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# Pure Links Between Graph Invariants and Large Cycle Structures

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## 1. Introduction

Hamiltonian graph theory is one of the oldest and attractive fields in discrete mathematics, concerning various path and cycle existence problems in graphs. These problems mainly are known to be NP-complete that force the graph theorists to direct efforts toward understanding the global and general relationship between various invariants of a graph and its path and cycle structure.

This chapter is devoted to large cycle substructures, perhaps the most important cycle structures in graphs: Hamilton, longest and dominating cycles and some generalized cycles including Hamilton and dominating cycles as special cases.

Graph invariants provide a powerful and maybe the single analytical tool for investigation of abstract structures of graphs. They, combined in convenient algebraic relations, carry global and general information about a graph and its particular substructures such as cycle structures, factors, matchings, colorings, coverings, and so on. The discovery of these relations is the primary problem of graph theory.

In the literature, eight basic (initial) invariants of a graph $G$ are known having significant impact on large cycle structures, namely order $n$, size $q$, minimum degree $\delta$, connectivity $\kappa$, independence number $\alpha$, toughness $\tau$ and the lengths of a longest path and a longest cycle in $G \setminus C$ for a given longest cycle $C$ in $G$, denoted by $\overline{p}$ and $\overline{c}$, respectively.

In this chapter we have collected 37 pure algebraic relations between $n, q, \delta, \kappa, \alpha, \tau, \overline{p}$ and $\overline{c}$ ensuring the existence of a certain type of large cycles. The majority of these results are sharp in all respects.

Focusing only on basic graph invariants, as well as on pure algebraic relations between these parameters, in fact, we present the simplest kind of relations for large cycles having no forerunners in the area. Actually they form a source from which nearly all possible hamiltonian results (including well-known Ore's theorem, Pósa's theorem and many other

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generalizations) can be developed further by various additional new ideas, generalizations, extensions, restrictions and structural limitations:

- **generalized and extended graph invariants** - degree sequences (Pósa type, Chvátal type), degree sums (Ore type, Fun type), neighborhood unions, generalized degrees, local connectivity, and so on,
- **extended list of path and cycle structures** - Hamilton, longest and dominating cycles, generalized cycles including Hamilton and dominating cycles as special cases, 2-factor, multiple Hamilton cycles, edge disjoint Hamilton cycles, powers of Hamilton cycles, $k$ - ordered Hamilton cycles, arbitrary cycles, cycle systems, pancyclic-type cycle systems, cycles containing specified sets of vertices or edges,  shortest cycles, analogous path structures, and so on,
- **structural (descriptive) limitations** - regular, planar, bipartite, chordal and  interval graphs, graphs with forbidden subgraphs, Boolean graphs, hypercubes,  and so on,
- **graph extensions** - hypergraphs, digraphs and orgraphs, labeled and weighted graphs, infinite graphs, random graphs, and so on.

We refer to (Bermond, 1978) and (Gould, 1991, 2003) for more background and general surveys.

The order $n$ , size $q$ and minimum degree $\delta$ clearly are easy computable graph invariants. In (Even & Tarjan, 1975), it was proved that connectivity $\kappa$ can be determined in polynomial time, as well. Determining the independence number $\alpha$ and toughness $\tau$ are shown in (Garey & Johnson, 1983) and (Bauer et al., 1990a) to be $CD_\lambda$ -hard problems. Moreover, it was proved (Bauer et al., 1990a) that for any positive rational number $t$ , recognizing $t$ -tough graphs (in particular 1-tough graphs) is an $NP$ -hard problem.

The order $n$ and size $q$ are neutral with respect to cycle structures. Meanwhile, they become more effective combined together (Theorem 1). The minimum degree $\delta$ having high frequency of occurrence in different relations is, in a sense, a more essential invariant than the order and size, providing some dispersion of the edges in a graph. The combinations between order $n$ and minimum degree $\delta$ become much more fruitful especially under some additional connectivity conditions. The impact of some relations on cycle structures can be strengthened under additional conditions of the type $\delta \geq \alpha \pm i$ for appropriate integer $i$ . By many graph theorists, the connectivity $\kappa$ is at the heart of all path and cycle questions providing comparatively more uniform dispersion of the edges. An alternate connectedness measure is toughness $\tau$ - the most powerful and less investigated graph invariant introduced by Chvátal (Chvátal, 1973) as a means of studying the cycle structure of graphs. Chvátal (Chvátal, 1973) conjectured that there exists a finite constant $t_0$ such that every $t_0$ -tough graph is hamiltonian. This conjecture is still open. We have omitted a number of results involving toughness $\tau$ as a parameter since they are far from being best possible.

Large cycle structures are centered around well-known Hamilton (spanning) cycles. Other types of large cycles were introduced for different situations when the graph contains no Hamilton cycles or it is difficult to find it. Generally, a cycle $C$ in a graph $G$ is a large cycle if it dominates some certain subgraph structures  in $G$ in a sense that every such structure

has a vertex in common with $C$. When $C$ dominates all vertices in $G$ then $C$ is a Hamilton cycle. When $C$ dominates all edges in $G$ then $C$ is called a dominating cycle introduced by Nash-Williams (Nash-Williams, 1971). Further, if $C$ dominates all paths in $G$ of length at least some fixed integer $\lambda$ then $C$ is a $PD_\lambda$ (path dominating)-cycle introduced by Bondy (Bondy, 1981). Finally, if $C$ dominates all cycles in $G$ of length at least $\lambda$ then $C$ is a $CD_\lambda$ (cycle dominating)-cycle, introduced in (Zh.G. Nikoghosyan, 2009a). The existence problems of generalized $PD_\lambda$ and $CD_\lambda$-cycles are studied in (Zh.G. Nikoghosyan, 2009a) including Hamilton and dominating cycles as special cases.

Section 2 is devoted to necessary notation and terminology. In Section 3, we discuss pure relations between various basic invariants of a graph and Hamilton cycles. Next sections are devoted to analogous pure relations concerning dominating cycles (Section 4), $CD_\lambda$-cycles (Section 5), long cycles (Section 6), long cycles with Hamilton cycles (Section 7), long cycles with dominating cycles (Section 8) and long cycles with $CD_\lambda$-cycles (Section 9). In Section 10 we present the proofs of Theorems 6, 21 and 27. Concluding remarks are given in Section 11.

## 2. Terminology

We consider only finite undirected graphs without loops or multiple edges. A good reference for any undefined terms is (Bondy & Murty, 1976). The set of vertices of a graph $G$ is denoted by $V(G)$ and the set of edges by $E(G)$. For $S$ a subset of $V(G)$, we denote by $G \setminus S$ the maximum subgraph of $G$ with vertex set $V(G) \setminus S$. For a subgraph $H$ of $G$ we use $G \setminus H$ short for $G \setminus V(H)$. Denote by $N(x)$ the neighborhood of a vertex $x$ in $G$. Put $d(x) = |N(x)|$

A simple cycle (or just a cycle) $C$ of length $t$ is a sequence $v_1 v_2 ... v_t v_1$ of distinct vertices $v_1, v_2, ..., v_t$ with $v_i v_{i+1} \in E(G)$ for each $i \in \{1, ..., t\}$, where $v_{t+1} = v_1$. When $t = 2$, the cycle $C = v_1 v_2 v_1$ on two vertices $v_1, v_2$ coincides with the edge $v_1 v_2$, and when $t = 1$, the cycle $C = v_1$ coincides with the vertex $v_1$. So, all vertices and edges in a graph can be considered as cycles of lengths 1 and 2, respectively. A graph $G$ is hamiltonian if $G$ contains a Hamilton cycle, i.e. a cycle containing all vertices of G. Let $\lambda$ be an integer. A cycle $C'$ in $G$ is a $PD_\lambda$-cycle if $|P| \le \lambda - 1$ for each path $P$ in $G \setminus C'$ and is a $CD_\lambda$-cycle if $|C''| \le \lambda - 1$ for each cycle $C''$ in $G \setminus C'$. In particular, $PD_0$-cycles and $CD_1$-cycles are well-known Hamilton cycles and $PD_1$-cycles and $CD_2$-cycles are often called dominating cycles.

