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Nonclassical Properties of Superpositions of Coherent and Squeezed States for Electromagnetic Fields in Time-Varying Media

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1. Introduction

In spite of remarkable advance of quantum optics, there would be many things that are yet to be developed regarding the properties of light. One of them is the behavior of light propagating or confined in time-varying media. If the characteristic parameters of medium such as electric permittivity, magnetic permeability, and electric conductivity are dependent on time, the medium is classified as time-varying media. After the publication of a seminal paper by Choi and Yeon (Choi & Yeon, 2005), there has been a surge of renewed research for electromagnetic field quantization in time-varying media and for the properties of corresponding quantized fields (Budko, 2009; Choi, 2010a; Choi, 2010b). Some important examples that the theory of optical wave propagation in time-varying media is applicable are magnetoelastic delay lines (Rezende & Morgenthaler, 1969), wave propagation in ionized plasmas (Kozaki, 1978), the modulation of microwave power (Morgenthaler, 1958), and novel imaging algorithms for dynamical processes in time-varying physical systems (Budko, 2009).

To study the time behavior of light rigorously, it may be crucial to quantize it. The purpose of this chapter is to analyze nonclassical properties of superpositions of quantum states for electromagnetic fields in time-varying linear media. The methods for quantization of a light propagating in free space or in transparent material is well known, since each mode of the field in that case acts like a simple harmonic oscillator. However, the quantization procedure for a light in a time-varying background medium is somewhat complicate and requires elaborate technic in accompanying mathematical treatments. One of the methods that enable us to quantize fields in such situation is to introduce an invariant operator theory (Lewis & Riesenfeld, 1969) in quantum optics. The invariant operator theory which employs Lewis-Riesenfeld invariants is very useful in deriving quantum solutions for time-dependent Hamiltonian systems in cases like this. The light in homogeneous conducting linear media which have time-dependent parameters will be quantized and their quantum properties will be investigated on the basis of invariant operator theory. The exact wave functions for the system with time-varying parameters will be derived in Fock, coherent, and squeezed states in turn.

For several decades, much attention has been devoted to the problem of superposed quantum states (the Schrödinger cat states) of an optical field (Choi & Yeon, 2008; Ourjoumtsev et

al., 2006; Yurke & Stoler, 1986). The superpositions in both coherent states and squeezed states of electromagnetic field are proved to be quite interesting and their generation has been an important topic in quantum optics thanks to their nonclassical properties such as high-order squeezing, subpoissonian photon statistics, and oscillations in the photon-number distribution (Richter & Vogel, 2002; Schleich et al., 1991). Moreover, it is shown that the Schrödinger cat states provide an essential tool for quantum information processing (Ourjoumtsev et al, 2006).

It may be interesting to study a phase space distribution function so-called Wigner distribution function (WDF) (Wigner, 1932) for Schrödinger cat states for fields in time-varying media. The propagation of a signal through optical systems is well described by means of the WDF transformations (Bastiaans, 1991), which results in accompaniment of the reconstruction of the propagated signal. A convolution of the WDF allows us to know the phase space distribution connected to a simultaneous measurement of position and momentum. Due to its square integrable property, the WDF always exists and can be employed to evaluate averages of Hermitian observables that are essential in the quantum mechanical theory. The WDF is regarded as 'quasiprobability distribution function', since it can be negative as well as positive on subregions of phase space. Gaussian is the only pure state for which the WDF is positive everywhere. In view of quantum optics, Bastiaans showed that the WDF provides a link between Fourier optics and the geometrical optics (Bastiaans, 1980). The WDF has been widely used in explaining intrinsic quantum features which have no classical analogue in various branches of physics, such as decoherence (Zurek, 1991), Fourier quantum optics (Bartelt et al, 1980), and interference of quantum amplitudes (Bužek et al., 1992). The nonclassical properties of superpositions of quantum states for electromagnetic fields with time-dependent parameters will be studied here via WDF.

