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Topological Electromagnetism: Knots and Quantization Rules

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1. Introduction

In this chapter, we revise the main features of a topological model of electromagnetism, also called the model of electromagnetic knots, that was presented in 1989 (Rañada, 1989) and has been developed in a number of references. Some of them are (Arrayás & Trueba, 2010; 2011; Irvine & Bouwmeester, 2008; Rañada, 1990; 1992; Rañada & Trueba, 1995; 1997; 2001; Rañada, 2003). One of the main characteristics of this model is that it allows to obtain interesting topological quantization rules for the electric charge (Rañada & Trueba, 1998) and the magnetic flux through a superconducting ring (Rañada & Trueba, 2006). We will pay special attention to these features.

An electromagnetic knot is defined as a standard electromagnetic field with the property that any pair of its magnetic lines, or any pair of its electric lines, is a link with linking number ℓ. This number is a measure of how much the force lines curl themselves the ones around the others. These lines coincide with the level curves of a pair of complex scalar fields $\phi(\mathbf{r},t)$, $\theta(\mathbf{r},t)$. In the model of electromagnetic knots, the physical space and the complex plane are compactified to S^3 and S^2, so that the scalars can be interpreted as maps $S^3 \mapsto S^2$, which are known to be classified in homotopy classes characterized by the integer value of the Hopf index n, which is related to the linking number ℓ.

The topological model of electromagnetism is locally equivalent to Maxwell's standard theory in the sense that the set of electromagnetic knots coincides locally with the set of the standard radiation fields (radiation fields are electromagnetis fields such that the magnetic field is orthogonal to the electric field at any point and at any instant of time). In other words, standard radiation fields can be understood as patched together electromagnetic knots. This can still be expressed as the statement that, in any bounded domain of space-time, any standard radiation fields can be approximated arbitrarily enough by electromagnetic knots.

It is remarkable that the standard Maxwell's equations are the *exact linearization*, by change of variables *not by truncation*, of a set of nonlinear equations referring to the complex scalar fields $\phi(\mathbf{r},t)$ and $\theta(\mathbf{r},t)$. The fact that this change is not completely invertible has the surprising consequence that the linearity of the Maxwell's equations is compatible with the existence of topological constants of the motion which are nonlinear in the magnetic and electric fields. In this chapter we will see how to find some of these topological constants.

2. Electromagnetic knots

As said before, the topological model of electromagnetic knots makes use of two fundamental complex scalar fields $\phi(\mathbf{r},t)$ and $\theta(\mathbf{r},t)$, the level curves of which coincide with the magnetic and electric lines, respectively. This means that each one of these lines are labelled by the constant value of the corresponding scalar. These complex scalar fields are assumed to have only one value at infinity, which is equivalent, from the mathematical point of view, to compactify the three-space to the sphere S^3. Moreover, the complex plane C is also compactified to the sphere S^2. Both compactifications imply that the scalars ϕ and θ can be interpreted (via stereographic projection) as maps $S^3 \rightarrow S^2$, which can be classified in homotopy classes and, as such, be characterized by the value of the Hopf index n. It can be shown that the two scalars have the same Hopf index and that the magnetic (resp. electric) lines are generically linked with the same Gauss linking number ℓ. If μ is the multiplicity of the level curves (i.e. the number of different magnetic (resp. electric) lines that have the same label ϕ (resp. θ)), then $n = \ell\mu^2$; the Hopf index can thus be interpreted as a generalized linking number if we define a line as a level curve with μ disjoint components.

From the dimensionless scalars $\phi(\mathbf{r},t)$ and $\theta(\mathbf{r},t)$, one can construct a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$ as

$$\begin{aligned}\mathbf{B}(\mathbf{r},t) &= \frac{\sqrt{a}}{2\pi i}\frac{\nabla\phi\times\nabla\bar{\phi}}{(1+\bar{\phi}\phi)^2},\\ \mathbf{E}(\mathbf{r},t) &= \frac{\sqrt{a}c}{2\pi i}\frac{\nabla\bar{\theta}\times\nabla\theta}{(1+\bar{\theta}\theta)^2},\end{aligned} \tag{1}$$

where $\bar{\phi}$ and $\bar{\theta}$ are the complex conjugates of ϕ and θ respectively, i is the imaginary unit, a is a constant introduced so that the magnetic and electric fields have correct dimensions, and c is the speed of light in vacuum. In the SI of units, a can be expressed as a pure number times the Planck constant $\hbar$ times the speed of light c times the vacuum permeability μ_0.

In order to obtain a solution of the Maxwell's equations in vacuum from the fields given by Equations (1), they also have to satisfy

$$\begin{aligned}\mathbf{B}(\mathbf{r},t) &= \frac{\sqrt{a}}{2\pi i c(1+\bar{\theta}\theta)^2}\left(\frac{\partial\bar{\theta}}{\partial t}\nabla\theta - \frac{\partial\theta}{\partial t}\nabla\bar{\theta}\right),\\ \mathbf{E}(\mathbf{r},t) &= \frac{\sqrt{a}}{2\pi i(1+\bar{\phi}\phi)^2}\left(\frac{\partial\phi}{\partial t}\nabla\bar{\phi} - \frac{\partial\bar{\phi}}{\partial t}\nabla\phi\right).\end{aligned} \tag{2}$$

Equations (1) and (2) constitute the definition of an electromagnetic knot, and the magnetic and the electric fields resulting from these equations satisfy exactly Maxwell's equations in vacuum.

