We are IntechOpen,
the world's leading publisher of
Open Access books
Built by scientists, for scientists

4,800
Open access books available

122,000
International authors and editors

135M
Downloads

Our authors are among the

154
Countries delivered to

TOP 1%
most cited scientists

12.2%
Contributors from top 500 universities

BOOK CITATION INDEX
CLARIVATE ANALYTICS
INDEXED

WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com

Some Aspects of the Sentinel Method for Pollution Problems

A. Omrane[*]

Laboratoire CEREGMIA, EA 2440, I.E.S Guyane, Campus Universitaire Trou-Biran,
Route de Baduel BP 792, 97337 Cayenne Cedex
French Guiana

1. Introduction

Modelling environmental problems leads to mathematical systems with missing data: Weather problems have generally missing initial conditions. This chapter is concerned with identifying pollution terms arising in the state equation of some dissipative system with incomplete initial condition.

To this aim the so-called sentinel method is used. We explain how the problemn of determining a sentinel is equivalent to a null-controllability problem for which Carleman inequalities are revisited.

In a second part of the chapter, we use the same techniques to discuss of how to get instantaneous information (at fixed $t = T \in [0, +\infty[$) on pollution terms in distributed systems of incomplete data in some ecology and/or meteorology problems.

We consiter a fixed final time $T > 0$, and Ω an open subset of $\mathbb{R}^d$ of smooth boundary $\partial\Omega$, and we denote by $Q = \Omega \times]0, T[$ the space-time cylinder. We are intersted with systems partially known; we consider here the state equation :

$$\begin{cases} y' - \Delta y + f(y) = \xi + \lambda \hat{\xi} & \text{in } Q, \\ \quad\quad y(0) = y^0 + \tau \hat{y}^0 & \text{in } \Omega, \\ \quad\quad\quad y = 0 & \text{on } \Sigma_1, \\ \quad\quad \dfrac{\partial y}{\partial \nu} = 0 & \text{on } \Sigma_2, \end{cases} \tag{1.1}$$

where $y = y(x, t; \lambda, \tau)$, and where Σ_1 is a piece of the boundary $\Sigma = \partial\Omega \times]0, T[$ and $\Sigma_2 = \Sigma \backslash \Sigma_1$. We assume here that $f : \mathbb{R} \to \mathbb{R}$ is of class C^1, the functions ξ and y^0 are known with $\xi \in L^2(Q)$ and $y^0 \in L^2(\Omega)$. But, the terms : $\lambda \hat{\xi}$ (so-called pollution term) and $\tau \hat{y}^0$ (so-called perturbation term) are unknown, $\hat{\xi}$ and $\hat{y}^0$ are renormalized and represent the size of pollution and perturbation

$$\|\hat{\xi}\|_{L^2(Q)} \le 1, \quad \|\hat{y}^0\|_{L^2(\Omega)} \le 1, \quad\quad \text{so that the reals} \quad \lambda, \tau \quad \text{are small enough.}$$

Some non empty open subset $O \subset \Omega$, is called observatory set. The observation is y on O, for the time T. We denote by y_{obs} this observation

$$y_{obs} = m_o \ \in \ L^2(O \times (0, T)). \tag{1.2}$$

We suppose that (1.1) has a unique solution denoted by $y(\lambda, \tau) := y(x, t; \lambda, \tau)$ in some relevant space. The question is

$$\left| \begin{array}{l} \text{how to calculate the pollution term } \lambda \hat{\xi} \text{ in} \\ \text{the state equation, independently from the} \\ \text{variation } \tau \hat{y}^0 \text{ around the initial data ?} \end{array} \right. \tag{1.3}$$

Least squares. Question (1.3) is natural and leds to some developments; some answer is given by the least squares method. The method consists in considering the unknowns $\{\lambda \hat{\xi}, \tau \hat{y}^0\} = \{v, w\}$ as control variables, then the state $y(x, t; v, w)$ has to be driven as close as possible to m_o.

This comes to some optimal control problem. By this way we look for the pair (v, w), there is then no real possibility to find v or w independently.

Sentinels. The sentinel method of Lions Lions J.-L. (1992) is a particular least squares method which is adapted to the identification of parameters in ecosystems with incomplete data; many models can be found in litterature. The sentinel concept relies on the following three objects : some state equation (for instance (1.1)), some observation function (1.2), and some control function w to be determined.

Many papers use the definition of Lions in the theoretical aspect (see for example Bodart Bodart O. (1994), Bodart-Fabre Bodart O. & Fabre C. (1993)Bodart O. & Fabre C. (1995)), as well as in the numerical one (see Bodart-Demeestere Bodart O. & Demeestere P. (1997), Demeestere Demeestere P. (1997) and Kernevez Kernevez J.-P. (1997)).

We use the techniques in Miloudi Y. et al. (2007) to give an answer to the question (1.3). Let h_0 be some function in $L^2(O \times (0, T))$. Let on the other hand ω be some open and non empty subset of Ω. For a control function $w \in L^2(\omega \times (0, T))$, we define the functional

$$S(\lambda, \tau) = \int_0^T \int_O h_0 \, y(\lambda, \tau) \, dxdt + \int_0^T \int_\omega w \, y(\lambda, \tau) \, dxdt. \tag{1.4}$$

We say that S defines a sentinel for the problem (1.1) if there exists w such that S is insensitive (at first order) with respect the to missing terms $\tau \hat{y}^0$, which means

$$\frac{\partial S}{\partial \tau}(0, 0) = 0 \tag{1.5}$$

for any $\hat{y}^0$ where here $(0, 0)$ corresponds to $\lambda = \tau = 0$, and if w minimizes the norm $\|v\|_{L^2(\omega \times (0, T))}$.

Important Remark. The Lions sentinels S assume $\omega = O$. In this case, the observation and the control share the same support, and the solution $w = -h_0$ is trivial.

The definition (1.4) extends the one by Lions to the case where the observation and the control have different supports. This point of view (where $\omega \neq O$) has been considered for the first time in Nakoulima O. (2004) and Miloudi Y. et al. (2007).

1.1 The informations given by the sentinel

Because of (1.5) we can write

$$S(\lambda, \tau) \simeq S(0,0) + \lambda \frac{\partial S}{\partial \lambda}(0,0), \qquad \text{for} \quad \lambda, \tau \quad \text{small.}$$

In (1.4), $S(\lambda, \tau)$ is observed and using (1.2),

$$S(\lambda, \tau) = \int_Q (h_0 \chi_O + \mathsf{w} \chi_\omega) m_o \, dx dt$$

so that (1.5) becomes

$$\lambda \frac{\partial S}{\partial \lambda}(0,0) \simeq \int_Q (h_0 \chi_O + \mathsf{w} \chi_\omega)(m_o - y_0) \, dx dt, \tag{1.6}$$

with

$$\frac{\partial S}{\partial \lambda}(0,0) = \int_Q (h_0 \chi_O + \mathsf{w} \chi_\omega) y_\lambda \, dx dt,$$

where here χ_O and χ_ω denote the characteristic functions of O and ω respectively.

The derivative $y_\lambda = (\partial y / \partial \lambda)(0,0)$ only depends on $\hat{\xi}$ and other known data. Consequently, the estimates (1.6) contains the informations on $\lambda \hat{\xi}$ (see for details remark 1 below).

2. Null-controllability problem

The existence of a sentinel is equivalent to a null-controllability property. Indeed, we begin by transforming the insensibility condition (1.5).

Denote by

$$y_\tau = \frac{d}{d\tau} y(\lambda, \tau) \Big|_{\lambda=\tau=0}.$$

Then the function y_τ is solution of

$$\begin{cases} y'_\tau - \Delta y_\tau + f'(y_0) y_\tau = 0 & \text{in } Q, \\ y_\tau(0) = \hat{y}^0 & \text{in } \Omega, \\ y_\tau = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial y_\tau}{\partial v} = 0 & \text{on } \Sigma_2, \end{cases} \tag{2.1}$$

where $y_0 = y(0,0)$. Problem (2.1) is linear and has a unique solution y_τ under mild assumptions on f.