2. Quantization of light in time-varying media

The characteristics of electromagnetic fields in media are determined in general by the parameters of media such as electric permittivity ϵ, magnetic permeability μ, and electric conductivity σ. If $\sigma = 0$ and other two parameters are real constants, the electromagnetic fields behave like simple harmonic oscillators. The electromagnetic fields propagating along a medium that have non-zero conductivity undergo dissipation that entails their energy loss. In case that the value of one or more parameters of media is complex and/or time-dependent, the mathematical description of optical fields may be not an easy task. We suppose that the parameters are time-dependent and use invariant operator theory to quantize the electromagnetic fields in such medium. The relations between fields and current in linear media are

$$\mathbf{D} = \epsilon(t)\mathbf{E}, \qquad \mathbf{H} = \frac{\mathbf{B}}{\mu(t)}, \qquad \mathbf{J} = \sigma(t)\mathbf{E}. \tag{1}$$

The Maxwell's equations in media that have no charge source can be written in SI unit as

$$\nabla \cdot \mathbf{D} = 0, \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \tag{5}$$

A fundamental relation between electromagnetic fields and potentials are

$$E = -\nabla\Phi - \frac{\partial A}{\partial t}, \tag{6}$$

$$B = \nabla \times A, \tag{7}$$

where Φ is a scalar potential and A is a vector potential. We take Coulomb gauge due to its usefulness in this situation. In particular Coulomb gauge is more advantageous in describing a purely transverse wave. The scalar potential then vanishes since we assumed that there is no net charge source. As a consequence, both the electric and the magnetic fields can be expanded only in terms of vector potential.

By solving Eqs. (2)-(5) considering Eqs. (6) and (7), we obtain a time-dependent damped wave equation such that

$$\nabla^2 A - [\sigma(t) + \dot{\epsilon}(t)]\mu(t)\frac{\partial A}{\partial t} - \epsilon(t)\mu(t)\frac{\partial^2 A}{\partial t^2} = 0. \tag{8}$$

To decouple the vector potential into position and time functions, it is necessary to put

$$A(r, t) = \sum_l u_l(r)q_l(t), \tag{9}$$

where particular modes are denoted by subscript l. The substitution of Eq. (9) into Eq. (8) leads to

$$\nabla^2 u_l(r) + k_l^2 u_l(r) = 0, \tag{10}$$

$$\frac{\partial^2 q_l(t)}{\partial t^2} + \frac{\sigma(t) + \dot{\epsilon}(t)}{\epsilon(t)}\frac{\partial q_l(t)}{\partial t} + c^2(t)k_l^2 q_l(t) = 0, \tag{11}$$

where k_l are separation constants and $c(t)$ is the time-dependent velocity of light which is given by $c(t) = 1/\sqrt{\epsilon(t)\mu(t)}$. Actually k_l are wave numbers that can be represented as

$$k_l = \frac{\omega_l(t)}{c(t)}, \tag{12}$$

where $\omega_l(t)$ are time-dependent natural angular frequencies. From the fact that k_l are constants, we have

$$\frac{\omega_l(t)}{c(t)} = \frac{\omega_l(0)}{c(0)}. \tag{13}$$

From now on, let us omit under subscript l from notations for the shake of convenience.

Using fundamental theory of dynamics, we can construct the Hamiltonian of the system associated with Eq. (11) to be

$$\hat{H}(\hat{q}, \hat{p}, t) = \frac{1}{2\epsilon_0}e^{-\Lambda(t)}\hat{p}^2 + \frac{1}{2}e^{\Lambda(t)}\epsilon_0\omega^2(t)\hat{q}^2, \tag{14}$$

where $\epsilon_0 = \epsilon(0)$ and $\Lambda(t) = \int_0^t \{[\sigma(t') + \dot{\epsilon}(t')]/\epsilon(t')\}dt'$. If we consider that this Hamiltonian is a time-varying form, the introduction of a suitable invariant operator $\hat{K}$ may enable us to

obtain quantum solutions of the system. The invariant operator can be evaluated from

$$\frac{d\hat{K}}{dt} = \frac{\partial \hat{K}}{\partial t} + \frac{1}{i\hbar}[\hat{K}, \hat{H}] = 0, \tag{15}$$

which is known as Liouville-von Neumann equation. Execution of some algebra after inserting Eq. (14) into the above equation gives

$$\hat{K} = \left(\frac{\Omega}{2\rho(t)}\hat{q}\right)^2 + [\rho(t)\hat{p} - \epsilon_0 e^{\Lambda(t)}\dot{\rho}(t)\hat{q}]^2, \tag{16}$$

where Ω is an arbitrary real positive constant and $\rho(t)$ is some real time-function that satisfies the following differential equation

$$\ddot{\rho}(t) + \frac{\sigma(t) + \dot{\epsilon}(t)}{\epsilon(t)}\dot{\rho}(t) + \omega^2(t)\rho(t) - \frac{\Omega^2}{4\epsilon_0^2}e^{-2\Lambda(t)}\frac{1}{\rho^3(t)} = 0. \tag{17}$$