It is possible to write Equations (1) and (2) in a more compact way by using the language of differential forms (a nice reference in which Electromagnetism is written in this language is (Hehl & Obukhov, 2003)). If $\mu, \nu = 0, 1, 2, 3$ are space-time indices and $i, j, = 1, 2, 3$ are purely space indices, $A^\mu = (V/c, \mathbf{A})$ (in which V is the electrostatic potential and $\mathbf{A}$ is the vector potential) is the 4-vector potential of the electromagnetic field, so that the electromagnetic tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3}$$

in which $x_0 = ct$. From this tensor one finds the components of the electric field as $\mathbf{E}_i = c\, F^{i0}$, and the magnetic field as $\mathbf{B}_i = -\epsilon_{ijk} F^{jk}/2$ as usual. Moreover, the dual to the electromagnetic tensor is defined as

$$G_{\mu\nu} = {}^{*}F_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}, \tag{4}$$

with components $\mathbf{B}_i = G^{0i}$, $\mathbf{E}_i = -c\,\epsilon_{ijk}\, G^{jk}/2$. Now, the Faraday 2-form is defined as

$$\mathcal{F} = \frac{1}{2}\, F_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{5}$$

and its dual 2-form is defined as

$$* \mathcal{F} = \frac{1}{2}\, G_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{6}$$

Because of clarity, we will use in this work *natural units*, in which the speed of light c, the Planck constant $\hbar$, the vacuum permittivity ε_0 and the vacuum permeability μ_0 are chosen as $c = \hbar = \varepsilon_0 = \mu_0 = 1$. In this system of units, the constant a in Equations (1) and (2) is a pure number. In the language of differential forms, Equations (1) and (2) simply and remarkably mean that the Faraday form $\mathcal{F}$ and its dual $*\mathcal{F}$ of any electromagnetic knot are the two pull-backs of σ, the area 2-form in S^2, by the maps ϕ and θ from S^3 to S^2, i. e.

$$\begin{aligned} \mathcal{F} &= -\sqrt{a}\,\phi^*\sigma, \\ *\mathcal{F} &= \sqrt{a}\,\theta^*\sigma. \end{aligned} \tag{7}$$

As a consequence the two maps are dual to one another in the sense that

$$* \left(\phi^*\sigma\right) = -\theta^*\sigma, \tag{8}$$

$*$ being the Hodge or duality operator. The existence of two maps satisfying Equation (8) guarantees that both $\mathcal{F}$ and $*\mathcal{F}$ obey the Maxwell equations in empty space without the need of any other requirement. The electromagnetic fields obtained as in Equations (7) are electromagnetic knots. They are radiation fields , i. e. they verify the condition $\mathbf{E} \cdot \mathbf{B} = 0$. Note that, because of the Darboux theorem, any electromagnetic field in empty space can be expressed locally as the sum of two radiation fields.

As stated before, the model of electromagnetic knots is locally equivalent to Maxwell's standard theory (Rañada & Trueba, 1998; Rañada, 2003). However, its difference from the global point of view has interesting consequences, as are the following topological quantizations:

- The electromagnetic helicity $\mathcal{H}$ is quantized. In natural units,

$$\mathcal{H} = \frac{1}{2} \int_{R^3} \left(\mathbf{A} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{E}\right)\, d^3r = na, \tag{9}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$, $\mathbf{E} = \nabla \times \mathbf{C}$, the integer n being equal to the common value of the Hopf indices of ϕ and θ. Note that $\mathcal{H} = N_R - N_L$, where N_R and N_L are the classical expressions of the number of right- and left-handed photons contained in the field (i.e. $\mathcal{H} = N_R - N_L = \int d^3k(\bar{a}_R a_R - \bar{a}_L a_L)$, $a_R(\mathbf{k}), a_L(\mathbf{k})$ being Fourier transforms of A_μ in the classical theory, but creation and annihilation operator in the quantum version). This implies that, if we take the constant a to be $a = 1$,

$$n = N_R - N_L, \tag{10}$$

which is a curious relation between the Hopf index (i.e. the generalized linking number) of the classical field and the classical limit of the difference $N_R - N_L$. This difference has a clear topological meaning, what is attractive from the intuitive physical point of view.

- The topology of the model of electromagnetic knots implies also the quantization of the electromagnetic energy in a cavity, as studied in Reference (Rañada, 2003). More precisely, the model predicts that its energy $\mathcal{E}$ in a cubic cavity verifies

$$\mathcal{E} = n\omega, \tag{11}$$

with $n = d/4$, d being an integer, equal to the degree of a certain map between two orbifolds, and ω is the angular frequency of the electromagnetic radiation. This rule is different from the Planck-Einstein law but very similar.

- The model of electromagnetic knots explains the discretization of the values of the electric charge and the magnetic flux through a superconducting ring. These properties will be studied in the next sections of this work.

3. The problem of the quantization of the electric charge

It is a experimental fact that electric charge is discrete. The theoretical prediction of this fact has been linked to the existence of magnetic monopoles. So far there is not any evidence of the existence of monopoles, although some modern unified theories of cosmology and fundamental interactions imply the existence of magnetic monopoles.

In the next section we will present a theoretical argument for the quantization of the electric charge where there is not need for the existence of a magnetic charge or quantum mechanics. However, in this section we also present the standard arguments of the electric charge quantization. We advice to consult the bibliography, specially (Jackson, 1998) and (Schwinger et al., 1998) for more details.

3.1 Thomson's calculation of the angular momentum

J. J Thomson considered in (Thomson, 1904) the electromagnetic field of a system consisting in a magnetic pole and an electric charge. He calculated the momentum and the angular momentum of the electromagnetic field. Then, from its conservation he deduced the magnetic part of the Lorentz force.

Let us assume that we have a magnetic pole g at the point A and a electric charge e at the point B both at rest. We have then that at an arbitrary point P of the space the electric field and magnetic fields are given, in natural units, by

$$\mathbf{E} = \frac{e}{4\pi}\frac{\mathbf{r}_1}{r_1^3},$$

$$\mathbf{B} = \frac{g}{4\pi}\frac{\mathbf{r}_2}{r_2^3} \tag{12}$$

where $\mathbf{r}_1 = \mathbf{r}_P - \mathbf{r}_A$ and $\mathbf{r}_2 = \mathbf{r}_P - \mathbf{r}_B$. It is very interesting to see how Thomson assumed in this work that the magnetic field produced by a magnetic pole was of the Coulomb type. In (Thomson, 1904), the author cites Coulomb and Gauss to provide the experimental proof. The fact that he assumed that, in a magnet, the total magnetic charge has to be zero led him to get the right answers.