We reserve $n, q, \delta, \kappa$ and $\alpha$ to denote the number of vertices (order), number of edges (size), minimum degree, connectivity and independence number of a graph, respectively. Let $c$ denote the circumference - the length of a longest cycle in a graph. In general, $c \ge 1$. For $C$ a longest cycle in $G$, denote by $\bar{p}$ and $\bar{c}$ the lengths of a longest path and a longest cycle in $G \setminus C$, respectively. Let $s(G)$ denote the number of components of a graph $G$. A graph $G$ is $t$-tough if $|S| \ge t \cdot s(G \setminus S)$ for every subset $S \subseteq V(G)$ with $s(G \setminus S) > 1$. The toughness of $G$, denoted $\tau(G)$, is the maximum value of $t$ for which $G$ is $t$-tough (taking $\tau(K_n) = \infty$ for all $n \ge 1$).

An $(x,y)$-path is a path with end vertices $x$ and $y$. Given an $(x,y)$-path $L$ of $G$, we denote by $\overrightarrow{L}$ the path $L$ with an orientation from $x$ to $y$. If $u,v \in V(L)$ then $u\overrightarrow{L}v$ denotes the consecutive vertices on $\overrightarrow{L}$ from $u$ to $v$ in the direction specified by $\overrightarrow{L}$. The same vertices, in reverse order, are given by $v\overleftarrow{L}u$. For $\overrightarrow{L} = x\overrightarrow{L}y$ and $u \in V(L)$, let $u^+\left(\overrightarrow{L}\right)$ (or just $u^+$) denotes the successor of $u$ $(u \neq y)$ on $\overrightarrow{L}$, and $u^-$ denotes its predecessor $(u \neq x)$. If $A \subseteq V(L) \setminus \{y\}$ then we denote $A^+ = \{v^+ \mid v \in A\}$. Similar notation is used for cycles. If $Q$ is a cycle and $u \in V(Q)$, then $u\overrightarrow{Q}u = u$.

Let $a, b, t, k$ be integers with $k \leq t$. We use $H(a,b,t,k)$ to denote the graph obtained from $tK_a + \overline{K}_t$ by taking any $k$ vertices in subgraph $\overline{K}_t$ and joining each of them to all vertices of $K_b$. Denote by $L_\delta$ the graph obtained from $3K_\delta + K_1$ by taking one vertex in each of three copies of $K_b$ and joining them each to other. For odd $n$, where $n \geq 15$, construct the graph $G_n$ from $\overline{K}_{(n-1)/2} + K_\delta + K_{(n+1)/2-\delta}$, where $n/3 \leq \delta \leq (n-5)/2$, by joining every vertex in $K_\delta$ to all other vertices and by adding a matching between all vertices in $K_{(n+1)/2-\delta}$ and $(n+1)/2 - \delta$ vertices in $\overline{K}_{(n-1)/2}$. It is easily seen that $G_n$ is 1-tough but not hamiltonian. A variation of the graph $G_n$, with $K_\delta$ replaced by $\overline{K}_\delta$ and $\delta = (n-5)/2$, will be denoted by $G_n^*$.

## 3. Pure relations for Hamilton cycles

We begin with a pure algebraic relation between order $n$ and size $q$ insuring the existence of a Hamilton cycle based on the natural idea that if a sufficient number of edges are present in the graph then a Hamilton cycle will exist.

**Theorem 1** (Erdös & Gallai, 1959). Let $G$ be an arbitrary graph. If

$$q \geq \frac{n^2 - 3n + 5}{2}$$

then $G$ is hamiltonian.

*Example for sharpness.* To see that the size bound $(n^2 - 3n + 5)/2$ in Theorem 1 is best possible, note that the graph formed by joining one vertex of $K_{n-1}$ to $K_1$, contains $(n^2 - 3n + 4)/2$ edges and is not hamiltonian.

The next pure algebraic relation links the size $q$ and minimum degree $\delta$ insuring the existence of a Hamilton cycle. In view of Theorem 1, it seems a little surprising, providing, in fact, a contrary statement.

**Theorem 2** (Zh.G. Nikoghosyan, 2011). Let $G$ be an arbitrary graph. If

$$q \leq \delta^2 + \delta - 1$$

then $G$ is hamiltonian.

*Example for sharpness.* The bound $\delta^2 + \delta - 1$ in Theorem 2 can not be relaxed to $\delta^2 + \delta$ since the graph $K_1 + 2K_\delta$ consisting of two copies of $K_{\delta+1}$ and having exactly one vertex in common, has $\delta^2 + \delta$ edges but is not hamiltonian.

The earliest sufficient condition for a graph to be hamiltonian is based on the order $n$ and minimum degree $\delta$ ensuring the existence of a Hamilton cycle with sufficient number of edges by keeping the minimum degree at a fairly high level.

**Theorem 3** (Dirac, 1952). Let $G$ be an arbitrary graph. If

$$\delta \geq \frac{n}{2}$$

then $G$ is hamiltonian.

*Example for sharpness*: $2K_\delta + K_1$.

The graph $2K_\delta + K_1$ shows that the bound $n/2$ in Theorem 3 can not be replaced by $(n-1)/2$.

The minimum degree bound $n/2$ in Theorem 3 can be slightly relaxed for graphs under additional 1-tough condition.

**Theorem 4** (Jung, 1978). Let $G$ be a graph with $n \geq 11$ and $\tau \geq 1$. If

$$\delta \geq \frac{n-4}{2}$$

then $G$ is hamiltonian.

*Examples for sharpness*: Petersen graph; $K_{\delta,\delta+1}$; $G_n^*$.

This bound $(n-4)/2$ itself was lowered further to $(n-7)/2$ under stronger conditions $n \geq 30$ and $\tau > 1$.

**Theorem 5** (Bauer et al., 1991a). Let $G$ be a graph with $n \geq 30$ and $\tau > 1$. If

$$\delta \geq \frac{n-7}{2}$$

then G is hamiltonian.

Furthermore, the bound $n/2$ was essentially lowered to $(n+\kappa)/3$ (when $k < n/2$) by incorporating connectivity $\kappa$ into the minimum degree bound.

**Theorem 6** (Zh.G. Nikoghosyan, 1981). Let $G$ be a graph with $\kappa \geq 2$. If

$$\delta \geq \frac{n + \kappa}{3}$$

then G is hamiltonian.

*Examples for sharpness*: $2K_\delta + K_1; H(1, \delta - \kappa + 1, \delta, \kappa)$ $(2 \leq \kappa < n/2)$.

A short proof of Theorem 6 was given by Häggkvist (Häggkvist & Nicoghossian, 1981).

The minimum degree bound $(n + \kappa)/3$ in Theorem 6 was slightly lowered to $(n + \kappa - 2)/3$ for 1-tough graphs.

**Theorem 7** (Bauer & Schmeichel, 1991b). Let $G$ be a graph with $\tau \geq 1$. If

$$\delta \geq \frac{n + \kappa - 2}{3}$$

then $G$ is hamiltonian.

*Examples for sharpness*: $K_{\delta, \delta+1}; L_\delta$.

Another essential improvement of Dirac's bound $n/2$ was established for 2-connected graphs under additional strong condition $\delta \geq \alpha$.

**Theorem 8** (Nash-Williams, 1971). Let $G$ be a graph with $\kappa \geq 2$. If

$$\delta \geq \max\left\{\frac{n+2}{3}, \alpha\right\}$$

then $G$ is hamiltonian.

*Examples for sharpness*:
$(\lambda + 1)K_{\delta - \lambda + 1} + K_\lambda$ $(\delta \geq 2\lambda); (\lambda + 2)K_{\delta - \lambda} + K_{\lambda+1}$ $(\delta \geq 2\lambda + 1); H(\lambda, \lambda + 1, \lambda + 3, \lambda + 2)$.