If we introduce annihilation and creation operators of the form

$$\hat{a} = \sqrt{\frac{1}{\hbar\Omega}}\left[\left(\frac{\Omega}{2\rho(t)} - i\epsilon_0 e^{\Lambda(t)}\dot{\rho}(t)\right)\hat{q} + i\rho(t)\hat{p}\right], \tag{18}$$

$$\hat{a}^\dagger = \sqrt{\frac{1}{\hbar\Omega}}\left[\left(\frac{\Omega}{2\rho(t)} + i\epsilon_0 e^{\Lambda(t)}\dot{\rho}(t)\right)\hat{q} - i\rho(t)\hat{p}\right], \tag{19}$$

the invariant operator can be rewritten as

$$\hat{K} = \hbar\Omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{20}$$

Note that Eqs. (18) and (19) are different from those of simple harmonic oscillator.

If we denote two linearly independent homogeneous solutions of Eq. (11) as $\rho_1(t)$ and $\rho_2(t)$, $\rho(t)$ is given by (Eliezer & Gray, 1976)

$$\rho(t) = [h_1\rho_1^2(t) + h_2\rho_1(t)\rho_2(t) + h_3\rho_2^2(t)]^{1/2}, \tag{21}$$

where h_1, h_2, and h_3 are constants that follow some relation imposed between them. In terms of an Wronskian w which is a time-constant and has the form

$$\mathsf{w} = \epsilon_0 e^{\Lambda(t)}[\rho_1(t)\dot{\rho}_2(t) - \dot{\rho}_1(t)\rho_2(t)], \tag{22}$$

h_1, h_2, and h_3 yield

$$4h_1h_3 - h_2^2 = \Omega^2/\mathsf{w}^2. \tag{23}$$

As an example, we can take the electromagnetic parameters to be

$$\epsilon(t) = \epsilon_0 e^{\gamma t}, \qquad \mu(t) = \mu(0), \qquad \sigma(t) = 0, \tag{24}$$

where γ is a real constant. Then, the Wronskian can be rewritten as

$$\mathsf{w} = \epsilon(t)[\rho_1(t)\dot{\rho}_2(t) - \dot{\rho}_1(t)\rho_2(t)], \tag{25}$$

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where $\rho_1(t)$ and $\rho_2(t)$ are given by

$$\rho_1(t) = \rho_{1,0}\, e^{-\gamma t/2} J_1[\xi(t)], \tag{26}$$

$$\rho_2(t) = \rho_{2,0}\, e^{-\gamma t/2} N_1[\xi(t)]. \tag{27}$$

Here, $\rho_{1,0}$ and $\rho_{2,0}$ are arbitrary real constants and $\xi(t) = [2\omega(0)/\gamma]e^{-\gamma t/2}$.

If we consider the asymptotic behavior of Bessel functions for $x \gg 1$:

$$J_m(x) \simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - m\frac{\pi}{2} - \frac{\pi}{4}\right), \tag{28}$$

$$N_m(x) \simeq \sqrt{\frac{2}{\pi x}} \sin\left(x - m\frac{\pi}{2} - \frac{\pi}{4}\right), \tag{29}$$

$\rho_1(t)$ and $\rho_2(t)$, in the limit $\xi \gg 1$ with a selection of $\rho_{1,0} = -\rho_{2,0} = \sqrt{\pi\Omega/(2\epsilon_0\gamma)}$, becomes

$$\rho_1(t) \simeq \sqrt{\frac{\Omega}{\epsilon_0\gamma\xi(t)}} e^{-\gamma t/2} \cos\left(\xi(t) - \frac{3\pi}{4}\right), \tag{30}$$

$$\rho_2(t) \simeq -\sqrt{\frac{\Omega}{\epsilon_0\gamma\xi(t)}} e^{-\gamma t/2} \sin\left(\xi(t) - \frac{3\pi}{4}\right). \tag{31}$$

Then, $\Omega/(2w) = 1$ and, as a consequence, we can choose $h_1 = h_3 = 1$ and $h_2 = 0$ so that Eq. (21) reduces to $\rho(t) = [\rho_1^2(t) + \rho_2^2(t)]^{1/2}$ which is a well used relation in the literature (Choi, 2010b).