The insensibility condition (1.5) holds if and only if

$$\int_Q \left(h_0 \chi_O + \mathsf{w} \chi_\omega \right) y_\tau \, dx dt = 0. \tag{2.2}$$

We can transform (2.2) by introducing the classical adjoint state. More precisely, we define the function $q = q(x,t)$ as the solution of the backward problem :

$$\begin{cases} -q' - \Delta q + f'(y_0)q = h_0\chi_O + \text{w}\chi_\omega & \text{in } Q, \\ \quad q(T) \qquad\qquad = 0 & \text{in } \Omega, \\ \quad q \qquad\qquad\quad = 0 & \text{on } \Sigma_1, \\ \quad \dfrac{\partial q}{\partial \nu} \qquad\qquad = 0 & \text{on } \Sigma_2. \end{cases} \qquad (2.3)$$

As for the problem (2.1), the problem (2.3) has a unique solution q (under mild assumptions on $f'(y_0)$). The function q depends on the control w that we shall determine :
Indeed, if we multiply the first equation in (2.3) by y_τ, and we integrate by parts over Q, we obtain

$$\int_Q \left(h_0\chi_O + \text{w}\chi_\omega \right) y_\tau \, dxdt = \int_\Omega q(0)\hat{y}^0 dx \quad \forall \hat{y}^0, \quad \|\hat{y}^0\|_{L^2(\Omega)} \le 1.$$

So, the condition (1.5) (or (2.2)) is equivalent to

$$q(0) = 0. \qquad (2.4)$$

This is a null-controllability problem.

Remark 1. *The knowledge of the optimal control* w *provides informations about the pollution term $\lambda\hat{\xi}$. Indeed, denote by*

$$L = \frac{\partial}{\partial t} - \Delta + f'(y_0) I_d$$

and let $y_\lambda = \dfrac{\partial y}{\partial \lambda}(0,0)$ be the solution of

$$\begin{cases} L\, y_\lambda = \hat{\xi} \text{ in } Q, \\ y_\lambda(0) = 0 \text{ in } \Omega, \\ \quad y_\lambda = 0 \text{ on } \Sigma_1, \\ \quad \dfrac{\partial y_\lambda}{\partial \nu} = 0 \text{ on } \Sigma_2. \end{cases}$$

Integrating by parts, we then obtain

$$\int_Q y_\lambda \, L^*q \, dxdt = \int_Q q\hat{\xi} \, dxdt$$

with

$$L^* = -\frac{\partial}{\partial t} - \Delta + f'(y_0) I_d.$$

So that from (2.3) and (1.6) we deduce

$$\int_Q \lambda\hat{\xi} \, dxdt = \int_Q (h_0\chi_O + \text{w}\chi_\omega)(m_o - y_0) \, dxdt.$$

3. Existence of a sentinel

We begin with some observability inequality, which will be proved in detail in the last section. Denote by

$$\mathcal{V} = \left\{ v \in \mathcal{C}^\infty(\overline{Q}) \text{ such that } : v_{|\Sigma_1} = \left.\frac{\partial v}{\partial t}\right|_{\Sigma_1} = 0 \text{ and } \left.\frac{\partial v}{\partial \nu}\right|_{\Sigma_2} = 0 \right\}. \qquad (3.1)$$

Then we have :

Theorem 1. *Let be $u \in \mathcal{V}$, then there exists a positive constant $C = C(\Omega, \omega, O, T, f'(y_o))$ such that*

$$\int_Q \frac{1}{\theta^2} |u|^2 \, dxdt \leq C \Big[\int_Q |L u|^2 \, dxdt + \int_0^T \int_\omega |u|^2 \, dxdt \Big], \tag{3.2}$$

where $\theta \in \mathcal{C}^2(Q)$ positive with $\frac{1}{\theta}$ bounded.

According to the RHS of (3.2), we consider the space $\mathcal{V}$ endowed with the bilinear form $a(.,.)$ defined by :

$$a(u, v) \quad = \quad \int_Q Lu \, Lv \, dxdt + \int_0^T \int_\omega u \, v \, dxdt.$$

Let V be the completion of $\mathcal{V}$ with respect to the norm

$$v \mapsto \|v\|_V = \sqrt{a(v, v)},$$

then, V is a Hilbert space for the scalar product $a(v, \hat{v})$ and the associated norm.

Remark 2. *We can precise the structure of the elements of V. Indeed, let $H_\theta(Q)$ be the weigthed Hilbert space defined by*

$$H_\theta(Q) = \Big\{ v \in L^2(Q) \quad \text{such that :} \quad \int_Q \frac{1}{\theta^2} |v|^2 \, dxdt < \infty \Big\},$$

endowed with the natural norm $\|v\|_\theta = (\int_Q \frac{1}{\theta^2} |v|^2 \, dxdt)^{\frac{1}{2}}$. This shows that V is imbedded continuously $\|v\|_\theta \leq C \|v\|_V$.

Now if $h_0 \in L^2(Q)$ and $\theta h_0 \in L^2(Q)$ (i.e. : $h_0 \in L^2_\theta(Q)$), then from (3.2) and the Cauchy-Schwartz inequality, we deduce that the linear form defined on V by

$$v \mapsto \int_Q h_0 \chi_O \, v \, dxdt$$

is continuous. Therefore, from the Lax-Milgram theorem there exits a unique u in V solution of the variational equation :

$$a(u, v) = \int_Q h_0 \chi_O \, v \, dxdt \qquad \forall v \in V.$$

Theorem 2. *Assume that $h_0 \in L^2_\theta(Q)$, and let u be the unique solution of (3.3). We set*

$$\mathsf{w} = -u \chi_\omega$$

and

$$q = Lu.$$

Then, the pair (w, q) is such that (2.3)-(2.4) hold (i.e there is some insensitive sentinel defined by (1.4)-(1.5)).

4. Proof of theorem 1

The proof for the observability inequality in theorem 1 will hold from Carleman estimates that we carefully show in the following results.

Lemma 1. *Let be ω_0 an open set such that $\overline{\omega_0} \subset \omega$. Then there is $\psi \in C^2(\overline{\Omega})$ such that*

$$\begin{cases} \psi(x) & > 0 \ \forall \ x \in \Omega, \\ \psi(x) & = 0 \ \forall \ x \in \Gamma, \\ |\nabla \psi(x)| & \neq 0 \ \forall \ x \in \overline{\Omega - \omega_0}. \end{cases}$$

See Imanuvilov O. Yu. Imanuvilov (1995)

We now use a function ψ as given by the previous lemma, to define convenient *weight* functions. For $\lambda > 0$, we set

$$\varphi(x,t) = \frac{e^{\lambda \psi(x)}}{t(T-t)},$$

and

$$\eta(x,t) = \frac{e^{2\lambda |\psi|_\infty} - e^{\lambda \psi(x)}}{t(T-t)}.$$

Then

$$\nabla \varphi = \lambda \varphi \nabla \psi, \ \nabla \eta = -\lambda \varphi \nabla \psi.$$

We also notice the following properties :

$$\left| \frac{\partial \varphi}{\partial t} \right| \leq T\varphi^2, \ \left| \frac{\partial^2 \varphi}{\partial t^2} \right| \leq T^2 \varphi^3, \tag{4.1}$$

$$\left| \frac{\partial \eta}{\partial t} \right| \leq T\varphi^2, \ \left| \frac{\partial^2 \eta}{\partial t^2} \right| \leq T^2 \varphi^3. \tag{4.2}$$

Remark 3. *Note that η increases to $+\infty$ when $t \to T$ or $t \to 0$, but η is uniformly bounded on $\Omega \times [\delta, T - \delta]$ for any $\delta > 0$.*
On the other hand, for fixed $s > 0$ the function $e^{-s\eta(x,t)}$ goes to 0 when $t \to T$ or $t \to 0$.

The following theorem states the Carleman inequalities concerning (3.1) :

Proposition 1. *There exist constants $s_0 > 0$, $\lambda_0 > 0$ and $C > 0$ depending on Ω, ω, ψ, and T, such that for all $s \geq s_0$, $\lambda \geq \lambda_0$, and for any function $u \in \mathcal{V}$ given by (3.1), we have*

$$2s^3\lambda^4 \int_Q \varphi^3 e^{-2s\eta} |u|^2 \, dxdt + 4s^2\lambda \int_{\Sigma_2} \varphi \frac{\partial \eta}{\partial t} \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 \, d\gamma dt$$

$$-4s^3\lambda^3 \int_{\Sigma_2} \varphi^3 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 \, d\gamma dt - 4s^2\lambda^3 \int_{\Sigma_2} \varphi^2 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 \, d\gamma dt$$

$$-2s\lambda \int_{\Sigma_2} \varphi \frac{\partial \psi}{\partial \nu} e^{-2s\eta} \frac{\partial u}{\partial t} u \, d\gamma dt - 4s\lambda \int_{\Sigma_1} \varphi \nabla \psi e^{-2s\eta} \nabla u \frac{\partial u}{\partial \nu} \, d\gamma dt \tag{4.3}$$

$$+2s\lambda \int_{\Sigma} \varphi \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |\nabla u|^2 \, d\gamma dt$$

$$\leq \ C \left(\int_Q e^{-2s\eta} \left| \frac{\partial u}{\partial t} - \Delta u \right|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 e^{-2s\eta} |u|^2 \, dxdt \right).$$

We use the method by Fursikov and Imanuvilov A. Fursikov & O. Yu. Imanuvilov (1996), Imanuvilov O. Yu. Imanuvilov (1995), Lebeau and Robbiano Lebeau G. & Robbiano L. (1995), and Puel Puel J.-P. (2001) (case $\Sigma_2 = \varnothing$ and $\Sigma_1 = \Sigma$).

