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S-Duality in Topological Supergravity

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1. Introduction

One of the main motivations for constructing a model of topological gravity in *three dimensions* (3D) is that it might serve as a 'laboratory' for applying techniques appearing rather awkward or even intractable in four dimensions. This stems from the fact that a Riemannian spacetime is Ricci-flat, i.e., the Ricci tensor determines the Riemann tensor in 3D and as a result, the only vacuum solutions of the Einstein equations with vanishing cosmological constant are flat. This result implies that the dynamical properties may not be attributed to the metric but rather to the coframe. When matter is included there are nontrivial solutions to the Einstein equations and if topological terms are included, these may induce dynamical properties in 3D. Such a 'laboratory' may no longer be a suitable testing ground for higher–dimensional models of Einsteinian gravity [5, 10, 18, 36].

There are other reasons for studying the dynamical aspects of topological gravity in three dimensions: Some problems in 4D gravity reduce to an effective 3D theory, such as cosmic strings, the high–temperature behavior of 4D theories and some membrane models of extended relativistic systems. Moreover, many aspects of black hole thermodynamics can be effectively reduced to problems in 3D, cf. Refs. [6, 7].

Outside of quantum gravity, the continuum theory of lattice defects in crystal physics can be regarded as 'analogue gravity' with Cartan's torsion in 3D, where such defects are modeled by connections in the orthonormal frame bundle and the Chern-Simons type free-energy integral by Riemann–Cartan (RC) spaces with constant torsion [11, 26]. Recently, flexural modes of graphene have also been considered as membranes with a 'gravitational' metric [25] or coframe induced from its embedding into three-dimensional spacetime.

Our paper is organized as follow: In Section 2, we give a brief introduction to the Mielke-Baekler (MB) model of toplogical gravity in 3D, in which the Einstein-Cartan Lagrangian is substituted by a *mixed* topological term, the so-called *mix*-model. The coupling of Rarita-Schwinger fields to topological gravity is presented in Section 3, whereas in Section 4 we deduce the restrictions on the coupling parameters in order to ensure that the model is supersymmetric. The particular dynamical symmetry of the MB model, in Ref. [32] dubbed "S–duality", is generalized in Section 5 to our topological supergravity model. In Section 6 and in an Outlook, we consider the still speculative applicability of this model to the

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	objects	p-forms	components	n=4	3	2
ϑ^α	vector	1	n^2	16	9	1
$\Gamma^\star_\alpha$	vector	1	n^2	16	9	1
T^α	vector	2	$n^2(n-1)/2$	24	9	2
$R^{\alpha\beta}$	bivector	2	$n^2(n-1)^2/4$	36	9	1
Σ_α	vector	$n-1$	n^2	16	9	4
$\tau_{\alpha\beta}$	bivector	$n-1$	$n^2(n-1)/2$	24	9	2
η_α	vector	$n-1$	n^2	16	9	4

Table 1. Geometrical objects and fields

flexural modes of corrugated surfaces (2D membranes) embedded in 3D spacetime, as recently realized by the rather prospective new material called *graphene*.

2. Topological gravity with torsion

In three spacetime dimensions, the basic gravitational variables in the Riemann-Cartan (RC) formalisms are the one–forms of the coframe and the Lie dual of the (Lorentz-) rotational connection $\Gamma^{\beta\gamma} = \Gamma_j{}^{\beta\gamma} dx^j$, i.e.,

$$\vartheta^\alpha = e_i{}^\alpha dx^i \quad \text{and} \quad \Gamma^\star_\alpha := \frac{1}{2}\eta_{\alpha\beta\gamma}\Gamma^{\beta\gamma}. \tag{1}$$

The related field strengths are the two–forms of torsion

$$T^\alpha := d\vartheta^\alpha - (-1)^s \eta^{\alpha\beta} \wedge \Gamma^\star_\beta \tag{2}$$

and curvature

$$R^\star_\alpha = \frac{1}{2}\eta_{\alpha\beta\gamma}R^{\beta\gamma} := d\Gamma^\star_\alpha + \frac{(-1)^s}{2}\eta_{\alpha\beta\gamma}\Gamma^\star_\beta \wedge \Gamma^\star_\gamma, \tag{3}$$

respectively, cf. the Appendices of Ref. [31]. Table 1 contains a summary of these basic variables and their components in various dimensions. Observe that only for $n = 3$ all fields have the same number of components. After converting bivectors into vectors via the Lie dual, a linear combination of all variables could pave the way for a better understanding of topological models.

In 3D, the Einstein-Cartan (EC) Lagrangian

$$L_{\text{EC}} := -\frac{\chi}{\ell}\vartheta^\alpha \wedge R^\star_\alpha \equiv -\chi\,C_{\text{TL}} - \frac{\chi}{\ell}\,d(\Gamma^\star_\alpha \wedge \vartheta^\alpha) \tag{4}$$

merely gives rise to a locally trivial dynamics [38]. This is due to its equivalence to a 'mixed' Chern-Simons type term C_{TL} plus a total divergence, as indicated above.

In this paper, we generalize this trivial dynamics by adding Chern-Simons (CS) type terms, following the lead of Witten [43]. By gauging the Poincaré group $\mathbb{R}^3 \rtimes SO(1,2)$, we arrive at the Mielke and Baekler (MB) model [2, 28] which is at most *linear* in the field strengths. This is slightly modified here by replacing L_{EC} via the 'mixed' Chern-Simons type term C_{TL}, which is simulating, in 3D, to some extend Einstein's theory with 'cosmological' term, as is indicated above. Thereby, we are able to depart from a completely topological theory.

Allowing for arbitrary "vacuum angles" θ_T, θ_L and $\theta_{TL} = -\chi$, the most general purely *topological* gravity Lagrangian in 3D, in first order formalism, takes the form

$$L_{MB}(\vartheta^\alpha, \Gamma^\star_\alpha) = \theta_T C_T + \theta_L C_L + \theta_{TL} C_{TL}, \tag{5}$$

where

$$C_T := \frac{1}{2\ell^2} \vartheta^\alpha \wedge T_\alpha, \qquad C_L := (-1)^s \Gamma^{\star\alpha} \wedge R^\star_\alpha - \frac{1}{3!} \eta_{\alpha\beta\gamma} \Gamma^{\star\alpha} \wedge \Gamma^{\star\beta} \wedge \Gamma^{\star\gamma} \tag{6}$$

and

$$C_{TL} := \frac{1}{\ell} \left(\Gamma^{\star\alpha} \wedge T_\alpha - \frac{(-1)^s}{2} \eta_{\alpha\beta\gamma} \Gamma^{\star\alpha} \wedge \Gamma^{\star\beta} \wedge \vartheta^\gamma \right), \tag{7}$$

respectively, are the translational, rotational and 'mixed' Chern-Simons type three forms of gauge type $C = \mathrm{Tr}\{A \wedge F\}$ in RC spacetime [8, 12, 43]. The equation (5) is the known topological Lagrangian of the Mielke-Baekler (MB) *mix*-model [28, 31]. Since the translational term C_T is covariant, it appears that the MB model is semi-topological, with interesting consequence on the degrees of propagating modes, cf. Ref. [4, 32, 36].