Theorem 8 was slightly improved by replacing the condition $\kappa \geq 2$ with a stronger condition $\tau \geq 1$.

**Theorem 9** (Bigalke & Jung, 1979). Let $G$ be a graph with $\tau \geq 1$. If

$$\delta \geq \max\left\{\frac{n}{3}, \alpha - 1\right\}$$

then $G$ is hamiltonian.

*Examples for sharpness*: $K_{\delta, \delta+1}$ $(n \geq 3); L_\delta$ $(n \geq 7); K_{\delta, \delta+1}$ $(n \geq 3)$.

For $\lambda$ a positive integer, the bound $(n+2)/3$ in Theorem 8 was essentially lowered under additional condition of the type $\delta \geq \alpha + \lambda$, including Theorem 8 as a special case.

**Theorem 10** (Fraisse, 1986). Let $G$ be a graph, $\lambda$ a positive integer and

$$\delta \geq \max\left\{\frac{n+2}{\lambda+2}+\lambda-1, \alpha+\lambda-1\right\}.$$

If $\kappa \geq \lambda+1$ then $G$ is hamiltonian.

*Examples for sharpness*:
$(\lambda+1)K_{\delta-\lambda+1}+K_\lambda \ (\delta \geq 2\lambda); (\lambda+2)K_{\delta-\lambda}+K_{\lambda+1} \ (\delta \geq 2\lambda+1); H(\lambda, \lambda+1, \lambda+3, \lambda+2).$

Later, Theorem 8 was essentially improved for 3-connected graphs by incorporating the connectivity $\kappa$ into the minimum degree bound.

**Theorem 11** (Zh.G. Nikoghosyan, 1985a). Let $G$ be a graph with $\kappa \geq 3$. If

$$\delta \geq \max\left\{\frac{n+2\kappa}{4}, \alpha\right\}$$

then $G$ is hamiltonian.

*Examples for sharpness*: $3K_2+K_2; 4K_2+K_3; H(1,2,\kappa+1,\kappa).$

The graph $4K_2+K_3$ shows that for $\kappa=3$ the minimum degree bound $(n+2\kappa)/4$ in Theorem 11 can not be replaced by $(n+2\kappa-1)/4$.

Finally, the bound $(n+2\kappa)/4$ in Theorem 11 was reduced to $(n+\kappa+3)/4$ without any additional limitations providing a best possible result for each $\kappa \geq 3$.

**Theorem 12** (Yamashita, 2008). Let $G$ be a graph with $\kappa \geq 3$. If

$$\delta \geq \max\left\{\frac{n+\kappa+3}{4}, \alpha\right\}$$

then $G$ is hamiltonian.

*Examples for sharpness*: $3K_{\delta-1}+K_2; H(2, n-3\delta+3, \delta-1, \kappa); H(1,2,\kappa+1,\kappa).$

The first pure relation between graph invariants involving connectivity $\kappa$ as a parameter was developed in 1972.

**Theorem 13** (Chvátal and Erdös, 1972). Let $G$ be an arbitrary graph. If

$$\kappa \geq \alpha$$

then $G$ is hamiltonian.

*Example for sharpness*: $K_{\delta,\delta+1}.$

## 4. Pure relations for dominating cycles

In view of Theorem 2, the following upper size bound is reasonable for dominating cycles.

**Conjecture 1.** Let $G$ be a graph with $\kappa \geq 2$. If

$$q \leq \frac{3(\delta^2 + \delta - 2) - 1}{2}$$

then each longest cycle in $G$ is a dominating cycle.

In 1971, it was proved that the minimum degree bound $(n+2)/3$ insures the existence of dominating cycles.

**Theorem 14** (Nash-Williams, 1971). Let $G$ be a graph with

$$\delta \geq \frac{n+2}{3}.$$

If $\kappa \geq 2$ then each longest cycle in $G$ is a dominating cycle.

*Examples for sharpness*: $2K_3 + K_1$; $3K_{\delta-1} + K_2$; $H(1,2,4,3)$.

The graph $2K_3 + K_1$ shows that the connectivity condition $\kappa \geq 2$ in Theorem 14 can not be replaced by $\kappa \geq 1$. The second graph shows that the minimum degree condition $\delta \geq (n+2)/3$ can not be replaced by $\delta \geq (n+1)/3$ and the third graph shows that the conclusion "is a dominating cycle" can not be strengthened by replacing it with "is a Hamilton cycle".

The condition $\delta \geq (n+2)/3$ in Theorem 14 can be slightly relaxed under stronger 1-tough condition instead of $\kappa \geq 2$.

**Theorem 15** (Bigalke & Jung, 1979). Let $G$ be a graph with $\tau \geq 1$. If

$$\delta \geq \frac{n}{3}$$

then each longest cycle in $G$ is a dominating cycle.

*Examples for sharpness*: $2(\kappa+1)K_2 + \kappa K_1$; $L_3$; $G_n^*$.

The bound $(n+2)/3$ in Theorem 14 can be lowered to $(n+2\kappa)/4$ by incorporating $\kappa$ into the minimum degree bound.

**Theorem 16** (Lu et al., 2005). Let $G$ be graph with $\kappa \geq 3$. If

$$\delta \geq \frac{n+2\kappa}{4}$$

then each longest cycle in $G$ is a dominating cycle.

*Examples for sharpness*: $3K_2 + K_2$; $4K_2 + K_3$; $H(1, 2, \kappa + 1, \kappa)$.

The graph $4K_2 + K_3$ shows that for $\kappa = 3$ the minimum degree bound $(n + 2\kappa)/4$ in Theorem 16 can not be replaced by $(n + 2\kappa - 1)/4$.

In 2008, the bound $(n + 2\kappa)/4$ itself was essentially reduced to $(n + \kappa + 3)/4$ without any additional limitations, providing a best possible result for each $\kappa \geq 3$.

**Theorem 17** (Yamashita, 2008). Let $G$ be graph with $\kappa \geq 3$. If

$$\delta \geq \frac{n + \kappa + 3}{4}$$

then each longest cycle in $G$ is a dominating cycle.

*Examples for sharpness*: $3K_{\delta-1} + K_2$; $H(2, n - 3\delta + 3, \delta - 1, \kappa)$; $H(1, 2, \kappa + 1, \kappa)$.

## 5. Pure relations for $CD_\lambda$ -cycles

In 1990, the exact analog of Theorems 3 and 14 was established In terms of generalized $CD_3$ - cycles.

**Theorem 18** (Jung, 1990). Let $G$ be a graph with

$$\delta \geq \frac{n + 6}{4} .$$

If $\kappa \geq 3$ then each longest cycle in $G$ is a $CD_3$ -cycle.

*Examples for sharpness*:
$\lambda K_{\lambda+1} + K_{\lambda-1}$ $(\lambda \geq 2)$; $(\lambda + 1)K_{\delta-\lambda+1} + K_\lambda$ $(\lambda \geq 1)$; $H(\lambda - 1, \lambda, \lambda + 2, \lambda + 1)$ $(\lambda \geq 2)$.

In 2009, a common generalization of Theorems 3, 14 and 18 was proved by covering $CD_\lambda$ - cycles for each integer $\lambda \geq 1$.

**Theorem 19** (Zh.G. Nikoghosyan, 2009a). Let $G$ be a graph, $\lambda$ a positive integer and

$$\delta \geq \frac{n + 2}{\lambda + 1} + \lambda - 2 .$$

Then each longest cycle in $G$ is a $CD_{\min\{\lambda, \delta - \lambda + 1\}}$ -cycle.

*Examples for sharpness*:
$\lambda K_{\lambda+1} K_{\lambda-1}$ $(\lambda \geq 2)$; $(\lambda + 1)K_{\delta-\lambda+1} + K_\lambda$ $(\lambda \geq 1)$; $H(\lambda - 1, \lambda, \lambda + 2, \lambda + 1)$ $(\lambda \geq 2)$.

