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Quantum Mechanics on Surfaces

Bjørn Jensen
Departement of Micro and Nano Systems Technology
Vestfold University College
Norway

1. Introduction

Quantum theory in curved spaces has received much attention over the years. It has been applied in the study of black holes and large scales structures in the universe as well as in the study of the Casimir effect, e.g. However, transferring our existing formulations of quantum theory to curved spaces is not straightforward and any approach will be hampered with a number of issues (1). Nanostructures provide an experimental arena which potentially can provide direct evidence for the interplay between geometry and quantum theory. The ability to manufacture micro- and nanosized surfaces has open up new vistas which should be utilized in order to get a firmer grip on the quantum theory of particles living on these curved structures.

In the following a brief account of the most widely accepted formulation (the 'standard' formulation) of Shrödinger theory on surfaces and linear structures in ordinary three-dimensional Euclidean space is given. We then apply this framework to derive a quantum theory on the catenoid in three-dimensional Euclidean space. This will highlight some important features connected with the interplay between quantum theory and geometry. Then follows a partial framework for an alternative formulation of Shrödinger theory on a surface in which we utilize the unique conformal properties of two-dimensional surfaces. Even though most work connected with quantum theory on structures embedded in three-dimensional Euclidean space so far have been concerned with surfaces, wire structures are also of great obvious interest. Next we therefore point to the possible importance of employing ideas from supersymmetric quantum mechanics in order to enhance our understanding of these structures. Workers in the field of quantum mechanics on lower-dimensional structures in flat space have mainly concerned themselves with Shrödinger theory. In the remaining part of this brief account we will concern ourselves with Dirac theory on surfaces in three-dimensional Euclidean space. We look at differences between the first and second order formulations, and device the proper framework for formulating Dirac theory on surfaces and linear structures in a way which makes contact with the standard formulation of Shrödinger theory on these structures. We then explore different issues, including the question of whether it is 'sufficient' to employ an intrinsically defined quantum theory in a surface compared to the standard approach in the context of Dirac theory. This issue should be of particular relevance when formulating effective theories for charge carriers in graphene. No effort has been made to provide an exhaustive list of

references. Those references which are cited, and the references they contain, are those which have been of particular importance for this author.

2. Dimensional reduction

Quantum mechanics has come to age. We can claim that its basic formulation, either in the form of the canonical quantization procedure or in terms of the Dirac quantization program, is well understood in the sense that its formulation is transparent, even though its consequences continue to surprise and baffle us all. However, this claim is only true as long as the theory is formulated in an Euclidean space which is charted with Cartesian coordinates. It was early recognized in the Dirac quantization programe that problems generally arise when non-Euclidean coordinates are used (2). This is a signal that the interplay between quantum theory and geometry is a deep and fundamental one. This interplay took center stage in physics when it was shown that black holes radiate (3). Curved spacetime geometries are not of immediate importance for quantum theory in the laboratory setting, but curved surfaces and linear structures are. The Casimir effect (4) is a well known example which proves this. Aspects of the challenges met in quantum theory in 'exotic' space-times might therefore also appear in more everyday settings. The coordinate challenge in the Dirac quantization program (e.g.) definitely does since curved surfaces in ordinary space can generally not be completely charted with Cartesian coordinates. The most generally accepted adaption of the canonical quantization procedure to curved surfaces and linear structures was developed in (5–7). What follows is a brief account of this adaptation. We will not systematically discuss the Dirac quantization procedure and possible adaptations of it to lower dimensional structures in space in this exposition, but we will briefly comment on an important aspect of the latter in Section 4.

Consider a smooth two dimensional static surface $\mathcal{S}$ in ordinary three dimensional space. We follow the parametrization in (6) and chart the three dimensional embedding space with coordinates X^i. We write the metric as (6; 8)

$$\begin{aligned} ds^2 &= -dt^2 + G_{ij}(X^i)dX^i dX^j + (dX^3)^2 = \\ &= -dt^2 + G_{ab}(x^a)dx^a dx^b + (dx^3)^2 \,, \end{aligned} \tag{1}$$

where $G_{ab}(x^a)$ is the metric in the surface $\mathcal{S}$ defined by coordinates x^a. We assume that we can define a normal vector field $\vec{N}$ everywhere on $\mathcal{S}$. The coordinate direction x^3 is assumed to be along $\vec{N}$ in the immediate vicinity of $\mathcal{S}$. Our conventions will be such that indices at the beginning of the alphabet will refer to the coordinates in the surface x^a, while indices in the middle of the alphabet refer to the global coordinates X^i. It follows that (8)

$$G_{ab}(X^i) = g_{ab}(x^a) - 2K_{ab}(x^a)x^3 + K^k{}_a(x^a)g_{km}(x^a)K^m{}_b(x^a)(x^3)^2 \tag{2}$$

$$G(X^i) = \det G_{ab}(X^i) = g(x^a)(1 - 8M(x^a)x^3 + (2K(x^a) + 8M(x^a)^2)(x^3)^2 + ...) \tag{3}$$

$$\sqrt{G(X^i)} \equiv \sqrt{g(x^a)}\xi(X^i)\,,\ \xi(X^i) = 1 - 4M(x^a)x^3 + K(x^a)(x^3)^2 + ... \tag{4}$$

where $g_{ab}(x^a)$ is the induced metric in the surface, $g(x^a) = \det g_{ab}(x^a)$ and $K^i{}_j(x^a)$ is the extrinsic curvature tensor associated with $\mathcal{S}$. $K = \det K^i{}_j$ and $M(x^a) \equiv G^{ij}(X^i)K_{ij}(X^i)$ is the mean curvature in $\mathcal{S}$.

Central to the approach developed in (5–7) is the assumption of the presence of forces which constrain the particle to $\mathcal{S}$. It is assumed that these forces act everywhere normal to $\mathcal{S}$ and that they can be derived from a potential $V_\lambda(X^3)$. λ is a parameter which measures the strength of the potential. The Schrödinger equation describing an electrically neutral particle in the embedding space within this framework is then given by (we use units such that $c \equiv \hbar \equiv 1$)

$$i\partial_t \psi = -\frac{1}{2m} G^{ij} \nabla_i (\nabla_j \psi) + V_\lambda(X^3) \,. \tag{5}$$

m denotes the particle mass. In order to derive a quantum theory in $\mathcal{S}$ we need to *dimensionally reduce* the Schrödinger equation. We therefore decompose the covariant derivative in a coordinate gauge invariant manner as a sum of one part which acts along the surface($||$), and one part which acts normal to the surface ($\perp$)

$$\nabla_i = \nabla_{||i} + \nabla_{\perp i} \,. \tag{6}$$

The purely kinetic term in the Schrödinger equation can then be written

$$G^{ij} \nabla_i \nabla_j \psi \equiv (\nabla_{||}^2 + \nabla_\perp^2)\psi = (\partial^a \partial_a \psi + G^{ab}\, \Gamma^c{}_{ab} \partial_c \psi) + (\partial^3 \partial_3 \psi + G^{ab}\, \Gamma^3{}_{ab} \partial_3)\psi \,. \tag{7}$$

In the last relation we have used the coordinate gauge Eq.(1). $\Gamma^i{}_{jk}$ represents the Christoffel symbols of the second kind. We will assume that the wave function is normalizable in the three dimensional embedding space, such that the norm is given by

$$N = \int d^3X \sqrt{G} |\psi|^2 = \int d^3x \sqrt{g} |\chi|^2 \,. \tag{8}$$

Probability conservation requires that $\psi(X^i) = \xi(x^i)^{-1/2} \chi(x^i)$. We use this relation to compute the kinetic term and rewrite the Schrödinger equation in terms of χ. Clearly,

$$\lim_{x^3 \to 0} \nabla_{||}^2 \psi = \nabla_{||}^2 \chi \,. \tag{9}$$

We also find that

$$\lim_{x^3 \to 0} \nabla_\perp^2 \psi = \lim_{x^3 \to 0} \frac{1}{\sqrt{G}} \partial_3 (\sqrt{G} \partial^3 \psi) = \lim_{x^3 \to 0} \xi^{-1} \partial_3 (\xi \partial_3 (\xi^{-1/2} \chi)) \equiv \partial_3^2 \chi - V_0 \chi \,. \tag{10}$$

Using these relations we find in the limit $x^3 \to 0$ that the Schrödinger equation becomes

$$i\partial_t \chi = -\frac{1}{2m} \nabla_{||}^2 \chi - \frac{1}{2m} \partial_3^2 \chi + V_0 \chi + V_\lambda \chi \,, \tag{11}$$

where V_0 is given by (6)

$$V_0 = -\frac{1}{2m} (M^2 - K) \,. \tag{12}$$

We see that an effective potential has emerged depending on scalars characterizing the extrinsic curvature of $\mathcal{S}$. V_0 is clearly non-positive on any surface. If χ is separable into one part which is independent of x^3 and one part which only depends on this coordinate we have effectively deduced a quantum theory in the surface $\mathcal{S}$. This program can also be adapted to

linear structures by continuing the dimensional reduction procedure above. The result is (6)

$$-\frac{\hbar^2}{2m}\partial_x^2\psi - \frac{\hbar^2}{8m}\kappa^2(x)\psi = E\psi\,. \tag{13}$$

x is the coordinate along the structure, and $\kappa(x)$ is its local curvature. We will later return to this result in Section 5.

