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The Properties of Graphs of Matroids*

Ping Li[1] and Guizhen Liu[2,†]
[1]Department of Mathematics, West Virginia University, Morgantown, WV
[2]School of Mathematics and System Science, Shandong University, Jinan
[1]USA
[2]China

1. Introduction

Let E be a finite set of elements. For $S_1, S_2 \subseteq E$, set $S_1 - S_2 = \{x | x \in S_1 \, and \, x \notin S_2\}$. Let $\mathcal{C}$ be a collection of non-null subsets of E which satisfies the following two axioms.

(**C1**) A proper subset of a member of $\mathcal{C}$ is not a member of $\mathcal{C}$.

(**C2**) If $a \in C_1 \cap C_2$ and $b \in C_1 - C_2$ where $C_1, C_2 \in \mathcal{C}$ and $a, b \in E$, then there exists a $C_3 \in \mathcal{C}$ such that $b \in C_3 \subseteq (C_1 \cup C_2) - \{a\}$.

Then $M = (E, \mathcal{C})$ is called a matroid on E. We refer to the members of $\mathcal{C}$ as circuits of matroid M. The set of bases of a matroid M is a nonempty collection $\mathcal{B}$ of subsets of E such that the following condition is satisfied. For any $B, B' \in \mathcal{B}$, $|B| = |B'|$ and for any $e \in B \setminus B'$, there exists $e' \in B' \setminus B$ such that $(B \setminus \{e\}) \cup \{e'\} \in \mathcal{B}$, We also write as $M = (E, \mathcal{B})$. Each member of $\mathcal{B}$ is called a base of M. The rank r of a matroid is the number of elements in a base and the co-rank r^* is the number of its basic circuits. If $e \in E \setminus B$, then $B \cup \{e\}$ contains a unique basic circuit and denoted by $C(e, B)$. For a given base B of M, the set of basic circuits with respect to B is denoted by $\mathcal{C}_B$. We use $\mathcal{B}_e$ and $\overline{\mathcal{B}_e}$ to denote the set of bases containing e and avoiding e, respectively. Let $M = (E, \mathcal{C})$ be a matroid. If $X \subseteq E$, then the matroid on $E - X$ whose circuits are those of M which are contained in $E - X$ is called the restriction of M to $E - X$ (or the matroid obtained by deleting X from M) and is denoted by $M \backslash X$ or $M|(E - X)$. There is another derived matroid of importance. If $X \subseteq E$, then the family of minimal non-empty intersections of $E - X$ with circuits of M is the family of circuits of a matroid on $E - X$ called the contraction of M to $E - X$. If $X = \{e\}$, we use $M \backslash e$ and M/e to denote the matroid obtained from M by deleting and contracting e, respectively. A matroid obtained from M by limited times of contractions and limited times of deletions is called a minor of M. A subset S of E is called a separator of M if every circuit of M is either contained in S or $E - S$. Union and intersection of two separators of M is also a separator of M. If $\emptyset$ and E are the only separators of M, then M is said to be connected. The minimal non-empty separators of M are called the

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†Corresponding Author

components of M. The base graph of matroid M is a graph $G_B(M)$ with vertex set $V(G_B)$ and edge set $E(G_B)$ such that $V(G_B) = \mathcal{B}$ and $E(G_B) = \{BB' \mid B, B' \in \mathcal{B}, \mid |B - B'| = 1\}$.

Let G be a graph. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. If $A \subseteq V(G)$, then $G[A]$ denotes the induced subgraph of G by A. A k-path is a path of k-edges and denoted by P_k. A k-circuit is a circuit of k-edges and denoted by C_k. K_n denotes the complete graph of order n. A graph is *Hamiltonian connected* if for any two vertices there is a Hamilton path connects them. A graph is *Hamiltonian* if it contains a Hamilton circuit. A graph G is positively Hamiltonian, written $G \in H^+$, if for every edge of G, there is a Hamilton circuit containing it. G is negatively Hamiltonian, written $G \in H^-$, if for every edge of G, there is a Hamilton circuit avoiding it. When $G \in H^+$ and $G \in H^-$, we say that G is *uniformly Hamiltonian*. If for every edge e of G, there is a k-circuit containing it for any k, $3 \leq k \leq |V(G)|$, then G is called edge-pancyclic. A graph G is called E_2-Hamiltonian if every two edges of G are contained in a Hamilton cycle of G. Let G be a simple graph of order at least 3 vertices. Then graph G is called p_3-Hamilton, if for any path P with 3 vertices, there exists a Hamilton cycle of G which contains P. If for any two vertices v_1 and v_2 and any edge v_2v_3 where $v_1 \neq v_3$, graph G has a Hamilton path from v_1 to v_2 and such that edge v_2v_3 in this path, then we say that graph G is 1-Hamilton connected. Terminology and notations not defined here can be found in [1] and [2].

Maurer defined the base graph of a matroid , and discussed the graphical properties of the base graph of a matroid [3-4]. Cummins showed that every matroid base graph with at least three vertices has a Hamilton circuit [5]. Holzmann and Harary showed that for every edge in a base graph there is a Hamilton circuit containing it and another Hamilton circuit avoiding it [6]. Alspach and Liu studied the properties of paths and circuits in base graphs of matroids [7]. The connectivity of the base graph of matroids is investigated by Liu [8]. The other graphical properties of the base graphs of matroid have also been investigated by Liu [9-16].

Now we give a new concept as follows. The circuit graph of a matroid M is a graph $G_C = G(M)$ with vertex set $V(G_C)$ and edge set $E(G_C)$ such that $V(G_C) = \mathcal{C}$ and $E(G_C) = \{CC' \mid C, C' \in \mathcal{C}, \mid C \cap C' \mid \neq 0\}$, where the same notation is used for the vertices of G and the circuits of M. We give another new graph related to the bases of matroids as follows. The intersection graph of bases of matroid $M = (E, \mathcal{B})$ is a graph $G_I(M)$ with vertex set $V(G_I)$ and edge set $E(G_I)$ such that $V(G_I) = \mathcal{B}$ and $E(G_I) = \{BB' : |B \cap B'| \neq 0, B, B' \in \mathcal{B}(M)\}$, where the same notation is used for the vertex of G_I and the base of M. The properties of paths , cycles and the connectivity of circuit graphs of matroids are discussed in this chapter. In particular, some new results obtained by us are given.