Varying the Lagrangian (5) with respects to ϑ^α and $\Gamma^{\star\alpha}$ and employing the results of Appendix A, yields the topological field equations

$$-\theta_{TL} R^\star_\alpha - \frac{\theta_T}{\ell} T_\alpha = \ell \Sigma_\alpha, \tag{8}$$

and

$$-(-1)^s \theta_{TL} T_\alpha - \frac{\theta_T}{2\ell} \eta_\alpha - \theta_L \ell R^\star_\alpha = \ell \tau^\star_\alpha, \tag{9}$$

cf. Eq. (6.9) of Ref. [2]. Observe that the translational CS term proportional to θ_T induces in the second field equation a constant term, familiar in 4D from Einstein's equation with cosmological constant Λ.

Thereby, combining the *vacuum* field equations (9) and (8) yield for the torsion and the RC curvature the constrictions:

$$T_\alpha = \frac{2\kappa}{\ell} \eta_\alpha, \qquad R^\star_\alpha = \frac{\rho}{\ell^2} \eta_\alpha \tag{10}$$

where the contortional constants $\kappa = \theta_{TL}\theta_T/2A$ and $\rho = -\theta_T^2/A$ are related to the vacuum angles. A singular case is exclude by assuming that $A =: -(-1)^s\theta_{TL}^2 + 2\theta_T\theta_L \neq 0$.

When including matter couplings, we explicitly find for the torsion

$$T_\alpha - \frac{2\kappa}{\ell} \eta_\alpha = \frac{2}{A} \ell \left(\theta_{TL}\tau^\star_\alpha - \theta_L \ell \Sigma_\alpha \right), \tag{11}$$

and the RC curvature

$$R^\star_\alpha - \frac{\rho}{\ell^2} \eta_\alpha = \frac{2}{A} \left(\theta_{TL} \ell \Sigma_\alpha - \theta_T \tau^\star_\alpha \right), \tag{12}$$

cf. Ref. [31].

3. Rarita–Schwinger Lagrangian in 3D

Commonly, *supergravity* [15, 19] with one supersymmetry generator, i.e. $\mathcal{N}=1$, represents the simplest consistent coupling of a Rarita–Schwinger (RS) spin–$\frac{3}{2}$ field [35] to gravity.

The Rarita-Schwinger [35] type spinor-valued one-form[1]

$$\Psi = \Psi_i dx^i = \Psi_\alpha \vartheta^\alpha \tag{13}$$

can be written holononically and anholononically. However, it does not depend on the coframe, inasmuch $\Psi_\alpha := e_\alpha \rfloor \Psi$ involves the inverse tetrad. In 3D, we adhere to the conventions that the holonomic indices run from $i, j, k, \ldots = 0, 1, 2$, whereas $\alpha, \beta, \ldots = \hat{0}, \hat{1}, \hat{2}$ for the anholonomic indices.

We are going to provide a brief summary of the spinors that will be used in three dimensions: As well known, the covering group of the rotation group $SO(3)$ is isomorphic to the unitary group $SU(2)$. Since an element of $SU(2)$ can be parameterized by three numbers, the most convenient basis of the Lie algebra are the familiar Pauli spin matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

These matrices satisfy the following Lie algebra:

$$\left[\sigma^\alpha, \sigma^\beta \right] = 2i\eta^{\alpha\beta\gamma}\sigma_\gamma. \tag{15}$$

However, for Lorentzian signature $s = 1$, the covering group of $SO(1, 2)$ is isomorphic to the real group $SL(2, \mathbb{R})$. Then the generators of $SL(2, \mathbb{R})$ may be realized by the matrices

$$\gamma_0 = i\sigma^2, \quad \gamma_1 = \sigma^1, \quad \gamma_2 = \sigma^3. \tag{16}$$

These real matrices [27] satisfying

$$\gamma_\alpha \gamma_\beta = g_{\alpha\beta} + \eta_{\alpha\beta\nu}\gamma^\nu \tag{17}$$

also provide a realization of the Clifford algebra

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2g_{\alpha\beta} \tag{18}$$

in 3D. In addition, the coframe basis ϑ^α converts into one Clifford algebra value one-form

$$\gamma = \gamma_\alpha \vartheta^\alpha \tag{19}$$

Then Ψ will become real two-component spinors, with the Dirac adjoint defined by $\overline{\Psi} := \Psi^\dagger \gamma^0$.

---

[1] In four dimensions (4D), the Rarita–Schwinger field $\Psi := \Psi_\alpha \vartheta^\alpha$ entering Eq. (13) is a *Majorana spinor* valued one-form. As it is well known [34], it satisfies the Majorana condition, i.e. $\Psi = C\overline{\Psi}^t$, where C is the charge conjugation matrix given by $C = -i\gamma_0$ satisfying $C^\dagger = C^{-1}$, $C^t = -C$ and $C^{-1}\gamma^\alpha C = -(\gamma^\alpha)^t$. Consequently,
$$\overline{\Psi} \wedge \Psi = 0, \quad \overline{\Psi} \wedge \gamma_5 \gamma^\alpha \Psi = 0, \quad \overline{\Psi} \wedge \gamma_5 \Psi = 0$$
For the real Majorana representation all γ^α are purely imaginary and the components of the gravitino vector–spinor consequently are all real [30].

The corresponding *manifestly Hermitian* RS type Lagrangian three–form of Howe and Tucker [23] reads

$$L_{\text{RS}} = \frac{i}{4}\left(\overline{\Psi} \wedge D\Psi - \Psi \wedge \overline{D\Psi}\right) + \frac{i}{4}\,m\overline{\Psi} \wedge \gamma \wedge \Psi, \tag{20}$$

including, however, a mass term. Here minimal coupling to gravity is achieved via

$$D\Psi = d\Psi - \frac{1}{2}\gamma_\alpha \Gamma^{\star\alpha} \wedge \Psi, \tag{21}$$

which is nothing more than the gauge covariant derivative of a spinor-valued one-form Ψ.

Only in 3D, however, there exists a generalization given by the following expression

$$L_\Psi = L_{\text{RS}} + s_1\,\overline{D\Psi} \wedge {}^*(D\Psi) + s_2\,\overline{D\Psi} \wedge \gamma \wedge {}^*(\gamma \wedge D\Psi). \tag{22}$$

As in the case of the Rarita-Schwinger Lagrangian L_{RS}, it is manifestly Hermitian when the additional quadratic derivative terms carry s_1 and s_2 as dimensionless coupling constants.

In order to supersymmetrize this action, it will be coupled to topological gravity later on.