3. The catenoid

Let us apply the dimensional reduction approach above to a concrete surface. We will in particular consider the catenoid surface (9). This is a classical minimal surface. It can conceivably be realized in a bilayer of honeycomb lattices with radially arranged dislocations or in bilayer graphene (10). We choose the following parametrization for the catenoid $x = R\cosh(z/R)\cos\phi$, $y = R\cosh(z/R)\sin\phi$ and $z = z$, with $\phi \in [0, 2\pi]$ where x, y, z represents the canonical Cartesian coordinates in ordinary three-dimensional space (Fig. 1). The local radius $\rho = R\cosh(z/R)$ and the metric is thus given by

$$g_{\rho\rho} = \frac{\rho^2}{\rho^2 - R^2}\ ,\ g_{\phi\phi} = \rho^2. \tag{14}$$

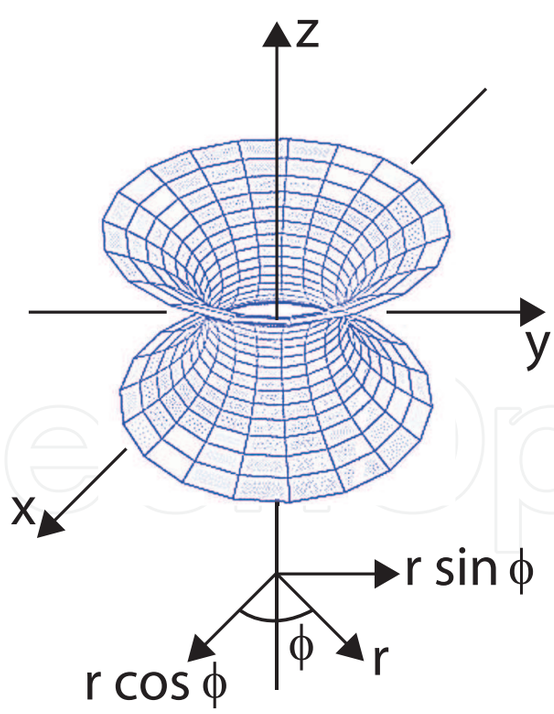

Fig. 1. A two-dimensional section (catenoid) of a three dimensional worm hole geometry with its axis along z and the throat radius R.

It is interesting to note that slices of a wormhole geometry is geometrically equivalent to a catenoid. In cylindrical coordinates (z, r, ϕ) a two-dimensional section of a wormhole is given by (11)

$$z(r) = \pm b_0 \ln \left[\frac{r}{b_0} + \sqrt{\frac{r^2}{b_0^2} - 1} \right], \tag{15}$$

with $l = \pm\sqrt{r^2 - b_0^2}$. The spatial part of the wormhole geometry is given by the following expression (11)

$$ds^2 = dl^2 + (b_0^2 + l^2)(d\theta^2 + \sin^2\theta d\phi^2), \tag{16}$$

where $l \in [-\infty, +\infty]$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. b_0 is the shape function of the wormhole [in general $b = b(l)$ and for $l = 0, b = b(0) = b_0 = const.$ represents the radius of the throat of the wormhole]. Here l is a radial coordinate measuring proper radial distance; θ and ϕ are the spherical polar coordinates. Here we will consider the slice $\theta = \pi/2$ which represents an equatorial section of the wormhole geometry (at constant coordinate time). For this particular slice we thus get the following line element

$$ds^2 = dl^2 + (b_0^2 + l^2)d\phi^2, \tag{17}$$

which is precisely equivalent to the line element on a catenoid (since $l^2 = r^2 - b_0^2$)

$$ds^2 = \frac{r^2}{r^2 - b_0^2} dr^2 + r^2 d\phi^2. \tag{18}$$

Note that if we consider any other section of the three dimensional wormhole, say for $\theta = \theta_0$, the line element will change to

$$ds^2 = \frac{r^2}{r^2 - b_0^2} dr^2 + a^2 r^2 d\phi^2, \tag{19}$$

where $a^2 = \sin^2\theta_0$, and (obviously) $a^2 \in [0, 1]$. For the catenoid this will only mean a rescaling of the radius of the catenoid at the throat (the circle with least local radius) from R to aR. The line element Eq.(19) corresponds to a catenoid with $x = aR\cosh(z/aR)\cos\phi$, $y = aR\cosh(z/aR)\sin\phi$ and $z = z$. Thus all θ-sections of the physical wormhole at constant time coordinates represent a catenoid with radius aR. The catenoid with the biggest radius corresponds to the equatorial section $\theta = \frac{\pi}{2}$ and the one with zero radius to the section $\theta = \pi$. Returning to the catenoid and focusing on the (z, ϕ) coordinates (instead of (ρ, ϕ)), the line element is given by

$$ds^2 = \cosh^2(z/R)dz^2 + R^2\cosh^2(z/R)d\phi^2, \tag{20}$$

with the principal curvatures given by

$$\kappa_1 = \frac{1}{R}\text{sech}^2(z/R), \quad \kappa_2 = -\frac{1}{R}\text{sech}^2(z/R). \tag{21}$$

This implies that the mean curvature $M = (\kappa_1 + \kappa_2)/2 = 0$ (meaning that the surface is a minimal surface) and the Gaussian curvature $K = \kappa_1\kappa_2 = -(1/R^2)\text{sech}^4(z/R)$. The

corresponding curvature induced potential on a catenoid is then given by

$$V(z) = -\frac{\hbar^2}{2m_0}(H^2 - K) = -\frac{\hbar^2}{2m_0 R^2}\text{sech}^4(z/R). \tag{22}$$

Note that for $a^2 \ll 1$ the potential becomes very deep at the origin. The corresponding Schrödinger equation is given by

$$-\frac{\hbar^2}{2m_0 R\cosh^2(z/R)}\left[R\frac{\partial^2\psi}{\partial z^2} + \frac{1}{R}\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{\hbar^2}{2m_0 R^2}\text{sech}^4(z/R)\psi = E\psi, \tag{23}$$

or simplifying

$$-R\frac{\partial^2\psi}{\partial z^2} - \frac{1}{R}\frac{\partial^2\psi}{\partial\phi^2} - \frac{\text{sech}^2(z/R)}{R}\psi = \frac{2m_0 R}{\hbar^2}\cosh^2(z/R)E\psi. \tag{24}$$

Using the cylindrical symmetry along the z-axis we set $\psi = e^{imz}\Phi$. We thus get the following equation for Φ

$$\Phi_{zz} - \frac{m^2}{R^2}\Phi + \frac{\text{sech}^2(z/R)}{R^2}\Phi + \frac{2m_0 E\cosh^2(z/R)}{\hbar^2}\Phi = 0. \tag{25}$$

Defining a dimensionless length $\zeta = z/R$ and energy $\epsilon = 2m_0 E R^2/\hbar^2$ we get the following effective Schrödinger equation

$$-\Phi_{\zeta\zeta} + V(\zeta)\Phi(\zeta) = 0, \tag{26}$$

where the potential now reads

$$V(\zeta) = [m^2 - \epsilon\cosh^2(\zeta)] - \text{sech}^2(\zeta). \tag{27}$$

This potential for $m \neq 0$ bears some similarity to the corresponding geometric potential for the physical wormhole (12). Note that in the ground state ($\epsilon = 0$, also called a *critically bound state* (13)) $V(\zeta)$ becomes the reflectionless Bargmann's potential (14) and the Schrödinger equation becomes the hypergeometric equation with the ground state wavefunction (the 'Goldstone mode') given by $\Phi(\zeta) = \text{sech}(\zeta)$. This result is remarkable since this implies that the catenoid surface enables complete transmission across it for a quantum particle. This does not seem to be the case for the physical wormhole geometry (12). For non-zero values on ϵ the above potential is an inverted double well potential shown in Fig. 2.
Let us consider Eq.'s(26-27) in some more detail. We see in particular that

$$\lim_{\zeta\to\pm\infty}|V| \to \infty\,. \tag{28}$$

This behavior of the potential at infinity is strange since the geometry on the catenoid in these regions approaches the usual Euclidean one. This feature can be traced to the coordinates used since the proper length per unit in the ζ direction diverges when $\zeta \to \pm\infty$. This can be remedied by introducing another set of coordinates on the catenoid.

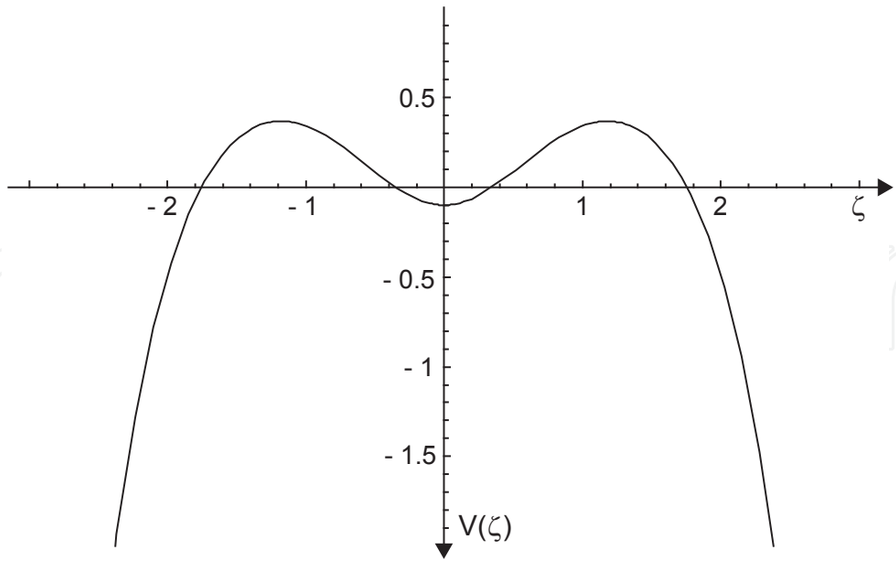

Fig. 2. The inverted double well potential $V(\zeta)$ with $m = \pm 1$ and $\epsilon = 0.1$.