We can directly check that the ladder operators satisfy the boson commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. Therefore, it is possible to obtain zero-point eigenstate $\langle q|\phi_0(t)\rangle$ of $\hat{K}$ from $\hat{a}\langle q|\phi_0(t)\rangle = 0$ and nth order eigenstate by operating $\hat{a}^\dagger$ n times into $\langle q|\phi_0(t)\rangle$. Thus we finally get

$$\langle q|\phi_n(t)\rangle = \sqrt[4]{\frac{\Omega}{2\rho^2(t)\hbar\pi}} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{\Omega}{2\rho^2(t)\hbar}}\, q\right)$$
$$\times \exp\left[-\frac{1}{2\rho(t)\hbar}\left(\frac{\Omega}{2\rho(t)} - i\epsilon_0 e^{\Lambda(t)}\dot{\rho}(t)\right)q^2\right], \tag{32}$$

where H_n is nth order Hermite polynomial. For the time-dependent Hamiltonian systems in cases like this, the Schrödinger solutions are different from the eigenstates $\langle q|\phi_n(t)\rangle$ by only time-dependent phase factors (Lewis & Riesenfeld, 1969):

$$\langle q|\psi_n(t)\rangle = \langle q|\phi_n(t)\rangle \exp\left[i\theta_n(t)\right]. \tag{33}$$

If we insert this equation together with Eq. (14) into Schrödinger equation, we obtain the phases $\theta_n(t)$ as:

$$\theta_n(t) = -\left(n + \frac{1}{2}\right)\frac{\Omega}{2\epsilon_0}\int_0^t \frac{e^{-\Lambda(t')}}{\rho^2(t')}dt'. \tag{34}$$

The probability of finding the real photons is given by probability density that is squared modulus of the wave function, while the wave function itself in general has no physical reality.

The probability density $|\langle q|\psi_n(t)\rangle|^2$ in number state is illustrated in Fig. 1 as a function
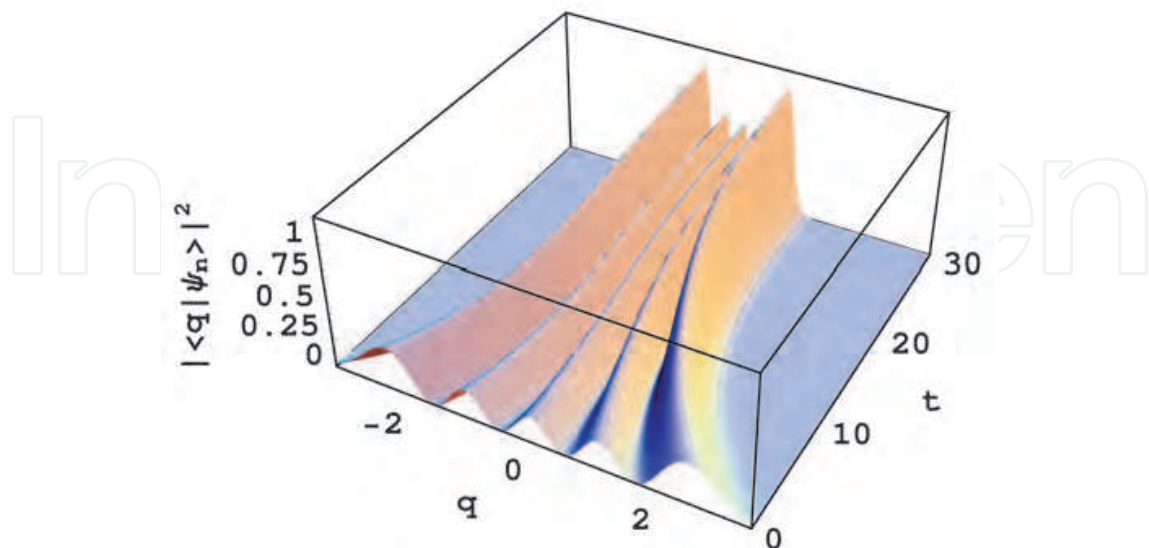

Fig. 1. Probability density in number state as a function of q and t. We used $n = 4$, $\hbar = 1$, $\gamma = 0.3$, $\epsilon_0 = 1$, $\mu(0) = 1$, $\Omega = 1$, $c_1 = c_2 = 25$, $\omega(0) = 1$, $h_1 = h_3 = \Omega/(2\text{w})$, $h_2 = 0$, and $\rho_{1,0} = \rho_{2,0} = 0.5$. All values are taken to be dimensionless for convenience (This convention will be used in all figures in this chapter).