In (Zh.G. Nikoghosyan, 2009a), an analogous generalization has been conjectured in terms of $PD_\lambda$ -cycles.

**Conjecture 1** (Zh.G. Nikoghosyan, 2009a). Let $G$ be a graph, $\lambda$ a positive integer and $\kappa \geq \lambda$. If

$$\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$$

then each longest cycle in $G$ is a $PD_{\min\{\lambda-1,\delta-\lambda\}}$ -cycle.

In view of Theorems 6 and 17, the next generalization seems reasonable.

**Conjecture 2** (Yamashita, 2008). Let $G$ be graph, $\lambda$ an integer and $\kappa \geq \lambda \geq 2$. If

$$\delta \geq \frac{n + \kappa + \lambda(\lambda-2)}{\lambda+1}$$

then each longest cycle in $G$ is a $PD_{\lambda-2}$ and $CD_{\lambda-1}$ -cycle.

## 6. Pure relations for long cycles

The earliest and simplest hamiltonian result links the circumference c and minimum degree $\delta$.

**Theorem 20** (Dirac, 1952). In every graph,

$$c \geq \delta + 1.$$

*Example for sharpness*: Join two copies of $K_{\delta+1}$ by an edge.

For $C$ a longest cycle in a graph $G$, a lower bound for $|C|$ was developed based on the minimum degree $\delta$ and $\overline{p}$ - the length of a longest path in $G \setminus C$.

**Theorem 21** (Zh.G. Nikoghosyan, 1998). Let $G$ be a graph and $C$ a longest cycle in $G$. Then

$$|C| \geq (\overline{p} + 2)(\delta - \overline{p}).$$

*Example for sharpness*: $(\kappa + 1)K_{\delta-\kappa+1} + K_{\kappa}$.

The next similar bound is based On the minimum degree $\delta$ and $\overline{c}$ - the length of a longest cycle in $G \setminus C$.

**Theorem 22** (Zh.G. Nikoghosyan, 2000a). Let $G$ be a graph and $C$ a longest cycle in $G$. Then

$$|C| \geq (\overline{c} + 1)(\delta - \overline{c} + 1).$$

*Example for sharpness*: $(\kappa + 1)K_{\delta-\kappa+1} + K_{\kappa}$.

In 2000, Theorem 22 was improved involving connectivity $\kappa$ as a parameter combined with $\overline{c}$ and $\delta$.

**Theorem 23** (Zh.G. Nikoghosyan, 2000b). Let $G$ be a graph with $\kappa \geq 2$ and $C$ a longest cycle in $G$. If $\overline{c} \geq \kappa$ then

$$|C| \geq \frac{(\bar{c}+1)\kappa}{\bar{c}+\kappa+1}(\delta+2).$$

Otherwise,

$$|C| \geq \frac{(\bar{c}+1)\bar{c}}{2\bar{c}+1}(\delta+2).$$

*Example for sharpness*: $(\kappa+1)K_{\delta-\kappa+1} + K_\kappa$.

In view of Theorem 23, the following seems reasonable for $PD_\lambda$-cycles.

**Conjecture 3** (Zh.G. Nikoghosyan, 2009a). Let $G$ be a graph with $\kappa \geq 2$ and $C$ a longest cycle in $G$. If $\bar{p} \geq \kappa - 1$ then

$$|C| \geq \frac{(\bar{p}+2)\kappa}{\bar{p}+\kappa+2}(\delta+2).$$

Otherwise,

$$|C| \geq \frac{(\bar{p}+2)\bar{p}}{2\bar{p}+2}(\delta+2).$$

## 7. Pure relations for Hamilton cycles and long cycles

The following direct generalization includes Theorem 3 as a special case.

**Theorem 24** (Alon, 1986). Let $G$ be a graph and $\lambda$ a positive integer. If $\delta \geq n/(\lambda+1)$ then

$$c \geq \frac{n}{\lambda}.$$

*Examples for sharpness*: $(\lambda+1)K_\lambda + K_1$; $\lambda K_{\lambda+1}$.

In 1952, a relationship was established linking the minimum degree $\delta$, circumference $c$ and Hamilton cycles for 2-connexted graphs.

**Theorem 25** (Dirac, 1952). Let $G$ be a graph with $\kappa \geq 2$. Then

$$c \geq \min\{n, 2\delta\}.$$

*Examples for sharpness*:
$(\lambda+1)K_{\lambda+1} + K_\lambda \ (\lambda \geq 1)$; $(\lambda+3)K_{\lambda-1} + K_{\lambda+2} \ (\lambda \geq 2)$; $(\lambda+2)K_\lambda + K_{\lambda+1} \ (\lambda \geq 1)$.

For 1-tough graphs the bound $2\delta$ in Theorem 25 was slightly enlarged.

**Theorem 26** (Bauer and Schmeichel, 1986). Let $G$ be a graph with $\tau \geq 1$. Then

$$c \geq \min\{n, 2\delta + 2\}.$$

*Examples for sharpness*: $K_{\delta, \delta+1}$; $L_2$.

The first essential improvement of Theorem 25 was achieved by incorporating connectivity $\kappa$ into the relation without any essential limitation.

**Theorem 27** (Zh.G. Nikoghosyan, 1981). Let $G$ be a graph with $\kappa \geq 3$. Then

$$c \geq \min\{n, 3\delta - \kappa\}.$$

*Examples for sharpness:* $3K_{\delta-1} + K_2$; $H(1, \delta - \kappa + 1, \delta, \kappa)$.

In (Voss & Zuluaga, 1977), it was proved that the bound $\min\{n, 2\delta\}$ in Theorem 25 can be essentially enlarged under additional condition $\delta \geq \alpha$ combined with $\kappa \geq 3$.

**Theorem 28** (Voss and Zuluaga, 1977). Let $G$ be a graph with $\kappa \geq 3$. If $\delta \geq \alpha$ then

$$c \geq \min\{n, 3\delta - 3\}.$$

*Examples for sharpness*: $(\lambda + 2)K_{\lambda+2} + K_{\lambda+1}$; $(\lambda + 4)K_\lambda + K_{\lambda+3}$; $(\lambda + 3)K_{\lambda+1} + K_{\lambda+2}$.

In 2009, a direct generalization was established for each positive integer $\lambda$, including Theorem 28 as a special case $(\lambda = 1)$.

**Theorem 29** (Zh.G. Nikoghosyan, 2009a). Let $G$ be a graph and $\lambda$ a positive integer. If $\kappa \geq \lambda + 2$ and $\delta \geq \alpha + \lambda - 1$ then

$$c \geq \min\{n, (\lambda + 2)(\delta - \lambda)\}.$$

*Examples for sharpness*: $(\lambda + 2)K_{\lambda+2} + K_{\lambda+1}$; $(\lambda + 4)K_\lambda + K_{\lambda+3}$; $(\lambda + 3)K_{\lambda+1} + K_{\lambda+2}$.

In 1985, the bound $3\delta - \kappa$ in Theorem 27 was enlarged to $4\delta - 2\kappa$ under additional condition $\delta \geq \alpha$ combined with $\kappa \geq 4$.

**Theorem 30** (Zh.G. Nikoghosyan, 1985b). Let $G$ be a graph with $\kappa \geq 4$ and $\delta \geq \alpha$. Then

$$c \geq \min\{n, 4\delta - 2\kappa\}.$$

*Examples for sharpness*: $4K_2 + K_3$; $H(1, n - 2\delta, \delta, \kappa)$; $5K_2 + K_4$.

The bound $4\delta - 2\kappa$ in Theorem 30 is sharp for $\kappa = 4$.

Furthermore, the bound $4\delta - 2\kappa$ in Theorem 30 was essentially improved to $4\delta - \kappa - 4$ without any additional limitations providing a best possible result for each $\kappa \geq 4$.