The linear momentum of the field is

$$\mathbf{P}_f = \int \mathbf{E} \times \mathbf{B}\, d^3 r = 0, \tag{13}$$

as the linear momentum field lines are circles with their centres along the line AB and their planes at right angles to it. The angular momentum of the electromagnetic field is defined as

$$\mathbf{L}_f = \int \mathbf{r} \times (\mathbf{E} \times \mathbf{B})\, d^3 r. \tag{14}$$

Since the total linear momentum is null, the total angular momentum will be independent of the point chosen to calculate it, according to Classical Mechanics. It will point in the direction of the line AB. To evaluate it, we take origin at the position of the magnetic pole A, the axis z as the line AB, and we have $\mathbf{L}_f = L_z\, \hat{\mathbf{z}}$ with

$$L_z = \frac{eg}{(4\pi)^2} \int \frac{\sin\theta}{r|\mathbf{r} - \mathbf{R}|^2} \sin\alpha\, d^3 r, \tag{15}$$

where $\mathbf{R} = R\,\hat{\mathbf{z}}$ is the position of the electric charge at B, $\theta = \angle PAB$ and $\alpha = \angle APB$. Using spherical coordinates and the law of sines, it turns out that

$$L_z = \frac{egR}{8\pi} \int_0^\infty \int_0^\pi \frac{r \sin^3\theta}{(r^2 + R^2 - 2Rr\cos\theta)^{3/2}}\, dr\, d\theta. \tag{16}$$

The integral can be calculated by different methods as can be seen in (Adawi, 1976). A change of variables $r = R(\cos\theta + \sin\theta \tan\gamma)$, $\gamma \in \left[\theta - \pi/2, \pi/2\right]$ solves the integral and yields $1/2$, so that

$$\mathbf{L}_f = \frac{eg}{4\pi}\frac{\mathbf{R}}{R}. \tag{17}$$

From the conservation of the total linear and angular momenta of the field plus the system of the pole and the charge, Thomson deduces then the magnetic part $e(\mathbf{v} \times \mathbf{B})$ of the Lorentz force over the charge. Note that Jackson follows the converse argument, starting from the Lorentz force between a monopole and a charge, to get the same result.

3.2 The semiclassical quatization rules by Saha and Wilson

Thomson result (17) was used by Saha (Saha, 1949) and independently by Wilson (Wilson, 1949) to get the same quantization condition that Dirac had obtained earlier (we will revise Dirac's argument below). The idea is that, from quantum mechanics, the angular momentum is quantized. Using Saha words, if we apply the *quantum logic*, identifying the angular momentum of the field created by a charge and a monopole with the quantum number for the angular momentum, we get the Dirac result in natural units,

$$eg = 2\pi n, \tag{18}$$

so the existence of a monopole implies the quantization of the charge. For further considerations of the role of the angular momentum and its conservation in the monopole problem, we will refer to the work (Goldhaber, 1965).

3.3 Dirac's argument

Now the turn for the source: Dirac's consideration about the wave function of a particle (Dirac, 1931; 1948). A particle in quantum mechanics is represented by a wave function

$$\psi = Ae^{i\gamma} \tag{19}$$

where A and γ are real functions of $\mathbf{r}$ and t, denoting the amplitude and the phase respectively. The physical meaning of the wave function, according to the quantum postulates, allows for an arbitrary numerical constant coefficient that we can choose to be of modulus unity. So we can add to the phase γ an arbitrary function β. This arbitrary function β does not have to be a unique value in each point $(\mathbf{r}, t)$, as if we go around a closed curve could change, but this change has to be the same for all the wave functions or vary for different wave functions in multiples of 2π, otherwise will have physical consequences such as interference between states. But it has to have definite derivatives as it has to be a solution of a quantum wave equation.

Following Dirac, we will introduce the four vector κ^μ as

$$\kappa_x = \frac{\partial \beta}{\partial x},\ \kappa_y = \frac{\partial \beta}{\partial y},\ \kappa_z = \frac{\partial \beta}{\partial z},\ \kappa_t = \frac{\partial \beta}{\partial t}, \tag{20}$$

and they have to be well defined as stated above. Thus the change in phase round a close curve in the 4-D space, where the vector κ^μ is defined, can be calculated as

$$\oint \kappa_\mu \, ds^\mu = 2\pi n. \tag{21}$$

If we take the close curve very small, the continuity of the wave function imposes the value $n = 0$ for a simple connected domain, as the integration domain reduces to a point. However, if there are points where the wave function vanishes, then the phase would have not meaning. Since the wave function is a complex number, we need two conditions for its vanishing, so we well have in general a nodal line. But now, if we take the closed curve for the integration in (21) around such a line, the continuity considerations are not longer able to tell us that the phase change must be zero. All we can say is that the change will be $2\pi n$ being n an integer, positive or negative depending on the defined orientation.

On the other hand, we can apply Stoke's theorem to the circulation in Equation (21) to write it as

$$\oint \kappa_\mu \, ds^\mu = \int_S (\mathrm{curl}\, \kappa)_j \; dS^j, \tag{22}$$

where the domain is any hypersurface bounded by the closed curve, and the $(\mathrm{curl}\, \kappa)_j$ is a 6-D vector that we can write in three dimensional vector notation as

$$\begin{aligned} \nabla \times \mathbf{k} &= e\, \mathbf{B} \\ \nabla \kappa_0 - \frac{\partial \mathbf{k}}{\partial t} &= e\, \mathbf{E}, \end{aligned} \tag{23}$$

where $\mathbf{k} = (\kappa_x, \kappa_y, \kappa_z)$. We can identify, as the notation in (23) suggests, this curl with an electromagnetic field given by the electromagnetic potentials $(V, \mathbf{A}) = (-\kappa_0, \mathbf{k})/e$. This can be seen clearer calculating the momentum using Equation (19) with the arbitrary phase β,

$$\mathbf{P} = -i \nabla \left(\psi e^{i\beta} \right) = e^{i\beta} (-i \, \nabla \psi + \mathbf{k}) = \mathbf{p} + e\mathbf{A}. \tag{24}$$

The interpretation of the phase curl as an electromagnetic field as far reaching consequences, as Dirac noted. If the close curve is taken in three-dimensional space, only the magnetic flux will come to play so, from Equations (21) and (22), one obtains

$$e \int_S \mathbf{B} \cdot d\mathbf{S} = 2\pi n. \tag{25}$$

So the magnetic flux through any surface bounded by the curve will be equal to the phase shift difference of the wave equation. We have seen that if there is not any nodal line across the surface defined by the curve, the phase difference is equal zero. If we take a closed surface around a nodal line, in the case that the nodal line comes in and out, and that difference should be again zero. But if the nodal line had an end, and we take the close surface around that end, then the phase shift will be nonzero. But that would mean that the there is a net magnetic flux crossing a closed surface, so there is a magnetic charge or monopole inside the surface. The magnetic flux can be written as g, being g the strength of the magnetic pole. Then we get, from Equation (25),

$$eg = 2\pi n, \tag{26}$$

which is the same condition as in Equation (18) that we got with the semiclassical rule. There is a nice account of bibliography related to the monopole problem in Reference (Goldhaber & Trower, 1990).