Quantum theory in curved spaces is generally a challenge since the theory is not generally covariant. Classical quantum theory is not even Lorentz invariant. This puts a severe constraint on the coordinate system in which one wishes to describe the physics in order to be able to extract the physical content of the theory. This challenge was even central in the early days of general relativity theory itself in connection with the physical interpretation of the Schwartzshild metric, e.g. Just like as in general relativity theory one is usually safe concerning the physical interpretation, as well as the definitions of physical quantities, when the manifold in question is asymptotically Minkowski (Euclidean). In such asymptotic regions we expect on physical grounds to rederive the usual flat space physical results. The asymptotic properties thus in some sense anchor the curved region and its physics to reality, as we know it. The catenoid is in this sense an asymptotic Euclidean object, thus making this manifold a space anchored to 'reality'.

The catenoid surface can in some sense be perceived as a deformation of the plane. Considering briefly the two-dimensional Schrödinger equation in the plane in polar coordinates we get the Bessel equation. Clearly, the boundary condition at the origin is suspect here. However, in our case we can as a first approximation consider a deformation of the plane in a region around the origin. In the deformed region the Schrödinger equation will generally be very complicated but the solutions of it must nevertheless be matched to the Bessel functions which survive sufficiently far from the deformed region. This reasoning goes *ad verbum* through also on the catenoid even though we here, in addition to curvature corrections, also have a topology change when compared to the plane since not any closed curve on the catenoid surface is null homotopic. Hence, we should seek coordinates on the catenoid such that the Schrödinger equation gives rise to the Bessel equation in the asymptotic region on the catenoid. The coordinates should thus in particular result in a metric which is reminiscent of polar coordinates at infinity. It is possible to find such coordinates if one covers the entire catenoid manifold with two coordinate patches. One patch covers the region $\zeta > 0$,

the other $\zeta < 0$. In the upper part we choose the new radially directed coordinate as

$$\eta^+ = e^{\zeta} - 1\,;\, \zeta > 0\,. \tag{29}$$

In the lower part we correspondingly choose

$$\eta^- = -(e^{-\zeta} - 1)\,;\, \zeta < 0\,. \tag{30}$$

Clearly $\eta^+ = \eta^-$ at $\zeta = 0$. The invariant line-element can then be written as

$$ds^2 = \frac{((\eta^\pm \pm 1)^2 + 1)^2}{4(\eta^\pm \pm 1)^4}(d\eta^\pm)^2 + \frac{1}{4}(\frac{(\eta^\pm \pm 1)^2 + 1}{\eta^\pm \pm 1})^2 d\phi^2\,. \tag{31}$$

In the limit $\eta^\pm \to \pm\infty$ the metric can be written as

$$ds^2 = \frac{1}{4}(d\eta^\pm)^2 + \frac{1}{4}(\eta^\pm)^2 d\phi^2\,. \tag{32}$$

Hence, the asymptotic form of this metric is very similar to the Euclidean metric expressed in ordinary polar coordinates. Indeed, they are exactly the same something which is easily seen by a simple rescaling of the radial coordinates. These new coordinates should therefore be well suited to explore the physical states of a quantum particle on the catenoid.
Let us now consider the Schrödinger equation (26). In terms of the new coordinates we have in particular that

$$\partial_\zeta^2 \Phi = (\eta^\pm \pm 1)\partial_\pm((\eta^\pm \pm 1)\partial_\pm \Phi)\,, \tag{33}$$

$$\cosh u = \pm\frac{1}{2}(\frac{(\eta^\pm \pm 1)^2 + 1}{\eta^\pm \pm 1})\,. \tag{34}$$

This gives rise to identical expressions for the Schrödinger equation in the two coordinate patches. In the upper patch the equation is explicitly given by

$$\partial_+^2 \Phi + \frac{1}{(\eta^+ + 1)}\partial_+ \Phi + \left[\frac{\epsilon}{4} - \frac{(m^2 - \epsilon/2)}{(\eta^+ + 1)^2} + \frac{1}{4}(\frac{\epsilon}{(\eta^+ + 1)^4} + \frac{16}{((\eta^+ + 1)^2 + 1)^2})\right]\Phi = 0\,. \tag{35}$$

Clearly, letting $\eta^+ \to \infty$ we easily get the Bessel equation, which is well behaved at infinity.
The stationary Schrödinger equation, and assuming a well defined energy E eigenvalue problem, is formally given by

$$(-\nabla^2 + V)\Psi = E\Psi\,. \tag{36}$$

Hence, we have that

$$V - E = -\left[\frac{\epsilon}{4} - \frac{(m^2 - \epsilon/2)}{(\eta^+ + 1)^2} + \frac{1}{4}\left(\frac{\epsilon}{(\eta^+ + 1)^4} + \frac{16}{((\eta^+ + 1)^2 + 1)^2}\right)\right]\,. \tag{37}$$

In the asymptotic region we find that

$$\lim_{\eta^+ \to \infty} V = E - \frac{1}{4}\epsilon > 0\,. \tag{38}$$

We have plotted the potential for $m = 0$ and $\epsilon = 2$ in Fig. 3. Note that in this s-channel the potential becomes negative sufficiently close to the origin.

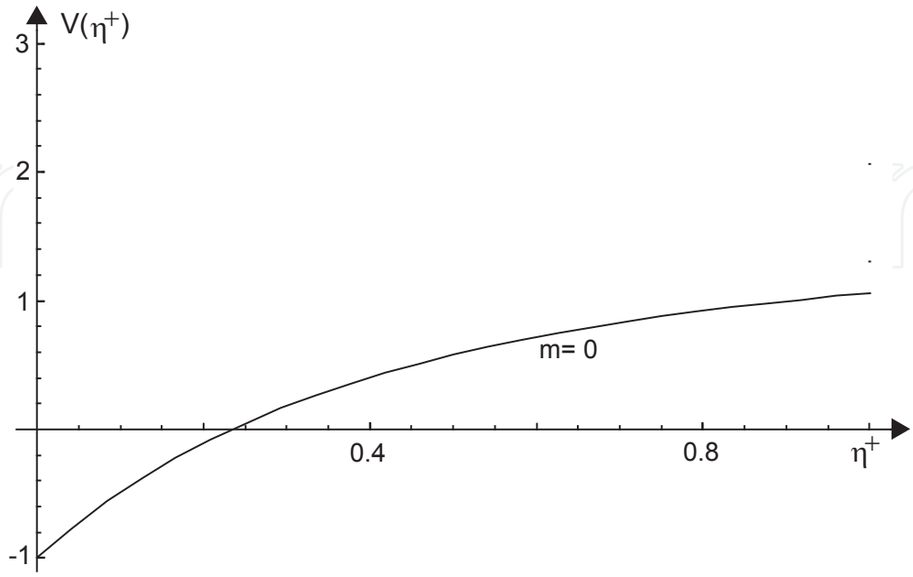

Fig. 3. The potential $V(\eta^+)$ with $m = 0$ and $\epsilon = 2$.

Clearly, the constant part of the potential can be renormalized to zero without any physical consequences. Hence, the renormalized potential V_r can be taken to be

$$V_r - E = -\left[-\frac{(m^2 - \epsilon/2)}{(\eta^+ + 1)^2} + \frac{1}{4}\left(\frac{\epsilon}{(\eta^+ + 1)^4} + \frac{16}{((\eta^+ + 1)^2 + 1)^2} \right) \right]. \tag{39}$$

4. Conformal deformations

The framework derived in (5–7) has become a standard one for analyzing quantum mechanics on surfaces and one-dimensional structures. The resulting theory is a theory framed in curved spaces. As such it is often difficult to deal with both from a purely computational perspective as well as also from an interpretational perspective. On the computational side the kinetic part of the Schrödinger equation will very often be highly complicated function of the metric tensor. This will typically give rise to complicating second order derivative terms which mix the spatial coordinates. The often lack of an 'asymptotically' flat region of a given surface (e.g.) often gives rise to all sorts of interpretational issues which also often appear in quantum theories in curved spaces (1). Our treatment of the catenoid above highlight this issue in particular. Here we sketch a framework which might be of utility in resolving some of these issues. We emphasize that it is a sketch which is presented, and not a complete and polished framework.

We entertain the following idea. It is well known that every open, connected and smooth surface in three-dimensional Euclidean space is conformally flat. Physically we can picture this as it is always possible to deform S in such a way that the local geometry, all the time expressed in the same coordinates, has a conformally flat form all the way to the point when S has become a two-dimensional plane P. We call such a process a *conformal deformation* of S. The

Schrödinger equation is not invariant under conformal transformations. Hence, is it possible in general to re-express the quantum theory on a given surface S on the two-dimensional plane P using conformal techniques in such a way as to bypass (some of) the issues mentioned above which arises in the more physically motivated dimensional reduction framework? We show that it is possible to reformulate the two-dimensional quantum theory on S in the plane such that it can be looked upon as standard free two-dimensional Schrödinger theory coupled to an external potential. The wave-function on P couples to a new potential V_P which exhibits a vector field $\vec{W}$ and a scalar field W, both of a purely geometric origin on S

$$V_P = -\frac{\hbar^2}{2m}(\vec{W} \cdot \nabla_P + W)\,. \tag{40}$$

∇_P denotes the metric compatible connection on P. Hence, certain computational issues is bypassed by this approach since the theory is formulated in a flat space which in Cartesian coordinates will give rise to a 'trivial' kinetic operator. The formulation will also be easier to interpret both because of the non-existing intrinsic curvature in the plane and its vanishing extrinsic curvature, and because V_P can be treated as a completely *external* potential not associated with the space P in which the theory is formulated. This might enhance our understanding of the physical picture on S.