of q and t. The parameters chosen in this figure are the same as those of Eq. (24). (This choice will also be hold in all subsequent figures without mentioning.) The probability density converges to origin ($q = 0$) as time goes by. This means that the amplitude of electromagnetic wave decreases with time. In general, the damping factor $(\sigma + \dot{\epsilon})/\epsilon$ appeared in Eq. (11) is responsible for the dissipation of amplitude. As you can see form Eq. (24), it is kept that $\sigma = 0$ in our model example but $\dot{\epsilon}$ is not zero. Therefore, the variation of $\epsilon(t)$ with time is the actual factor that leads to take place the dissipation of amplitude in this case. Note that $\epsilon(t)$ exponentially increases depending on γ. Thus, for large γ, the amplitude decreases more rapidly.

The density operator ϱ is defined in the form

$$\hat{\varrho} = \sum_{n,m} \varrho_{nm} |\psi_n\rangle\langle\psi_m|. \tag{35}$$

Then, the WDF is represented in terms of $\hat{\varrho}$ as

$$W(q, p, t) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} \langle q - x|\hat{\varrho}|q + x\rangle e^{2ipx/\hbar} dx. \tag{36}$$

The properties of superposition states are well understood from WDF representation. A little algebra gives

$$W(q, p, t) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} \hat{\varrho}(q - x, q + x, t) e^{2ipx/\hbar} dx. \tag{37}$$

It is well known that the WDF for number state consists of many concentric circles of ridge and valley which have different radii (Choi, 2004). The total number of ridge and valley is associated with the quantum number of the system and the value of WDF at a valley is negative. Whenever the WDF takes on negative values in parts of the phase, the corresponding state is regarded as nonclassical one. Due to its allowance of negative values, it is impossible to interpret WDF as the real distribution function. For this reason, there have been established several kinds of weighted WDF that takes non-negative value in phase space (Mogilevtsev & Kilin, 2000). The negative values of WDF have indeed been observed from lots of experimental measurements for a variety of states of optical field and matter (Leonhardt, 1997). However, integration of WDF over either of q and p makes it to be probability distribution for the other

$$\int_{-\infty}^{\infty} W(q, p, t) dq = |\psi(p, t)|^2, \tag{38}$$

$$\int_{-\infty}^{\infty} W(q, p, t) dp = |\psi(q, t)|^2. \tag{39}$$

These formulae guarantee WDF to be a quantum distribution function (but quasi) in spite of its singular properties.

3. Superposition of coherent states

The state engineering can also be achieved using a coherent-state expansion, instead of expanding number-state whose wave function is derived in the previous section. Coherent state for harmonic oscillator were firstly found by Schrödinger (Schrödinger, 1926) and rediscovered afterwards by Glauber (Glauber, 1963). Though coherent states are classical-like quantum states and hardly exhibit nonclassical effects, most class of superposition of coherent states can exhibit one or more nonclassical effects among various possible nonclassicality such as sub-Poissonian photon statistics and squeezing.

The coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{40}$$

If we consider Eq. (18), α is given by

$$\alpha = \sqrt{\frac{1}{\hbar\Omega}} \left[\left(\frac{\Omega}{2\rho(t)} - i\epsilon_0 e^{\Lambda(t)} \dot{\rho}(t) \right) q_{\mathrm{cl}}(t) + i\rho(t) p_{\mathrm{cl}}(t) \right], \tag{41}$$

where $q_{\mathrm{cl}}(t)$ and $p_{\mathrm{cl}}(t)$ are classical trajectories of variables q and p, which are given by

$$q_{\mathrm{cl}}(t) = c_1 \rho_1(t) + c_2 \rho_2(t), \tag{42}$$

$$p_{\mathrm{cl}}(t) = \epsilon_0 e^{\Lambda(t)} \frac{dq_{\mathrm{cl}}(t)}{dt} = \epsilon_0 e^{\Lambda(t)} [c_1 \dot{\rho}_1(t) + c_2 \dot{\rho}_2(t)], \tag{43}$$

where c_1 and c_2 are arbitrary real constants. If we divide α into real and imaginary parts such that

$$\alpha_{\mathrm{R}} = \sqrt{\frac{\Omega}{\hbar}} \frac{1}{2\rho(t)} [c_1 \rho_1(t) + c_2 \rho_2(t)], \tag{44}$$

$$\alpha_{\mathrm{I}} = \sqrt{\frac{1}{\hbar\Omega}} \epsilon_0 e^{\Lambda(t)} \{\rho(t)[c_1\dot{\rho}_1(t) + c_2\dot{\rho}_2(t)] - \dot{\rho}(t)[c_1\rho_1(t) + c_2\rho_2(t)]\}, \tag{45}$$