3.1 Energy-momentum and spin currents

By definition, the energy-momentum current two-form Σ_α of matter is given by

$$\Sigma_\alpha := \frac{\delta L_\Psi}{\delta \vartheta^\alpha} = \frac{\partial L_\Psi}{\partial \vartheta_\alpha} + D\frac{\partial L_\Psi}{\partial T^\alpha}, \tag{23}$$

where the second term accounts for the possibility of a non-minimal coupling to torsion via Pauli type terms, cf. Eq. (5.1.8) of Ref. [22]. According to the Noether theorem, the energy-momentum current two-form of matter Σ_α without Pauli terms can be rewritten as

$$\Sigma_\alpha := e_\alpha \rfloor L_\Psi - (e_\alpha\rfloor\Psi) \wedge \frac{\partial L_\Psi}{\partial \Psi} - (e_\alpha\rfloor\overline{\Psi}) \wedge \frac{\partial L_\Psi}{\partial \overline{\Psi}} - (e_\alpha\rfloor D\Psi) \wedge \frac{\partial L_\Psi}{\partial D\Psi} - (e_\alpha\rfloor D\overline{\Psi}) \wedge \frac{\partial L_\Psi}{\partial D\overline{\Psi}}, \tag{24}$$

see Eq. (5.4.11) of Ref. [22] for details. This equivalent equation often is more convenient, since it involves only partial derivatives of the matter fields and avoids the intricate treatment of a possible dependence of the matter Lagrangian on the Hodge dual. Taking into account the identities of Appendix B, we find

$$\Sigma_\alpha = -\frac{i}{4}m\overline{\Psi} \wedge \gamma_\alpha\Psi + s_1\left\{\overline{D\Psi} \wedge e_\alpha\rfloor{}^*(D\Psi) - (e_\alpha\rfloor D\Psi) \wedge {}^*\left(\overline{D\Psi}\right)\right\}$$
$$+ s_2\left[\overline{D\Psi} \wedge \gamma_\alpha \wedge {}^*(\gamma \wedge D\Psi) - (e_\alpha\rfloor D\Psi) \wedge {}^*\left(\overline{D\Psi} \wedge \gamma\right) \wedge \gamma\right]. \tag{25}$$

Since the kinetic terms in the Rarita-Schwinger type Lagrangian L_{RS} do not depend explicitly on the coframe ϑ^α, they provides no contribution to the energy-momentum current.

The 3-dual of the spin current is defined by

$$\tau_\alpha^\star := \frac{1}{2}\eta_{\alpha\beta\gamma}\tau^{\beta\gamma} = \frac{(-1)^s}{2}\frac{\delta L_\Psi}{\delta\Gamma_\alpha^\star}. \tag{26}$$

In view of the definition (21) of the covariant derivative, we find

$$\tau_\alpha^\star = \frac{(-1)^s}{2} \left\{ \frac{i}{4} \overline{\Psi} \gamma_\alpha \Psi + \frac{s_1}{2} \left[\overline{\Psi} \gamma_\alpha \wedge {}^*(D\Psi) + \gamma_\alpha \Psi \wedge {}^* \left(\overline{D\Psi} \right) \right] \right.$$
$$\left. + \frac{s_2}{2} \left[\overline{\Psi} \gamma_\alpha \wedge \gamma \wedge {}^*(\gamma \wedge D\Psi) + \gamma_\alpha \Psi \wedge \gamma \wedge {}^* (\gamma \wedge D\Psi) \right] \right\}. \tag{27}$$

Using the Hermetian properties of the spinor-valued p–forms, we finally obtain

$$\tau_\alpha^\star = \frac{(-1)^s}{2} \left[\frac{i}{4} \overline{\Psi} \wedge \gamma_\alpha \Psi + s_1 \overline{\Psi} \gamma_\alpha \wedge {}^*(D\Psi) + s_2 \overline{\Psi} \gamma_\alpha \wedge \gamma \wedge {}^*(\gamma \wedge D\Psi) \right], \tag{28}$$

cf. the identities of Appendix C.

It should be noted that for the pure Rarita-Schwinger Lagrangian with $s_1 = s_2 = 0$, the energy-momentum current is proportional to its dual spin, i.e.

$$\Sigma_\alpha = -(-1)^s 2m \tau_\alpha^\star . \tag{29}$$

4. Topological supersymmetry in 3D

Let us consider the first order topological Lagrangian

$$L_\infty = L_\infty(\vartheta^\alpha, \Gamma_\alpha^\star, \Psi) = L_{\mathrm{MB}} + L_\Psi \tag{30}$$

and verify if it is supersymmetric or not: The variation of its independent variables $(\vartheta^\alpha, \Gamma_\alpha^\star, \Psi)$ yields

$$\delta L = \delta \vartheta^\alpha \wedge \frac{\delta L}{\delta \vartheta^\alpha} + \delta \Gamma_\alpha^\star \wedge \frac{\delta L}{\delta \Gamma_\alpha^\star} + \delta \overline{\Psi} \wedge \frac{\delta L}{\delta \overline{\Psi}} \tag{31}$$

where, for convenience, it suffices to vary only for the Dirac adjoint $\overline{\Psi}$.

The supersymmetric transformation of Deser [13, 14] read in exterior form notation

$$\delta_{\mathrm{susy}} \vartheta^\alpha = i\overline{\sigma} \Psi \gamma^\alpha, \qquad \delta_{\mathrm{susy}} \Gamma_\alpha^\star = i\overline{\sigma} \gamma_\alpha^* D\Psi + ic\overline{\sigma} \left(\gamma_\alpha \Psi + e_\alpha \rfloor {}^* \Psi \right),$$
$$\delta_{\mathrm{susy}} \Psi = 2D\sigma + c\gamma\sigma, \tag{32}$$

where σ stands in for a spinor valued zero form and c a real constant. Inserting this into Eq. (31) yields

$$\delta_{\mathrm{susy}} L = i\overline{\sigma} \Psi \gamma^\alpha \wedge \frac{\delta L}{\delta \vartheta^\alpha} + \delta_{\mathrm{susy}} \Gamma_\alpha^\star \wedge \frac{\delta L}{\delta \Gamma_\alpha^\star} + \left(2\overline{D\sigma} + c\overline{\sigma}\gamma \right) \wedge \frac{\delta L}{\delta \overline{\Psi}}, \tag{33}$$

where we used $\overline{c\gamma\sigma} = c\overline{\sigma}\gamma$ for the Dirac adjoint.

In the following, we assume that the second field equation $\delta L/\delta \Gamma_\alpha^\star \cong 0$ is fulfilled "on shell", i.e., Eq. (9) of the 'mixed' MB model. Then, the SUSY transformation reduce to

$$\delta_{\mathrm{susy}} L \cong \overline{\sigma} \left(i\gamma^\alpha \Psi \wedge \frac{\delta L}{\delta \vartheta^\alpha} - 2D \wedge \frac{\delta L}{\delta \overline{\Psi}} + c\gamma \wedge \frac{\delta L}{\delta \overline{\Psi}} \right) + 2d \left(\overline{\sigma} \wedge \frac{\delta L}{\delta \overline{\Psi}} \right) \tag{34}$$