We consider a smooth two-dimensional surface S in ordinary three-dimensional Euclidean space. In the present work we will focus on *open* surfaces. We will briefly deal with compact surfaces at the end of this section. We will repeat some of the mathematical technicalities from section 2 in order to fix the needed notation. We assume that we can define a normal vector field $\vec{N}$ everywhere on S. We will choose coordinates in the embedding space in such a way that one coordinate basis vector is always parallel to $\vec{N}$ in the vicinity of S. We will denote the associated coordinate by x^3 in the following and the coordinates on S by x^1, x^2. We denote the metric tensor induced on S by G_{Sab} such that the indices refer to the coordinates in S. In the dimensional reduction approach it is assumed that the quantum particle is constrained to S by a constraining potential $V_{S\lambda}$ where λ is a measure of the strength of the potential. It is assumed that the constraining forces always act along $\vec{N}$. In order to derive a theory on S we decompose the connection ∇ in the three-dimensional Euclidean space into a component acting solely in the surface ∇_S and a component $\nabla_\perp$ acting along the normal vector

$$\nabla = \nabla_S + \nabla_\perp\,. \tag{41}$$

The potential $V_{S\lambda}$ can be thought of as an infinite well potential such that S is sandwiched between two potential 'walls' (6). In the limit when the width Δx^3 of the potential well goes to zero, when the particle is literally forced to follow the surface S, we get (6)

$$\lim_{\Delta x^3 \to 0} \nabla_\perp^2 = \partial_3^2 - V_S(x^1, x^2)\,. \tag{42}$$

In these calculations coordinates are chosen such that $G_{3i} = 0\,, G_{33} = 1\,, ; i \neq 3$. The Schrödinger equation can then be written as

$$i\partial_t \chi_S = -\frac{\hbar^2}{2m}\nabla_S^2 \chi_S - \frac{\hbar^2}{2m}\partial_3^2 \chi_S + V_S \chi_S + V_{S\lambda}\chi_S\,. \tag{43}$$

If this equation is separable we are left with a theory in the surface S. The theory in S carries with it a 'memory' of the three-dimensional embedding space through the potential V_S.
Any two-dimensional metric can locally be written in a conformally Euclidean form. This means explicitly that there exists a coordinate transformation, *a conformal diffeomorphism* $\mathcal{D}$: $(x^1, x^2) \to (X^1, X^2)$ such that

$$\begin{aligned} ds_S^2 &= G_{Sij} dx^i dx^j \\ &\equiv \omega^2(X^1, X^2)((dX^1)^2 + (dX^2)^2) \equiv \omega^2 ds_P^2 \equiv G_{Pij} dX^i dX^j \,. \end{aligned} \tag{44}$$

Here ds_S^2 and ds_P^2 are the metrics on S and P respectively, and $\omega^2(X^1, X^2)$ is a positive definite scalar function. We will be concerned with the situation when the metric ds_S^2 on S undergoes a *point wise* conformal transformation $\mathcal{T}$ such that

$$\mathcal{T} : ds_S^2 \to \Omega^2(X^1, X^2) ds_S^2 \equiv ds_P^2 \,. \tag{45}$$

$\Omega^2(X^1, X^2)$ is a positive definite function. In the case when none of the coordinates are periodic this represents deformations of S to a plane. We will primely assume this picture in the following. Physically we will picture the transformation $\mathcal{T}$ as either an adiabatic or an instantaneous one in order not to perturb the quantum system out of the quantum state it exhibits on S. For definiteness assume here an instantaneous process such that $\Omega = \omega^{-1}$.
The normal vector field $\vec{N}$ is assumed normal to the surface during the complete deformation process. Hence, the system of coordinates defined by (X^1, X^2, x^3) does therefore represent *comoving isothermal coordinates*. We also assume that the external confining potential V_λ always has a form such that the resulting force acting on the particle is along $\vec{N}$. We will furthermore assume that the particle never escapes the surface; we will assume conservation of probability during the deformation process, i.e. This implies in particular that the integral of the probability density on S must equal the same integral over the plane. If we denote the wave-function on P by χ_P it follows that

$$\int_S dS \sqrt{G_S} |\chi_S|^2 = \int_S dS \sqrt{G_P} |\chi_P|^2 \,, \tag{46}$$

where $G_S = \det(G_{Sij})$, $G_P = \det(G_{Pij})$. Since $\mathcal{T} : G_P = \Omega^4 G_S$, it follows that

$$\chi_S = \Omega \chi_P \,. \tag{47}$$

Hence, the wave-function must transform with conformal weight equal unity under the restricted class of conformal transformations which we deal with in this work. There exist previous studies in the literature of the properties of the Schrödinger equation under conformal transformations. Interestingly, these have to our knowledge only been concerned with space-time or space conformal transformations and not conformal transformations restricted to (hyper-) surfaces.
Two metric compatible connections ∇_S and ∇_P on two conformally related metrics ds_S^2 and ds_P^2 are related by

$$\nabla_{Pi} \omega_b = \nabla_{Si} \omega_j - C^k{}_{ij} \omega_k \tag{48}$$

where ω_k is some one-form, and

$$C^k{}_{ij} = 2\delta^k{}_{(i}\nabla_{Pj)}\ln\Omega - G_{Pij}G_P{}^{kl}\nabla_{Pl}\ln\Omega\,. \tag{49}$$

It follows that

$$\nabla_S^2 = \Omega^2(\nabla_P^2 + G_P{}^{ij}C^c{}_{ij}\nabla_{Pc}) \tag{50}$$

and

$$\nabla_S^2(\chi_S) = \nabla_S^2(\Omega\chi_P) = \Omega^2(\Omega\nabla_P^2\chi_P + V^i\nabla_{Pi}\chi_P + V\chi_P)\,, \tag{51}$$

where

$$V^i = 2G_P{}^{ji}\nabla_{Pj}\Omega + G_P{}^{jk}C^i{}_{jk}\Omega\,, \tag{52}$$

$$V \equiv \nabla_P^2\Omega + G_P{}^{ij}C^k{}_{ij}\nabla_{Pk}\Omega\,. \tag{53}$$

The extrinsic curvature of S is $K_{Si}^j = \nabla_{Si}N^j$. It follows that

$$K_{Sij} = K_{Pij} + C^3{}_{ij}N_3\,. \tag{54}$$

Identify P with the two-dimensional Euclidean plane. We then have that

$$K_{Pij} \equiv 0 \Rightarrow \begin{cases} K_S = \det(K_{Sij}) = \det(C^3{}_{ij}) \\ M_S = \mathrm{tr}(K_{Sij}) = -\mathrm{tr}(C^3{}_{ij}) \end{cases} \tag{55}$$

where

$$C^3{}_{ij} = (\delta^3{}_i\nabla_{Pj} + \delta^3{}_j\nabla_{Pi})\ln\Omega\,. \tag{56}$$

The $(C^3{}_{ij})$-matrix is explicitly given by

$$(C^a{}_{bc}) = \begin{pmatrix} 0 & 0 & C^3{}_{13} \\ 0 & 0 & C^3{}_{23} \\ C^3{}_{31} & C^3{}_{32} & 0 \end{pmatrix}\,. \tag{57}$$

Hence

$$K_S = 0\,,\ M_S = 0\,. \tag{58}$$

We note that this also implies that $G_P^{ij}C^3{}_{ij}\nabla_{P3} = 0$ since G_P^{ij} is diagonal. We then get the highly non-trivial result that the potential V_S transforms to zero under conformal deformations to P. Consequently, the Schrödinger equation on P reads

$$i\partial_t\chi_P = -\frac{\hbar^2}{2m}\Omega^2\nabla_P^2\chi_P - \frac{\hbar^2}{2m}\partial_3^2\chi_P - \frac{\hbar^2}{2m}(\vec{W}\cdot\nabla_P + W)\chi_P + V_{P\lambda}\chi_P\,, \tag{59}$$

where

$$W^i \equiv \Omega^{-1}V^i\ ,\ W \equiv \Omega^{-1}V\,. \tag{60}$$

We assume that we can separate the motion orthogonal to the surface from the motion in the surface. We will also assume stationary states such that the time-dependent part of the wave-equation can be written as a simple exponential $\sim e^{-iEt}$ where we identify E with the

total energy of the states. Hence, Eq. (59) can be decomposed into

$$-\frac{\hbar^2}{2m}\nabla^2_{||}\chi_P + \frac{1}{\Omega^2}(E - \kappa^2 + V_P)\chi_P = 0\,, \tag{61}$$

$$-\frac{\hbar^2}{2m}\partial^2_3\chi_P\chi_P + V_{P\lambda}\chi_P = \kappa^2\chi_P \tag{62}$$

where we have defined $V_P \equiv -\frac{\hbar^2}{2m}(\vec{W}\cdot\nabla_P + W)$ and introduced a separation constant κ. This constant can naturally be identified with the momentum in the direction perpendicular to the surface.

Clearly, the set of equations above can be interpreted as describing a particle interacting with two potentials. Equation (61) is in general simpler than Eq. (43). Even though Eq. (61) like Eq. (43) in addition to the second order kinetic terms also exhibits first order differential operator terms contained in the V_P-potential it will not contain differential operator cross terms as Eq. (43) in general will exhibit. Furthermore, note that the potentials W and $\vec{W}$ are functions of Ω. Since Ω does not describe P we can interpret these terms as *external* potentials which are applied to P in very much the same fashion as $V_{P\lambda}$.