For $s \geq s_0$ and $\lambda \geq \lambda_0$, we define

$$w(x,t) = e^{-s\eta(x,t)} u(x,t).$$

We easily notice that

$$w(x,0) = w(x,T) = 0.$$

Calculating $g = (\partial_t - \Delta)(e^{s\eta} w)$, with notation (4), we get

$$P_1 w + P_2 w = g_s,$$

where

$$P_1 w = \frac{\partial w}{\partial t} + 2s\lambda\varphi\,\nabla\psi\,\nabla w + 2s\lambda^2\varphi|\nabla\psi|^2 w,$$

$$P_2 w = -\Delta w - s^2\lambda^2\varphi^2|\nabla\psi|^2 w + s\frac{\partial\eta}{\partial t} w,$$

$$g_s = e^{-s\eta} g + s\lambda^2\varphi|\nabla\psi|^2 w - s\lambda\varphi\,\Delta\psi\,w.$$

Taking the L^2 norm we get :

$$\int_Q |P_1 w|^2\,dxdt + \int_Q |P_2 w|^2\,dxdt + 2\int_Q P_1 w P_2 w\,dxdt = \int_Q |g_s|^2\,dxdt.$$

We shall now calculate $\int_Q P_1 w P_2 w\,dxdt$. This will give 9 terms $I_{k,l}$.

In order to organize our calculus, we denote by A and B the quantities such that A contains all the terms which can be upper bounded by

$$c\left(s\lambda + \lambda^2\right) \int_Q \varphi\,|\nabla w|^2\,dxdt,$$

and by B all those which can be bounded by

$$c\left(s^2\lambda^4 + s^3\lambda^3\right) \int_Q \varphi^3\,|w|^2\,dxdt.$$

We denote by ν the outer normal on Γ. Note down that ψ cancels on Γ. We then have the following results :

$$I_{1,1} = -\int_Q \frac{\partial w}{\partial t}\Delta w\,dxdt$$

$$= -\int_\Sigma \frac{\partial w}{\partial t}\frac{\partial w}{\partial \nu}\,d\gamma dt + 0,$$

$$I_{1,2} = -s^2\lambda^2 \int_Q \frac{\partial w}{\partial t}\varphi^2|\nabla\psi|^2 w\,dxdt = B,$$

$$I_{1,3} = s \int_Q \frac{\partial w}{\partial t} \frac{\partial \eta}{\partial t} w \, dxdt = \frac{s}{2} \int_Q \frac{\partial \eta}{\partial t} \frac{\partial}{\partial t} \left(|w|^2 \right) dxdt$$

$$= -\frac{s}{2} \int_Q \frac{\partial^2 \eta}{\partial t^2} |w|^2 \, dxdt = B.$$

And,

$$I_{2,1} = -2s\lambda \int_Q \varphi \nabla \psi \nabla w \Delta w \, dxdt$$

$$= -2s\lambda \int_\Sigma \varphi \nabla \psi . \nabla w \frac{\partial w}{\partial \nu} \, d\gamma dt + 2s\lambda^2 \int_Q \varphi |\nabla \psi . \nabla w|^2 \, dxdt$$

$$+ s\lambda \int_\Sigma \varphi \frac{\partial \psi}{\partial \nu} |\nabla w|^2 \, d\gamma dt - s\lambda^2 \int_Q \varphi |\nabla \psi|^2 |\nabla w|^2 \, dxdt + A,$$

$$I_{2,2} = -2s^3\lambda^3 \int_Q \varphi^3 |\nabla \psi|^2 \nabla \psi \nabla w . w = -s^3\lambda^3 \int_Q \varphi^3 |\nabla \psi|^2 \nabla \psi \nabla |w|^2$$

$$= -s^3\lambda^3 \int_\Sigma \varphi^3 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} |w|^2 + 3s^3\lambda^4 \int_Q \varphi^3 |\nabla \psi|^4 |w|^2 + B,$$

$$I_{2,3} = 2s^2\lambda \int_Q \varphi \nabla \psi \, \nabla w \, \frac{\partial \eta}{\partial t} w \, dxdt$$

$$= s^2\lambda \int_\Sigma \varphi \frac{\partial \eta}{\partial t} \frac{\partial \psi}{\partial \nu} |w|^2 \, dxdt + B.$$

Finally :

$$I_{3,1} = -2s\lambda^2 \int_Q \varphi |\nabla \psi|^2 \Delta w . w \, dxdt$$

$$= -2s\lambda^2 \int_\Sigma \varphi |\nabla \psi|^2 \frac{\partial w}{\partial \nu} . w + 2s\lambda^2 \int_Q \varphi |\nabla \psi|^2 |\nabla w|^2 + A + B,$$

$$I_{3,2} = -2s^3\lambda^4 \int_Q \varphi^3 |\nabla \psi|^4 |w|^2 \, dxdt,$$

$$I_{3,3} = 2s^2\lambda^2 \int_Q \varphi \frac{\partial \eta}{\partial t} |\nabla \psi|^2 |w|^2 \, dxdt = B.$$

Summing all the terms, it follows :

$$2 \int_Q P_1 w P_2 w \, dxdt = A + B + 2s\lambda^2 \int_Q \varphi |\nabla \psi|^2 |\nabla w|^2 \, dxdt$$

$$+ 2s^3\lambda^4 \int_Q \varphi^3 |\nabla \psi|^4 |w|^2 \, dxdt + 4s\lambda^2 \int_Q \varphi |\nabla \psi . \nabla w|^2 \, dxdt$$

$$- 2 \int_\Sigma \frac{\partial w}{\partial t} \frac{\partial w}{\partial \nu} \, d\gamma dt - 4s\lambda \int_\Sigma \varphi \nabla \psi . \nabla w \frac{\partial w}{\partial \nu} \, d\gamma dt$$

$$+ 2s\lambda \int_\Sigma \varphi \frac{\partial \psi}{\partial \nu} |\nabla w|^2 \, d\gamma dt - 2s^3\lambda^3 \int_\Sigma \varphi^3 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} |w|^2 \, d\gamma dt$$

$$+ 2s^2\lambda \int_\Sigma \varphi \frac{\partial \eta}{\partial t} \frac{\partial \psi}{\partial \nu} |w|^2 \, d\gamma dt - 4s\lambda^2 \int_\Sigma \varphi |\nabla \psi|^2 \frac{\partial w}{\partial \nu} . w \, d\gamma dt.$$

But $|\nabla \psi| \neq 0$ on $\overline{\Omega - \omega_0}$, hence there is $\delta > 0$ such that

$$|\nabla \psi| \geq \delta \quad \text{on} \quad \overline{\Omega - \omega_0}.$$

On the other hand

$$\int_Q |g_s|^2 \, dxdt \leq \int_Q e^{-2s\eta} |g|^2 \, dxdt + B,$$

so that

$$\int_Q |P_1 w|^2 \, dxdt + \int_Q |P_2 w|^2 \, dxdt + 2 \int_Q P_1 w P_2 w \, dxdt$$

$$\leq \int_Q e^{-2s\eta} |g|^2 \, dxdt + B.$$

Consequently :

$$\int_Q |P_1 w|^2 \, dxdt + \int_Q |P_2 w|^2 \, dxdt + 2s\lambda^2 \delta^2 \int_Q \varphi \, |\nabla w|^2 \, dxdt$$

$$+ 2s^3 \lambda^4 \delta^4 \int_Q \varphi^3 \, |w|^2 \, dxdt - 2 \int_\Sigma \frac{\partial w}{\partial t} \frac{\partial w}{\partial \nu} \, d\gamma dt - 4s\lambda \int_\Sigma \varphi \nabla \psi . \nabla w \frac{\partial w}{\partial \nu} \, d\gamma dt$$

$$+ 2s\lambda \int_\Sigma \varphi \frac{\partial \psi}{\partial \nu} |\nabla w|^2 \, d\gamma dt - 2s^3 \lambda^3 \int_\Sigma \varphi^3 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} |w|^2 \, d\gamma dt$$

$$+ 2s^2 \lambda \int_\Sigma \varphi \frac{\partial \eta}{\partial t} \frac{\partial \psi}{\partial \nu} |w|^2 \, d\gamma dt - 4s\lambda^2 \int_\Sigma \varphi |\nabla \psi|^2 \frac{\partial w}{\partial \nu} . w \, d\gamma dt + A + B$$

$$\leq \int_Q e^{-2s\eta} |g|^2 \, dxdt + B + 2s\lambda^2 \delta^2 \int_0^T \int_{\omega_0} \varphi \, |\nabla w|^2 \, dxdt + 2s^3 \lambda^4 \delta^4 \int_0^T \int_{\omega_0} \varphi^3 \, |w|^2 \, dxdt.$$