Let us restrict for the moment to the usual Rarita-Schwinger Lagrangian L_{RS}, or equivalently to L_Ψ with $s_1 = s_2 = 0$. Then the Rarita-Schwinger equation

$$\frac{2}{i} \frac{\delta L}{\delta \Psi} = D\Psi + \frac{1}{2} m\gamma \wedge \Psi \cong 0 \tag{35}$$

becomes massive. Moreover, in Eq. (34) the term in brackets following form the supersymmetric transformations reads

$$i\gamma^\alpha \Psi \wedge \frac{\delta L}{\delta \vartheta^\alpha} + c\gamma \wedge \frac{\delta L}{\delta \overline{\Psi}} - 2D\frac{\delta L}{\delta \overline{\Psi}}$$

$$\cong i\gamma^\alpha \Psi \left(\frac{\theta_{\text{TL}}}{\ell} R_\alpha^\star + \frac{\theta_{\text{T}}}{\ell^2} T_\alpha + \Sigma_\alpha \right) + c\gamma \wedge \left(\frac{i}{2} D\Psi + \frac{i}{4} m\gamma \wedge \Psi \right)$$

$$- D\left(iD\Psi + \frac{i}{2} m\gamma \wedge \Psi \right)$$

$$= i\gamma^\alpha \Psi \left(\frac{\theta_{\text{TL}}}{\ell} R_\alpha^\star + \frac{\theta_{\text{T}}}{\ell^2} T_\alpha \right) + \gamma^\alpha \Psi \left(\frac{1}{4} m\overline{\Psi}\gamma_\alpha \Psi \right) \tag{36}$$

$$+ c\gamma \wedge \left(\frac{i}{2} D\Psi + \frac{i}{4} m\gamma \wedge \Psi \right) - iR_\alpha^\star \gamma^\alpha \Psi - \frac{i}{2} mT_\alpha \gamma^\alpha \Psi + \frac{i}{2} m\gamma \wedge D\Psi$$

By a Fierz rearrangement, i.e.,

$$\gamma^\alpha \Psi \wedge \overline{\Psi}\gamma_\alpha \Psi = 0, \tag{37}$$

terms arising from the energy-momentum current Σ_α, or likewise from the dual spin $\tau_\alpha^\star$, are vanishing.

Moreover, in our restricted model with $s_1 = s_2 = 0$ we have to put

$$c = -m, \tag{38}$$

in order to eliminate the kinetic $\gamma \wedge D\Psi$ terms. Then, using the formula

$$\gamma \wedge \gamma = -2\gamma^\alpha \eta_\alpha, \tag{39}$$

of Howe and Tucker [23], we find from Eq. (36) the requirement

$$i\left[\left(\frac{\theta_{\text{TL}}}{\ell} - 1 \right) R_\alpha^\star + \left(\frac{\theta_{\text{T}}}{\ell^2} - \frac{m}{2} \right) T_\alpha + \frac{m^2}{2} \eta_\alpha \right] \gamma^\alpha \Psi = 0, \tag{40}$$

in order that our Lagrangian becomes supersymmetric.

At first sight, it appears that there is no cosmological constant in order to compensate a similar one arising from the RS mass. However, one should compare the bracket with the second field equation (9) *inserted*, which indeed involves a cosmological term induced by the translational Chern-Simons term proportional to θ_{T}. In this insertion

$$i\left[\left(\theta_{\text{L}} + \frac{\theta_{\text{TL}}}{\ell} - 1 \right) R_\alpha^\star + \left((-1)^s \frac{\theta_{\text{TL}}}{\ell} + \frac{\theta_{\text{T}}}{\ell^2} - \frac{m}{2} \right) T_\alpha + \frac{1}{2} \left(\frac{\theta_{\text{T}}}{\ell^2} + m^2 \right) \eta_\alpha + \tau_\alpha^\star \right] \gamma^\alpha \Psi = 0, \tag{41}$$

the dual spin $\tau_\alpha^\star$ of the RS field will not contribute, again due to Fierz rearrangement (37). This finally leads to the "on shell" conditions

$$\theta_{\mathrm{T}} = -m^2 \ell^2, \qquad \theta_{\mathrm{TL}} = \frac{(-1)^s}{2} m(2m+1)\ell, \qquad \theta_{\mathrm{L}} = 1 - \frac{\theta_{\mathrm{TL}}}{\ell} = 1 - \frac{(-1)^s}{2} m(2m+1) \quad (42)$$

for the coupling constants of the bosonic part of our Lagrangian L_∞. Consequently, massless RS spinors do not require a translational nor a 'mixed' CS term in order to acquire supersymmetry.

5. Towards supersymmetric S–duality

There exists a continuous deformation [or a field redefinition (FR)] of the (Lorentz-) rotational connection by adding a tensor–valued one–form, similarly as in Eq. (3.11.1) of Ref. [22]. In 3D, the particular deformation

$$\widetilde{\Gamma}_\alpha^\star = \Gamma_\alpha^\star - (-1)^s \frac{\varepsilon}{2\ell} \vartheta_\alpha, \tag{43}$$

where ε is a continuous parameter, is involving the Lie dual $\Gamma_\alpha^\star = \frac{1}{2}\eta_{\alpha\beta\gamma}\Gamma^{\beta\gamma}$ of the connection. In view of the definitions (2) and (3) of torsion and curvature, respectively, this FR implies

$$\widetilde{T}_\alpha = T_\alpha - \frac{\varepsilon}{\ell}\eta_\alpha, \qquad \widetilde{R}_\alpha^\star = R_\alpha^\star - (-1)^s \frac{\varepsilon}{2\ell} T_\alpha + (-1)^s \frac{\varepsilon^2}{4\ell^2}\eta_\alpha \tag{44}$$

for the deformed torsion and curvature, respectively. In particular, there can arise two subcases: Riemannian spacetime with deformed torsion $\widetilde{T}_\alpha = 0$, or deformed *teleparallelism* in the gauge $\widetilde{\Gamma}_\alpha^\star \overset{*}{=} 0$, equivalent to the covariant constraint of vanishing modified RC curvature, i.e., $\widetilde{R}_\alpha^\star = 0$.

In the latter case, coframe and connection are *Lie dual* to each other, i.e.,

$$\Gamma_\alpha^\star = (-1)^s \frac{\varepsilon}{2\ell} \vartheta_\alpha \qquad \Leftrightarrow \qquad \vartheta_\alpha = (-1)^s \frac{2\ell}{\varepsilon} \Gamma_\alpha^\star. \tag{45}$$

Observe the inversion of the parameter ε, i.e., a small deformation ε of the connection will induce a large coframe proportional to $1/\varepsilon$ and vice versa, resembling strong/weak duality. Such a duality of the *strong/weak* coupling regime of gauge fields, is the so-called *S–duality*. For Chern-Simons (super-)gravity, some of its aspects have also been discussed in Ref. [16, 20].

There could also arise the seemingly trivial case of a completely *flat* deformed spacetime, i.e., $\widetilde{T}_\alpha = 0$ and $\widetilde{R}_\alpha^\star = 0$. This would correspond to configurations with constant axial torsion and constant RC curvature as originally envision by E. Cartan, i.e.,

$$T_\alpha = \frac{\varepsilon}{\ell}\eta_\alpha, \qquad R_\alpha^\star = \frac{\rho}{\ell^2}\eta_\alpha, \tag{46}$$

where $\rho = (-1)^s \varepsilon^2/4$ depends quadratically on the deformation parameter ε.

Let us extend such ideas to supergravity in 3D: Generalizing the peculiar dynamical symmetry of BMH [2], identified as *S–duality* in Ref. [31], we try the following Ansatz

$$\vartheta_\alpha = (-1)^s \ell\, \Gamma_\alpha^\star + \overline{\sigma}\, \gamma_\alpha \Psi, \tag{47}$$

where σ is again a spinor valued zero-form and ℓ a fundamental length.