In the analysis above we have for simplicity assumed that P is identified with the two-dimensional plane. Clearly, P must not by necessity be identified with the plane in order to have a target manifold P with vanishing intrinsic curvature. One natural example which is also of practical importance is the straight tube. Clearly, when P is a curved surface (with non-vanishing extrinsic curvature, but with vanishing intrinsic curvature) the effective geometric potential on P will become more complicated compared to the exact planar situation. This is easily seen from the transformation property of V_S under a conformal deformation. From the expression for K_{Sij} in terms of K_{Pij} and $C^k{}_{ij}$ it follows that the transformed of the potential V_S will exhibit products between the extrinsic curvature tensor on P and $C^3{}_{ij}$-terms. Hence, changing the *topology* of S away from the planar one (every closed curve is null homotopic) will alter the induced potential on S in a fundamental way.

Let us summarize. In (5–7) a physically motivated framework for dealing with quantum mechanics on surfaces and linear structures in ordinary three dimensional Euclidean space were developed. Here we have attempted to reformulate this framework for quantum mechanics on surfaces into a framework on the two-dimensional plane. On the plane a quantum particle is shown to interact with an external potential V_P in addition to the external potential $V_{P\lambda}$ which constrains the particle to the plane. Clearly, the conformal formulation presented in this work represents *a priori* a simplification computation wise. It also represents an interesting tool in the analysis of quantum mechanics on a given surface. The form of the geometric potential on the plane provides an immediate physical insight. Considering momentum eigen-states it follows that states with opposite sign interacts differently with the geometry since $\vec{W}\cdot\nabla_P \Rightarrow -\vec{W}\cdot\nabla_P$ when $\nabla_P \Rightarrow -\nabla_P$. Hence, a non-trivial geometry on S might lift a degeneracy which is present on the plane provided $\vec{W} \neq 0$. An analogy to the motion of an electrically charged particle moving on the plane with a magnetic field piercing the plane is immediate. Hence, the class of surfaces defined by $\vec{W} \neq 0$ is thus interesting to consider further in order to build a general physical picture of quantum dynamics on surfaces. The tentative framework presented here might furthermore also help shed light on a fundamental problem connected with the understanding of quantum mechanics on surfaces following Dirac's quantization prescription (2). It is well known that the Dirac quantization

scheme does not produce an unique expression for the induced quantum geometric potential V_S (15–17). It is claimed that this result is not related to improper choice of coordinates, but emerges solely due to operator ordering issues (15). Interestingly, considering a plane P it follows that no geometric potential of the kind stemming from the dimensional reduction approach will get induced. The coupling to the external potential V_P should not imply any operator ordering issues of the kind reported in (15). An unique theory should consequently ensue. However, the quantum theory on P is conformally related to a quantum theory on a certain surface S where the very same quantization procedure will not give rise to a unique theory. This apparent 'contradiction' between the '$P-$' and the '$S-$pictures' of the theory might thus hold a key for resolving the disturbing discrepancy in the quantum formulation of classical mechanics on surfaces following from Dirac's prescription.

5. Supersymmetric quantum mechanics

Let us turn to one-dimensional structures living in ordinary three-dimensional Euclidean space. On such strutures the effective theory stemming from the dimensional reduction approach is given by Eq. (13). Ideas from supersymmetry was incorporated into one-dimensional quantum mechanics by E. Witten in (18). This approach leads to a natural notion of isospectral deformations, deformations of the potential in the Schrödinger equation such that that the energy spectrum is identically preserved. It would be interesting to apply this approach to our subject. This should mean that we can set out with a particular linear configuration in space which is described by the local curvature of the structure $\kappa(x)$ (x denotes some coordinate along the structure). From this it should then be possible to generate another potential appearing in the Schrödinger equation which in our context must be related to another curvature configuration $\hat{\kappa}(x)$; to another linear structure in space, i.e. This line of approach, as the one in the previous section, has not been pursued in the previous literature. It seems to represent a promising approach in the work of getting a deeper understanding of the relation between quantum physics and geometry. Let us initiate this study by some relatively straightforward considerations. We will assume that the reader has a basic understanding of supersymmetric quantum mechanics. The recent book (19) represents a nice introduction to the subject. We will follow the notation in that book in the following.

Isospectral deformations in supersymmetric quantum mechanics are connected to the problem of generating another superpotential $\hat{W}(x)$, which gives rise to a new partner potential $\hat{V}_+(x)$, from a given superpotential $W(x)$ and partner potential $V_+(x)$ such that

$$\hat{V}_+(x) = \hat{W}^2 + \hat{W}' = V_+(x) = W^2 + W' . \tag{63}$$

$'$ indicates differentiation with respect to the space variable. From the knowledge of $\hat{W}(x)$ a new physical potential $\hat{V}_-(x)$ can be constructed

$$\hat{V}_-(x) = \hat{W}^2 - \hat{W}' , \tag{64}$$

which give rise to the same spectrum as the initial one generated by $V_-(x)$. In the literature the only type of deformation that has been studied so far has the form (19)

$$W(x) \to \hat{W}(x) = W(x) + f(x) . \tag{65}$$

$f(x)$ is some function to be determined from the condition Eq. (63). Finding the explicit expression for $f(x)$ is straightforward. Inserting Eq. (65) into Eq. (63) we get

$$\frac{d}{dx}f(x) + 2W(x)f(x) + f^2(x) = 0\,. \tag{66}$$

This is the Riccati equation. Further making the substitution $f(x) = 1/y(x)$ we get

$$\frac{d}{dx}y(x) - 2W(x)y(x) = 1\,. \tag{67}$$

This equation is easily integrated and we find

$$f(x;\lambda) = \frac{e^{-\int 2W(x)dx}}{\lambda - \int e^{-\int 2W(x)dx}dx}\,. \tag{68}$$

λ is an integration constant. We note that letting $\lambda \to \infty \Rightarrow f(x;\lambda) \to 0$ results in the identity deformation $\hat{W}(x) = W(x)$. The result Eq. (68) has been taken in the literature as an expression for the most general deformation of $W(x)$ stemming from Eq. (63) when a deformation scheme of the kind in Eq. (65) is employed (19). Let us now identify the physical potential V_- with the potential in Eq. (13)

$$V_-(x) = -\frac{\hbar^2}{8m}\kappa^2(x)\,. \tag{69}$$

This gives the following equation for the superpotential

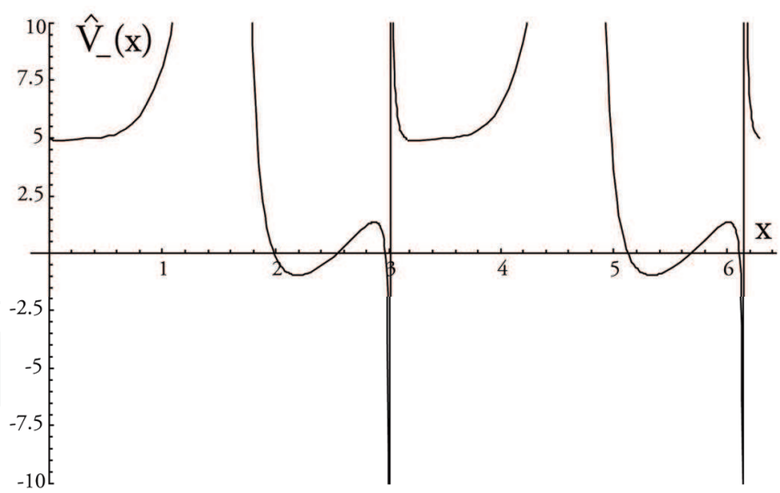

Fig. 4. The effecetive potential $\hat{V}_-(x)$ on the circle with $\lambda = 0.5$.

$$-\frac{dW}{dx} + W^2 + \kappa^2 = 0\,. \tag{70}$$

This is also a Riccati equation. It can be connected to ordinary second order differential equations by the substitution

$$W = \frac{1}{U(x)} \frac{dU(x)}{dx} \,. \tag{71}$$

In terms of $U(x)$ Eq.(70) reads

$$\frac{d^2U}{dx^2} + \kappa^2(x)U = 0 \,. \tag{72}$$

Let us consider the circle which is well understood in quantum theory. We can set $\kappa^2 = 1$ without loss of generality. We then easily get

$$U = A_0 \sin(\omega x) + B_0 \cos(\omega x) \tag{73}$$

where ω is some constant. Let us for simplicity set $A_0 = 0$ and $\omega = 1$. Then we get

$$W = -\tan(x) \,. \tag{74}$$

We are then in position to compute the deformed physical potential $\hat{V}_-(x)$. We find that

$$\begin{aligned} \hat{V}_-(x) &= \hat{W}^2 - \hat{W}' = \\ &= (-\tan(x) + \frac{1}{\lambda \cos^2(x) - 1})^2 - (-\frac{1}{\cos^2(x)} + \frac{2\cos(x)\sin(x)}{(\lambda \cos^2(x) - 1)^2}) \,. \end{aligned} \tag{75}$$

This potential is graphed in Fig. (4) with $\lambda = 0.5$. The potential exhibits singularities. It is not strictly non-positive but can also take positive values. When this happens the corresponding curvature $\hat{\kappa}$ appears to be imaginary. This last feature is clearly unphysical. However, we have only probed a particular solution of a vast solution space and physically acceptable deformations might very well exist. A more systematic study of the circle is left for future research.

Another configuration which is natural to study, and which also is easily tractable by analytical means, is a straight line which has a bent part somewhere along it. A model for such a structure is captured by a curvature function given by

$$\kappa^2 = 1 - \tanh^2(x) \,. \tag{76}$$

κ^2 is graphed in Fig. (5). This gives rise to U-functions given by

$$U(x) = C_1 P_{\frac{1}{2}(\sqrt{5}-1)}(\tanh(x)) + C_2 Q_{\frac{1}{2}(\sqrt{5}-1)}(\tanh(x)) \tag{77}$$

where the P- and Q-functions denote the Legendre functions. The deformed superpotential can formally be written

$$\hat{W} = \frac{1}{U(x)} \frac{dU(x)}{dx} + \frac{1}{\lambda U^2 - 1} \,. \tag{78}$$

From this expression we can deduce an infinite family of new linear structures as we did above. However, the study of this deformation is also left for the future. It is hoped that the relative easy one apparently can derive new physically realizable linear structures from known ones as demonstrated here will inspire more studies along these lines.