We can eliminate A and B by choosing s and λ large enough. And we observe that :

$$\int_0^T \int_\omega \varphi \theta^2 \, |\nabla w|^2 \, dxdt$$

$$\leq C \left(\int_Q \varphi P_2 w \theta^2 w dxdt + \int_0^T \int_\omega \varphi^{\frac{1}{2}} \theta^2 \, |\nabla w| \, \varphi^{\frac{1}{2}} w \, dxdt + s^2 \lambda^2 \int_0^T \int_\omega \varphi^3 w^2 dxdt \right),$$

for $\theta \in \mathcal{D}(\omega)$ such that $0 \leq \theta \leq 1$ and $\theta(x) = 1$ on ω_0 gives

$$2s\lambda^2 \delta^2 \int_0^T \int_{\omega_0} \varphi \, |\nabla w|^2 \, dxdt \leq \frac{1}{2} \int_0^T \int_\Omega |P_2 w|^2 \, dxdt + cs^3 \lambda^4 \int_0^T \int_\omega \varphi^3 \, |w|^2 \, dxdt,$$

Now, we should write the inequality below in terms of the solution u, since

$$|w|^2 = e^{-2s\eta} |u|^2.$$

So,

$$\frac{1}{2} \int_Q |P_1 w|^2 \, dxdt + \frac{1}{2} \int_Q |P_2 w|^2 \, dxdt$$

$$+2s\lambda^2 \int_Q \varphi \, |\nabla w|^2 \, dxdt + 2s^3\lambda^4 \int_Q \varphi^3 e^{-2s\eta} \, |u|^2 \, dxdt$$

$$+2s^2\lambda \int_\Sigma \varphi \frac{\partial \eta}{\partial t} \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 \, d\gamma dt - 2s^3\lambda^3 \int_\Sigma \varphi^3 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 \, d\gamma dt$$

$$-2\int_\Sigma \frac{\partial w}{\partial t} \frac{\partial w}{\partial \nu} \, d\gamma dt - 4s\lambda \int_\Sigma \varphi \nabla \psi . \nabla w \frac{\partial w}{\partial \nu} \, d\gamma dt$$

$$+2s\lambda \int_\Sigma \varphi \frac{\partial \psi}{\partial \nu} |\nabla w|^2 \, d\gamma dt - 4s\lambda^2 \int_\Sigma \varphi |\nabla \psi|^2 \frac{\partial w}{\partial \nu} . w \, d\gamma dt$$

$$\leq C \left(\int_Q e^{-2s\eta} \, |g|^2 \, dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 e^{-2s\eta} \, |u|^2 \, dxdt \right).$$

Now from

$$\nabla u = e^{s\eta} \left(\nabla w - s\lambda\varphi\nabla\psi w \right),$$

we deduce

$$\int_Q \varphi e^{-2s\eta} \, |\nabla u|^2 \, dxdt \leq C \left(\int_Q \varphi \, |\nabla w|^2 \, dxdt + s^2\lambda^2 \int_Q \varphi^3 \, |w|^2 \, dxdt \right).$$

We then use the explicit form of $P_1 w$ and $P_2 w$, and get

$$\frac{1}{s} \int_Q \frac{1}{\varphi} \left| \frac{\partial w}{\partial t} \right|^2 dxdt \leq C \left(\int_Q e^{-2s\eta} |g|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 |w|^2 dxdt \right),$$

and

$$\frac{1}{s} \int_Q \frac{1}{\varphi} \, |\Delta w|^2 \, dxdt \leq C \left(\int_Q e^{-2s\eta} |g|^2 dxdt + s^3\lambda^4 \int_0^T \int_\omega \varphi^3 |w|^2 dxdt \right).$$

We sum up to finally have

$$2s^3\lambda^4 \int_Q \varphi^3 e^{-2s\eta} \, |u|^2 \, dxdt + 4s^2\lambda \int_\Sigma \varphi \frac{\partial \eta}{\partial t} \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 \, d\gamma dt$$

$$-4s^3\lambda^3 \int_\Sigma \varphi^3 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 \, d\gamma dt - 4s^2\lambda^3 \int_\Sigma \varphi^2 |\nabla \psi|^2 \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |u|^2 \, d\gamma dt$$

$$+ \int_\Sigma \left(2s \frac{\partial \eta}{\partial t} - 4s\lambda^2 \varphi |\nabla \psi|^2 - 4s^2\lambda^2 \varphi^2 |\nabla \psi|^2 \right) e^{-2s\eta} \frac{\partial u}{\partial \nu} \, u$$

$$-2s\lambda \int_\Sigma \varphi \frac{\partial \psi}{\partial \nu} e^{-2s\eta} \frac{\partial u}{\partial t} \, u \, d\gamma dt - 4s\lambda \int_\Sigma \varphi \nabla \psi e^{-2s\eta} \nabla u \, \frac{\partial u}{\partial \nu} \, d\gamma dt$$

$$+2s\lambda \int_\Sigma \varphi \frac{\partial \psi}{\partial \nu} e^{-2s\eta} |\nabla u|^2 \, d\gamma dt - 2 \int_\Sigma e^{-2s\eta} \frac{\partial u}{\partial t} \frac{\partial u}{\partial \nu} \, d\gamma dt$$

$$\leq C \left(\int_Q e^{-2s\eta} \, |g|^2 \, dxdt + s^3\lambda^4 \int_0^T\!\!\int_\omega \varphi^3 e^{-2s\eta} \, |u|^2 \, dxdt \right).$$

Using the fact that $u = \left. \frac{\partial u}{\partial t} \right|_{\Sigma_1} = 0$, and $\left. \frac{\partial u}{\partial \nu} \right|_{\Sigma_2} = 0$, we obtain (4.3).

Now we proceed as the following. We define

$$\widetilde{\varphi}(x,t) = \frac{e^{-\lambda\psi(x)}}{t(T-t)},$$

and

$$\widetilde{\eta}(x,t) = \frac{e^{2\lambda|\psi|_{\infty}} - e^{-\lambda\psi(x)}}{t(T-t)}.$$

Then

$$\nabla\widetilde{\varphi} = -\lambda\widetilde{\varphi}\nabla\psi, \ \nabla\widetilde{\eta} = \lambda\widetilde{\varphi}\nabla\psi;$$

we still have the properties (4.1) and (4.2).

For $s \geq s_0$ and $\lambda \geq \lambda_0$, we define

$$\widetilde{w}\,(x,t) = e^{-s\widetilde{\eta}(x,t)}u\,(x,t). \tag{4.4}$$

We easily notice that

$$\widetilde{w}\,(x,0) = \widetilde{w}\,(x,T) = 0. \tag{4.5}$$

Calculating $\widetilde{P}\widetilde{w} = e^{-s\widetilde{\eta}}g = e^{-s\widetilde{\eta}}\left[(\partial_t - \Delta)(e^{s\widetilde{\eta}}\widetilde{w})\right]$, we set

$$\widetilde{P}_1\widetilde{w} + \widetilde{P}_2\widetilde{w} = \widetilde{g}_s, \tag{4.6}$$

where

$$\widetilde{P}_1\widetilde{w} = \frac{\partial\widetilde{w}}{\partial t} - 2s\lambda\widetilde{\varphi}\,\nabla\psi\,\nabla\widetilde{w} + 2s\lambda^2\widetilde{\varphi}|\nabla\psi|^2\widetilde{w},$$

$$\widetilde{P}_2\widetilde{w} = -\Delta\widetilde{w} - s^2\lambda^2\widetilde{\varphi}^2|\nabla\psi|^2\widetilde{w} + s\frac{\partial\widetilde{\eta}}{\partial t}\widetilde{w},$$

$$\widetilde{g}_s = e^{-s\widetilde{\eta}}g + s\lambda^2\widetilde{\varphi}|\nabla\psi|^2\widetilde{w} + s\lambda\widetilde{\varphi}\,\Delta\psi\,\widetilde{w}.$$