By exterior differentiation, we find

$$d\vartheta_\alpha = (-1)^s \ell d\Gamma_\alpha^\star + d(\overline{\sigma}\gamma_\alpha\Psi), \tag{48}$$

or, after separating the covariant two-forms of torsion and curvature,

$$T_\alpha - (-1)^s \eta_{\alpha\beta} \wedge \Gamma^{\star\beta} = (-1)^s \ell R_\alpha^\star - \frac{\ell}{2}\eta_{\alpha\beta\gamma}\Gamma^{\star\beta} \wedge \Gamma^{\star\gamma} + d(\overline{\sigma} \wedge \gamma_\alpha\Psi) \tag{49}$$

Let us reconstitute our Ansatz (47) in order to replace all the connection terms $\Gamma^{\star\beta}$. Then, using also the fundamental relation (18) for a Clifford algebra, we obtain

$$T_\alpha + \frac{2}{\ell}\eta_\alpha + \frac{1}{\ell}\eta_{\alpha\beta} \wedge \overline{\sigma}\,\gamma^\beta\Psi \tag{50}$$

$$= (-1)^s \ell R_\alpha^\star - \frac{1}{2\ell}\eta_{\alpha\beta\gamma}(\vartheta^\beta - \overline{\sigma}\gamma^\beta\Psi) \wedge (\vartheta^\gamma - \overline{\sigma}\gamma^\gamma\Psi) + d(\overline{\sigma}\gamma_\alpha\Psi).$$

Now we can eliminate torsion and RC curvature via (11) and (12) with the result

$$\frac{2}{A}\left[(\theta_{TL} + (-1)^s\theta_T)\tau_\alpha^\star - (\theta_L + (-1)^s\theta_{TL})\ell\Sigma_\alpha\right]\ell^2 + (3 + 2\kappa - (-1)^s\rho)\eta_\alpha$$

$$= -2\eta_{\alpha\beta} \wedge \overline{\sigma}\,\gamma^\beta\Psi - \frac{1}{2}\eta_{\alpha\beta\mu}\overline{\sigma}\gamma^\beta\Psi \wedge \overline{\sigma}\gamma^\mu\Psi + \ell d(\overline{\sigma}\gamma_\alpha\Psi). \tag{51}$$

Together with (29), this leads to

$$\frac{B}{4A}i\ell^2\overline{\Psi}\gamma_\alpha\Psi + \frac{C}{A}\eta_\alpha$$

$$= -2\eta_{\alpha\beta} \wedge \overline{\sigma}\,\gamma^\beta\Psi - \frac{1}{2}\eta_{\alpha\beta\mu}\overline{\sigma}\gamma^\beta\Psi \wedge \overline{\sigma}\gamma^\mu\Psi + \ell D(\overline{\sigma}\gamma_\alpha\Psi) \tag{52}$$

as a condition for S-duality, where

$$B = \theta_T + (-1)^s\theta_{TL} + 2m\ell[\theta_L + (-1)^s\theta_{TL}] \tag{53}$$

and

$$C = 3A + \theta_T[\theta_{TL} + (-1)^s\theta_T]. \tag{54}$$

In the case of vanishing B and C and in view of the massive Rarita-Schwinger equation (35), there remains a first order nonlinear differential equation for $\overline{\sigma}$ coupled to RS fields to be satisfied.

6. Membranes with torsion defects

As an example of a spacetime with torsion and/or curvature *defects* [9] or singularities, let us consider a *a planar graphene* solution within the 'mixed' MB model governed by the two Einstein-Cartan type field equations (11) and (12).
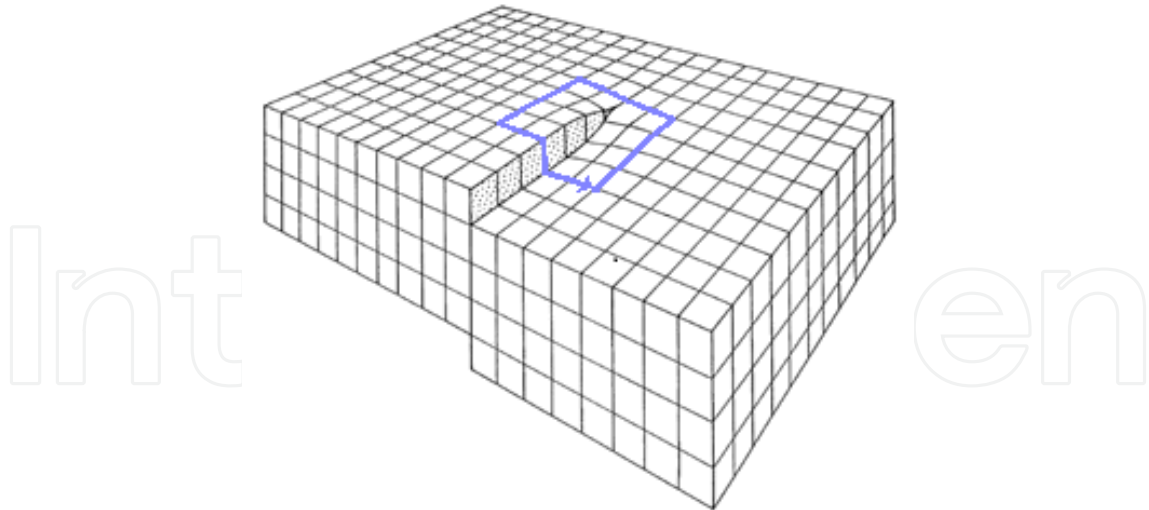
Fig. 1. 'Screw' dislocation with singular torsion in a cubic lattice. (The Cartan circuit is indicate in blue, cf. Ref. [26].]

Let us assume that the 2D membrane of a corrugated graphene is evolving in an *intrinsic* three-dimensional spacetime, suppressing for the moment the embedding of a real graphene into *flat* 4D Minkowski spacetime. Then we may adopt the convention that x^α together with y^α are spacelike orthogonal vectors which span the (x, y)–plane perpendicular to the time coordinate t, which itself is orthogonal to the world sheet of the graphene. The corresponding one–forms [29] are denoted by capital letters, i.e.

$$X := x_\alpha \, \vartheta^\alpha, \qquad Y := y_\alpha \, \vartheta^\alpha. \tag{55}$$

Moreover, the vector n^α is a timelike unit vector normal to the hypersurface with $n^\alpha n_\alpha = s$, the signature s of our 3D spacetime.

Following Soleng [37], cf. Anandan [1, 3, 22], we assume that the two–forms Σ_α and $\tau_\alpha^\star$ of the energy–momentum and spin current, respectively, vanish outside of the graphene sheet, whereas "inside" they are *constant*, i.e.

$$\Sigma_\alpha = \varepsilon \, x_\alpha \, X \wedge Y, \qquad \tau_\alpha^\star = \sigma \, y_\alpha \, X \wedge Y, \tag{56}$$

which satisfy

$$\vartheta^\alpha \wedge \Sigma_\alpha = 0, \qquad \vartheta^\alpha \wedge \tau_\alpha^\star = 0 \tag{57}$$

by construction. The constant parameters ε and σ of this *spinning string* type Ansatz are related to the exterior vacuum solution by appropriate matching conditions. For the related solution with *conical singularities* and torsion of Tod [40], we can infer that ε and σ are *delta distributions* [39] at the location of the defect, cf. Fig 1. From the specification (55) of the one–forms X and Y it can easily be inferred that the only nonzero components are $\Sigma_{\hat{0}} \neq 0$ and $\tau_{\hat{1}\hat{2}} = -\tau_{\hat{2}\hat{1}} \neq 0$.