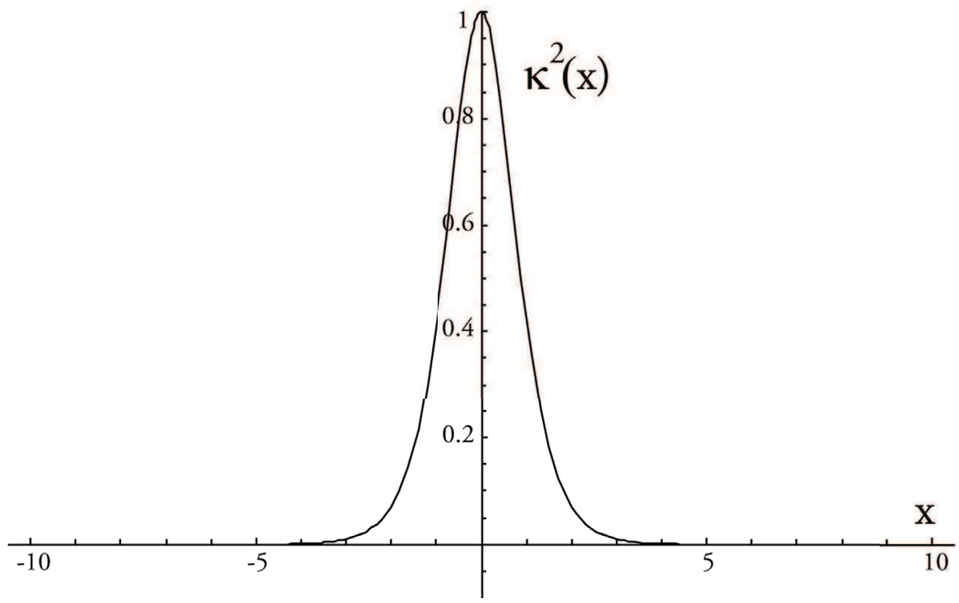

Fig. 5. The curvature $\kappa^2 = 1 - \tanh^2(x)$.

6. The Dirac equation

Let us turn to the Dirac equation. We start our discussion by considering the Dirac equation in a (3+1)-dimensional curved space-time. We apply a curved space-time formalism from the very outset in order to clearly exhibit general computational details which are not readily available elsewhere, but which are of fundamental importance in the discussion of the Dirac equation on general surfaces in three dimensional Euclidean space. Due to the spinor structure the Dirac equation is most easily formulated relative to a local vier-bein field. The massive equation relative to a given viel-bein field $e_A{}^a$ (capital latin letters denote viel-bein components while lower capital latin letters denote coordinate indices) is in general given by (1)

$$(\gamma^A(\partial_A - \Gamma_A) + m)\Psi \equiv (\gamma^A D_A + m)\Psi = 0\,. \tag{79}$$

Ψ represents the Dirac spinor field, and m represents the rest mass of the particle. The γ^A-matrices obey locally the Clifford algebra $\{\gamma^A, \gamma^B\} = 2\eta^{AB}$ with $\eta_{AB} = \mathrm{diag}(-1,1,1,1) = e_A{}^a e_{Ba}$. The matrix valued spin connection Γ_A is explicitly given by

$$\Gamma_i = -\frac{1}{4}\gamma^A\gamma^B e_A{}^j(\partial_i e_{Bj} - e_B{}^k\Gamma_{kji})\,. \tag{80}$$

$\Gamma_{\lambda\nu A}$ represents the Christoffel symbols of the second kind. The current $j^A \equiv \overline{\Psi}\gamma^A\Psi$, where $\overline{\Psi} = \Psi^\dagger\gamma^0$ ($\dagger$ signifies Hermitian conjugation), is covariantly conserved $\nabla_A j^A = 0$. The Dirac field is normalized with respect to the canonical integration measure. When $e^i_A\Gamma_i = 0$ we define $D_a = \nabla_a$. We are primarily interested in the second order form of the Dirac equation in order to make a direct comparison with Schrödinger theory on a two-dimensional surface S, or on a linear structure. However, let us begin with the first order formulation of the Dirac equation.

We may compute the Dirac equation in our coordinate system X relative to an orthonormal tetrade frame so that

$$\underline{\omega}^0 = \underline{d}t \tag{81}$$

$$\underline{\omega}^1 = G_{11}^{1/2}\underline{d}x^1 \tag{82}$$

$$\underline{\omega}^2 = G_{22}^{1/2}\underline{d}x^2 \tag{83}$$

$$\underline{\omega}^3 = \underline{d}x^3\,. \tag{84}$$

The beins may therefore be read off from the knowledge of the metric. Hence, under a change of coordinates from x to X probability conservation requires that the Dirac spinor $\psi(X) \to \psi(X)\xi^{-1/2}$ as for the Schrödinger field. We assume for simplicity that the surface has vanishing intrinsic curvature. We parametrize the surface by Cartesian coordinates. It is then easy to show that

$$\Gamma_3 = 0 \tag{85}$$

$$\Gamma_i = \frac{i}{2}\epsilon_{ij}K_{ii}\sigma_j \cdot I \tag{86}$$

where $i = 1, 2$ is summed and I is the identity for **4** spinor indices. We are now in a position to construct the covariant derivative explicitly. Since $\Gamma_3 = 0$ one sees that this is unnecessary however; there will be no contribution from the spin connection to the geometrical terms. We may therefore proceed to separate the Dirac equation into parallel and perpendicular parts. We have

$$(\gamma^A D_A + m)(\psi(X)\xi^{-1/2}) = 0 \tag{87}$$

where $A = 0, ..., 3$. Pulling ξ through the derivative we have

$$(\gamma^A D_A + \frac{1}{2}\gamma^3 \mathrm{Tr}K + m + O(x^3)^3)\psi(X) = 0\,. \tag{88}$$

We shall assume that the equation is separable and write

$$\psi(x) = \phi_\perp(x^3)\phi_{||}(x^1, x^2, t)\hat{\psi} \tag{89}$$

where $\hat{\psi}$ is a constant, four-component spinor. For later convenience we shall also define $\psi_\perp = \phi_\perp\hat{\psi}$, $\psi_{||} = \phi_{||}\hat{\psi}$, and $A^0 = A^0_\perp(x^3) + A^0_{||}(x^1, x^2, t)$ and $m = m + \Delta m(x^3)$. We then easily see that

$$\phi_{||}^{-1}(\gamma^0 D_0^{||} + \gamma^i D_i + \frac{1}{2}\gamma^3 \mathrm{Tr}K + m)\phi_{||}\hat{\psi} =$$

$$-\phi_\perp^{-1}(\gamma^0(-ieA_0^\perp) + \gamma^3\partial_3 + \Delta m(x^3) + ...O(x^3))\phi_\perp\hat{\psi}\,. \tag{90}$$

The usual argument for the separation of these equations is that the left-hand side is only a function of x^1, x^2, t while the right-hand side is only a function of x^3; thus both sides must be equal to some separation constant k. However, this is not quite correct owing to the terms $...O(x^3)^3$ which contain the curvature K_{ij}, which is a function of x^1, x^2. However, we are

assuming that the excursion into x^3 are small owing to the physics of the problem; thus we shall for the time being adopt the standard procedure and neglect these higher-order terms. To eliminate x^3 from the problem we identify $A_0^{\perp}$ with a squeezing potential $A_0^{\perp} = \frac{\lambda}{e}x^3$, so that nontrivial solutions, given by the vanishing of the determinant of the operator on the right-hand side of Eq.(90), are characterized by

$$\phi_{\perp}(x^3) \sim exp(-(\frac{\lambda}{2}(x^3)^2 + kx^3) - \int dx^3 m(x^3))\,. \tag{91}$$

The first term in the exponential falls off like a Gaussian function, i.e., a δ function in the limit of infinite λ. The presence of a nonzero separation constant spoils this falloff by introducing a straightforward exponential decay with characteristic length k^{-1}. The mass parameter $m(x^3)$, which is initially at least linear in x^3, will also give rise to a Gaussian falloff. Thus increasing the coupling of this parameter will also have the desired effect of squeezing the system. The residual Dirac equation in the surface S is thus

$$(\gamma^A D_A + \frac{1}{2}\gamma^3 \mathrm{Tr}K + (m-k))\psi_{||} = 0\,. \tag{92}$$

We have managed to derive a theory in a surface but with a number of assumptions and approximations. Let us next turn to the second order form of the Dirac equation.
We derive the second order form of the Dirac equation by 'squaring' the first order Dirac equation (20)

$$(\gamma^B D_B - m)(\gamma^A D_A + m)\Psi = 0 \tag{93}$$

$$\Rightarrow\ (\gamma^B \gamma^A D_B D_A - m^2)\Psi = 0\,. \tag{94}$$

Utilizing the algebraic identity

$$\gamma^B \gamma^A = \frac{1}{2}(\{\gamma^B, \gamma^A\} + [\gamma^B, \gamma^A])\,, \tag{95}$$

and the fundamental Clifford algebra, we can write the second order Dirac equation as

$$((\eta^{BA} + \frac{1}{2}[\gamma^B, \gamma^A])D_B D_A - m^2)\Psi = 0\,. \tag{96}$$

Note that it is not possible to extract a term corresponding to the antisymmetric part above in a purely bosonic scalar theory like the Schrödinger theory. The antisymmetric part can be written as

$$\frac{1}{2}[\gamma^B, \gamma^A]D_B D_A = \frac{1}{4}[\gamma^B, \gamma^A][D_B, D_A]\,. \tag{97}$$