**Theorem 31** (M.Zh. Nikoghosyan & Zh.G. Nikoghosyan, 2011). Let $G$ be a graph with $\kappa \geq 4$ and $\delta \geq \alpha$. Then

$$c \geq \min\{n, 4\delta - \kappa - 4\}.$$

*Examples for sharpness*: $4K_{\delta-2} + K_3$; $H(1,2,\kappa+1,\kappa)$; $H(2,n-3\delta+3,\delta-1,\kappa)$.

The next theorem provides a lower bound for the circumference in terms of $n, \delta$ and $\alpha$ under the hypothesis of Theorem 14.

**Theorem 32** (Bauer et al., 1990b). Let $G$ be a graph with $\kappa \geq 2$. If $\delta \geq (n+2)/3$ then

$$c \geq \min\{n, n+\delta - \alpha\}.$$

*Examples for sharpness:* $2K_\delta + K_1$; $3K_{\delta-1} + K_2$; $K_{2\delta-2,\delta}$.

An analogous bound was established for 1-tough graphs.

**Theorem 33** (Bauer et al., 1988). Let $G$ be a graph with $\tau \geq 1$. If $\delta \geq n/3$ then

$$c \geq \min\{n, n+\delta - \alpha + 1\}.$$

*Examples for sharpness:* $K_{\delta,\delta+1}$; $L_\delta$; $G_n^*$.

## 8. Pure relations for dominating cycles and long cycles

The exact analog of Theorem 25 for dominating cycles can be formulated as follows.

**Theorem 34** (Voss & Zuluaga, 1977). Let $G$ be a graph with $\kappa \geq 3$. Then either

$$c \geq 3\delta - 3$$

or each longest cycle in $G$ is a dominating cycle.

*Examples for sharpness*:
$(\lambda+1)K_{\lambda+1} + K_\lambda$ $(\lambda \geq 1)$; $(\lambda+3)K_{\lambda+1} + K_{\lambda+2}$ $(\lambda \geq 2)$; $(\lambda+2)K_\lambda + K_{\lambda+1}$ $(\lambda \geq 1)$.

The bound $3\delta - 3$ in Theorem 34 was enlarged to $4\delta - 2\kappa$ by incorporating connectivity $\kappa$ into the bound.

**Theorem 35** (Zh.G. Nikoghosyan, 2009b). Let $G$ be a graph with $\kappa \geq 4$. Then either

$$c \geq 4\delta - 2\kappa$$

or $G$ has a dominating cycle.

*Examples for sharpness*: $4K_2 + K_3$; $5K_2 + K_4$; $H(1, n-2\delta, \delta, \kappa)$.

Theorem 35 is sharp only for $\kappa = 4$ as can be seen from $5K_2 + K_4$. Further, the bound $4\delta - 2\kappa$ in Theorem 35 was essentially improved to $4\delta - \kappa - 4$ without any limitation providing a sharp bound for each $\kappa \geq 4$.

**Theorem 36** (M.Zh. Nikoghosyan & Zh.G. Nikoghosyan, 2011). Let $G$ be a graph with $\kappa \geq 4$. Then either

$$c \geq 4\delta - \kappa - 4$$

or each longest cycle in $G$ is a dominating cycle.

*Examples for sharpness*: $4K_{\delta-2} + K_3$; $H(2, \delta - \kappa + 1, \delta - 1, \kappa)$; $H(1, 2, \kappa + 1, \kappa)$.

## 9. Pure relations for $CD_\lambda$-cycles and long cycles

TThe following theorem can be considered as a common generalization of Theorems 25 and 34 by covering $CD_\lambda$-cycles for all $\lambda \geq 1$ including Hamilton and dominating cycles as special cases.

**Theorem 37** (Zh.G. Nikoghosyan, 2009a). Let $G$ be a graph and $\lambda$ a positive integer. If $\kappa \geq \lambda + 1$ then either

$$c \geq (\lambda + 1)(\delta - \lambda + 1)$$

or each longest cycle in $G$ is a $CD_{\min\{\lambda, \delta-\lambda\}}$-cycle.

*Examples for sharpness*:
$(\lambda + 1)K_{\lambda+1} + K_\lambda$ $(\lambda \geq 1)$; $(\lambda + 3)K_{\lambda-1} + K_{\lambda+2}$ $(\lambda \geq 2)$; $(\lambda + 2)K_\lambda + K_{\lambda+1}$ $(\lambda \geq 1)$.

In (Zh.G. Nikoghosyan, 2009a), another version of Theorem 37 was conjectured in terms of $PD_\lambda$-cycles, instead of $CD_\lambda$-cycles.

**Conjecture 4** (Zh.G. Nikoghosyan, 2009a). Let $G$ be a graph and $\lambda$ a positive integer. If $\kappa \geq \lambda + 1$ then either

$$c \geq (\lambda + 1)(\delta - \lambda + 1)$$

or each longest cycle in $G$ is a $PD_{\min\{\lambda-1, \delta-\lambda-1\}}$-cycle.

In view of Theorems 27 and 36, the following common generalization seems quite reasonable.

**Conjecture 5.** Let $G$ be a graph and $\lambda \geq 2$ an integer. If $\kappa \geq \lambda + 1$ then either

$$c \geq (\lambda + 1)\delta - \kappa - (\lambda + 1)(\lambda - 2)$$

or each longest cycle in $G$ is a $PD_{\lambda-2}$ and $CD_{\lambda-1}$-cycle.

## 10. Proofs of theorems 6, 21 and 27

In proofs of theorems and lemmas, the end of the proof is marked by □. In proofs of claims, the end of the proof is marked by $\Delta$.

**Proof of Theorem 27** (Mosesyan et al., 2009). Let $G$ be a 3-connected graph and $S$ a minimum cut-set in $G$. Choose a longest cycle $C$ in $G$ so as to maximize $|V(C) \cap S|$. The result holds immediately if $|C| \geq 3\delta - 3$, since $3\delta - 3 \geq 3\delta - \kappa$. Otherwise, by Theorem 34, $C$ is a dominating cycle. Assume first that $S \not\subseteq V(C)$ and let $v \in S \setminus V(C)$. Since $\overrightarrow{C}$ is dominating, $N(v) \subseteq V(C)$. Let $\xi_1, ..., \xi_t$ be the elements of $N(v)$, occurring on $\overrightarrow{C}$ in a consecutive order. Put

$$M_1 = \left\{ \xi_i \mid V\left( \xi_i^+ \overrightarrow{C} \xi_{i+1}^- \right) \cap S \neq \varnothing \right\}, \quad M_2 = N(v) \setminus M_1.$$

Since $v \in S$, we have

$$|M_1| \leq \kappa - 1, \qquad |M_2| = |N(v)| - |M_1| \geq \delta - \kappa + 1.$$

Further, since $C$ is extreme and $|V(C) \cap S|$ is maximum,

$$N(v) \cap N^+(v) \cap M_2^{++} = \varnothing.$$

Hence

$$\begin{aligned}
|C| &\geq |N(v)| + |N^+(v)| + |M_2^{++}| \\
&= 2|N(v)| + |M_2| \geq 3\delta - \kappa + 1.
\end{aligned}$$