2. Preliminary results

To prove the main theorem we need the following preliminary results.

Lemma 2.1. [17] A matroid M is connected if and only if for every pair e_1, e_2 of distinct elements of E, there is a circuit containing both e_1 and e_2.

Lemma 2.2. [17] If M is a connected matroid, then for every $e \in E$, either M/e or $M \backslash e$ is also connected.

Lemma 2.3. [17] Let C and $C^\star$ be any circuit and co-circuit of a matroid M. Then $|C \cap C^\star| \neq 1$.

Lemma 2.4. [1] If $a \in C_1 \cap C_2$ and $b \in C_1 - C_2$ where $C_1, C_2 \in \mathcal{C}$, then there exists a $C_3 \in \mathcal{C}$ such that $b \in C_3 \subseteq (C_1 \cup C_2) - \{a\}$.

Let $M = (E, \mathcal{C})$ be a connected matroid. An element e of E is called an essential element if $M \backslash e$ is disconnected. Otherwise it is called an inessential element. A connected matroid each of whose elements is essential is called a critically connected matroid or simply a critical matroid.

Lemma 2.5. [17] A critical matroid of rank 2 contains a co-circuit of cardinality two.

A matroid M is trivial if it has no circuits. In the following matroids will be nontrivial. Next we will discuss the properties of the matroid circuit graph. To prove the main results we firstly present the following result which is clearly true.

Lemma 2.6. [17] Let M be any nontrivial matroid on E and $e \in E$. If G_C and G_{Ce} are circuit graphs of M and $M \backslash e$, respectively, then G_{Ce} is a subgraph of G induced by V_1 where $V_1 = \{C \mid C \in \mathcal{C}, e \notin C\}$.

Obviously the subgraph G_{C2} of G induced by $V_2 = V - V_1$ is a complete graph. By Lemma 2.6 G_{C1} and G_{C2} are induced subgraphs of G and $V(G_{C1})$ and $V(G_{C2})$ partition $V(G)$.

Lemma 2.7. [17] For any matroid $M = (E, \mathcal{C})$ which has a 2-cocircuit $\{a, b\}$, the circuit graph of M is isomorphic to that of M/a.

Proof. Since $|C \cap \{a, b\}| \neq 1$ for any circuit C, by Lemma 2.3, the circuits of M can be partitioned into two classes, those circuits containing both a and b and those circuits containing neither a nor b. Likewise, the circuits of M/a can be partitioned into two classes: those containing b and those not containing b, clearly there is a bijection between $\mathscr{C}(M)$ and $\mathscr{C}(M/a)$. Hence $G(M) \cong G(M/a)$, the lemma is proved.

Lemma 2.8. [17] Suppose that $M = (E, \mathcal{C})$ is a connected matroid with an element e such that the matroid $M \backslash e$ is connected and $G = G_C(M)$ is the circuit graph of matroid M. Let $G_1 = G(M \backslash e)$ be the circuit graph of $M \backslash e$ and G_2 be the subgraph of G induced by V_2 where $V_2 = \{C \mid C \in \mathcal{C}, e \in C\}$. If the matroid $M \backslash e$ has more than one circuit, then for any edge $C_1C_2 \in E(G)$, there exists a 4-cycle $C_1C_2C_3C_4$ in graph G such that one edge of the 4-cycle belongs to $E(G_1)$ and one belongs to $E(G_2)$ and C_1, C_2 are both adjacent to C_3.

Proof. By Lemma 2.6, $V(G_1)$ and $V(G_2)$ partition $V(G)$. There are three cases to distinguish.

Case 1. $e \in E - (C_1 \cup C_2)$. Thus C_1C_2 is an edge of $M \backslash e$. By Lemma 2.1, there are at least three vertices in $G(M \backslash e)$. There is an element e_1 such that $e_1 \in C_1 \cap C_2$. Let G_1 and G_2 be the graphs defined as above. Note that G_2 is a complete graph. By Lemma 2.1, there is a vertex C_3 in G_2 containing both e_1 and e. Thus in G, C_3 is adjacent to both C_1 and C_2. Since $C_1 \not\subset C_3$, there exists e_2 such that $e_2 \in C_1$, but $e_2 \notin C_3$. By Lemma 2.1, there is a circuit C_4 in G_2 containing e_2 and e and $C_3 \neq C_4$. Thus C_4 is adjacent to C_1.

Case 2. $e \in C_1 - C_2$ or $e \in C_2 - C_1$. Suppose that $e \in C_2 - C_1$, $e_1 \in C_1 \cap C_2$. By Lemma 2.4, there is a circuit $C_3 \subseteq (C_1 \cup C_2) - \{e_1\}$ containing e. We assume that $e_2 \in C_1 \cap C_3$, $e_3 \in E - (C_1 \cup \{e\})$. Note that e_3 exists because, by hypothesis, $M \backslash e$ has more than one circuit. By Lemma 2.1, in $G_1 = G(M\backslash e)$ there is a circuit C_4 containing e_2 and e_3. $C_1C_2C_3C_4$ is the 4-cycle we wanted. C_1 and C_2 are both adjacent to C_3.

Case 3. $e \in C_1 \cap C_2$. C_1 and C_2 are both in G_2. If there are only two circuits containing e, it is easy to see that $C_1 \cup C_2 = E(M)$ by Lemma 2.6. We prove that $C_1 \cap C_2 = \{e\}$. If $C_1 \cap C_2 = \{e, e'\}$, then $\{e, e'\}$ is a co-circuit of M because by Lemma 2.4, if there is a circuit containing e' does not contain e, then there is a circuit containing e does not contain e'; which is a contradiction to the hypothesis. Thus $C_1 \cap C_2 = \{e\}$. Then there is only one circuit $C_3' = (C_1 \cup C_2) - \{e\}$ in $M\backslash e$. For if there is a circuit $C_4' \neq C_3'$ in $M\backslash e$, then there exists $e_1 \in E(M) - (\{e\} \cup C_4')$ and $e_1 \in C_1 - C_2$ or $e_1 \in C_2 - C_1$. We assume that $e_1 \in C_1 - C_2$. By Lemma 2.4, there is a circuit C_5 such that $e \in C_5 \subseteq (C_2 \cup C_4') - \{e_2\}$ where $e_2 \in C_2 \cap C_4'$. $e_1 \notin C_5$, $C_5 \neq C_1$, thus there are at least three circuits containing e, a contradiction. So there are more than two circuits containing e, we assume that $e_3 \in C_1 - C_2$. By Lemma 2.4, there is C_3 in G_1 such that $e_3 \in C_3 \subseteq (C_1 \cup C_2) - \{e\}$. By Lemma 2.1, there is a vertex C_4 in G_1 containing e_3 and e_4 where $e_4 \in E - (C_3 \cup \{e\})$. We get the 4-cycle $C_1C_2C_3C_4$. The proof is completed.