4. Quantization of the electric charge in the model of electromagnetic knots

A topological mechanism for the quantization of the charge in the model of electromagnetic knots can be seen in Reference (Rañada & Trueba, 1998). Quantization of charge is usually stated by saying that the electric charge of any particle is an integer multiple of a fundamental value e, the electron charge, whose value in the International System of Units is $e = 1.6 \times 10^{-19}$ C. The Gauss theorem allows a different, although fully equivalent, statement of this property:

the electric flux across any closed surface Σ which does not intersect any charge is always an integer multiple of e. This can be written as

$$\int_{\Sigma} \mathbf{E} \cdot \mathbf{n}\, dS = ne, \tag{27}$$

where $\mathbf{n}$ is a unit vector orthogonal to the surface, $\mathbf{E}$ is the electric field and dS the surface element. We could as well write Equation (27) as

$$\int_{\Sigma} *\mathcal{F} = ne, \tag{28}$$

$*\mathcal{F}$ being the dual to the Faraday 2-form $\mathcal{F} = 1/2\, F_{\mu\nu} dx^{\mu} \wedge dx^{\nu}$. Stating in this way the discretization of the charge is interesting because it shows a close similarity with the expression of the topological degree of a map. Assume that we have a regular map θ of Σ on a 2-sphere S^2 and let σ be the normalized area 2-form in S^2. It then happens that

$$\int_{\Sigma} \theta^* \sigma = n, \tag{29}$$

$\theta^*\sigma$ being the pull-back of σ and n an integer called the degree of the map, which gives the number of times that S^2 is covered when one runs once through Σ (equal to the number of points in Σ in which θ takes any prescribed value). The comparison of Equations (28) and (29) shows that there is a close formal similarity between the dual to the Faraday 2-form and the pull-back of the area 2-form of a sphere S^2.

Suppose that an electromagnetic field is given, such that its form $*\mathcal{F}$ is regular except at the positions of some point charges. Suppose also that we have a map $\theta : R^3 \mapsto S^2$ which is regular except at some point singularities where its level curves converge or diverge. It happens then that Equations (28) and (29) are simultaneously satisfied for all the closed surfaces Σ which do not intersect any charge or singularity. This means that the electric charge will be automatically and topologically quantized in a model in which these two forms $*\mathcal{F}$ and $\theta^*\sigma$ are proportional, the fundamental charge being equal to the proportionality coefficient and the number of fundamental charges in a volume having then the meaning of a topological index.

This is exactly what happens in the topological model of electromagnetic knots. In it, the dual to the Faraday 2-form is expressed as

$$*\mathcal{F} = \sqrt{a}\, \theta^* \sigma, \tag{30}$$

where a is a normalizing constant, that is a pure number in natural units, or proportional to the product $\hbar c \mu_0$ in the International System of Units. The electric field is then $\mathbf{E} = \sqrt{a}\, c\, (2\pi i)^{-1} (1 + \bar{\theta}\theta)^{-2} \nabla\bar{\theta} \times \nabla\theta$, the electric lines being therefore the level curves of θ. The degree of the map $\Sigma \mapsto S^2$ induced by θ is given by Equation (29). Therefore,

$$\int_{\Sigma} *\mathcal{F} = n\sqrt{a}. \tag{31}$$

As this is equal to the charge Q inside Σ, it does happen that $Q = n\sqrt{a}$, what implies that there is then a fundamental charge $q_0 = \sqrt{a}$, the degree n being the number of fundamental charges inside Σ. This gives a topological interpretation of n, the number of fundamental charges inside any volume.

It is easy to understand that $n = 0$ if θ is regular in the interior of Σ. This is because each level curve of θ, i. e. each electric line, is labeled by its value along it (a complex number) and, in the regular case, any one of these lines enters into this interior as many times as it goes out of it. But assume that θ has a singularity at point P, from which the electric lines diverge or to which they converge. If Σ is a sphere around P, we can identify R^3 except P with $\Sigma \times R$, so that the induced map $\theta : \Sigma \mapsto S^2$ is regular. In this case, n need not vanish and is equal to the number of times that θ takes any prescribed complex value in Σ, with due account to the orientation. Otherwise stated, among the electric lines diverging from or converging to P, there are $|n|$ whose label is equal to any prescribed complex number.

To understand better this mechanism of discretization, let us take the case of a Coulomb potential as in Reference (Rañada & Trueba, 1997): $\mathbf{E} = Q\mathbf{r}/(4\pi r^3)$, $\mathbf{B} = 0$. The corresponding scalar is

$$\theta = \tan\left(\frac{\vartheta}{2}\right) \exp\left(i\frac{Q}{\sqrt{a}}\varphi\right), \tag{32}$$

where φ and ϑ are the azimuth and the polar angle. The scalar (32) is well defined only if $Q = n\sqrt{a}$, n being an integer. The lines diverging from the charge are labeled by the corresponding value of θ, so that there are $|n|$ lines going in or out of the singularity and having any prescribed complex number as their label. If $n = 1$, it turns out that $\theta = (x + iy)/(z + r)$.

This mechanism has a very curious aspect: it does not apply to the source but to the electromagnetic field itself. This is surprising since one would expect that the topology should operate restricting the fields of the charged particles. However, in this model, it is the field who mediates the force the one which is submitted to a topological condition. It must be emphasized furthermore that the maps $S^3 \mapsto S^2$, given by the two scalars ϕ, θ are regular except for singularities at the position of point charges, either electrical or magnetic (if the latter do exist). At these points, the level curves (the electric lines) converge or diverge.