Now assume that $S \subseteq V(C)$. Let $H_1, ..., H_h$ be the connected components of $G \setminus S$. If $V(G \setminus C) = \varnothing$ then $|C| = n$ and we are done. Let $x \in V(G \setminus C)$. Assume without loss of generality that $x \in V(H_1)$. Since $C$ is dominating, $N(x) \subseteq V(C)$. Put $Y_1 = N(x) \cup N^+(x)$. Clearly $|Y_1| \geq 2\delta$ and to prove that $|C| \geq 3\delta - \kappa$, it remains to find a subset $Y_2$ in $V(C)$ such that $Y_1 \cap Y_2 = \varnothing$ and $|Y_2| \geq \delta - \kappa$. Abbreviate, $V_1 = V(H_1) \cup S$. Suppose first that $Y_1 \subseteq V_1$. If $V(H_2) \subseteq V(C)$ then $Y_2 = V(H_2)$ since $|V(H_2)| \geq \delta - \kappa + 1$. Otherwise, there exist $y \in V(H_2 \setminus C)$. Since $C$ is dominating, $N(y) \subseteq V(C)$ and we can take $Y_2 = N(y) \setminus S$. Now let $Y_1 \not\subseteq V_1$. Assume without loss of generality that $Y_1 \cap V(H_2) \neq \varnothing$. Since $N(x) \subseteq V_1$, we have $N^+(x) \cap V(H_2) \neq \varnothing$. Let $z \in N^+(x) \cap V(H_2)$. If $N(z) \subseteq V(C)$ then take $Y_2 = N(z) \setminus S$, since $N^+(x)$ is an independent set of vertices (by standard arguments) and therefore, $N(z) \cap N^+(x) = \varnothing$. Otherwise, choose $w \in N(z) \setminus V(C)$. Clearly

$$N(w) \subseteq V(C), w \in V(H_2), N(w) \cap N^+(x) = \{z\}.$$

Then by taking

$$Y_2 = \left( N(w) \setminus \{z\} \right) \setminus \left( S \setminus \{z^-\} \right)$$

we complete the proof of Theorem 27. □

**Proof of Theorem 6** (Mosesyan et al., 2009). Let $G$ be a 2-connected graph with $\delta \geq (n+\kappa)/3$ and let $S$ be a minimum cut-set in $G$. Since $\delta \geq (n+\kappa)/3 \geq (n+2)/3$, by Theorem 14, every longest cycle in $G$ is a dominating cycle. As in proof of Theorem 27, we can show that either $G$ is hamiltonian or $c \geq 3\delta - \kappa$. Since $n \leq 3\delta - \kappa$ (by the hypothesis), it follows from $c \geq 3\delta - \kappa$ that $c = 3\delta - \kappa = n$. So, in any case, $G$ is hamiltonian. □

To prove Theorem 21, we need some special definitions. Let $G$ be a graph, $C$ a longest cycle in $G$ and $\overrightarrow{M}$ a longest path (or a cycle) in $G \setminus C$. Further, let $u_1,...,u_m$ be the elements of $V(M)$ occurring on $\overrightarrow{M}$ in a consecutive order.

**Definition 1** $\{ M_C$-spreading; $\overrightarrow{\Upsilon}(u)$; $\dot{u}$; $\ddot{u} \}$. An $M_C$-spreading $\Upsilon$ is a family of pairwise disjoint paths $\overrightarrow{\Upsilon}(u_1),...,\overrightarrow{\Upsilon}(u_m)$ in $G \setminus C$ with $\overrightarrow{\Upsilon}(u_i) = u_i \overrightarrow{\Upsilon}(u_i) \ddot{u}_i$ $(i = 1,...,m)$. If $u \neq \ddot{u}$ for some $\overrightarrow{\Upsilon}(u)$, then we use $\dot{u}$ to denote the successor of $u$ along $\overrightarrow{\Upsilon}(u)$.

**Definition 2** $\{ \Phi_u; \varphi_u; \Psi_u; \psi_u \}$. Let $\Upsilon$ be any $M_C$-spreading. For each $u \in V(M)$, put

$$\Phi_u = N(\ddot{u}) \cap V(\Upsilon), \quad \varphi_u = |\Phi_u|,$$

$$\Psi_u = N(\ddot{u}) \cap V(H), \quad \psi_u = |\Psi_u|.$$

**Definition 3** $\{ U_0; \bar{U}_0; U_1; U^* \}$. For $\Upsilon$ an $M_C$-spreading, put

$$U_0 = \{ u \in V(M) \mid u = \ddot{u} \}, \quad \bar{U}_0 = V(M) \setminus U_0,$$
$$U^* = \{ u \in \bar{U}_0 \mid \Phi_u \subseteq V(\Upsilon(u)) \}, \quad U_1 = V(M) \setminus (U_0 \cup U^*).$$

**Definition 4** $\{ (U_0)$-minimal $M_C$-spreading$\}$. An $M_C$-spreading $\Upsilon$ is said to be $(U_0)$-minimal, if it is chosen such that $|U_0|$ is minimum.

**Definition 5** $\{ B_u; B_u^*; b_u; b_u^* \}$. For $\Upsilon$ an $M_C$-spreading and $u \in V(M)$, set

$$B_u = \{ v \in U_0 \mid v\dot{u} \in E \}, \quad b_u = |B_u|.$$

Further, for each $u \in U_0$, set

$$B_u^* = \{ v \in \bar{U}_0 \mid u\dot{v} \in E \}, \quad b_u^* = |B_u^*|.$$

**Lemma 1**. Let $C$ be a longest cycle in a graph $G$ and $M$ a path in $G \backslash C$. Let $\overrightarrow{L}_1,...,\overrightarrow{L}_r$ be vertex disjoint paths in $G \backslash C$ with $\overrightarrow{L}_i = v_i \overrightarrow{L}_i w_i$ $(i=1,...,r)$ having only $v_1,...,v_r$ in common with $M$ and let $. Z_i = N(w_i) \cap V(C)$ $(i=1,...,r)$ .. Then

$$|C| \geq \sum_{i=1}^{r} |Z_i| + \left| \bigcup_{i=1}^{r} Z_i \right|.$$

**Proof.** We can assume without loss of generality that $v_i = w_i$ $(i=1,...,r)$, since otherwise, we can use the same arguments. If $\bigcup_{i=1}^{r} Z_i = \varnothing$, then there is nothing to prove. Let $\bigcup_{i=1}^{r} Z_i \neq \varnothing$ and let $\xi_1,...,\xi_t$ be the elements of $\bigcup_{i=1}^{r} Z_i$ occurring on $\overrightarrow{C}$ in a consecutive order. Set

$$F_i = N(\xi_i) \cap \{w_1,...,w_r\} \quad (i=1,...,t).$$

Suppose first that $t=1$. If $|F_1| = 1$ then $\sum_{i=1}^{r} |Z_i| = \left| \bigcup_{i=1}^{r} Z_i \right| = 1$ and the result follows from $|C| \geq 2$ immediately. If $|F_1| \geq 2$, then choosing a largest segment $u \overrightarrow{M} v$ on $M$ with $u,v \in F_1$, we get a cycle $C' = \xi_1 u \overrightarrow{M} v \xi_1$ satisfying

$$|C| \geq |C'| \geq \sum_{i=1}^{r} |Z_1| + 1 = \sum_{i=1}^{r} |Z_1| + \left| \bigcup_{i=1}^{r} Z_i \right|.$$

Now assume $t \geq 2$. Put

$$f(\xi_i) = \left| \xi_i \overrightarrow{C} \xi_{i+1} \right| \quad (i=1,2,...,t),$$

where $\xi_{i+1} = \xi_1$. Then it is easy to see that

$$|C| = \sum_{i=1}^{t} f(\xi_i), \quad \sum_{i=1}^{t} |F_i| = \sum_{i=1}^{r} |Z_i|, \quad t = \left| \bigcup_{i=1}^{r} Z_i \right|. \tag{1}$$

For each $i \in \{1,...,t\}$, let $x_i \overrightarrow{M} y_i$ be the largest segment on $\overrightarrow{M}$ with $x_i, y_i \in F_i \cup F_{i+1}$ (indices mod $t$). Now we need to show that

$f(\xi_i) \geq (|F_i| + |F_{i+1}| + 2) / 2$. Indeed, if $x_i \in F_i$ and $y_i \in F_{i+1}$, then $f(\xi_i) \geq \left| \xi_i x_i \overrightarrow{M} y_i \xi_{i+1} \right|$ since $C$ is extreme. It means that

$$f(\xi_i) \geq \max\{|F_i|, |F_{i+1}|\} + 1 \geq \frac{1}{2}(|F_i| + |F_{i+1}| + 2).$$