Lemma 2.9. [17] Suppose that $M = (E, \mathcal{C})$ is a connected matroid with an element e such that the matroid $M\backslash e$ is connected and $G = G(M)$ is the circuit graph of matroid M. Let $G_1 = G(M\backslash e)$ be the circuit graph of $M\backslash e$ and G_2 be the subgraph of G induced by V_2 where $V_2 = \{C \mid C \in \mathcal{C}, e \in C\}$. If $C_1 \in V(G_1)$, $C_2 \in V(G_2)$ and $d(C_1, C_2) = 2$, there exists a 3-path $C_1C_3C_4C_2$ in graph G such that $C_3 \in V(G_1)$, $C_4 \in V(G_2)$ and C_1, C_2 are both adjacent to C_3 and C_4.

Proof. By Lemma 2.6, $V(G_1)$ and $V(G_2)$ partition $V(G)$. By assumption, $|C_1 \cap C_2| = 0$. We assume that $e_1 \in C_1$, $e_2 \in C_2$. By Lemma 2.1, there is C_4 in G_2 containing both e_1 and e. Thus in G, C_4 is adjacent to C_1 and C_2. By Lemma 2.1, there is C_3 in G_1 containing both e_1 and e_2. Thus in G, C_3 is adjacent to C_1, C_2 and C_4

Let P_4 be the Hamilton path connecting C_1 and C_2 in $G(M\backslash e)$ which traverses $C_1'C_2'$ and P_2 be the m-path connecting C_3 and C_4 in G_2 ($1 \leq m \leq n_2 - 1$), respectively. $P_3 + C_1'C_3 + C_3C_2' + P_4$ is a n_1-path of $G(M)$ that joins C_1 and C_2. $P_3 + C_1'C_4 + P_2 + C_3C_2' + P_4$ is a $n_1 + m$-path of $G(M)$ that joins C_1 and C_2.

Subcase 1.2. $e \in C_1 - C_2$ or $e \in C_2 - C_1$. Suppose that $e \in C_2 - C_1$. Thus $C_1 \in V(G_1)$, $C_2 \in V(G_2)$. If $d(C_1, C_2) = 1$, by Lemma 2.8, there is a 4-cycle $C_1C_2C_3C_4$ in G such that $C_3 \in V(G_2)$ and $C_4 \in V(G_1)$ and $C_1C_2C_3$ is a 3-cycle of G. By induction, for any k_1, $1 \leq k_1 \leq n_1 - 1$, there is a k_1-path in G_1 connecting C_4 and C_1. Note that G_2 is a complete graph. For any k_2, $1 \leq k_2 \leq n_2 - 1$, there is a k_2-path in G_2 connecting C_2 and C_3. Let P_1 be the k_1-path in G_1 connecting C_1 and C_4 and P_2 be the k_2-path in G_2 connecting C_3 and C_2, respectively. $P_1 + C_4C_3 + P_2$ is a $k_1 + k_2 + 1$-path of $G(M)$ that connects C_1 and C_2. If $d(C_1, C_2) = 2$ and $e \in C_2 - C_1$, by Lemma 2.9, there is a 3-path $C_1C_3C_4C_2$ in G such that $C_3 \in V(G_1)$ and $C_4 \in V(G_2)$ and $C_1C_3C_2$ is a 2-path of G. Then the proof is similar to the case when $d(C_1, C_2) = 1$.

Subcase 1.3. $e \in C_1 \cap C_2$. Thus $d(C_1, C_2) = 1$. If there are only two circuits containing e, then there is only one circuit in $M\backslash e$. The result holds obviously because $G = K_3$. Assume that there are more than two circuits containing e. Note that G_2 is a complete graph. For any m, $1 \leq m \leq n_2 - 1$, there is a path of length m connecting C_1 and C_2. Choose $C_3 \in V(G_2)$

such that $C_3 \neq C_1$ and $C_3 \neq C_2$. By Lemma 2.8, there is a 4-cycle $C_1C_3C_4C_5$ in G such that $C_4C_5 \in E(G_1)$ and $C_1C_3C_4$ is a 3-cycle of G. By induction, for any k, $1 \leq k \leq n_1 - 1$, there is a k-path in G_1 connecting C_4 and C_5. Note that G_2 is a complete graph. Let P_1 be the k-path in G_1 connecting C_4 and C_5 and P_2 be the Hamilton path in G_2 connecting C_1 and C_2 which traverses the edge C_1C_3. Let $P_2 = C_1C_3 + P_3$. $C_1C_4 + C_4C_3 + P_3$ is a n_2-path that connects C_1 and C_2. $C_1C_5 + P_1 + C_4C_3 + P_3$ is the $n_2 + k$-path we wanted.

Case 2. The matroid M is critically connected. By Lemma 2.2, for any element e in M, M/e is connected. By Lemma 2.5, M has a 2-cocircuit $\{a, b\}$. By Lemma 2.7, the circuit graph of M/a is isomorphic to that of M. By induction hypothesis, the result holds.

Thus the theorem follows by induction.

Lemma 2.10. [2] A graph G is k-edge-connected if and only if any two distinct vertices of G are connected by at least k edge- disjoint paths.

3. The properties of circuit graphs of matroids

We have known that a matroid M is connected if and only if for every pair e_1, e_2 of distinct elements of E, there is a circuit containing both e_1 and e_2. The following results are the new results obtained by Li and Liu.

Theorem 3.1. [17] For any connected matroid $M = (E, \mathcal{C})$ which has at least three circuits, the circuit graph $G = G(M)$ is positively Hamiltonian, that is, for every edge of G, there is a Hamilton circuit containing it.

We shall prove the theorem by induction on $|E|$. When $|E| = 3$, each element in M is parallel to another. It is easy to see that $G = K_3$. The theorem is clearly true. Suppose that the result is true for $|E| = n - 1$. We prove that the result is also true for $|E| = n > 3$. Let C_1C_2 be any edge in G.