Due to the identities (57), contractions of the second field equation (12) with x^α and y^α reveal that $x^{[\alpha} y^{\beta]} R_{\alpha\beta} = R_{\hat{1}\hat{2}} = -R_{\hat{2}\hat{1}} \neq 0$ are the only nonvanishing components of the RC curvature. From its covariant expression

$$R^{\alpha\beta} = \varepsilon \ell^2 \, x^{[\alpha} y^{\beta]} \, X \wedge Y \tag{58}$$

there follows the identity

$$R_\beta{}^\alpha \wedge \vartheta^\beta = \frac{\varepsilon\ell^2}{2}(x^\alpha\, Y \wedge X \wedge Y - y^\alpha\, X \wedge X \wedge Y) = 0\,. \tag{59}$$

Recalling that $N^\alpha = n\rfloor\vartheta^\alpha$ is the lapse and shift vector in the (2+1)–decomposition a la ADM, the corresponding coframe and connection can now explicitly be obtained by applying a finite boost to the usual conical metric of a defect simulated by a cosmic string:

$$\vartheta^{\hat{0}} = dt + \ell^2\sigma\rho^{*2}[1 - \cos(\rho/\rho^*)]d\phi$$
$$\vartheta^{\hat{1}} = d\rho\,, \qquad \vartheta^{\hat{2}} = \rho^*\sin(\rho/\rho^*)d\phi\,,$$
$$\Gamma^{\hat{1}\hat{2}} = \cos(\rho/\rho^*)d\phi = -\Gamma^{\hat{2}\hat{1}}\,. \tag{60}$$

From the Cartan type relation (11) and the identities (57) we can infer that the *axial* torsion

$$\mathcal{A} = {}^*(\vartheta^\alpha \wedge T_\alpha) = -(-1)^s\frac{2\kappa}{\ell^2} \tag{61}$$

of such a membrane defect is a non-vanishing constant. Thus, in 3D there is no contribution to the Pointrjagin type term $d(\mathcal{A} \wedge d\mathcal{A})$ from the axial torsion.

Moreover, the Nieh–Yan term dC_T proportional to $d\,{}^*\mathcal{A}$ *vanishes identically* for this example of a spinning cosmic string exhibiting a *torsion line defect*.

7. Outlook: Graphene and supersymmetry

Fundamental interactions are rather successful formulated in terms of Yang-Mills theories with large gauge groups, stipulating that symmetry breaking is occurring in the ground state. The idea of supersymmetry or supergravity, anticipated to some extent already by Hermann Weyl [42], goes in the same direction but so far lacks empirical support in particle physics.

Recently, graphene [33] as a new material has attracted a lot of attention because its charge carriers can be described by massless Dirac fields, cf. Ref. [41], whereas the flexural models of the 2D membrane of graphene have been tentatively considered as membranes, cf. Ref. [25], evolving in $2 + 1$ dimensional curved, but conformally flat spacetime [24]. There are also indications of dislocations [9] related to torsion.

A related topological framework with a coupling to Dirac fields in 3D has been considered before by Lemke and Mielke [27]. It seems to be feasible to enlarge the dynamical framework of the theory by including supersymmetry, cf. Ref. [17] and apply the topological ideas developed to some extent in this paper.

8. Acknowledgments

9. Appendices

A: Variations of Chern–Simons terms

Gauging the Poincaré group in (2+1) dimensions, local translations and (Lorentz-) rotations give rise to two type of gauge potentials, the coframe ϑ^γ and the dual of Lorentz-connection $\Gamma^\star_\alpha$. Then the two Bianchi identities of Riemann-Cartan geometry can be rewritten as

$$DT^\alpha \equiv (-1)^s \, \eta^{\alpha\beta} \wedge R^\star_\beta, \tag{62}$$

$$DR^\star_\alpha \equiv 0. \tag{63}$$

In 3D the corresponding Chern–Simons three–forms of gauge type $C = \mathrm{Tr}\{A \wedge F\}$, are

$$C_{\mathrm{T}} := \frac{1}{2\ell^2} \vartheta^\alpha \wedge T_\alpha = -\frac{(-1)^s}{\ell^2} \eta^\alpha \wedge K^\star_\alpha, \quad C_{\mathrm{L}} := (-1)^s \, \Gamma^{\star\alpha} \wedge R^\star_\alpha - \frac{1}{3!} \eta_{\alpha\beta\gamma} \, \Gamma^{\star\alpha} \wedge \Gamma^{\star\beta} \wedge \Gamma^{\star\gamma}. \tag{64}$$

and

$$C_{\mathrm{TL}} := \frac{1}{\ell} \left(\Gamma^{\star\alpha} \wedge T_\alpha - \frac{(-1)^s}{2} \eta_{\alpha\beta\gamma} \, \Gamma^{\star\alpha} \wedge \Gamma^{\star\beta} \wedge \vartheta^\gamma \right). \tag{65}$$

The variational derivatives of these terms lead us to the following expressions

$$\frac{\delta C_{\mathrm{T}}}{\delta \vartheta^\alpha} = \frac{1}{\ell^2} \, T_\alpha \qquad \frac{\delta C_{\mathrm{T}}}{\delta \Gamma^{\star\alpha}} = \frac{(-1)^s}{\ell^2} \, \eta_\alpha, \tag{66}$$

$$\frac{\delta C_{\mathrm{L}}}{\delta \vartheta^\alpha} = 0, \qquad \frac{\delta C_{\mathrm{L}}}{\delta \Gamma^{\star\alpha}} = (-1)^s \, 2 R^\star_\alpha, \tag{67}$$

$$\frac{\delta C_{\mathrm{TL}}}{\delta \vartheta^\alpha} = \frac{1}{\ell} R^\star_\alpha, \qquad \frac{\delta C_{\mathrm{TL}}}{\delta \Gamma^{\star\alpha}} = \frac{1}{\ell} T_\alpha, \tag{68}$$

respectively. Note that these three–forms are uniquely related to the torsion T_α, the curvature $R^\star_\alpha$, and the cosmological term η_α, as developed in much more detail in Ref. [21].

B: The η–basis for exterior forms in 3D

The symbol $\wedge$ denotes the exterior product of forms, the symbol $\rfloor$ the interior product of a vector with a form and * the Hodge star (or left dual) operator which maps a p–form into a $(3-p)$–form. It has the property that

$$^{**}\Phi^{(p)} = (-1)^{p(3-p)+s} \Phi^{(p)}, \tag{69}$$

where p is the degree of the form Φ and s denotes the number of negative *eigenvalues* of the metric, i.e., the signature of spacetime.