In the case that the value of a in natural units is $a = 1$ (in order to obtain the right quantization of the electromagnetic helicity), the topological model of electromagnetic knots predicts that the fundamental charge has the value

$$q_0 = 1, \tag{33}$$

which is about 3.3 times the electron charge. Note that this applies both to the electron charge and to the hypothetical monopole charge. This property can be stated saying that, in the topological model, the electromagnetic fields can only be coupled to point charges which are integer multiple of the fundamental charge $q_0 = 1$. Note that the same discretization mechanism would apply to the hypothetical magnetic charges (located at singularities of ϕ), their fundamental value being also $q_0 = 1$.

5. Quantization of the magnetic flux in the model of electromagnetic knots

Electromagnetic knots are compatible with the quantization of the magnetic flux of a superconducting ring, which in standard theory is always an integer multiple of $g/2$, where $g = \sqrt{a}$ (or $g = 1$ in natural units) is the value of the magnetic monopole in the topological model of electromagnetic knots. The mechanism of quantization was stablished in Reference (Rañada & Trueba, 2006). To understand how this mechanism of quantization works, let us begin with the case of an infinite solenoid.

5.1 Flux quantization in an infinite solenoid

Consider again the equations for any electromagnetic knot. The Faraday 2-form and its dual generated by the pair of complex scalar fields ϕ and θ can be written, with the constant a fixed to $a = 1$ in natural units, as

$$\begin{aligned} \mathcal{F} &= ds \wedge dp, \quad \text{with } \; p = 1/(1+|\phi|^2), \; s = \arg(\phi)/2\pi \\ {}^*\mathcal{F} &= dv \wedge du, \quad \text{with } \; v = 1/(1+|\theta|^2), \; u = \arg(\theta)/2\pi, \end{aligned} \tag{34}$$

so that $\phi = \sqrt{(1-p)/p}\, e^{i2\pi s}$ and $\theta = \sqrt{(1-v)/v}\, e^{i2\pi u}$. This implies that the magnetic and electric fields have the form

$$\begin{aligned} \mathbf{B} &= \nabla p \times \nabla s = (\partial_0 u \nabla v - \partial_0 v \nabla u)\,, \\ \mathbf{E} &= \nabla u \times \nabla v = (\partial_0 s \nabla p - \partial_0 p \nabla s)\,. \end{aligned} \tag{35}$$

The quantities (p, s) and (v, u) are called *Clebsch variables* of the fields **B** and **E**, respectively (or of the scalars ϕ and θ). Note that ϕ and θ are not uniquely determined by the magnetic and electric fields. Indeed, a different pair defines the same fields **E**, **B** if the corresponding Clebsch variables (P, S), (V, U) can be obtained through a canonical transformation $(p, s) \to (P, S)$ or $(v, u) \to (V, U)$. However, the canonical transformation must satisfy two conditions: (i) $0 \leq P, V \leq 1$, and (ii) S, U must be arguments of complex numbers in units of 2π, i. e. they can be multivalued but their change along a closed curve must be an integer.

Let us turn to our physical problem. Consider an infinite perfect solenoid around the z-axis with N turns per unit length and intensity I (perfect means that no flux escapes through the coils). This can happen exactly only in a superconducting ring. Indeed, from the purposes of the present study, perfect solenoids and superconducting rings can be considered synonymous. The magnetic field vanishes outside and is equal to $B = \mu_0 NI$ inside. Now let us ask what can be the scalar ϕ (which gives a map $S^3 \mapsto S^2$) that corresponds to that magnetic field if we restrict ourselves to the model of electromagnetic knots. With the configuration of the magnetic lines of that solenoid, it is impossible that ϕ be regular in all the sphere S^3. However, we may consider the 3-space as $S^2 \times R$ and require that ϕ be regular in the induced map $S^2 \mapsto S^2$, the first S^2 being the plane (x, y), the second the complex plane, both completed with the point at infinity. If $\phi = |\phi| \exp(2\pi i s)$ and $p = 1/(1 + |\phi|^2)$, then

$$\mathbf{B} = \nabla p \times \nabla s. \tag{36}$$

As $B = 0$ outside the solenoid, p and s can not be independent functions there. This may happen in three ways:

1. The first possibility is $s = f(p)$, f being a nontrivial function. We can change s to $s - f(p)$. This is a canonical transformation of the variables (s, p) which does not affect the value of B in view of Equation (36). The new expression of ϕ is real outside the solenoid, but not inside in general. Consequently, the magnetic flux across a section of the solenoid is

topologically quantized, being equal to the area of the set $\phi(S)$ in the sphere S^2, where S is any surface that cuts the solenoid and is bordered by a circuit outside it. Indeed its value is necessarily Flux $= n/2$, because any curve contained in a great circle of a sphere encircles a integer multiple of semispheres.

2. The second possibility is $s = s_0 =$ constant. The situation is similar to and gives the same flux quantization as in the previous case (outside the solenoid, ϕ takes values also in a great circle of S^2). That is Flux $= n/2$.
3. The last possibility is $p = p_0 =$ constant. Let $p = p_0$ outside and s variable. Then the scalar would be

$$\phi = \sqrt{\frac{1-p_0}{p_0}}\, e^{i2\pi s(r,\varphi)}, \tag{37}$$

where $r = (x^2+y^2)^{1/2}$ and φ is the azimuth. Moreover,

$$\int_0^{2\pi} \frac{\partial s}{\partial \varphi}\, d\varphi = m, \tag{38}$$

where m is an integer number. In order for ϕ to be a regular map, there are two possibilities: $s = s_0 =$ constant, and s = function of φ but with either p_0 (and $\phi = \infty$) or $p_0 = 1$ (and $\phi = 0$). In both cases, it turns out that Flux $= n/2$.