There are two cases to distinguish.

Case 1. There is an element e in M such that $M \backslash e$ is connected. Let G_1 and G_2 be the graphs defined as above. We assume that $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$.

There are three subcases to distinguish.

Subcase 1.1. $e \in E - (C_1 \cup C_2)$. Thus C_1C_2 is an edge of $M \backslash e$. By Lemma 2.4, there are at least three vertices in $G(M \backslash e)$. By induction, $G(M \backslash e)$ is edge-pancyclic. For any m, $3 \leq m \leq n_1$, there is a cycle of length m containing C_1C_2. By Lemma 2.8, for any edge $C_1'C_2'$ in the Hamilton cycle of $G(M \backslash e)$ containing C_1C_2 where $C_1'C_2' \neq C_1C_2$, there is a 4-cycle $C_1'C_2'C_3C_4$ in G such that $C_3C_4 \in E(G_2)$ and $C_1'C_2'C_3$ is a 3-cycle in G. If there are only three vertices in $G(M \backslash e)$, let $C_1 = C_1'$. Note that G_2 is a complete graph. Let P_1 be the Hamilton path connecting C_1' and C_2' in $G(M \backslash e)$ which traverses C_1C_2 and P_2 be the k-path connecting C_3 and C_4 in G_2 ($1 \leq k \leq n_2 - 1$), respectively. $P_1 + C_2'C_3 + C_3C_1'$ is a $n_1 + 1$-cycle of $G(M)$ that contains C_1C_2. $P_1 + C_2'C_3 + P_2 + C_4C_1'$ is a $n_1 + k + 1$-cycle of $G(M)$ that contains C_1C_2.

Subcase 1.2. $e \in C_1 - C_2$ or $e \in C_2 - C_1$. Suppose that $e \in C_2 - C_1$. Thus $C_1 \in V(G_1)$, $C_2 \in V(G_2)$. By Lemma 2.8, there is a 4-cycle $C_1C_2C_3C_4$ in G such that $C_3 \in V(G_2)$ and $C_4 \in V(G_1)$ and $C_1C_2C_3$ is a 3-cycle of G. By induction, for any k_1, $1 \leq k_1 \leq n_1 - 1$, there is a k_1-path in G_1 connecting C_4 and C_1. Note that G_2 is a complete graph. For any k_2, $1 \leq k_2 \leq n_2 - 1$, there is a k_2-path in G_2 connecting C_2 and C_3. Let P_1 be the k_1-path in G_1 connecting C_1 and C_4 and P_2 be the k_2-path in G_2 connecting C_3 and C_2, respectively. $P_1 + C_4C_3 + P_2 + C_2C_1$ is a $k_1 + k_2 + 2$-cycle of $G(M)$ that contains C_1C_2.

Subcase 1.3. $e \in C_1 \cap C_2$. If there are only two circuits containing e, then there is only one circuit in $M \backslash e$. The result holds obviously because $G = K_3$. Assume that there are more than two circuits containing e. Note that G_2 is a complete graph. For any m, $3 \leq m \leq n_2$, there is a cycle of length m containing C_1C_2. Choose $C_3 \in V(G_2)$ such that $C_3 \neq C_1$ and $C_3 \neq C_2$. By Lemma 2.8, there is a 4-cycle $C_1C_3C_4C_5$ in G such that $C_4C_5 \in E(G_1)$ and $C_1C_3C_4$ is a 3-cycle of G. By induction, for any k, $1 \leq k \leq n_1 - 1$, there is a k-path in G_1 connecting C_4 and C_5. Note that G_2 is a complete graph. Let P_1 be the k-path in G_1 connecting C_4 and C_5 and P_2 be the Hamilton path in G_2 connecting C_3 and C_1 which traverses the edge C_2C_1. $C_4C_3 + P_2 + C_1C_4$ is a $n_2 + 1$-cycle that contains C_1C_2. $P_1 + C_4C_3 + P_2 + C_1C_5$ is the $n_2 + k + 1$-cycle we wanted.

Case 2. The matroid M is critically connected. By Lemma 2.2, for any element e in M, M/e is connected. By Lemma 2.5, M has a 2-cocircuit $\{a, b\}$. By Lemma 2.7, the circuit graph of M/a is isomorphic to that of M. By induction hypothesis, the result holds.

Thus the theorem follows by induction.

Theorem 3.2. [21] Let M be any connected matroid which has at least four circuits and let $G = G(M)$ be the circuit graph of M. Then for each edge of $G = G(M)$ there is a Hamilton cycle avoiding it. that is, the circuit graph $G = G(M)$ is negatively Hamiltonian.

Proof. We prove the theorem by induction on $|E|$. It is easy to see that $|E| \geqslant 4$. When $|E| = 4$ and $r(M) = 1$, $M = U_{1,4}$ [1]. It is easy to see that $G(M) \in H^-$. When $|E| = 4$ and $r(M) = 2$, M has at most three circuits except when $M = U_{2,4}$. Obviously $G(U_{2,4}) = K_4$ and $K_4 \in H^-$. Then suppose that the theorem is true for $|E| = n - 1$. We shall prove the theorem holds for $|E| = n > 4$. Let C_1C_2 be any edge of G. There are two cases to consider.

Case 1. There exists an element e such that $M \backslash e$ is connected. Let G_1 and G_2 be the graphs defined as above.

There are three subcases to consider.

Subcase 1.1. If $e \in E - (C_1 \cup C_2)$, then C_1C_2 is an edge of $G(M \backslash e)$. By Theorem 3.1, there is a Hamilton cycle in $G(M \backslash e)$ containing C_1C_2. By the proof of Lemma 2.8, there is a 4-cycle $C_1C_2C_3C_4$ in G such that $C_3C_4 \in E(G_2)$. As in the proof of Theorem 3.1, let P_1 be a Hamilton path in G_1 connecting C_1 and C_2 and P_2 be a Hamilton path in G_2 connecting C_3 and C_4, respectively. Then the cycle $P_1 + C_2C_3 + P_2 + C_4C_1$ is a Hamilton cycle of G avoiding C_1C_2.