The commutator between the connection components is per definition proportional to the components of the Riemannian curvature tensor R_{ABCD}

$$[D_B, D_A] = \frac{1}{8}R_{BACD}[\gamma^C, \gamma^D]\,. \tag{98}$$

Relative to the local viel-bein it is furthermore always straightforward to locally decompose the kinetic operator $D^2 = \eta^{AB}D_B D_A$ into a tangential surface component $D_{||}^2$, an normal

component $D_\perp^2$ as well as a time component. Since

$$\frac{1}{8}[\gamma^B,\gamma^A]R_{BACD}[\gamma^C,\gamma^D] = \frac{1}{2}R_{BACD}\gamma^A\gamma^B\gamma^C\gamma^D \tag{99}$$

the squared Dirac equation can thus be written as

$$(-D_t^2 + D_{||}^2 + D_\perp^2 + \frac{1}{8}R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D - m^2)\Psi = 0\,. \tag{100}$$

In the dimensional reduction approach (6) one introduces a set of coordinates adapted to the lower dimensional structure S such that the normal field to the surface is always parallel to the tangent vector field to one of the coordinates. $D_\perp^2$ will act in this direction. If one assumes that the dynamic equation in question is separable in parts depending on the coordinates in the surface and the perpendicular coordinate respectively one derives a theory which is living solely in the surface. The corresponding result in Schrödinger theory exhibits a scalar potential which depends on the Gaussian and mean curvatures.

Since we are considering a theory which is living in a space with vanishing intrinsic curvature it follows that Eq. (100) reduces to Schrödinger theory by triviality if we neglect any spin-orbit interactions stemming from the presence of the position dependent spin connection. This conclusion stands out in sharp contrast to the conclusions drawn from our elaborations of the first order form of the Dirac equation. Does this conclusion hold if we follow the intrinsically (2+1)-dimensional approach in which one defines the theory in the surface without taking the embedding space into account? This approach is employed in most current treatments of graphene, e.g. Let us explore this issue.

The standard theory for the charge carriers in graphene in the intrinsically (2+1)-dimensional approach is formulated in a three dimensional Minkowski space with the dynamic equation equal the standard three dimensional Dirac equation. We choose the $\mathbf{4}\times\mathbf{4}$-representation of the γ^A-matrices because the alternative $\mathbf{2}\times\mathbf{2}$-representation, which is available in (2+1)-dimensions, breaks parity invariance (8). In order to adapt this formulation to curved space we simply make the theory generally covariant in the usual fashion. Then the second order formulation is again Eq. (100) but with no $D_\perp^2$-term. The resulting theory will be valid at low energies and in principle on large molecules. In the case of graphene it is well known that a $U(1)$-gauge field which couples to the Dirac spinor is induced on curved surfaces due to the intrinsic curvature (se (21) e.g.). We neglect this field in the following. However, we cannot discard the intrinsic curvature tensor contribution as we could in the dimensional reduction approach. Working in (2+1)-dimensions and taking the symmetries of the curvature tensor into account it follows from some algebraic manipulations that

$$R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D = -2R\,, \tag{101}$$

where R is the Ricci curvature scalar in the static surface. In two-dimensional surfaces the Ricci scalar equals twice the Gaussian extrinsic curvature $R = 2K$. Clearly, this Ricci contribution will add to the effective geometrically induced potential in a surface formed from graphene. Inserting the relation Eq. (101) into Eq. (100) we rederive the classical Lichnerowicz formula for the Dirac operator (22).

Let us make a direct comparison with Schrödinger theory by looking at the low energy limit of the intrinsic massive Dirac theory. We neglect all the spin-connection terms. Without these

terms the Dirac equation reduces to

$$(-\partial_t^2 + \nabla_{||}^2 - \frac{1}{2}K - m^2)\psi = 0\,, \tag{102}$$

where ψ is assumed to be a definite spin-state, and $\nabla_{||}$ is the usual covariant derivative acting on a scalar function. We assume energy-eigenstates and denote the total energy of the particle by E. The equation above then takes the following form to first order in a standard $1/m$-expansion (reinstating $\hbar$)

$$E_c\psi \equiv (E-m)\psi = (-\frac{\hbar^2}{2m}\nabla_{||}^2 + \frac{\hbar^2}{2m}K)\psi\,. \tag{103}$$

This is the Schrödinger equation corresponding to the second order Dirac equation, and E_c is the classical energy measure. An effective potential V_D has emerged. It is given by (reinstating $\hbar$)

$$V_D = \frac{\hbar^2}{2m}K\,. \tag{104}$$

Comparing V_D and V_S we see a huge formal difference. Schrödinger theory predicts a negative, or an attractive, potential on any curved surface. However, the sign of K is not restricted. Hence, contrary to Schrödinger theory, intrinsic second order Dirac theory predicts both attractive as well as repulsive geometrically induced quantum potentials depending on the surface in question.

In (23) it was pointed out that on a carbon nanotube the *winding* states in the angular direction will give rise to an effective mass. It is straightforward to see how this will work in our case. Consider first the massless Dirac theory with $m = 0$. This theory does not have a natural low energy limit and the relation between the Dirac and the Schrödinger approaches becomes highly problematic in general. However, assume a perfectly *straight* carbon nanotube with an intrinsic geometry given by $ds^2 = dz^2 + R^2 d\theta^2$, where $-\infty \leq z \leq +\infty$ and $\theta \in [0, 2\pi\rangle$. Assuming momentum eigen-states in the angular direction $\psi \sim e^{in\theta}$ (where n represents integers) Eq. (102) (with $m = 0$) becomes

$$(-\partial_t^2 + \partial_z^2 - \frac{1}{2}K - (\frac{n}{R})^2)\psi = 0\ ,\ |n| \in 0, 1, 2, 3, \ldots\,. \tag{105}$$

Clearly, the last term in Eq. (105) can be identified as a mass parameter so that

$$m = \frac{|n|}{R}\,. \tag{106}$$

Since R is on the nanoscale, $1/m$ so defined ($n \neq 0$) is an effective parameter which can be used in the series expansion leading to Eq. (103). Of course, this scheme can also be employed in the dimensional reduction scheme. The effective theory then becomes

$$E_c\psi = (-\frac{1}{2m}\partial_z^2 + \frac{1}{2mR^2}(n^2-1))\psi\,. \tag{107}$$

Here m is the ordinary rest mass of the particle. These results can be utilized as tools to discriminate between the dimensional reduction scheme and the intrinsic Dirac approach. Let us pursue this topic a little bit further.

Assume that the ends of a straight carbon nanotube is welded to two metal slabs (20). Let us for simplicity neglect the physics at the welding areas such that we can assume a free Dirac theory in the metal and the welding areas. Since $K = 0$ on the tube the Dirac theory implies that a current with all the electrons in the lowest winding state $n = 0$ can be transmitted from one of the metal pieces and through the tube without backscattering (reflection). This is not the case according to the dimensional reduction approach. The $n = 0$ states will form bound states implying a net reduction in the net current in the tube. Clearly, when $n = \pm 1$ the dimensional reduction approach also implies a vanishing net potential. However, the dominant transmission channel should correspond to the lowest lying winding mode $n = 0$ due to the relatively large mass scale in the system. Hence, to leading order the two formulations will imply different currents through a straight carbon nanotube connecting two metal slabs. This argument is a very simple one, but the conclusion is robust. In particular, even though one cannot neglect the physics in the welding areas in general (24) the relative difference between the dimensional reduction and the intrinsic Dirac approaches can nevertheless be deduced since the effective potential on the tube in the dimensional reduction picture scales as $1/R^2$. Hence, making the very reasonable assumption that the physics in the welding areas is the same for tubes with differing radii the presence of a geometry induced potential can be extracted. Clearly, for a tube with a large radius $R \to \infty$ the effective potential will vanish while it blows up as $R \to 0$. The intrinsic Dirac theory will not exhibit a similar scaling behavior in any state. Furthermore, note that the effective potential in Eq. (107) changes sign at $n^2 = 1$. Hence, for $|n| \geq 2$ the effective potential becomes repulsive. If real, these differences should be readily observable experimentally.

Of course, the discussion above is not confined to graphene. It should be valid for any surface. As another application let us briefly consider the rolled up nanotubes (RUNTs) discussed in (25). RUNTs are generally made of bi- or multilayer thin films of various materials. A flat thin film might, due to the relaxation of the elastic stresses, curl up and form a RUNT described by an Archimedes spiral. Physically it has the same symmetries as the straight carbon nanotubes considered above, but instead of curling up to form a closed cylinder the film curls to form a structure similar to a book scroll. In (25) the charge carriers were described as an exact two dimensional electron gas within the dimensional reduction framework. Clearly, the Gaussian curvature on the Archimedes spiral vanishes, but not the mean curvature. It was shown in (25) that the non-vanishing mean curvature will result in a number of atomic-like bound states in the spiral surface. The number of bound states equals the number of windings of the spiral. How does this relate to an an intrinsic Dirac theory description? In thin films we are not dealing with massless theory so that $m \neq 0$ in Eq. (102). We are therefore not dependent upon either a doped material, nor a periodic structure in order to deduce a low energy limit as on the fullerenes. We can thus employ Eq. (103) directly. Hence, no geometric potential will appear on the RUNT, and consequently no bound states, since $K = 0$. Just as with the straight carbon nanotube above, measuring for the existence or non-existence of bound states on rolled up nanotubes could prove a veritable tool for discriminating between the different descriptions of quantum mechanics on surfaces.

Even though much work have been done on the quantum mechanics on many different surfaces a complete analytical analysis of quantum mechanics on the simple torus is still missing, even though this surface is an important one. Next we will therefore provide some basic equations as a start for such an analysis.