The same inequality holds from $f(\xi_i) \geq \left| \xi_i y_i \overleftarrow{M} x_i \xi_{i+1} \right|$ if $x_i \in F_{i+1}$ and $y_i \in F_i$, by a similar argument. Now suppose that either $x_i, y_i \in F_i$ or $x_i, y_i \in F_{i+1}$. Assume without loss of generality that $x_i, y_i \in F_i$. In addition, we have $x_i, y_i \notin F_{i+1}$, since otherwise we are in the previous case. Let $x_i' \overrightarrow{M} y_i'$ be the largest segment on $\overrightarrow{M}$ with $x_i', y_i' \in F_{i+1}$. If $\left| x_i \overrightarrow{M} x_i' \right| \geq (|F_i| - |F_{i+1}|)/2$, then $f(\xi_i) \geq \left| \xi_i x_i \overrightarrow{M} y_i' \xi_{i+1} \right|$ and hence

$$f(\xi_i) \geq \frac{1}{2}(|F_i| - |F_{i+1}|) + |F_{i+1}| + 1 = \frac{1}{2}(|F_i| + |F_{i+1}| + 2).$$

Finally, if $\left| x_i \overrightarrow{M} x_i' \right| \leq (|F_i| - |F_{i+1}| - 1)/2$, then

$$f(\xi_i) \geq \left| \xi_i y_i \overleftarrow{M} x_i' \xi_{i+1} \right| = \left| x_i' \overrightarrow{M} y_i \right| + 2 = \left| x_i \overrightarrow{M} y_i \right| - \left| x_i \overrightarrow{M} x_i' \right| + 2$$

$$\geq |F_i| - 1 - \frac{1}{2}(|F_i| - |F_{i+1}| - 1) + 2 > \frac{1}{2}(|F_i| + |F_{i+1}| + 2).$$

So, $f(\xi_i) \geq (|F_i| + |F_{i+1}| + 2)/2$ $(i = 1, \ldots, t)$ in any case, implying that

$$\sum_{i=1}^{t} f(\xi_i) \geq \sum_{i=1}^{t} \frac{1}{2}(|F_i| + |F_{i+1}| + 2) = \sum_{i=1}^{t} |F_i| + t$$

and the result follows from (1).   □

**Lemma 2**. Let $C$ be a longest cycle in a graph $G$ and $M$ a longest cycle in $G \setminus C$ with a $U_0$-minimal $M_C$-spreading $\Upsilon$. Then for each $u \in U_1$, $|M| \geq \varphi_u + b_u + 1$.

**Proof**. Let $u \in U_1$. For each $x \in V(M)$, put

$A_u(x) = (\Phi_u \cup B_u) \cap V(\Upsilon(x))$. By the definition,

$$|\Phi_u \cup B_u| = \sum_{x \in V(M)} |A_u(x)|. \tag{2}$$

If $A_u(x) \neq \varnothing$ for some $x \in V(M)$, then we choose a vertex $\rho_u(x)$ in $A_u(x)$ such that $|x \overrightarrow{\Upsilon}(x) \rho_u(x)|$ is maximum. By the definition, $\rho_u(u) = (\ddot{u})^-$. Put $\overline{\rho}_u(x) = \ddot{u}$ if $\rho_u(x) \in \Phi_u$, and $\overline{\rho}_u(x) = \dot{u}$ if $\rho_u(x) \in B_u \setminus \Phi_u$. Clearly $\overline{\rho}_u(u) = \ddot{u}$.

Let $\Lambda_u = \{x \in V(M) \mid A_u(x) \neq \varnothing\}$. Further, for each distinct $x, y \in \Lambda_u$, put $\Lambda_u(x, y) = x \dot{u} y$ if either $x = u, y \in U_0$ or $y = u, x \in U_0$. Otherwise,

$$\Lambda_u(x,y) = x\overrightarrow{\Upsilon}(x)\rho_u(x)\overline{\rho}_u(x)\Upsilon(u)\overline{\rho}_u(y)\rho_u(y)\overleftarrow{\Upsilon}(y)y.$$

Let $\xi_1, ..., \xi_f$ be the elements of $\Lambda_u$, occurring on $\overrightarrow{M}$ in a consecutive order with $\xi_1 = u$. For each integer $i$ $(1 \le i \le f)$, set

$$M_i = \xi_i \overrightarrow{M} \xi_{i+1}, \quad \omega_i = |A_u(\xi_i)| + |A_u(\xi_{i+1})| \quad (indices \bmod f)$$

**Claim 1**. $\sum_{i=1}^{f} |M_i| \ge \sum_{i=1}^{f} \omega_i$.

**Proof**. Since $M$ is extreme, for each $i \in \{2, ..., f-1\}$,

$$|M_i| \ge |\Lambda_u(\xi_i, \xi_{i+1})| \ge |A_u(\xi_i)| + |A_u(\xi_{i+1})| = \omega_i.$$

If $\Phi_u \cap V(\Upsilon(\xi_2)) \ne \varnothing$ and $\Phi_u \cap V(\Upsilon(\xi_f)) \ne \varnothing$, then the inequality $|M_i| \ge \omega_i$ $(i = 1, f)$ holds as in previous case and we are done. Now let $\Phi_u \cap V(\Upsilon(\xi_2)) = \varnothing$ and $\Phi_u \cap V(\Upsilon(\xi_f)) = \varnothing$. It means that $A_u(\xi_2) = \{\xi_2\}$ and therefore, $|M_1| \ge 2 = |A_u(\xi_2)| + 1 = \omega_1 - |A_u(u)| + 1$. Analogously, $|M_f| \ge \omega_f - |A_u(u)| + 1$. By the definition of $\Lambda_u$, $\dot{u}\xi_2 \in E$ and $\dot{u}\xi_f \in E$. Since $u \in U_1$, we have $\Phi_u \cap V(\Upsilon(\xi_s)) \ne \varnothing$ for some $3 \le s \le f-1$. Then we can choose $i, j$ such that $2 \le i \le s-1$ and $s \le j \le f-1$ with $|M_i| \ge \omega_i + |A_u(u)| - 1$ and $|M_j| \ge \omega_j + |A_u(u)| - 1$, and the result follows. Finally, because of the symmetry, we can suppose that $\Phi_u \cap V(\Upsilon(\xi_2)) = \varnothing$ and $\Phi_u \cap V(\Upsilon(\xi_f)) \ne \varnothing$. Clearly $|M_1| \ge \omega_1 - |A_u(u)| + 1$. By the definition of $\Lambda_u$, $\dot{u}\xi_2 \in E$. Then we can choose $i \in \{2, ..., f-1\}$ such that $|M_i| \ge \omega_i + |A_u(u)| - 1$ and again the result follows. $\Delta$

**Claim 2**. If $|\Upsilon(u)| \ge 2$ then $\Phi_u \cap U_0 = \varnothing$.

**Proof.** Suppose to the contrary and let $v \in \Phi_u \cap U_0$. Then replacing $\Upsilon(u)$ and $\Upsilon(v)$ by $u\overrightarrow{\Upsilon}(u)(\ddot{u})^-$ and $v\ddot{u}$, respectively, we can form a new $M_H$-spreading, contradicting the $(U_0)$-minimality of $\Upsilon$. $\Delta$

By (2) and Claim 1,

$$|M| = \sum_{i=1}^{f} |M_i| \ge \sum_{i=1}^{f} \omega_i = \sum_{i=1}^{f} \left( |A_u(\xi_i)| + |A_u(\xi_{i+1})| \right)$$

$$(3)$$

$$= 2\sum_{i=1}^{f} |A_u(\xi_i)| = 2 \sum_{x \in V(M)} |A_u(x)| = 2|\Phi_u \cup B_u|.$$

If $|\Upsilon(u)| \geq 2$ then by Claim 2, $|\Phi_u \cup B_u| = \varphi_u + b_u$, which by (3) gives $|M| \geq 2(\varphi_u + b_u) \geq \varphi_u + b_u + 1$.