Subcase 1.2. Let $e \in C_1 - C_2$ or $e \in C_2 - C_1$. It is easy to see that there are more than two vertices in $G(M \backslash e)$, and there are at least three vertices in G_2. We assume that $e \in C_2 - C_1$ and $e_1 \in C_1 - C_2$. By Lemma 2.1, there is C_3 in G_2 containing e and e_1. By the proof of Lemma 2.8, in G there is a 4-cycle $C_1C_3C_4C_5$ such that $C_4 \in V(G_2)$, $C_5 \in V(G_1)$. Let P_1 be a Hamilton path in $G(M \backslash e)$ connecting C_1 and C_5 and P_2 be a Hamilton path in G_2 connecting C_4 and C_3, respectively. $P_1 + C_5C_4 + P_2 + C_3C_1$ is a Hamilton cycle avoiding C_1C_2.

Subcase 1.3. If $e \in C_1 \cap C_2$, as in the proof of subcase 1.3 in Theorem 3.1, let $C_2 = C_3$ and P_2 be a Hamilton path connecting C_1 and C_2 in G_2, then $P_1 + C_4C_2 + P_2 + C_1C_5$ is a Hamilton cycle we wanted.

Case 2. For every $e \in E(M)$, $M\backslash e$ is disconnected. Then the matroid M is critically connected. By Lemma 2.2, for any element e in M, M/e is connected. By Lemma 2.5, M has a cocircuit $\{a, b\}$. By Lemma 2.7, $G(M) \cong G(M/a)$. By induction hypothesis, the result holds.

The proof of the theorem is completed.

Corollary 3.3. For any connected matroid M, the circuit graph $G_C(M)$ is uniformly Hamilton whenever $G_C(M)$ contains at least four vertices. That is for any edge e of $G_C(M)$, there is a Hamilton cycle containing e and there is another Hamilton cycle excluding e.

Theorem 3.4. [20] Let $G = G(M)$ be the circuit graph of a connected matroid $M = (E, \mathcal{C})$. If $|V(G)| = n$ and $C_1, C_2 \in V(G)$ with $d(C_1, C_2) = r$, then there is a path of length k joining C_1 and C_2 for any k satisfying $r \leq k \leq n - 1$.

Proof. We shall prove the theorem by induction on $|E|$. When $|E| = 3$, each element in M is parallel to another. It is easy to see that $G = K_3$. The theorem is clearly true. Suppose that the result is true for $|E| = n - 1$. We prove that the result is also true for $|E| = n > 3$. It is easy to see that for any vertices C_1, C_2 in $V(G)$, we have $d(C_1, C_2) = r$ where $r = 1$ or $r = 2$ by the definition of the circuit graph of a matroid.

There are two cases to distinguish.

Case 1. There is an element e in M such that $M\backslash e$ is connected. Let G_1 and G_2 be the graphs defined as above. We assume that $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$.

There are three subcases to distinguish.

Subcase 1.1. $e \in E - (C_1 \cup C_2)$. Thus $C_1 \in V(G_1)$, $C_2 \in V(G_1)$. Clearly, there are at least three vertices in $G(M\backslash e)$. By induction, for any k, $r \leq k \leq n_1 - 1$, there is a path of length k joining C_1 and C_2 in $G(M\backslash e)$. By Lemma 2.8, for any edge $C_1'C_2'$ in the Hamilton path connecting C_1 and C_2 in $G(M\backslash e)$, there is a 4-cycle $C_1'C_2'C_3C_4$ in G such that $C_3C_4 \in E(G_2)$ and $C_1'C_2'C_3$ is a 3-cycle in G. If there are only three vertices in $G(M\backslash e)$, let $C_1 = C_1'$. Note that G_2 is a complete graph. Let $P_1 = P_3 + C_1'C_2' + P_4$ be the Hamilton path connecting C_1 and C_2 in $G(M\backslash e)$ which traverses $C_1'C_2'$ and P_2 be the m-path connecting C_3 and C_4 in G_2 ($1 \leq m \leq n_2 - 1$), respectively. $P_3 + C_1'C_3 + C_3C_2' + P_4$ is a n_1-path of $G(M)$ that joins C_1 and C_2. $P_3 + C_1'C_4 + P_2 + C_3C_2' + P_4$ is a $n_1 + m$-path of $G(M)$ that joins C_1 and C_2 .

Subcase 1.2. $e \in C_1 - C_2$ or $e \in C_2 - C_1$. Suppose that $e \in C_2 - C_1$. Thus $C_1 \in V(G_1)$, $C_2 \in V(G_2)$. If $d(C_1, C_2) = 1$, by Lemma 2.8, there is a 4-cycle $C_1C_2C_3C_4$ in G such that $C_3 \in V(G_2)$ and $C_4 \in V(G_1)$ and $C_1C_2C_3$ is a 3-cycle of G. By induction, for any k_1, $1 \leq k_1 \leq n_1 - 1$, there is a k_1-path in G_1 connecting C_4 and C_1. Note that G_2 is a complete graph. For any k_2, $1 \leq k_2 \leq n_2 - 1$, there is a k_2-path in G_2 connecting C_2 and C_3. Let P_1 be the k_1-path in G_1 connecting C_1 and C_4 and P_2 be the k_2-path in G_2 connecting C_3 and C_2, respectively. $P_1 + C_4C_3 + P_2$ is a $k_1 + k_2 + 1$-path of $G(M)$ that connects C_1 and C_2.

If $d(C_1, C_2) = 2$ and $e \in C_2 - C_1$, by Lemma 2.9, there is a 3-path $C_1C_3C_4C_2$ in G such that $C_3 \in V(G_1)$ and $C_4 \in V(G_2)$ and $C_1C_3C_2$ is a 2-path of G. Then the proof is similar to the case when $d(C_1, C_2) = 1$.

Subcase 1.3. $e \in C_1 \cap C_2$. Thus $d(C_1, C_2) = 1$. If there are only two circuits containing e, then there is only one circuit in $M\backslash e$. The result holds obviously because $G = K_3$. Assume that there are more than two circuits containing e. Note that G_2 is a complete graph. For any m, $1 \leq m \leq n_2 - 1$, there is a path of length m connecting C_1 and C_2. Choose $C_3 \in V(G_2)$ such that $C_3 \neq C_1$ and $C_3 \neq C_2$. By Lemma 2.8, there is a 4-cycle $C_1C_3C_4C_5$ in G such that $C_4C_5 \in E(G_1)$ and $C_1C_3C_4$ is a 3-cycle of G. By induction, for any k, $1 \leq k \leq n_1 - 1$, there is a k-path in G_1 connecting C_4 and C_5. Note that G_2 is a complete graph. Let P_1 be the k-path in G_1 connecting C_4 and C_5 and P_2 be the Hamilton path in G_2 connecting C_1 and C_2 which traverses the edge C_1C_3. Let $P_2 = C_1C_3 + P_3$. $C_1C_4 + C_4C_3 + P_3$ is a n_2-path that connects C_1 and C_2. $C_1C_5 + P_1 + C_4C_3 + P_3$ is the $n_2 + k$-path we wanted.