The volume three–form is defined by

$$\eta := \frac{1}{3!} \eta_{\alpha\beta\gamma} \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma, \tag{70}$$

where $\eta_{\alpha\beta\gamma} := \sqrt{|\det g_{\mu\nu}|}\,\epsilon_{\alpha\beta\gamma}$, and $\epsilon_{\alpha\beta\gamma}$ is the Levi–Civita symbol. The forms $\{\eta, \eta_\alpha, \eta_{\alpha\beta}, \eta_{\alpha\beta\gamma}\}$ span a *dual basis* for the algebra of arbitrary p–forms in 3D, where

$$\eta_\alpha := e_\alpha \rfloor \eta = \frac{1}{2}\eta_{\alpha\beta\gamma}\vartheta^\beta \wedge \vartheta^\gamma = {}^*\vartheta_\alpha,$$
$$\eta_{\alpha\beta} := e_\beta \rfloor \eta_\alpha = \eta_{\alpha\beta\gamma}\vartheta^\gamma = {}^*(\vartheta_\alpha \wedge \vartheta_\beta),$$
$$\eta_{\alpha\beta\gamma} := e_\gamma \rfloor \eta_{\alpha\beta}. \tag{71}$$

In 3D, the following relations for the η–basis hold:

$$\eta^{\alpha\beta\gamma}\eta_{\alpha\beta\gamma} = (-1)^s 3!,$$
$$\eta^{\alpha\beta\gamma}\eta_{\alpha\beta\nu} = (-1)^s 2\delta_\nu^\gamma,$$
$$\eta^{\alpha\beta\gamma}\eta_{\alpha\mu\nu} = (-1)^s \delta_\mu^\beta \delta_\nu^\gamma = (-1)^s 2\delta_{[\mu}^\beta \delta_{\nu]}^\gamma,$$
$$\eta^{\alpha\beta\gamma}\eta_{\rho\mu\nu} = (-1)^s \delta_{\rho\mu\nu}^{\alpha\beta\gamma}, \tag{72}$$

and

$$\eta_\beta \wedge \eta^{\alpha\beta} = e_\beta \rfloor (\eta \wedge \eta^{\alpha\beta}) + \eta \wedge e_\beta \rfloor \eta^{\alpha\beta} \equiv 0 \tag{73}$$

due to the antisymmetry of $\eta^{\alpha\beta}$ and the fact that $\eta \wedge \eta^{\alpha\beta}$ would already be a four-form in 3D.

C: Identities for spinor–valued forms

Now some relations of special importance are presented which take care of the order of the forms in the exterior products and its Dirac adjoint: We would like to remind the reader that Φ is a p–form and Ψ a q–form with the spinor indices suppressed:

$$\Phi^p \wedge \Psi^q = (-1)^{p \cdot q}\Psi^q \wedge \Phi^p, \tag{74}$$
$$\overline{\Phi^p \wedge \Psi^q} = (-1)^{p \cdot q}\,\overline{\Psi^q} \wedge \overline{\Phi^p}, \tag{75}$$
$$\Phi^p \wedge {}^*\Psi^p = \Psi^p \wedge {}^*\Phi^p, \tag{76}$$
$$e_\alpha \rfloor (\Phi^p + \Psi^q) = e_\alpha \rfloor \Phi^p + e_\alpha \rfloor \Psi^q, \tag{77}$$
$$e_\alpha \rfloor (\Phi^p \wedge \Psi^q) = (e_\alpha \rfloor \Phi^p) \wedge \Psi^q + (-1)^p \Phi^p \wedge (e_\alpha \rfloor \Psi^q), \tag{78}$$
$$\vartheta^\alpha \wedge (e_\alpha \rfloor \Phi) = p\Phi \tag{79}$$
$${}^*(\Phi \wedge \vartheta_\alpha) = e_\alpha \rfloor {}^*\Phi, \tag{80}$$
$$\overline{\gamma} = \gamma. \tag{81}$$

D: No axial torsion restrictions in 3D

Spaces of constant curvature deserve special attention in General Relativity, in particular in the cosmological context. In particular, when the RC curvature is constant as in Eq. (10), i.e.

$$R_\alpha^\star = \frac{\rho}{\ell 2}\eta_\alpha = \frac{\rho}{2\ell 2}\eta_{\alpha\beta\gamma}\vartheta^\beta \wedge \vartheta^\gamma, \tag{82}$$

the Bianchi identities (62) and (63) could lead to constraints on the admissible torsion T^α, as in 4D and higher dimensions. However, in 3D the situation is different: Using Appendix B, the first Bianchi identity yields

$$(-1)^s \eta^{\alpha\beta} \wedge R^*_\beta = (-1)^s \frac{\rho}{\ell^2} \eta_\beta \wedge \eta^{\alpha\beta} = (-1)^s \frac{\rho}{2\ell^2} \left(\eta_{\beta\mu\nu} \eta^{\alpha\beta\gamma} \right) \vartheta_\gamma \wedge \vartheta^\mu \wedge \vartheta^\nu \qquad (83)$$
$$= -(-1)^s 2\delta^\alpha_{[\mu} \delta^\gamma_{\nu]} \vartheta_\gamma \wedge \vartheta^\mu \wedge \vartheta^\nu = 0.$$

Furthermore, the exterior covariant derivative of Eq. (10) provides the identity

$$DT_\alpha = \frac{2\kappa}{\ell} D\eta_\alpha = \frac{2\kappa}{\ell} T^\beta \wedge \eta_{\alpha\beta} = \frac{4\kappa^2}{\ell^2} \eta_{\alpha\beta} \wedge \eta^\beta \equiv 0. \qquad (84)$$

Thus the first Bianchi identity does not give any further information. The second Bianchi identity (63) yields

$$DR^\star_\alpha = \frac{\rho}{\ell^2} D\eta_\alpha = \frac{2\kappa\rho}{\ell^3} \eta_{\alpha\beta} \wedge \eta^\beta \equiv 0 \qquad (85)$$

which is identically zero by a similar argument, or by employing Eq. (73). Consequently, the Bianchi identities impose *no* restrictions on the axial torsion given by (10) in 3D, a fact which has allowed us to construct something non-trivial from the MB model.

10. References

[1] Anandan, J. (1994). Topological and geometrical phases due to gravitational field with curvature and torsion, *Physics Letters* A195, 284–292; (1996). Gravitational phase operator and cosmic strings, *Physical Review* D53, 779–786.

[2] Baekler, P., Mielke, E. W. & Hehl, F. W. (1992). Dynamical symmetries in topological 3D gravity with torsion, *Il Nuovo Cimento* B107, 91–110.

[3] Bakke, K., Furtado, C. & Nascimento, J. R. (2009). Gravitational geometric phase in the presence of torsion, *The European Physical Journal* C60, 501–507.

[4] Cacciatori, S. L., Caldarelli, M. M., Giacomini, A., Klemm, D. & Mansi, D. S. (2006). Chern-Simons formulation of three-dimensional gravity with torsion and nonmetricity, *Journal Geometrical Physics* 56, 2523–2543.

[5] Carlip, S. (1995). Lectures on (2+1)–dimensional gravity, *Journal of Korean Physical Society* 28, S447–S467.

[6] Carlip, S. (1998). *Quantum gravity in* $(2 + 1)$*–dimensions*, Cambridge University Press.

