_i = \frac{1-\cos\left[(i-1)\pi/(n-1)\right]}{2} \quad ; \; i = 1,2,\ldots,n$$

where n is the number of discrete points or nodes and x_i is the coordinate of node i.

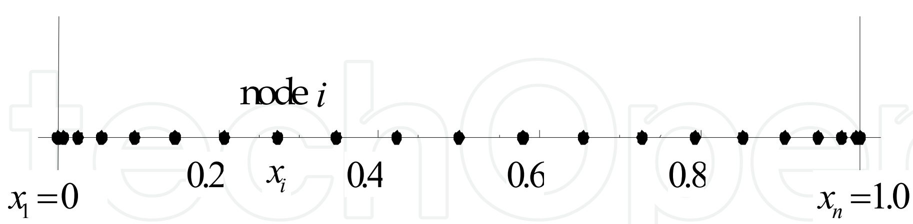


Fig. A1. Grid of n points

The weighting coefficients $A_{ij}^{(1)}$ and $A_{ij}^{(2)}$, which appeared in the linear algebraic equations of quadrature (28-35), were determined using the explicit expressions cited by (Bert & Malik, 1996).

The coefficients $A_{ij}^{(1)}$ correspond to first order derivatives and can be arranged in a square matrix of order n.

The matrix elements $A_{ij}^{(1)}$ with $i \neq j$, are determined by:

$$A_{ij}^{(1)} = \frac{\Pi(x_i)}{(x_i - x_j)\,\Pi(x_j)}$$

where

$$\Pi(x_i) = \prod_{\nu=1 \;\; \nu \neq i}^{n} (x_i - x_\nu) \;;\; \Pi(x_j) = \prod_{\nu=1 \;\; \nu \neq i}^{n} (x_j - x_\nu) \;;$$

The coefficients $A_{ij}^{(1)}$ with $i = j$, will tend to infinity and need to be calculated in another way.

The coefficients $A_{ij}^{(2)}$ correspond to second-order derivatives and are obtained from

$$A_{ij}^{(2)} = 2\left[A_{ii}^{(1)} * A_{ij}^{(1)} - \frac{A_{ij}^{(1)}}{x_i - x_j} \right]$$

with $i \neq j$ and $i,\ j = 1, 2, 3, \ldots, n$.
Because the sum of the weighting coefficients of a row of the matrix is zero, it is easy to calculate the diagonal terms of derivatives of any order q, using the following expression:

$$A_{ii}^{(q)} = -\sum_{j=1 \;\; j \neq i}^{n} A_{ij}^{(q)}$$

And the equations for q equal to 1 and 2, corresponding to first and second order derivatives, are:

$$A_{ii}^{(1)} = -\sum_{j=1 \;\; j \neq i}^{n} A_{ij}^{(1)} \;;\; A_{ii}^{(2)} = -\sum_{j=1 \;\; j \neq i}^{n} A_{ij}^{(2)}$$

8. Acknowledgment

The authors gratefully acknowledge the support of the Universidad Nacional del Sur (UNS) and the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina.

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