We easily deduce the following result :

Proposition 2. *There exists $s_0 > 0$, $\lambda_0 > 0$ and C a positive constant depending on Ω, ω, ψ, and T, such that for all $s \geq s_0$, $\lambda \geq \lambda_0$, and for any fuction u of (3.1), we have*

$$2s^3\lambda^4\int_Q\widetilde{\varphi}^3e^{-2s\widetilde{\eta}}\,|u|^2\,dxdt - 4s^2\lambda\int_{\Sigma_2}\widetilde{\varphi}\frac{\partial\widetilde{\eta}}{\partial t}\frac{\partial\psi}{\partial\nu}e^{-2s\widetilde{\eta}}|u|^2\,d\gamma dt$$

$$+4s^3\lambda^3\int_{\Sigma_2}\widetilde{\varphi}^3|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}e^{-2s\widetilde{\eta}}|u|^2\,d\gamma dt + 4s^2\lambda^3\int_{\Sigma_2}\widetilde{\varphi}^2|\nabla\psi|^2\frac{\partial\psi}{\partial\nu}e^{-2s\widetilde{\eta}}|u|^2\,d\gamma dt$$

$$+2s\lambda\int_{\Sigma_2}\widetilde{\varphi}\frac{\partial\psi}{\partial\nu}e^{-2s\widetilde{\eta}}\frac{\partial u}{\partial t}\,u\,d\gamma dt + 4s\lambda\int_{\Sigma_1}\widetilde{\varphi}\nabla\psi e^{-2s\widetilde{\eta}}\nabla u\frac{\partial u}{\partial\nu}\,d\gamma dt \tag{4.7}$$

$$-2s\lambda\int_{\Sigma}\widetilde{\varphi}\frac{\partial\psi}{\partial\nu}e^{-2s\widetilde{\eta}}|\nabla u|^2\,d\gamma dt$$

$$\leq\ C\left(\int_Q e^{-2s\widetilde{\eta}}\left|\frac{\partial u}{\partial t} - \Delta u\right|^2\,dxdt + s^3\lambda^4\int_0^T\!\!\int_\omega\widetilde{\varphi}^3e^{-2s\widetilde{\eta}}\,|u|^2\,dxdt\right).$$

The proof is similar to the one of Proposition 1, we let it to the reader. We obtain from the above Propositions 1 and 2 the following observability inequality :

Corollary 1. *There is a positive constant $C = C(\Omega, \omega, \psi, T)$ such that we have*

$$\int_Q\frac{1}{\theta^2}\,|u|^2\,dxdt \leq C\Big[\int_Q\left|\frac{\partial u}{\partial t} - \Delta u\right|^2\,dxdt + \int_0^T\!\!\int_\omega|u|^2\,dxdt\Big], \tag{4.8}$$

where $\frac{1}{\theta^2} = \varphi^3e^{-2s\eta} + \widetilde{\varphi}^3e^{-2s\widetilde{\eta}}$ is a bounded weight function.

Summing the terms in (4.3) and (4.7) we get the following :

$$2s^3\lambda^4 \int_Q \left(\varphi^3 e^{-2s\eta} + \widetilde{\varphi}^3 e^{-2s\widetilde{\eta}}\right) |u|^2 \, dxdt$$

$$-4s^2\lambda \int_{\Sigma_2} \left(\varphi\frac{\partial\eta}{\partial t}e^{-2s\eta} - \widetilde{\varphi}\frac{\partial\widetilde{\eta}}{\partial t}e^{-2s\widetilde{\eta}}\right) \frac{\partial\psi}{\partial\nu}|u|^2 \, d\gamma dt$$

$$-4s^3\lambda^3 \int_{\Sigma_2} \left(\varphi^3 e^{-2s\eta} - \widetilde{\varphi}^3 e^{-2s\widetilde{\eta}}\right) |\nabla\psi|^2 \frac{\partial\psi}{\partial\nu}|u|^2 \, d\gamma dt$$

$$-4s^2\lambda^3 \int_{\Sigma_2} \left(\varphi^2 e^{-2s\eta} - \widetilde{\varphi}^2 e^{-2s\widetilde{\eta}}\right) |\nabla\psi|^2 \frac{\partial\psi}{\partial\nu}|u|^2 \, d\gamma dt$$

$$+2s\lambda \int_{\Sigma_2} \left(\varphi e^{-2s\eta} - \widetilde{\varphi}e^{-2s\widetilde{\eta}}\right) \frac{\partial\psi}{\partial\nu}\frac{\partial u}{\partial t} u \, d\gamma dt$$

$$-4s\lambda \int_{\Sigma_1} \left(\varphi e^{-2s\eta} - \widetilde{\varphi}e^{-2s\widetilde{\eta}}\right) \nabla\psi\nabla u \frac{\partial u}{\partial\nu} \, d\gamma dt$$

$$+2s\lambda \int_{\Sigma} \left(\varphi e^{-2s\eta} - \widetilde{\varphi}e^{-2s\widetilde{\eta}}\right) \frac{\partial\psi}{\partial\nu}|\nabla u|^2 \, d\gamma dt$$

$$\leq C\Big[\int_Q \left(e^{-2s\eta} + e^{-2s\widetilde{\eta}}\right) \left|\frac{\partial u}{\partial t} - \Delta u\right|^2 \, dxdt$$

$$+s^3\lambda^4 \int_0^T \int_\omega \left(\varphi^3 e^{-2s\eta} + \widetilde{\varphi}^3 e^{-2s\widetilde{\eta}}\right) |u|^2 \, dxdt\Big].$$

Now, it suffices to notice that $\varphi = \widetilde{\varphi}$ and $\eta = \widetilde{\eta}$ on Σ.

5. Instantaneous sentinels

In this part, we discuss of how to get instantaneous information (at fixed $t = T \in [0, +\infty[$) on pollution terms in systems of incomplete data in ecology and/or meteorology problems.

Here, the ecological system is affected by pollution in the boundary of the domain (a border of an air pollution cloud for example). We verify that if the initial data is completely unknown, the sentinel is nul. If there is some information on the initial data, the *instantaneous* sentinel naturally exists if the control set is bigger than the one where the observation may be defined.

Finally, we give the characterization of the instantaneous sentinel with some remarks.

Using the techniques of the sections below and those in Miloudi Y. et al. (2009). We prove the existence and characterization of a sentinel, which permits to identify the pollution parameters at fixed time T. The problem we consider now is the following :

$$\frac{\partial y}{\partial t} + Ay + f(y) = 0 \qquad \text{in } Q :=]0, T[\times \Omega, \tag{5.1}$$

where $\Omega \subset \mathbb{R}^d$ is an open domain of regular boundary $\Gamma = \partial\Omega$ for instance, A represents a second order elliptic operator such that $a_{kl} = a_{lk}$, $a_{kl} \in \mathcal{C}^2(\bar{\Omega})$, and $\sum_{i,j=1}^d a_{ij}\eta_i\eta_j \geq c^2|\eta|^2$, and $f : \mathbb{R} \to \mathbb{R}$ is a (nonlinear) $\mathcal{C}^1$ function.

To (5.1) we should add initial and boundary conditions. It is in these conditions that we can meet incomplete data. We assume that we know that $y_{|t=0}$ is in a ball of $L^2(\Omega)$ of center y^o, say $\|y(0,.) - y^o(.)\|_{L^2(\Omega)} \leq \tau$, that we should write :

$$y(0) = y^o + \tau\hat{y}^o, \tag{5.2}$$

where $\|\hat{y}^o\| \leq 1$, and where $\tau \hat{y}^o$ is the missing data. To the boundary condition ξ is added a pollution term $\lambda \hat{\xi}$ which is unknown, and which *we want* to identify. It appears on a part $\Sigma_0 = \Gamma_0 \subset \Sigma$ of the total boundary $\Sigma =]0, T[\times \Gamma$, as we have :

$$y = \begin{cases} \xi + \lambda \hat{\xi} & \text{on } \Sigma_0, \\ 0 & \text{on } \Sigma \backslash \Sigma_0. \end{cases} \tag{5.3}$$

The question is to obtain information on the pollution $\lambda \hat{\xi}$, not affected by the missing term $\tau \hat{y}^o$ of the initial data.