[7] Carlip, S. (1995). The $(2 + 1)$–dimensional black hole, *Classical Quantum Gravitation* 12, 2853–2880; (2005). Conformal field theory, (2+1)-dimensional gravity, and the BTZ black hole, *Classical Quantum Gravitation* 22, R85–R124.

[8] Chern, S. S. & Simons, J. (1971). Some cohomology classes in principal fiber bundles and their application to Riemannian geometry, *Proceedings of Natural Academy of Science* 68, 791–794.

[9] de Juan, F. , Cortijo, A. & Vozmediano, M. A. H. (2010). Dislocations and torsion in graphene and related systems, *Nuclear Physics* B828, 625–637 .

[10] Dereli, T. & Tucker, R. W. (1988). Gravitational interactions in 2+1 dimensions, *Classical Quantum Gravitation* 5, 951–959.

[11] Dereli, T. & Verçin, A. (1991). A gauge model of amorphous solids containing defects II. Chern-Simons free energy, *Philosophy Magnetics* B64, 509.

[12] Deser, S., Jackiw, R. & Templeton, S. (1982). Topologically massive gauge theories, *Annals of Physics* 140, 372–411.

[13] Deser, S. (1984). Cosmological topological supergravity, *Quantum Theory of Gravity*, essays in honor of the 60th Birthday of Bryce S. DeWitt. S. M. Chistensen, ed. Adam Hilger, Bristol, 374.

[14] Deser, S. (1989). Three topics in three dimensions, *Supermembranes and Physics in 2+1 Dimensions*. Proc of the Trieste Conference. ICTP, Trieste, M. J. Duff, C. N. Pope and E. Sezgin, eds. World Scientific, Singapore, p.239.

[15] Deser, S. & Zumino, B. (1976). Consistent supergravity, *Physics Letters* B62, 335.

[16] Deser, S. & McCarthy, J. G. (1990). Selfdual formulations of $D = 3$ gravity theories, *Physics Letters* B246, 441. [Addendum-ibid. B248, 473].

[17] Ezawa, M. (2008). Supersymmetric structure of quantum Hall effects in graphene, *Physics Letters* A 372, 924.

[18] Falbo–Kenkel, M. K. & Mansouri, F. (1993). Nonperturbative two–body dynamics in 2+1–dimensional gravity, *Journal of Mathematical Physics*. 34, 139–153.

[19] Freedman, D. Z. (1993). Some beautiful equations of mathematical physics, *Dirac Lecture Trieste*, preprint CERN, TH 7367/94.

[20] Garcia-Compean, H., Obregon, O., Ramirez, C. & Sabido, M. (2001). On S-duality in (2+1)-Chern-Simons supergravity, *Physical Review* D64, 024002.

[21] Hehl, F. W., McCrea, J. D., Kopczynski, W. & Mielke, E. W. (1991). Chern–Simons terms in metric–affine spacetime: Bianchi identities as Euler–Lagrange equations, *Journal of Mathematical Physics* 32, 2169–2180.

[22] Hehl, F. W., McCrea, J. D., Mielke, E. W. & Ne'eman, Y. (1995). Metric–affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance, *Physics Report* 258, 1–171.

[23] Howe, P. S. & Tucker, R. W. (1978). Local supersymmetry in (2+1)-dimensions. 1. Supergravity and differential forms, *Journal of Mathematical Physics* 19, 869–873.

[24] Iorio, A. (2011). Weyl-gauge symmetry of graphene, *Annals of Physics* 326, 1334–1353.

[25] Kerner, R. & Naumis, G. (2011). Interaction of flexural phonons with electrons in graphene: A generalized Dirac equation in corrugated surfaces, [arXiv:1102.0746 [cond-mat.mtrl-sci]].

[26] Lazar, M. & Hehl, F. W. (2010). Cartan's spiral staircase in physics and, in particular, in the gauge theory of dislocations, *Foundations of Physics* 40, 1298–1325.

[27] Lemke, J. & Mielke, E. W. (1993). Gravitational moments of spin one–half particles and of topological massive photons and gravitons in $2 + 1$ dimensions, *Physics Letters* A175, 277–281.

[28] Mielke, E. M. & Baekler, P. (1991). Topological gauge model of gravity with torsion, *Physics Letters* A156, 399.

[29] Mielke, E. M. & Kreimer, D. (1998). Chiral anomaly in Ashtekar's approach to canonical gravity, *International Journal of Modern Physics* D7, 535–548.

[30] Mielke, E. W. & Macías, A. (1999). Chiral supergravity and anomalies, *Annalen Physik* 8, 301.

[31] Mielke, E. W. & Maggiolo, A. R. (2003). Rotating black hole solution in a generalized topological 3D gravity with torsion, *Physical Review* D68, 104026.

[32] Mielke, E. W. & Maggiolo, A. R. (2007). S-duality in 3D gravity with torsion, *Annals of Physics* (N.Y.) 322, 341–362.

[33] Novoselov, K. S., Geim, A. K., Morozov, S. V., Jiang, D., Katsnelson, M. I., Grigorieva, I. V., Dubonos, S. V. & Firsov, A. A. (2005). Two-dimensional gas of massless Dirac fermions in graphene, *Nature* 438, 197.

[34] Nieuwenhuizen, P. van (1981). Supergravity, *Physics Reports* 68, 189.

[35] Rarita, W. & Schwinger, J. S. (1941). On a theory of particles with half integral spin, *Physical Review* 60, 61.

[36] Schroers, B. J. (2011). Quantum gravity and non-commutative spacetimes in three dimensions: a unified approach, arXiv:1105.3945v1 [gr-qc] 1–22.

[37] Soleng, H. H. (1992). A spinning string, *General Relativity and Gravitation* 24, 111–117.

[38] Staruszkiewicz, A. (1963). Gravitation theory in three-dimensional space, *Acta Physica Polonica* 24, 735–740.

[39] Taub, A. H. (1980). Space-times with distribution valued curvature tensors, *Journal of Mathematical Physics* 21, 1423–1431.

[40] Tod, K. P. (1994). Conical singularities and torsion, *Classical Quantum and Gravitation* 11, 1331.

[41] Villalba, V. M. & Maggiolo, A. R. (2001). Energy spectrum of a 2-D Dirac oscillator in the presence of a constant magnetic field, *The European Physical Journal* B22, 31–35.

[42] Weyl, H. (1931). Geometrie und Physik, *Naturwissenschaften* 19, 49.

[43] Witten, E. (1988). (2+1)-dimensional gravity as an exactly soluble system, *Nuclear Physics* B311, 46.
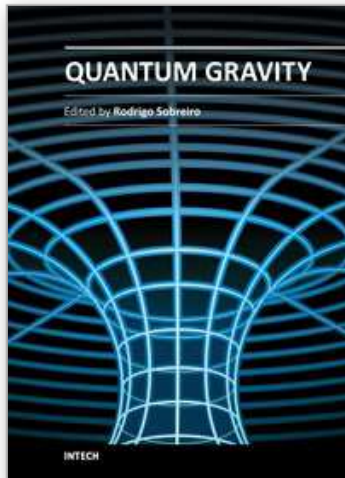
Quantum Gravity

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