So, in conclusion, in the topological model of electromagnetism based on electromagnetic knots, the magnetic flux in an infinite perfect solenoid is always an semi-integer multiple of the fundamental magnetic charge q_0 (with $q_0 = 1$ in natural units),

$$\text{Flux } = \frac{n}{2}\, q_0. \tag{39}$$

This is interesting, because it says that the flux in the solenoid is necessarily quantized, the fundamental fluxoid being half the fundamental magnetic charge q_0 (as the real fluxoid is half the Dirac monopole). This quantity, however, is $q_0/2 = 1/2$ in natural units, as compared with $g/2 = 10.37$ for the Dirac monopole.

5.2 Flux quantization in a finite solenoid

Let us consider now the case of a superconducting ring, i. e. of a perfect but finite solenoid. We can imagine it as a cylinder around de z-axis between $z = -L/2$ and $z = L/2$ and the radii r_0 and $r_0 + h$, although these magnitudes are quite irrelevant in this case. Since the magnetic field does not enter inside the superconductor, $B = 0$ inside it. If the superconductor is infinitely think (i.e. $h = \infty$), the topology of the problem is the same as in the previous case of infinite solenoid, and all the results are also the same. In the realistic case in which h is finite, there are also three cases. In the two first cases, the result would be the same. However, it is not clear that the same could be said of the third possibility. We have to take a different way for the third possibility.

Consider the quantization of the magnetic flux across a superconducting ring in the standard theory (Feynman et al., 1965). In this case the wave function can be treated as a classical macroscopic field $\psi = \sqrt{\rho}\, e^{i\vartheta}$, the following equation being satisfied

$$\hbar \nabla \vartheta = Q\mathbf{A}, \tag{40}$$

where Q is the charge of a Cooper pair of electrons, equal to $2e$. The flux is thus

$$\text{Flux } = \oint \mathbf{A} \cdot d\mathbf{s} = \frac{2\pi n'}{Q}, \tag{41}$$

where n' is an integer. We see that the fundamental unit of flux is then $2\pi/Q$.

Let us take a finite superconducting ring of cylindrical shape, with axis along the z-axis, between the planes $z = \pm L/2$ and radii r_0 and $r_0 + h$. The interior magnetic field created by the superconducting ring at the central plane $z = 0$ can be written as

$$\mathbf{B} = B(r)\, \hat{\mathbf{z}}, \tag{42}$$

r being the radial coordinate. The magnetic flux across the ring is

$$\text{Flux} = \int_{C_0} B(r)\, r dr d\varphi, \tag{43}$$

where C_0 is the circle of radius r_0. Because of the symmetry of the problem, we can take a scalar $\phi(r, \varphi)$, with $p = 1/(1 + |\Phi|^2)$ and $s = \arg(\Phi)/2\pi$, such that

$$\mathbf{B} = \frac{1}{r} \left(\frac{\partial p}{\partial r} \frac{\partial s}{\partial \varphi} - \frac{\partial p}{\partial \varphi} \frac{\partial s}{\partial r} \right) \hat{\mathbf{z}}. \tag{44}$$

It is convenient to define the dimensionless radial coordinate

$$R = \frac{r}{r_0}, \tag{45}$$

so that, in each plane (R, φ), ϕ can be taken as a map $\phi : C_1 \to S^2$, where C_1 is the circle of unit radius, and

$$\mathbf{B} = \frac{1}{r_0^2} \frac{1}{R} \left(\frac{\partial p}{\partial R} \frac{\partial s}{\partial \varphi} - \frac{\partial s}{\partial \varphi} \frac{\partial s}{\partial R} \right) \hat{\mathbf{z}}. \tag{46}$$

The magnetic flux across the superconductor results

$$\text{Flux } = \int_{C_1} \left(\frac{\partial p}{\partial R} \frac{\partial s}{\partial \varphi} - \frac{\partial p}{\partial \varphi} \frac{\partial s}{\partial R} \right) dR d\varphi. \tag{47}$$

The quantity between brackets in (47) is the Jacobian of the change of variables $(p, s) \to (R, \varphi)$, so that

$$\text{Flux } = \int_{\phi(C_1)} dp ds, \tag{48}$$

where $\phi(C_1)$ is the image in S^2 of the unit circle C_1.

In the framework of London's theory of type II superconductors (the case in which the magnetic flux is quantized), the magnetic field in the superconductor satisfies a phenomenological equation in the transition layer in which the magnetic field goes to zero.

This is the second London equation,

$$\mathbf{A} = -\lambda^2 \nabla \times \mathbf{B}, \tag{49}$$

where $\mathbf{A}(\mathbf{r})$ obeys Coulomb gauge and λ is the penetration length of the magnetic field inside the superconductor material (in practice, λ is about ten Angstroms, much shorter than the inner radius of the superconductor ring r_0).

From Equation (44), the vector potential $\mathbf{A}(\mathbf{r})$ for the magnetic field $\mathbf{B}(\mathbf{r})$ in the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) can be written as

$$\mathbf{A} = \frac{p}{r} \frac{ds}{d\varphi} \mathbf{u}_\varphi. \tag{50}$$

It follows that $s = s(\varphi)$ and $p = p(r)$. Furthermore, the quantity $\int_0^{2\pi} A_\varphi r \, d\varphi$ has to be independent of r inside the superconductor. From these considerations one obtains

$$p = p_0, \; s = n \frac{\varphi}{2\pi}. \tag{51}$$

Inserting Equation (50) into London Equation (49), we obtain the following ordinary differential equation for $p(r)$,

$$\lambda^2 \left(\frac{d^2 p}{dr^2} - \frac{1}{r} \frac{dp}{dr} \right) - p = 0. \tag{52}$$

Up to first order in λ / r_0, we can neglect the first term in (52) to obtain

$$p(r) = 0, \; r \geq r_0, \tag{53}$$

characterizing p inside the superconductor. As the Clebsch variable p has to be continuous and constant inside the superconductor, with a value $p = p_0$, we obtain $p_0 = 0$, i.e. $\phi = \infty$. In the model of electromagnetic knots, if an electromanetic field is generated by the scalar field ϕ and the Clebsch variables (p, s), it is also generated by the scalar $1/\bar{\phi}$ and the Clebsch variables $(1 - p, -s)$. In the latter case, Equation (53) would be $1 - p(r) = 0$, $r \geq r_0$, so that $p_0 = 1$ and $\phi = 0$ inside the superconductor.