Still, one can use the least square method : In the context of the above problem, it consists in considering the unknowns $\{\lambda \hat{\xi}, \tau \hat{y}^o\} = \{v, w\}$ as two control variables. At time T, the state is then $y(T, x; v, w)$, and we want this solution to be *as close as possible* to some measurement m_o. We consider then the distance $J(v, w) = \|y(v, w) - m_o\|$ in an appropriate norm and we search for

$$\inf_{\{v, w\}} J(v, w)$$

where v and w are arbitrary. As we know, with this method, we can not seperate v and w. The more suitable is then the Sentinel method (see Lions J.-L. (1992), Miloudi Y. et al. (2009), Kernevez J.-P. (1997) and the references therein).

The state solution $y(\lambda, \tau) := y(t, x; \lambda \hat{\xi}, \tau \hat{y}^o)$. The observation is given in an observatory $\mathcal{O} \subset \Omega$, but here we observe y at fixed *instant* T :

$$y_{\text{obs}} = m_o. \tag{5.4}$$

Now, we are given

$$h_o \in L^2(\mathcal{O}) \tag{5.5}$$

and

$$\mathsf{w} \in L^2(\omega) \tag{5.6}$$

where $\omega \subset \Omega$ the open set of controls. We search for w, such that the mean function

$$\begin{aligned} \mathcal{S}(\lambda, \tau) &= \int_{\mathcal{O}} h_o \, y(T, x; \lambda, \tau) \, dx + \int_{\omega} \mathsf{w} \, y(T, x; \lambda, \tau) \, dx \\ &= \int_{\Omega} \left(h_o \chi_{\mathcal{O}} + \mathsf{w} \chi_{\omega} \right) y(T, x; \lambda, \tau) \, dx \end{aligned} \tag{5.7}$$

(here $\chi_{\mathcal{O}}$ and χ_{ω} denote the characteristic functions of $\mathcal{O}$ and ω respectively), is insensitive to the missing data at the first order; i.e.

$$\frac{\partial \mathcal{S}}{\partial \tau}(0, 0) = 0,$$

for $\lambda = \tau = 0$, and such that

$$\|\mathsf{w}\|_{L^2(\omega)} \quad \text{is of minimal norm.} \tag{5.8}$$

The integral (5.7) is called *sentinel*, following the definition of Lions Lions J.-L. (1992).
The pollution sources in (5.3) can be considered as functionals whose position and nature are known, but their amplitudes are however unknown. The goal is to estimate these amplitudes; the other missing terms (in the initial data (5.2)) do not interest us.

Remark 4. *In Lions Lions J.-L. (1992), $\omega = \mathcal{O}$, so there exists always a sentinel (at least the one where* w $= -h_o$*). Then the problem is only to find* w *solution to (5.8). At last, we have to be sure that* w $\neq -h_o$*.*

Below, we use the sentinel method as formulated in Miloudi Y. et al. (2007) and Miloudi Y. et al. (2009) and Nakoulima O. (2004), where the observation and the control have their support in two different open sets. Indeed, one can observe somewhere and control elsewhere !

6. Instantaneous sentinel. Case of no information on the missing data

In this section, we extend the method of sentinels to the case of observation and control having their supports in two different open sets. Moreover, we want information at precise time T which is a difficult problem.

Denote by $y(t, x; \lambda, \tau) := y(\lambda, \tau)$, the state solution of (5.1)-(5.3). We begin by noticing that the solution $y = y(\lambda, \tau)$ satisfies the system :

$$\begin{cases} \mathcal{L} y_\tau = 0 & \text{in } Q, \\ y_\tau(0) = \hat{y}^o & \text{in } \Omega, \\ y_\tau = 0 & \text{on } \Sigma, \end{cases}$$

where y_τ denotes the derivative

$$y_\tau = \frac{\partial y}{\partial \tau}(0, 0)$$

and where $\mathcal{L}$ is the linear differential operator defined by

$$\mathcal{L} = \frac{\partial}{\partial t} + A + f'(y_o) I_d$$

with $y_o = y(t, x; 0, 0)$ for $\lambda = \tau = 0$. Then

$$\frac{\partial \mathcal{S}}{\partial \tau}(0, 0) = \int_\Omega \left(h_o \chi_{\mathcal{O}} + \text{w} \chi_\omega \right) y_\tau(T) \, dx,$$

Remark 5. *From above, the insensitive criterion (5) to the missing term $\tau \hat{y}^o$ is given by :*

$$\int_\Omega \left(h_o \chi_{\mathcal{O}} + \text{w} \chi_\omega \right) y_\tau(T) \, dx = 0.$$

Lemma 2. *Let q be the solution to the following well-posed backward problem :*

$$\mathcal{L}^* q = 0 \qquad\qquad in \quad Q, \qquad\qquad (6.1)$$

$$q(T) = h_o \chi_{\mathcal{O}} + \text{w} \chi_\omega \quad in \quad \Omega, \qquad\qquad (6.2)$$

$$q = 0 \qquad\qquad on \quad \Sigma, \qquad\qquad (6.3)$$

where $\mathcal{L}^* = -\frac{\partial}{\partial t} + A^* + f'(y_o) I_d$ is the adjoint operator. Then, the existence of an instantaneous sentinel insensitive to the missing data (i.e. such that (5) holds), is equivalent to the null-controllability problem (6.1)-(6.3) with

$$q(0) = 0 \qquad in \quad \Omega. \qquad\qquad (6.4)$$

Proof - Multiplying (6.1) by y_τ and integrating by parts we obtain :

$$\int_Q (\mathcal{L}^* q) \, y_\tau \, dxdt = \int_Q q \, \mathcal{L} y_\tau \, dxdt - \int_\Omega q(T) y_\tau(T) \, dx$$

$$+ \int_\Omega q(0) y_\tau(0) \, dx - \int_\Sigma \frac{\partial q}{\partial \nu} y_\tau \, d\sigma + \int_\Sigma \frac{\partial y_\tau}{\partial \nu} q \, d\sigma = 0. \qquad (6.5)$$

But, y_τ is solution to (6), and q verifies (6.2) and (6.3). Hence

$$- \int_\Omega \left(h_o \chi_{\mathcal{O}} + \mathsf{w} \chi_\omega \right) y_\tau(T) \, dx + \int_\Omega q(0) \, \hat{y}^o \, dx = 0.$$

If a sentinel exists, then we have (5). So it remains :

$$\int_\Omega q(0) \, \hat{y}^o \, dx = 0 \qquad \text{for every} \quad \hat{y}^o \in L^2(\Omega),$$

and consequently $q(0) = 0$ in Ω. The converse is obvious.   ∎

Corollary 2. *If there is no information on the missing data, then the instantaneous sentinel is nul.*

Proof - The proof is easy and lies on the backward uniqueness property (see Lions-Malgrange Lions J.-L. & Malgrange B. (1960)) : If an instantaneous sentinel exists, then from the above Lemma we have (6.1), (6.3) and (6.4). We deduce that

$$q \equiv 0 \qquad \text{in} \quad Q.$$

Thus, in particular for $t = T$, we have $q(T) = 0$. So that $h_o \chi_{\mathcal{O}} + \mathsf{w} \chi_\omega = 0$ in Ω, and hence -as in Lions Lions J.-L. (1992)- the sentinel is nul.   ∎

6.0.0.1 Information given by the sentinel

We briefly show how (5) and (5.8) are sufficient to get information on the pollution term $\lambda \hat{\xi}$. We write

$$S(\lambda, \tau) \simeq S(0,0) + \lambda \frac{\partial S}{\partial \lambda}(0,0), \qquad \text{for} \quad \lambda, \tau \quad \text{small}.$$

Using (5.4), we have

$$\lambda \frac{\partial S}{\partial \lambda}(0,0) = \int_Q (h_0 \chi_{\mathcal{O}} + \mathsf{w} \chi_\omega) y_\lambda \, dxdt \simeq \int_Q (h_0 \chi_{\mathcal{O}} + \mathsf{w} \chi_\omega)(m_o - y_0) \, dxdt, \qquad (6.6)$$

where the derivative

$$y_\lambda = (\partial y / \partial \lambda)(0,0)$$

only depends on $\hat{\xi}$ and other known data. Consequently, the estimates (6.6) contain the information on $\lambda \hat{\xi}$. Indeed, y_λ is solution to the well-posed problem

$$\mathcal{L} y_\lambda = 0 \text{ in } Q, \quad y_\lambda(0) = 0 \text{ in } \Omega, \quad \text{and } y_\lambda = \hat{\xi} \text{ on } \Sigma. \qquad (6.7)$$

Multiplying by q and integrating by parts, we obtain

$$\int_Q \lambda \hat{\xi} \, dxdt = \int_Q (h_0 \chi_O + \mathsf{w} \chi_\omega)(m_o - y_0) \, dxdt.$$

Remark 6. *From the previous corollary, we see that in order to get a non nul sentinel, we need more information (on the structure) of the missing initial data. We will need here the assumption (7) below.*

7. Instantaneous sentinel. Case of partial information on the missing data

In this section, we consider the case of the *incomplete* initial data :

$$y(0) = y^o + \sum_{i=1}^{N} \tau_i \hat{y}_i^o \tag{7.1}$$

where $\hat{y}_i^o$, $1 \le i \le N$ are linearly independent in $L^2(\Omega)$, and belong to a vector subspace of N dimension, which we denote by $Y = \langle \hat{y}_1^o, \hat{y}_2^o, \cdots, \hat{y}_N^o \rangle$. The parameters τ_i are *unknown* and are supposed small.