Case 2. The matroid M is critically connected. By Lemma 2.2, for any element e in M, M/e is connected. By Lemma 2.5, M has a 2-cocircuit $\{a, b\}$. By Lemma 2.7, the circuit graph of M/a is isomorphic to that of M. By induction hypothesis, the result holds.

Thus the theorem follows by induction.

Theorem 3.5. [20] Suppose that $G = G_C(M)$ is the circuit graph of a connected matroid M and C_1 and C_2 are distinct vertices of G. Then C_1 and C_2 are connected by $d = min\{d(C_1), d(C_2)\}$ edge -disjoint paths where $d(C_1)$ and $d(C_2)$ denote the degree of vertices C_1 and C_2 in G, respectively.

Proof. We shall prove the theorem by induction on $|E|$. When $|E| = 3$, each element in M is parallel to another. It is easy to see that $G = K_3$. The theorem is clearly true. Suppose that the result is true for $|E| = n - 1$. We prove that the result is also true for $|E| = n > 3$. Let C_1 and C_2 be any two vertices in G.

There are two cases to distinguish.

Case 1. $(C_1 \cup C_2) = E$. It is easy to see that C_1 and C_2 are both adjacent to any circuit in $\mathscr{C} - \{C_1 \cup C_2\}$ and the conclusion is obviously true.

Case 2. $(C_1 \cup C_2) \neq E$.

There are two subcases to distinguish.

Subcase 2.1. There is an element $e \in E - (C_1 \cup C_2)$ such that $M\backslash e$ is connected. Let $G_1 = G(M\backslash e)$ be the circuit graph of $M\backslash e$ and G_2 be the subgraph of G induced by V_1 where $V_1 = \{C \mid C \in \mathscr{C}, e \in C\}$. Thus C_1 and C_2 are in G_1. By induction, in G_1, C_1, C_2 are connected by $d_1 = min\{d_1(C_1), d_1(C_2)\}$ edge-disjoint paths where $d_1(C_1)$ and $d_1(C_2)$ denote the degree of vertices C_1 and C_2 in G_1, respectively. Let $\mathscr{P}_1 = \{P_1, P_2, \cdots, P_{d_1}\}$ be the family of shortest edge-disjoint paths connecting C_1 and C_2 in G_1. Without loss of generality, we may assume that $d_1(C_1) \geq d_1(C_2)$. There are two subcases to distinguish.

Subcase 2.1 a. $d_1(C_1) = d_1(C_2)$. Thus $d_1 = min\{d_1(C_1), d_1(C_2)\} = d_1(C_1) = d_1(C_2)$. We assume that there are m vertices $A_1, A_2, \cdots, A_m$ in G_2 that are adjacent to C_1 and n vertices $D_1, D_2, \cdots, D_n$ in G_2 that are adjacent to C_2 where m, n are integers. G_2 is a complete graph, so A_i is adjacent to $D_j(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$. Here maybe $A_i = D_j$ for some $1 \le i \le m; 1 \le j \le n$. Let $q = min\{m, n\}$. $C_1 A_i D_i C_2 (i = 1, 2, \cdots, q)$ are q edge-disjoint paths in G. It is easy to see that $d(C_1) = d_1(C_1) + m, d(C_2) = d_1(C_2) + n$ and $d = min\{d(C_1), d(C_2)\} = min\{d_1(C_1) + m, d_1(C_2) + n\} = d_1(C_1) + min\{m, n\} = d_1(C_1) + q$. $\mathscr{P} = \mathscr{P}_1 \cup \{C_1 A_1 D_1 C_2, C_1 A_2 D_2 C_2, \cdots, C_1 A_q D_q C_2\}$ are d edge-disjoint paths connecting C_1 and C_2 in G.