Finally, if $|\Upsilon(u)| = 1$, i.e. $\ddot{u} = \dot{u}$, then $|\Phi_u \cup B_u| = |B_u| + |\{u\}| = b_u + 1 = \varphi_u$ and again by (3)

$$|M| \geq 2|\Phi_u \cup B_u| \geq 2\varphi_u \geq \varphi_u + b_u + 1. \qquad \square$$

**Lemma 3**. Let $C$ be a longest cycle in a graph $G$ and $L$ a longest path in $G \backslash C$ with a $U_0$-minimal $L_C$-spreading $\Upsilon$. Then for each $u \in \bar{U}_0$, $|L| \geq \varphi_u + b_u$.

**Proof**. Put $L = u_1 ... u_m$. Let $\Lambda_u$, $\Lambda_u(x, y)$ and $\omega_i$ be as defined in proof of Lemma 2. Let $\xi_1, ..., \xi_f$ be the elements of $\Lambda_u$ occurring on $\overrightarrow{L}$ in a consecutive order. Set

$$\overrightarrow{M'} = u_1 \overrightarrow{L} \xi_1, \quad \overrightarrow{M''} = \xi_f \overrightarrow{L} u_m, \quad \overrightarrow{M_i} = \xi_i \overrightarrow{L} \xi_{i+1} \quad (i = 1, ..., f-1).$$

Let $G'$ be the graph obtained from $G$ by adding an extra edge $u_m u_1$. Set $\overrightarrow{M} = u_1 ... u_m u_1$ and $M_f = \xi_f \overrightarrow{M} \xi_1$. Let $\Lambda'_u(\xi_f, \xi_1)$ and $\Lambda''_u(\xi_f, \xi_1)$ be the paths obtained from $\Lambda_u(\xi_f, \xi_1)$ by deleting the first and the last edges, respectively. Since $L$ is extreme, $|M_i| \geq |\Lambda_u(\xi_i, \xi_{i+1})|$ $(i = 1, ..., f-1)$. As for $M_f$, observe that

$$|M'| \geq |\Lambda'_u(\xi_f, \xi_1)| = |\Lambda_u(\xi_f, \xi_1)| - 1,$$
$$|M''| \geq |\Lambda''_u(\xi_f, \xi_1)| = |\Lambda_u(\xi_f, \xi_1)| - 1,$$

Implying that

$$|M_f| = |M'| + |M''| + 1 \geq 2|\Lambda_u(\xi_f, \xi_1)| - 1 \geq |\Lambda_u(\xi_f, \xi_1)|.$$

So, $|M_i| \geq |\Lambda_u(\xi_i, \xi_{i+1})|$ for each $i \in \{1, ..., f\}$. Further, for each $u \in U_1$, we can argue exactly as in proof of Lemma 2 to get $|L| = |M| - 1 \geq \varphi_u + b_u$. Now let $u \in U^*$. By the definition, $\Phi_u \subseteq V(\Upsilon(u))$ and therefore, $|\Upsilon(u)| \geq |\Phi_u| = \varphi_u$. Since $L$ is extreme, $|L| \geq 2(|B_u| + |\{u\}|) - 2 = 2b_u$. Hence,

$$|L| \geq |\Upsilon(u)| + \frac{1}{2}|\overrightarrow{L}| \geq \varphi_u + b_u.$$

**Proof of Theorem 21** (Zh.G. Nikoghosyan, 1998). Let $M$ be a longest path in $G \backslash C$ of length $\bar{p}$ with a $(U_0)$-minimal $M_C$-spreading $\Upsilon$. If $\bar{p} = -1$, i.e. $M$ is a Hamilton cycle, then $|C| \geq \delta + 1 = (\bar{p} + 2)(\delta - \bar{p})$. Let $\bar{p} \geq 0$. We claim that

**(a1)** if $u \in U_0$ and $v \in \bar{U}_0$ then $\Phi_u \cap V(\Upsilon(v)) \subseteq \{v, \dot{v}\}$,

**(a2)** if $u \in U_0$ then $\varphi_u \leq \bar{p} + b_u^*$,

**(a3)** if $v \in \bar{U}_0$ then $\varphi_u \leq \bar{p} - b_u$.

Let $u \in U_0$. If $v \in \bar{U}_0$ then to prove (a1) we can argue exactly as in proof of Claim 2 (see the proof of Lemma 2). The next claim follows immediately from (a1). To prove (a3), let $v \in \bar{U}_0$. Since $M$ is extreme, by Lemma 3, $\bar{p} \geq \varphi_u + b_u$ for each $u \in \bar{U}_0$, and (a3) follows.

Observing that

$$\sum_{u \in U_0} b_u^* = \sum_{u \in \bar{U}_0} b_u$$

and using (a2) and (a3), we get

$$\sum_{u \in V(M)} \varphi_u \leq \bar{p}(\bar{p} + 1) + \sum_{u \in U_0} b_u^* - \sum_{u \in \bar{U}_0} b_u = \bar{p}(\bar{p} + 1).$$

Since $\Upsilon$ is extreme, we have

$$\psi_u = d(\ddot{u}) - \varphi_u \geq \delta - \varphi_u$$

for each $u \in V(M)$.

By summing, we get

$$\sum_{u \in V(M)} \psi_u = (\bar{p} + 1)\delta - \sum_{u \in V(M)} \varphi_u \geq (\bar{p} + 1)(\delta - \bar{p}).$$

In particular,

$$\max_u \psi_u \geq \delta - \bar{p}.$$

By Lemma 1,

$$|C| \geq \sum_{u \in V(M)} \psi_u + \max_u \psi_u$$
$$\geq (\bar{p} + 1)(\delta - \bar{p}) + \delta - \bar{p} = (\bar{p} + 2)(\delta - \bar{p}).$$

## 11. Conclusions

Graph invariants provide a powerful and maybe the single analytical tool for investigation of abstract structures of graphs. They, combined in convenient algebraic relations, carry global and general information about a graph and its particular substructures such as cycle

structures, factors, matchings, colorings, coverings, and so on. The discovery of these relations is the primary problem of graph theory.

We focus on large cycle substructures, perhaps the most important cycle structures in graphs: Hamilton, longest and dominating cycles and some generalized cycles including Hamilton and dominating cycles as special cases.

In the literature, eight basic (initial) invariants of a graph $G$ are known having significant impact on large cycle structures, namely order $n$, size $q$, minimum degree $\delta$, connectivity $\kappa$, independence number $\alpha$, toughness $\tau$ and the lengths of a longest path and a longest cycle in $G \setminus C$ for a given longest cycle $C$ in $G$.

We have collected 37 pure algebraic relations between $n, q, \delta, \kappa, \alpha, \tau, \overline{p}$ and $\overline{c}$ ensuring the existence of a certain type of large cycles. The majority of these results are sharp in all respects.

Focusing only on basic graph invariants, as well as on pure algebraic relations between these parameters, in fact, we present the simplest kind of relations for large cycles having no forerunners in the area. Actually they form a source from which nearly all possible hamiltonian results (including well-known Ore's theorem, Pósa's theorem and many other generalizations) can be developed further by various additional new ideas, generalizations, extensions, restrictions and structural limitations.

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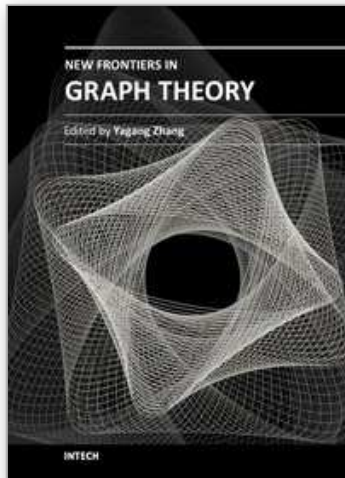
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