If we denote by $y_{\tau_i} = \dfrac{\partial y}{\partial \tau_i}(0, 0)$ for $\lambda = \tau_i = 0$, $1 \le i \le N$, then y_{τ_i} is solution with $y_{\tau_i}(0) = \hat{y}_i^o$. Now, we assume the following :

$$\omega \cap \mathcal{O} \ne \varnothing$$

and we define the instantaneous sentinel by :

$$\mathcal{S}(\lambda, \tau) = \int_{\Omega} \Big(h_o \chi_{\mathcal{O} \cap \omega} + w \chi_\omega \Big) y(T; \lambda, \tau)\, dx = 0$$

with $\tau = (\tau_1, \cdots, \tau_N)$. Hence, looking for w such that the sentinel $\mathcal{S}$ is insensitive to the missing terms $\tau_i \hat{y}_i^o$, is finding w such that :

$$\frac{\partial \mathcal{S}}{\partial \tau_i}(0, \tau_i = 0) = \int_{\Omega} \Big(h_o \chi_{\mathcal{O} \cap \omega} + w \chi_\omega \Big) y_{\tau_i}(T)\, dx = 0, \quad 1 \le i \le N. \tag{7.2}$$

Lemma 3. *The existence of an instantaneous sentinel insensitive to the missing terms is equivalent to the existence of a unique pair* (w, q) *such that we have :*

$$\begin{cases} \mathcal{L}^* q = 0 & \text{in } Q, \\ q(T) = h_o \chi_{\mathcal{O} \cap \omega} + w \chi_\omega & \text{in } \Omega, \\ q = 0 & \text{on } \Sigma. \end{cases} \tag{7.3}$$

and such that

$$q(0) \in Y^{\perp}. \tag{7.4}$$

Proof - Multiplying the first equation in (7.3) by y_{τ_i} and integrating by parts, we find :

$$-\int_{\Omega} q(T)\, y_{\tau_i}(T)\, dx + \int_{\Omega} q(0) \hat{y}_i^o\, dx = 0,$$

since y_{τ_i} is solution of (6). And so,

$$-\int_{\Omega} \Big(h_o \chi_{\mathcal{O} \cap \omega} + w \chi_\omega \Big) y_{\tau_i}(T)\, dx + \int_{\Omega} q(0) \hat{y}_i^o\, dx = 0, \qquad 1 \le i \le N.$$

But the sentinel should satisfy (7.2). Hence,

$$\int_{\Omega} q(0) \hat{y}_i^o\, dx = 0, \quad \text{for all } \quad 1 \le i \le N.$$

So that

$$q(0) \perp \hat{y}_i^o, \qquad 1 \le i \le N \qquad \text{if and only if} \qquad q(0) \in Y^{\perp},$$

where $Y^{\perp}$ is the orthogonal of Y in $L^2(\Omega)$. ∎

Remark 7. *The controllability problem (7.3)-(7.4) has at least a solution: Given $h_o \in L^2(\mathcal{O})$, the control 'solution' w is given by*

$$w = \left\{ \begin{array}{ll} -h_o & \text{in } \mathcal{O} \cap \omega, \\ 0 & \text{in } \omega \setminus (\mathcal{O} \cap \omega), \end{array} \right. \tag{7.5}$$

and so, the set of solutions to (7.3)-(7.4) is not empty.

8. Penalization

Here we are interested in the problem (5.8). We consider the optimization problem :

$$(\mathcal{P}) \qquad \min_{(\mathsf{w},z) \in \mathcal{A}} \|\mathsf{w}\|_{L^2(\omega)}$$

with

$$\mathcal{A} = \left\{ (\mathsf{w},q) \text{ such that } \left| \begin{array}{ll} \mathcal{L}^* q = 0 & \text{in } Q, \ q(0) \perp \widehat{y}_i^o \ 1 \le i \le N, \\ q(T) = h_o \chi_{\mathcal{O} \cap \omega} + \mathsf{w}\chi_\omega \text{ in } \Omega, & q = 0 \qquad \text{on } \Sigma \end{array} \right. \right\}.$$

Theorem 3. *There is a unique pair $(\widehat{\mathsf{w}}, \widehat{q})$ solution to the problem $(\mathcal{P})$.*

Proof - The domain $\mathcal{A}$ is no empty and is closed. The mapping : $\mathsf{w} \longrightarrow \|\mathsf{w}\|_{L^2(\omega)}$ is continuous, coercitive and strictly convex. We deduce that there exists a unique solution to $(\mathcal{P})$ denoted by $(\widehat{\mathsf{w}}, \widehat{q}) \in \mathcal{A}$, which satisfies

$$\|\widehat{\mathsf{w}}\|_{L^2(\omega)} \le \|\mathsf{w}\|_{L^2(\omega)} \qquad \forall \, (\mathsf{w},q) \in \mathcal{A}.$$

∎

We now use the penalization method in order to characterize the optimal control $(\widehat{\mathsf{w}}, \widehat{q})$. Let be $\varepsilon > 0$, we introduce the function

$$J_\varepsilon(\mathsf{w},q) = \frac{1}{2} \|\mathsf{w}\|^2_{L^2(\omega)} + \frac{1}{2\varepsilon} \|\mathcal{L}^* q\|^2_{L^2(Q)},$$

and we consider the problem $(\mathcal{P}_\varepsilon)$ given by

$$(\mathcal{P}_\varepsilon) \qquad \min_{(\mathsf{w},q) \in \mathcal{U}} J_\varepsilon(\mathsf{w},q)$$

with

$$\mathcal{U} = \left\{ (\mathsf{w},q) \text{ such that } \left| \begin{array}{ll} \mathcal{L}^* q \in L^2(Q) & \text{in } Q, \ q(0) \perp \widehat{y}_i^o \ 1 \le i \le N, \\ q(T) = h_o \chi_{\mathcal{O} \cap \omega} + \mathsf{w}\chi_\omega \text{ in } \Omega, & q = 0 \qquad \text{on } \Sigma \end{array} \right. \right\}.$$

The following proposition gives the existence of a solution to the penalized problem $(\mathcal{P}_\varepsilon)$.

Proposition 3. *The problem $(\mathcal{P}_\varepsilon)$ has a unique solution denoted by $(\mathsf{w}_\varepsilon, q_\varepsilon)$.*

Proof - We have $\mathcal{A} \subset \mathcal{U}$. Moreover, $\mathcal{A}$ is nonempty by the previous theorem. Consequently, $\mathcal{U}$ is nonempty and closed. The cost function J_ε is continuous, coercitive and strictly convex. Hence, problem $(\mathcal{P}_\varepsilon)$ has a unique solution.

∎

We now prove the following proposition.

Proposition 4. *Let be* $(w_\varepsilon, q_\varepsilon)$ *the unique solution of* $(\mathcal{P}_\varepsilon)$*, then for* $\varepsilon \longrightarrow 0$ *we obtain*

$$\begin{cases} w_\varepsilon \rightharpoonup \widehat{w} & weakly\ in \quad L^2(\omega), \\ q_\varepsilon \rightharpoonup \widehat{q} & weakly\ in \quad \mathcal{W}(0,T), \end{cases}$$

where $\mathcal{W}(0,T) = \left\{ q \in \mathcal{C}(0,T); \quad q(t) \in L^2(\Omega) \right\}$.