Subcase 2.1 b. $d_1(C_1) > d_1(C_2)$. By induction, in G_1 there are $d_1 = min\{d_1(C_1), d_1(C_2)\} = d_1(C_2)$ edge -disjoint paths connecting C_1 and C_2. Let $\mathscr{P}_1 = \{P_1, P_2, \cdots, P_{d_1(C_2)}\}$ be the family of shortest edge-disjoint paths connecting C_1 and C_2 in G_1. It is obvious that each $P_i(i = 1, 2, \cdots, d_1(C_2))$ contains exactly one vertex adjacent to C_1 and one vertex adjacent to C_2. Let $A_1, A_2, \cdots, A_{d_1(C_1)-d_1(C_2)}$ be the vertices in G_1 that are adjacent to C_1 but not contained in d_1 edge-disjoint paths. By Lemma 2.1, for any element e' in A_i $(i = 1, 2, \cdots, d_1(C_1) - d_1(C_2))$ there is a circuit A'_i in G_2 containing e and e', thus $A_i A'_i$ is an edge in $G(M)$. Let $D_1, D_2, \cdots, D_m$ denote the vertices in G_2 that is adjacent to C_2. G_2 is a complete graph, so A'_i is adjacent to $D_j(i = 1, 2, \cdots, d_1(C_1) - d_1(C_2); j = 1, 2, \cdots, m)$. If $m \le d_1(C_1) - d_1(C_2)$, $C_1 A_i A'_i D_i C_2$ are m edge-disjoint paths connecting C_1 and C_2 where A'_i can be D_i $(i = 1, 2, \cdots, m)$. Here it is possible that $A'_i = A'_j (i \ne j; i, j = 1, 2, \cdots, d_1(C_1) - d_1(C_2))$. But it is forbidden that $D_i = D_j (i \ne j; i, j = 1, 2, \cdots, m)$. $d(C_2) = d_1(C_2) + m \le d_1(C_1) < d(C_1)$, thus $d = min\{d(C_1), d(C_2)\} = d(C_2)$. $\mathscr{P} = \mathscr{P}_1 \cup \{C_1 A_1 A'_1 D_1 C_2, C_1 A_2 A'_2 D_2 C_2, \cdots, C_1 A_m A'_m D_m C_2\}$ are d edge-disjoint paths connecting C_1 and C_2 in G. If $m > d_1(C_1) - d_1(C_2)$, let $\mathscr{P}_2 = \{C_1 A_1 A'_1 D_1 C_2, C_1 A_2 A'_2 D_2 C_2, \cdots, C_1 A_{d_1(C_1)-d_1(C_2)} A'_{d_1(C_1)-d_1(C_2)} D_{d_1(C_1)-d_1(C_2)} C_2\}$ be $d_1(C_1) - d_1(C_2)$ edge-disjoint paths connecting C_1 and C_2 where A'_i can be D_i $(i = 1, 2, \cdots, d_1(C_1) - d_1(C_2))$. Let $L_1, L_2, \cdots, L_n$ denote the vertices in G_2 that is adjacent to C_1. G_2 is a complete graph, so L_i is adjacent to $D_j (i = 1, 2, \cdots, n; j = d_1(C_1) - d_1(C_2) + 1, d_1(C_1) - d_1(C_2) + 2, \cdots, m)$. If $m > n + d_1(C_1) - d_1(C_2)$, $d(C_1) = d_1(C_1) + n \le d_1(C_2) + m < d(C_2)$, thus $d = min\{d(C_1), d(C_2)\} = d(C_1)$. $\mathscr{P}_3 = C_1 L_i D_{d_1(C_1)-d_1(C_2)+i} C_2$ are n edge-disjoint paths connecting C_1 and C_2 where L_i can be $D_{d_1(C_1)-d_1(C_2)+i}$ $(i = 1, 2, \cdots, n)$. Thus $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{P}_3$ are $d = d(C_1)$ edge-disjoint paths in G. If $d_1(C_1) - d_1(C_2) < m \le n + d_1(C_1) - d_1(C_2)$, $\mathscr{P}'_3 = C_1 L_i D_{d_1(C_1)-d_1(C_2)+i} C_2$ are $m - (d_1(C_1) - d_1(C_2)$ edge-disjoint paths connecting C_1 and C_2 where L_i can be $D_{d_1(C_1)-d_1(C_2)+i}$ $(i = 1, 2, \cdots, m - (d_1(C_1) - d_1(C_2))$ but $D_{d_1(C_1)-d_1(C_2)+i} \ne D_{d_1(C_1)-d_1(C_2)+j}$ $(i \ne j; i, j = 1, 2, \cdots, m - (d_1(C_1) - d_1(C_2))$. $d(C_2) = d_1(C_2) + m \le d_1(C_1) + n = d(C_1)$, thus $d = min\{d(C_1), d(C_2)\} = d(C_2)$. $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{P}'_3$ are $d = d(C_2)$ edge-disjoint paths connecting C_1 and C_2 in G. The conclusion holds.

Subcase 2.2. There is no element $e \in E - (C_1 \cup C_2)$ such that $M \backslash e$ is connected. If $E - (C_1 \cup C_2) = \{e\}$ and M e is disconnected, it is easy to see that $C_1 \cap C_2 = \emptyset$ and C_1, C_2 are the two components of $M \backslash e$. Thus any circuit of M intersecting both C_1 and C_2 contains e. Then C_1 and C_2 are both adjacent to any circuit in $\mathscr{C} - \{C_1 \cup C_2\}$ and the conclusion is obviously true. Suppose that $|E - (C_1 \cup C_2)| \ge 2$ and for any $e \in E - (C_1 \cup C_2)$, $M \backslash e$ is disconnected. By Lemma 2.5, M has a 2-cocircuit $\{a, b\}$. By Lemma 2.7, the circuit graph of M/a is isomorphic to that of M. By induction hypothesis, the result holds. Thus the theorem follows by induction.

By Theorem 3.5, we can get the following corollary.

Corollary 3.6. Suppose that $G = G(M)$ is the circuit graph of a connected matroid M with minimum degree $\delta(G)$. Then the edge connectivity $\kappa'(G) = \delta(G)$.

Proof. By Theorem 3.5, we know that $\kappa'(G) \geq \delta(G)$. Since for any graph G, we have $\kappa'(G) \leq \delta(G)$, then $\kappa'(G) = \delta(G)$.

Theorem 3.7. [20] Let G be the circuit graph of a connected matroid $M = (E, \mathscr{C})$. If $|V(G)| = n$ and $k_1 + k_2 + \cdots + k_p = n$ where k_i is an integer, $i = 1, 2, \ldots, p$, then there is a partition of $V(G)$ into p parts $V_1, V_2, \ldots, V_p$ such that $|V_i| = k_i$ and the subgraph H_i induced by V_i contains a k_i -cycle when $k_i \geq 3$, H_i is isomorphic to K_2 when $k_i = 2$, H_i is a single vertex when $k_i = 1$.

Proof. We shall prove the theorem by induction on $|E|$. When $|E| = 3$ and $|V(G)| = 1$, the result holds clearly. When $|E| = 3$ and $|V(G)| = 3$, $M = U_{1,3}[1]$. It is easy to see that $G = K_3$. The theorem is clearly true. Suppose that the result is true for $|E| = m - 1$. We prove that the result is also true for $|E| = m > 3$.

Case 1. There is an element e in M such that $M \backslash e$ is connected. Let G_1 and G_2 be the graphs defined as above. We assume that $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$. There exists an q such that $k_1 + k_2 + \cdots + k_{q-1} < n_1$ and $k_1 + k_2 + \cdots + k_q \geq n_1$. By induction, the vertices of $G(M \backslash e)$ can be partitioned into q parts $V_1, V_2, \ldots, V'_q$ such that $|V_1| = k_1, |V_2| = k_2, \ldots, |V_{q-1}| = k_{q-1}, |V'_q| = n_1 - (k_1 + k_2 + \cdots + k_{q-1}) = k'_q$ and the subgraph H_i (H'_q) induced by V_i ($i = 1, 2, \ldots, q-1$) (V'_q) contains a k_i (k'_q) -cycle when $k_i \geq 3$ ($k'_q \geq 3$), H_i (H'_q) is isomorphic to K_2 when $k_i = 2$ ($k'_q = 2$), H_i (H'_q) is a single point when $k_i = 1$ ($k'_q = 1$). When $k_q = k'_q$, the result holds clearly because G_2 is a complete graph. When $k_q > k'_q$, there are three subcases to consider.