Consequently, the value of the scalar field ϕ inside the superconductor is $\phi = \infty$ or $\phi = 0$. In both cases, the magnetic flux is

$$\text{Flux} \; = \int_{C_1} A(R) \, r_0 R \, d\varphi = \int_0^{2\pi} \frac{n}{2\pi} \, d\varphi = n. \tag{54}$$

If we consider the solutions given by the families 1 and 2 at the begining of this subsection, it results that the magnetic flux is quantized, being always an integer multiple of 1/2.

The previous argument relies on London's equation. However, the same conclusion can be reached considering the following. The radial derivative of p is in general discontinuous at r_0. However, this irregularity in the map ϕ is vanished if either $\phi = 0$ or $\phi = \infty$ inside the superconductor. Therefore, the requirement that the map is regular leads to the topological quantization of the flux, without taking into account the London equation.

5.3 The fine structure constant at infinite energy equal to $1/4\pi$?

Because the topological model here presented is classical, the electric charge $q_0 = \sqrt{\hbar c \epsilon_0}$ must be interpreted as the fundamental bare charge, both electric and magnetic (remember that we are using natural units). The corresponding fine-structure constant $\alpha = q_0^2/4\pi\hbar c\epsilon_0$ is clearly equal to $\alpha_0 = 1/4\pi$, which is certainly a nice and simple number. We will argue now that $1/4\pi$ is an appealing and interesting value for the non-renormalized fine-structure constant (i. e. neglecting the effect of the quantum vacuum). As we show now, the topological model seems to describe the electromagnetic field at infinite energy.

The argument goes as follows. Let us combine this topological quantization of the charge with the appealing and plausible idea that, in the limit of very high energies, the interactions of charged particles could be determined by their bare charges, i. e. the values of that their charges would have if they were not renormalized by the quantum vacuum; see e. g. Section 11.8 of Reference (Milonni, 1994). A warning is necessary, however. As the concept of bare charge is not simple, it is convenient to speak instead of charge at a certain scale. To avoid confusion and be precise, the expression “bare charge" will be used here as synonymous or equivalent of “infinite energy limit of the charge" or, more correctly, “charge at infinite momentum transfer Q", defined as $e_\infty = \sqrt{4\pi\hbar c\epsilon_0\alpha_\infty}$, where $\alpha_\infty = \lim \alpha(Q^2)$ when $Q^2 \to \infty$.

The possibility of a finite value for α_∞ is an interesting idea worth of consideration. In fact, it was discusseed by Gell-Mann and Low in their classical and seminal paper “QED at small distances", Reference (Gell-Mann & Low, 1954), in which they showed that it is something to be seriously studied. However, they could not decide from their analysis whether e_∞ is finite or infinite. The current wisdom idea that it is infinite was established later on the basis of perturbative calculations, but the alternative posed by Gell-Mann and Low has not been really settled. It is still open.

The infinite energy charge e_∞ of an electron is partially screened by the sea of virtual pairs that are continuously being created and destroyed in empty space. It is hence said that it is renormalized. Because the pairs are polarized, as are the molecules in a dielectric, a polarization cloud is formed around any charged particle, with the result that the observed value of the electron charge is smaller than e_∞. Moreover, the apparent electron charge increases as any probe goes deeper into the polarization cloud and is therefore less screened. This effect is difficult to measure, since it can be appreciated only at extremely short distances. However it has been observed in experiments of electron-positron scattering at high energies Reference (Levine *et al.*, 1997). This means that the vacuum is dielectric. On the other hand, it is paramagnetic because the effect of the magnetic field is due to the spin of the pairs. The consequence is that the hypothetical magnetic charge would be observed with a greater value at low energy than at very high energy, contrary to the electron charge.

It is easy to understand the reason for the expression “bare charge" to denote e_∞. When two electrons interact with very high momentum transfer, each one is located so deeply inside the polarization cloud around the other that very little space is left between them to screen the charges, so that the bare charges, namely e_∞, interact directly. As unification is is assumed to happen at very high energy, it is an appealing idea that $\alpha_\infty = \alpha_{\mathrm{GUT}}$ (GUT stands for “Grand Unified Theories", that include weak and gravitation ones. This suggests that a unified theory could be a theory of bare particles (i. e. in the sense that it neglects the effect of the vacuum.)

If this were the case, nature would have provided us with a natural cut-off, $\alpha_{\text{GUT}} = \alpha_\infty$. As a consequence, it can be argued that the topological model implies that $\alpha_{\text{GUT}} = \alpha_\infty = 1/4\pi$. The argument goes as follows.

1. The value of the fundamental charge predicted by this topological quantization, $e_0 = \sqrt{\hbar c \epsilon_0} = 5.28 \times 10^{-19}$ C is in the right interval to verify $e_0 = e_\infty = g_\infty$, in other words to be equal to the common value of both the fundamental electric and magnetic bare charges. This is so because, as the quantum vacuum is dielectric but paramagnetic, the following inequalities must be satisfied: $e < e_0 < g$, as it happens since $e = 0.3028$, $e_0 = 1$, $g = e/2\alpha = 20.75$, in natural units. Note that it is impossible to have a completely symmetry between electricity and magnetism simultaneously at low and high energies. The lack of symmetry between the charges of the electron and the Dirac monopole would be due to the vacuum polarization: according to the topological model, the electric and magnetic infinite energy charges are equal and verify $e_\infty g_\infty = e_0^2 = 1$, but they would be decreased and increased, respectively, by the sea of virtual pairs, until their current values that verify $eg = 2\pi$. This qualitative picture seems nice and appealing.

2. Let us admit as a working hypothesis that two charged particles interact with their bare charges at high energies. There could be then a conflict between (a) a unified theory of electroweak and strong forces in which $\alpha = \alpha_s$ and (b) an infinite value of α_∞. The reason is that unification implies that the curves of the running constants $\alpha(Q^2)$ and $\alpha_s(Q^2$ must converge asymptotically to the same value α_{GUT}. It could be argued that, to have unification at a certain scale, it would suffice that these two curves be close in an energy interval, even if they cross and separate afterwards. However in that case the unified theory would be just an approximate accident. On the other hand, the assumption that both running constants go asymptotically to the same finite value gives a much deeper meaning to the idea of unified theory. In that case, e_∞ must expected to be finite, and the equality $\alpha_{\text{GUT}} = \alpha_\infty$ must be satisfied.