Proof - Since $(w_\varepsilon, q_\varepsilon)$ is solution to $(\mathcal{P}_\varepsilon)$ then

$$J_\varepsilon(w_\varepsilon, q_\varepsilon) \leq J_\varepsilon(w, q) \quad \forall (w, q) \in \mathcal{A}.$$

In particular $(\widehat{w}, \widehat{q}) \in \mathcal{A} \subset \mathcal{U}$ is solution to $(\mathcal{P}_\varepsilon)$. Then :

$$\|w_\varepsilon\|_{L^2(\omega)} \leq \|\widehat{w}\|_{L^2(\omega)} \leq C \qquad \text{and so} \qquad \|\mathcal{L}^* q_\varepsilon\|_{L^2(Q)} \leq C\sqrt{\varepsilon},$$

where C is a positive constant which is not the same at each time.

Knowing that $(w_\varepsilon, q_\varepsilon) \in \mathcal{U}$, we deduce

$$\|q_\varepsilon\|_{H^{2,1}(Q)} \leq C.$$

Then, there is a subsequent $(w_\varepsilon, q_\varepsilon)$, and two functions $w_0 \in L^2(\omega)$ and $q_0 \in H^{2,1}(Q)$ such that

$$\begin{cases} w_\varepsilon \rightharpoonup w_0 & \text{in } L^2(\omega), \\ q_\varepsilon \rightharpoonup q_0 & \text{in } H^{2,1}(Q). \end{cases}$$

Now as $H^{2,1}(Q) \hookrightarrow L^2(Q)$ with compact injection, the pair (w_0, q_0) satisfies the following :

$$\mathcal{L}^* q_0 = h + k_0 \chi_\omega \text{ in } Q, \quad q_0(T) = q_0 = 0 \text{ in } \Omega, \quad q_0 = 0 \text{ in } \Sigma. \tag{8.1}$$

But,

$$\frac{1}{2}\|w_0\|_{L^2(\omega)}^2 \leq \liminf_{\varepsilon \to 0} J_\varepsilon(w_\varepsilon, q_\varepsilon) \leq J_\varepsilon(\widehat{w}, \widehat{q}) \leq \frac{1}{2}\|\widehat{w}\|_{L^2(\omega)}.$$

Since $(\widehat{w}, \widehat{q})$ is the unique solution of $(\mathcal{P})$, then $\widehat{w} = w_0$. Finally, as q_0 satisfies (8.1), we obtain $\widehat{q} = q_0$. ∎

9. Concluding remarks

From the numerical aspect, there where some results in the litterature. In B.-E. Ainseba et al. Ainseba B.E. et al. (1994a)Ainseba B.-E. et al. (1994b), the authors compare numerically the least square method and the sentinel one during time $[0, T[$ and find that they are equivalent in the linear case $f(y) = C\,y$. However, the sentinel approach has an advantage in case of several measures -as it is really the case in general-; Indeed, it suffices to calculate a simple integral to identify the parameter each time, since one has to minimize a quadratic cost functional for the least square approach in order to determine both the parameter and the missing terms.

In the nonlinear case and when the observation is noisy, the least square method costs a lot numerically and can fail after a large number of iterations, while the sentinel method is relatively robust face to the perturbations of the observation. Moreover, the calculus of w

does not depend on the observation and then makes the method efficient in case of several measures. We refer to Ainseba B.E. et al. (1994a)Ainseba B.-E. et al. (1994b) and to the book of J.-P. Kernevez Kernevez J.-P. (1997) for further information and numerical details.

As we have seen, the existence of an instantaneous sentinel is equivalent to controllability problems. In the case of no supplementary information on the missing data (that we do not want to identify), the sentinel is nul as in Lions Lions J.-L. (1992).

In the case where we have some more information on the structure of the missing data (which becomes incomplete data), we proved that the existence of the instantaneous sentinel is equivalent to a nontrivial controllability problem, which has a solution under the assumption $\mathcal{O} \cap \omega \neq \varnothing$.

10. References

Ainseba B.-E., Kernevez J.-P., Luce R.   *Identification de pollution dans une rivière*, ESAIM:Mathematical Modeling and Numerical Analysis M2AN, 1994, pp $819 - 823$.

Ainseba B.-E., Kernevez J.-P., Luce R. *Identification de paramètres dans les problèmes non linéaires à données manquantes*, ESAIM:Mathematical Modeling and Numerical Analysis M2AN, 1994, pp $313 - 328$.

Bodart O.,   Application de la méthode des sentinelles à l'identification de sources de pollution dans un système distribué. Thèse de Doctorat, Université de Technologie de Compiègne, 1994.

Bodart O., Demeestere P.,   Contrôle frontière de l'explosion en temps fini de la solution d'une équation de combustion. C.R. Acad. Sci., Série I, 325(8), 1997, pp $841 - 845$.

Bodart O., Fabre C.,   Contrôle insensibilisant la norme de la solution d'une équation de la chaleur semi-linéaire. C.R. Acad. Sci., Série I, 316(8), 1993, pp $789 - 794$.

Bodart O., Fabre C.,   Controls insensitizing the norm of a semilinear heat equation. Journ. of Math. Analysis and Applications, 195, 1995, pp $658 - 683$.

Brezis H.,   Analyse fonctionnelle, Collection mathématiques appliquées pour la maîtrise, Masson, 1984.

Demeestere P., Méthode des sentinelles : étude comparative et application à l'identification d'une frontière Contrôle du temps d'explosion d'une équation de combustion. Thèse de Doctorat, Université de Technologie de Compiègne, 1997.

A. Fursikov, O. Yu. Imanuvilov,   Controllability of evolution equations. Lecture Notes, Research Institute of Mathematics, Seoul National University, Korea (1996).

O. Yu. Imanuvilov   Controllability of parabolic equations. Sbornik Mathematics 186 : 6, 1995, pp $879 - 900$.

Kernevez J.-P.,   The Sentinel Method and its Application to Environmental Pollution Problems, CRC. Press, Boca Raton, FL, 1997.

Lebeau G., Robbiano L.,   Contrôle exacte de l'équation de la chaleur, Comm. P.D.E., 20, 1995, pp $335 - 356$.

Lions J.-L.,   Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles. Dunod, gauthier-Villars, Paris 1968.

Lions J.-L.   Sentinelles pour les systèmes distribués à données incomplètes. Masson, Paris, 1992.

Lions J.-L., Malgrange B.   *Sur l'unicité rétrograde dans les problèmes mixtes paraboliques*, Math. Scand., 8, pp 277-286 (1960).

Miloudi Y., Nakoulima O., Omrane A., A method of detection of pollution in dissipative systems of incomplete data, Proc. of the European Society of Appl. and Indust. Math., ESAIM:Proc Vol. 17, 2007, pp 67 − 79.

Miloudi Y., Nakoulima O., Omrane A., Instantaneous sentinels for distributed systems of incomplete data, Inverse Problems in Science and Engineering, Vol. 17, 4, 2009, pp 451 − 459.

Nakoulima O., Contrôlabilité à zéro avec contraintes sur le contrôle. C. R. Acad. Sci. Paris, Ser. I 339, 2004, pp 405 − 410.

Puel J.-P., Contrôlablité approchée et contrôlabilité exacte. Notes de cours de D.E.A. Université Pierre et Marie Curie, Paris, 2001.
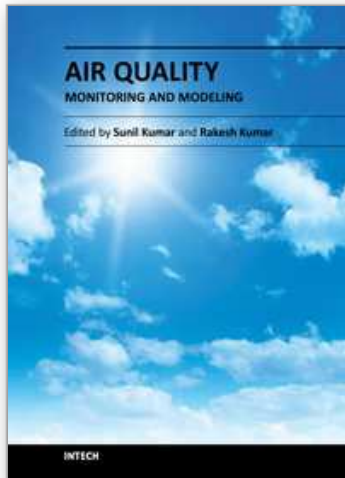
Air Quality - Monitoring and Modeling

Edited by Dr. Sunil Kumar

The book reports research on relationship between fungal contamination and its health effects in large Asian cities, estimation of ambient air quality in Delhi, a qualitative study of air pollutants from road traffic, air quality in low-energy buildings, some aspects of the Sentinel method for pollution problem, evaluation of dry atmospheric deposition at sites in the vicinity of fuel oil fired power, particles especially PM 10 in the indoor environment, etc.

How to reference

In order to correctly reference this scholarly work, feel free to copy and paste the following:

A. Omrane (2012). Some Aspects of the Sentinel Method for Pollution Problems, Air Quality - Monitoring and Modeling, Dr. Sunil Kumar (Ed.), ISBN: 978-953-51-0161-1, InTech, Available from:
http://www.intechopen.com/books/air-quality-monitoring-and-modeling/some-aspects-of-the-sentinel-method-for-pollution-problems

INTECH

open science | open minds