Subcase 1.1. $k'_q = 1$. Suppose that C_1 is the subgraph in $G(M \backslash e)$ induced by V'_q. When $k_q = 2$, obviously there is a vertex in G_2 that is adjacent to C_1. When $k_q \geq 3$, we prove that there is a 3-cycle $C_1C_2C_3$ in G such that $C_2C_3 \in E(G_2)$. For any $e_1 \in C_1$, by Lemma 2.1, there is a circuit $C_2 \in G_2$ containing e_1 and e. Let $e_2 \in C_1 - C_2$. There is a circuit $C_3 \in G_2$ containing e_2 and e. We get the 3-cycle $C_1C_2C_3$. Note that G_2 is a complete graph. In G_2 there is a $k_q - 2$ path P connecting C_2 and C_3. $C_1C_2 + P + C_3C_1$ is a k_q -cycle in G. Because G_2 is a complete graph, the subgraph induced by $V(G_2) - V(P)$ is also a complete graph. Thus the vertices of the subgraph induced by $V(G_2) - V(P)$ can be partitioned into $p - q$ parts $V_{q+1}, V_{q+2}, \ldots, V_p$ such that $|V_i| = k_i$, $i = q+1, q+2, \ldots, p$, and the subgraph H_i induced by V_i contains a k_i -cycle when $k_i \geq 3$, H_i is isomorphic to K_2 when $k_i = 2$, H_i is a single point when $k_i = 1$. The result holds.

Subcase 1.2. $k'_q = 2$. Suppose that C_1C_2 is the subgraph in $G(M \backslash e)$ induced by V'_q. When $k_q = 3$, by Lemma 2.8, the result holds. When $k_q \geq 4$, by Lemma 2.8, there is a 4-cycle $C_1C_2C_3C_4$ in G such that $C_3 \in V(G_2)$ and $C_4 \in V(G_2)$ and $C_1C_2C_3$ is a 3-cycle of G. In G_2 there is a $k_q - 3$ -path P connecting C_3 and C_4. $C_1C_2 + C_2C_3 + P + C_4C_1$ is a k_q -cycle in G. Because G_2 is a complete graph, the subgraph induced by $V(G_2) - V(P)$ is also a complete graph. Thus the result holds.

Subcase 1.3. $k'_q > 2$. The subgraph H'_q in $G(M \backslash e)$ induced by V'_q contains a k'_q -cycle. Let C_1C_2 be any edge in this cycle and P_1 be the Hamilton path in H'_q connecting C_1 and C_2. By Lemma 2.8, there is a 4-cycle $C_1C_2C_3C_4$ in G such that $C_3C_4 \in E(G_2)$ and $C_1C_2C_3$ is a 3-cycle of G. When $k_q - k'_q = 1$, $P_1 + C_2C_3 + C_3C_1$ is a k_q -cycle in G. When $k_q - k'_q \geq 2$, in G_2 there is a $k_q - k'_q - 1$ -path P_2 connecting C_3 and C_4. $P_1 + C_2C_3 + P_2 + C_4C_1$ is a k_q -cycle in G. Note that

G_2 is a complete graph. The subgraph induced by $V(G_2) - V(P_2)$ is also a complete graph. Thus the result holds.

Case 2. The matroid M is critically connected. By Lemma 2.2, for any element e in M, M/e is connected. By Lemma 2.5, M has a 2-cocircuit $\{a, b\}$. By Lemma 2.7, the circuit graph of M/a is isomorphic to that of M. By induction hypothesis, the result holds.

Thus the theorem follows by induction.

From Theorem 3.7 we have the following theorem holds.

Theorem 3.8. Let $G = G(M)$ be the circuit graph of a connected matroid $M = (E, \mathcal{C})$. If $|V(G)| = n$ and $k_1 + k_2 + \cdots + k_p = n$ where k_i is an integer and $k_i \geq 3$, $i = 1, 2, \ldots, p$, then G has a 2-factor F containing p vertex-disjoint cycles $D_1, D_2, \ldots, D_p$ such that the length of D_i is k_i $(i = 1, 2, \ldots, p)$.

By the similar methods we can prove that the above theorems also holds for the intersection graph of bases of matroids.

Finally, We present the following open problems to be considered.

Problem 1. Let $G = G(M)$ be the circuit graph of a connected matroid $M = (E, \mathcal{C})$. If $|V(G)| = n$ and $C_1, C_2 \in V(G)$ with $d(C_1, C_2) = r$, how many Hamilton paths connect C_1 and C_2 in G? Furthermore, how many k-paths connect C_1 and C_2 in G $(r \leq k \leq n-1)$?

Problem 2. Let $G = G(M)$ be the intersection graph of bases of matroid $M = (E, \mathcal{B})$. If $|V(G)| = n$ and $B_1, B_2 \in V(G)$ with $d(B_1, B_2) = r$, how many Hamilton paths connect B_1 and B_2 in G? Furthermore, how many k-paths connect B_1 and B_2 in G $(r \leq k \leq n-1)$?

Other related results of graphs on matroids can be found in [22-60].

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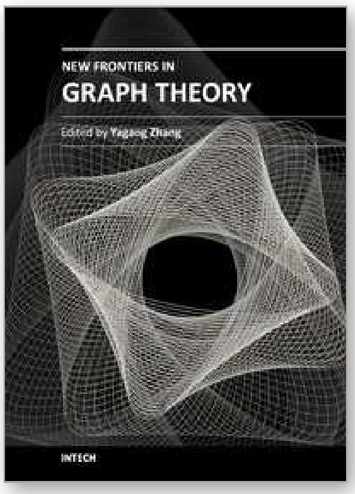

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Nowadays, graph theory is an important analysis tool in mathematics and computer science. Because of the inherent simplicity of graph theory, it can be used to model many different physical and abstract systems such as transportation and communication networks, models for business administration, political science, and psychology and so on. The purpose of this book is not only to present the latest state and development tendencies of graph theory, but to bring the reader far enough along the way to enable him to embark on the research problems of his own. Taking into account the large amount of knowledge about graph theory and practice presented in the book, it has two major parts: theoretical researches and applications. The book is also intended for both graduate and postgraduate students in fields such as mathematics, computer science, system sciences, biology, engineering, cybernetics, and social sciences, and as a reference for software professionals and practitioners.