3. The value $\alpha_0 = e_0^2/4\pi\hbar c \epsilon_0 = 1/4\pi = 0.0796$ for the infinite energy fine-structure constant is thought-provoking and fitting, since α_{GUT} is believed to be in the interval $(0.05, 0.1)$. This reaffirms the assertion that the fundamental value of the charge given by this topological mechanism e_0 could be equal to e_∞, the infinite energy electron charge (and the infinite energy monopole charge, as well). It also supports the statement that $\alpha_{\text{GUT}} = \alpha_0 = 1/4\pi$. All this is certainly curious and intriguing: indeed, the topological mechanism for the quantization of the charge here described is obtained simply by putting some topology in elementary classical low energy electrodynamics.

We believe, therefore, that the following three ideas must be studies carefully: (1) the complete symmetry between electricity and magnetism at the level of the infinite energy charges, where both are equal to $\sqrt{\hbar c \epsilon_0}$, this symmetry being broken by the dielectric and paramagnetic quantum vacuum; (2) That the topological model on which the topological mechanism of quantization is based could give a theory of high- energy electromagnetism at the unification scale and (3) that the value that this model predicts for the fine-structure constant $\alpha_0 = 1/4\pi$ could be equal to the infinite energy limit α_∞ and also to α_{GUT}, the constant of the unified theory of strong and electroweak interactions.

6. Conclusions

The topological model of electromagnetism constructed with electromagnetic knots is based on the existence of a topological structure which underlies the Maxwell's standard theory, in such a way that the Maxwell's equations in empty space are the exact linearization of some nonlinear equations with topological properties and constants of the motion. Although the model is classical, it embodies the topological quantizations of the helicity and the energy inside a cavity, which suggest that it offers a way to understand better the relation between the classical and quantum aspects of the electromagnetic theory. The model is locally equivalent to Maxwell's standard theory in empty space (but globally non-equivalent). This means that it can not enter in conflict with Maxwell's theory in experiments of local nature.

In the model of electromagnetic knots, the electric charge which is topologically quantized, its fundamental value being $q_0 = 1$ in natural units (or $q_0 = \sqrt{\hbar c \epsilon_0} = 5.28 \times 10^{-19}$ C in the International System of Units). Furthermore, the number of fundamental charges inside a volume is equal to the degree of a map between two spheres. It turns out that there are exactly $|m|$ electric lines going out or coming into a point charge $q = mq_0$, for which a complex scalar field is equal to any prescribed complex number (taking into account the orientation of the map).

The topological model is completely symmetric between electricity and magnetism, in the sense that it predicts that the fundamental hypothetical magnetic charge would be also q_0. Note that $q_0 = 3.3\,e$, where e is the electron charge, and that the corresponding fine structure constant is $\alpha_0 = 1/4\pi$. Hence, q_0 could be interpreted as the bare electron and monopole charge. As the quantum vacuum is dielectric but paramagnetic, the observed electric charge must be smaller than q_0, but the Dirac magnetic monopole must be greater (it is equal to $20.75\,q_0$). This suggests that α_0 could be the fine structure constant at infinite energy and, consequently, that the coupling constant of the Grand Unified Theory could be also $\alpha_s = \alpha_0 = 1/4\pi$.

The model of electromagnetic knots also predicts that the magnetic flux is quantized, the fundamental flux unit being $1/2$ in natural units. Consequently, the relation between the fundamental magnetic flux and electric charge in this model is the same as that between the Dirac monopole and the electron charge in standard theory. The quantum vacuum increases the value of the magnetic fields by a factor $2\pi/e$, according to the above mentioned interpretation. This can be represented by a relative permeability $\mu_r = 2\pi/e = 20.75$ (with respect to the state in which there is neither matter nor radiation and the effect of the zero point radiation has been discounted). The renormalized magnetic flux must be therefore equal to $\mu_r \times$ Flux , where Flux $= 1/2$ is the bare value. This implies that the flux is a multiple integer of π/e, either in standard theory or in the topological theory after multiplying by the permeability μ_r to take care of the effect of the quantum vacuum. Hence, the topological quantization of the magnetic flux coincides with the standard one after introducing a relative permeability to account for the effect of the quantum vacuum. This is fully coherent with the interpretation given in the previous paragraph that the topological model of electromagnetic knots gives a theory of high energy electromagnetism at the unification scale or a theory of bare electromagnetism.

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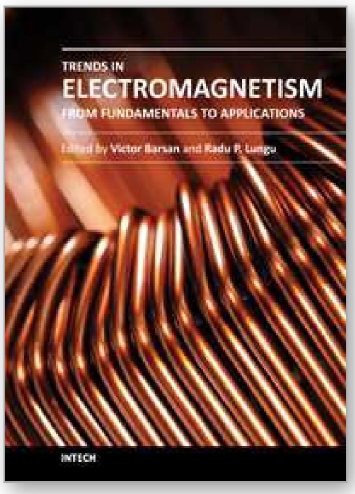

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Among the branches of classical physics, electromagnetism is the domain which experiences the most spectacular development, both in its fundamental and practical aspects. The quantum corrections which generate non-linear terms of the standard Maxwell equations, their specific form in curved spaces, whose predictions can be confronted with the cosmic polarization rotation, or the topological model of electromagnetism, constructed with electromagnetic knots, are significant examples of recent theoretical developments. The similarities of the Sturm-Liouville problems in electromagnetism and quantum mechanics make possible deep analogies between the wave propagation in waveguides, ballistic electron movement in mesoscopic conductors and light propagation on optical fibers, facilitating a better understanding of these topics and fostering the transfer of techniques and results from one domain to another. Industrial applications, like magnetic refrigeration at room temperature or use of metamaterials for antenna couplers and covers, are of utmost practical interest. So, this book offers an interesting and useful reading for a broad category of specialists.