We will assume the following parametrization of the torus

$$x = (c + a\cos v)\cos u\,, \tag{108}$$

$$y = (c + a\cos v)\sin u\,, \tag{109}$$

$$z = a\sin v\,, \tag{110}$$

where $u, v \in [0, 2\pi\rangle$. a signifies the 'small' radius and c the 'big' radius of the torus such that $c > a$. The coordinates are depicted in Fig. (6). The geometry on the torus can then be expressed as

$$ds^2 = (c + a\cos v)^2 du^2 + a^2 dv^2 \equiv A^2 du^2 + a^2 dv^2\,. \tag{111}$$

The extrinsic curvature is explicitly given by

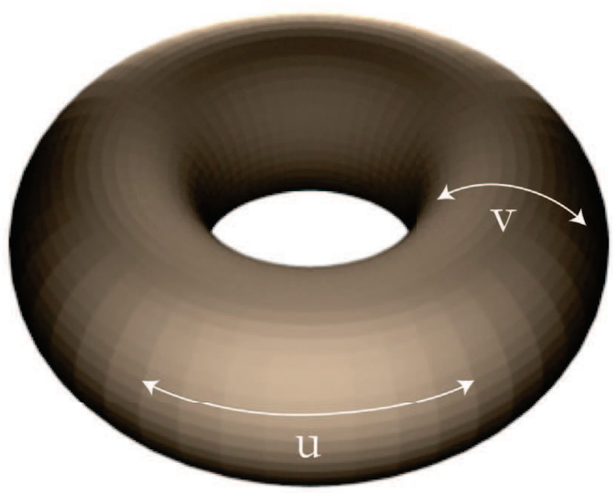

Fig. 6. The torus. The letters denote the angular coordinates.

$$K = \frac{\cos v}{a(c + a\cos v)}\,. \tag{112}$$

Let us consider stationary quantum energy eigenstates on the torus. Due to the non-dependence of the u-coordinate in the metric we also consider momentum eigenstates in the u-direction. We call these states winding states when this momentum is non-zero. Hence, we assume that the wave-function is on the form

$$\Phi \sim e^{-iEt} e^{imu} A^{-1/2} W(v)\ ,\ m \in \{0, \pm 1, \pm 2, ...\} \tag{113}$$

where W is a function to be determined by the wave-equation. With this ansatz the wave-equation reduces, quite remarkably, to the simple form

$$(-\frac{\partial^2}{\partial v^2} + V_e(v) - E^2) W(v) = 0\,, \tag{114}$$

where the effective potential V_e is given by

$$V_e(v) = -(\frac{1}{4}(\partial_v A)^2 + a^2 m^2)A^{-2} + \frac{1}{2}A^{-1}\partial_v^2 A$$
$$= \frac{-a^2}{(c + a\cos v)^2}((\frac{\sin v}{2})^2 - m^2) - \frac{a\cos v}{2(c + a\cos v)} - \frac{\cos v}{2a(c + a\cos v)}\,. \tag{115}$$

A plot of the effective potential is provided in Fig. (7). If the contribution from the extrinsic curvature is discarded in the expression for the effective potential it still has the same qualitative form as in Fig. (7).

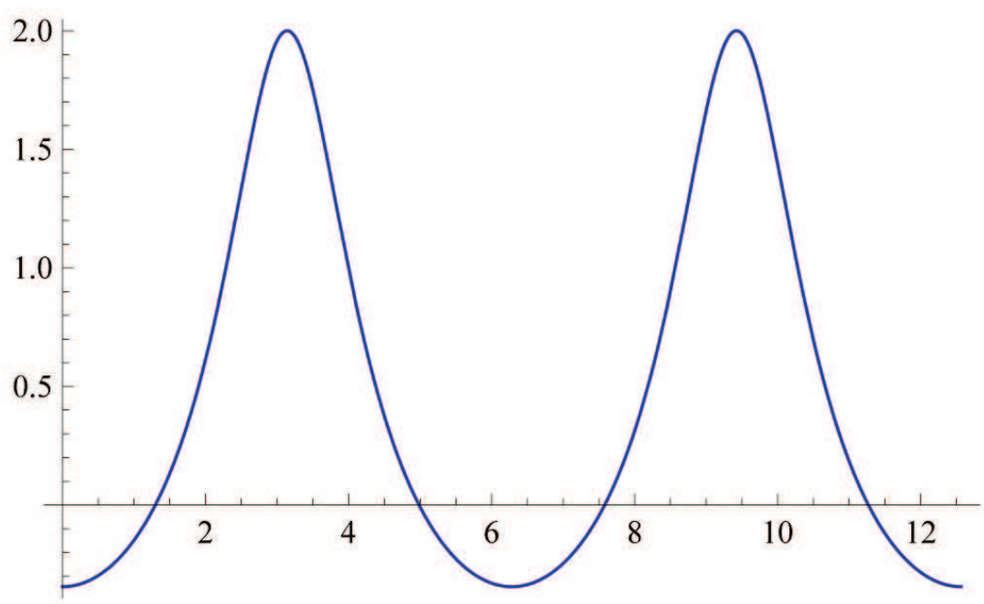

Fig. 7. The effective potential V_e with $m = \pm 1$ in the interval $u \in [0, 4\pi]$. It has the same qualitative features irrespective of the parameter values. Negative energy states seemingly exist only for states with $m \in \{0, \pm 1, \pm 2\}$.

Let us finally make a brief comment on Dirac theory on linear structures. Clearly, neglecting the spin-connection the extrinsically defined theory will reproduce Schrödinger theory as in the case of surfaces. Considering an intrinsically defined theory the second order formulation in Eq. (100) is again valid. However, now the Riemann curvature tensor is trivially identically zero when we consider static linear structures since the geometry will be time independent. This means that intrinsically defined fermions on static linear structures will not experience a spin induced potential. However, note that if the structure exhibits a time dependent geometry this is no longer true. This is easily seen already on the algebraic level since now the right hand side of Eq. (101) is non-zero with changed overall sign

$$R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D = 2R\,. \tag{116}$$

This sign flip is induced by the Clifford algebra. Explorations of the implications of this change of sign is left for future work.

7. Conclusion

In this brief account of quantum mechanics on surfaces we introduced the reader to the dimensional reduction approach which is currently the most widely accepted method to construct quantum theories on surfaces and linear structures embedded in ordinary three-dimensional Euclidean space. We generalized this framework to accommodate Dirac theory. We studied quantum theory on explicit geometries; the catenoid in Schrödinger theory and the cylinder in Dirac theory. In addition we presented novel approaches to the study of the interplay between quantum physics and geometry in the form of a conformal approach to defining quantum theory on a surface and by applying ideas from supersymmetric quantum mechanics. Both of these approaches need further elaboration and refinements, but it is the hope of this author that this material might inspire other workers in the field to pursue these new avenues.

This account of quantum mechanics on surfaces also makes an effort to put our subject at the center stage of current physics research. Quantum theory on surfaces might in particular be important in order to get a complete understanding of graphene. We have shown that there is a major discrepancy between the formulation of Dirac theory on surfaces depending on whether one employs an extrinsic or an intrinsic approach. One immediate consequence is that the intrinsic theory implies a new effective scalar potential proportional to the Gaussian curvature in the surface, while the extrinsic approach implies the usual potential stemming from the dimensional reduction approach (6). The new potential emerges due to the Clifford algebra and will thus not be present in any scalar theory defined on surfaces. We also looked at the low energy limit of the intrinsic Dirac theory and derived an effective potential V_D which corresponds to the effective potential stemming from the Schrödinger (or the Dirac) theory in the extrinsic approach (V_S). Clearly, while V_S is strictly non-positive the potential stemming from the low energy intrinsic Dirac theory V_D can carry any sign depending on the surface in question. Graphene is described by the massless Dirac equation near the Dirac points. Hence, near these points the charge carriers might respond very differently to the graphene surface geometry than one would expect from Schrödinger theory. This insight might be of great importance in the modeling of the charge carriers on graphene with consequent experimental and technical implications. We considered one particular technical implication for a system composed of a carbon nanotube bridging two metal slabs. We also considered a particular rolled up nanotube in the same vain with qualitatively the same conclusions as with the carbon tube. We emphasize that we have ignored the effect of the spin-connection. This will induce spin-orbit couplings which will add motion dependent potentials. The effect of these can be very pronounced (26). Clearly, much more work needs to be done in this field, but it is work which apparently has potential to further advance recent discoveries in physics.

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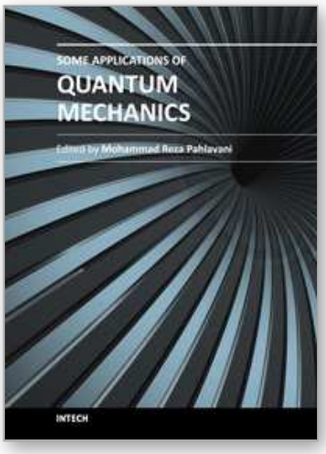

Some Applications of Quantum Mechanics

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Quantum mechanics, shortly after invention, obtained applications in different area of human knowledge. Perhaps, the most attractive feature of quantum mechanics is its applications in such diverse area as, astrophysics, nuclear physics, atomic and molecular spectroscopy, solid state physics and nanotechnology, crystallography, chemistry, biotechnology, information theory, electronic engineering... This book is the result of an international attempt written by invited authors from over the world to response daily growing needs in this area. We do not believe that this book can cover all area of application of quantum mechanics but wish to be a good reference for graduate students and researchers.

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University Campus STeP Ri
Slavka Krautzeka 83/A
51000 Rijeka, Croatia
Phone: +385 (51) 770 447
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