the eigenstate can be represented in terms of amplitude α_0 and phase φ:

$$\alpha = \alpha_0 e^{i\varphi}, \tag{46}$$

where

$$\alpha_0 = \sqrt{\alpha_{\mathrm{R}}^2 + \alpha_{\mathrm{I}}^2}, \tag{47}$$

$$\varphi = \tan^{-1}(\alpha_{\mathrm{I}}/\alpha_{\mathrm{R}}). \tag{48}$$

The substitution of Eqs. (44) and (45) into Eqs. (47) and (48) leads to

$$\alpha_0 = \sqrt{\frac{\Omega(c_1^2 h_3 - c_1 c_2 h_2 + c_2^2 h_1)}{\hbar(4h_1 h_3 - h_2^2)}}, \tag{49}$$

$$\varphi(t) = \tan^{-1}\left(\frac{2h_1 c_2 \rho_1(t) - h_2[c_1\rho_1(t) - c_2\rho_2(t)] - 2h_3 c_1 \rho_2(t)}{\sqrt{4h_1 h_3 - h_2^2}[c_1\rho_1(t) + c_2\rho_2(t)]}\right). \tag{50}$$

The time behavior of $\varphi(t)$ is illustrated in Fig. 2. The considered domain for $\varphi(t)$ in this figure is $-\pi/2 < \varphi(t) < \pi/2$, i.e., $\varphi(t) \equiv m\pi + \delta(t) \to \delta(t)$ where m is an integer and $-\pi/2 < \delta(t) < \pi/2$ at a given time. The direct differentiation of Eq. (50) with respect to time gives

$$\frac{d\varphi(t)}{dt} = -\frac{\Omega e^{-\Lambda(t)}}{2\epsilon_0 \rho^2(t)}. \tag{51}$$

Thus, we can represent φ in another way such that

$$\varphi(t) = -\frac{\Omega}{2\epsilon_0} \int_0^t \frac{e^{-\Lambda(t')}}{\rho^2(t')} dt' + \varphi(0). \tag{52}$$

It may be instructive to compare this equation with Eq. (34). The time behavior of $\varphi(t)$ is the same as that of $\theta_n(t)$ when we neglect some constants.

By operating $\langle q|$ from left in Eq. (40), the coherent state in configuration space is obtained. Then, a suitable choice of phase leads to (Choi & Yeon, 2008)
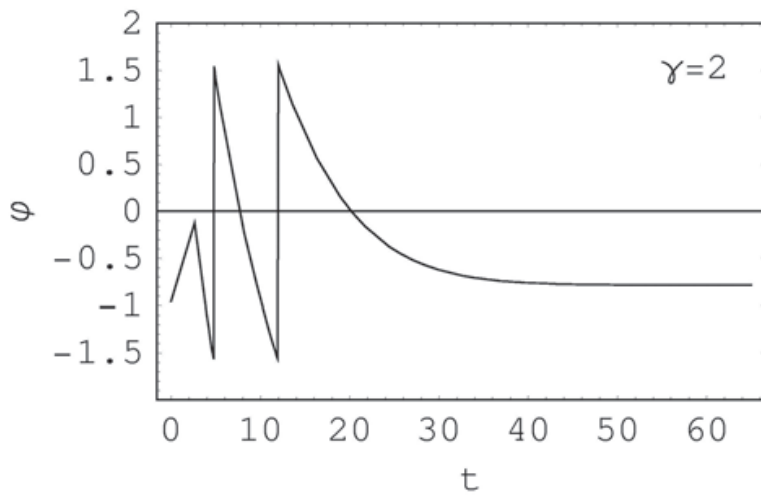
$$\langle q|\alpha\rangle = \sqrt[4]{\frac{\Omega}{2\rho^2(t)\hbar\pi}} \exp\left[\alpha\sqrt{\frac{\Omega}{\rho^2(t)\hbar}}q - \frac{1}{4\rho(t)\hbar}\right.$$

$$\left. \times \left(\frac{\Omega}{\rho(t)} - 2i\epsilon_0 e^{\Lambda(t)}\dot{\rho}(t)\right)q^2 - \frac{1}{2}\alpha_0^2 - \frac{1}{2}\alpha^2\right]. \tag{53}$$

The relation between coherent state and number state eigenfunction is given by

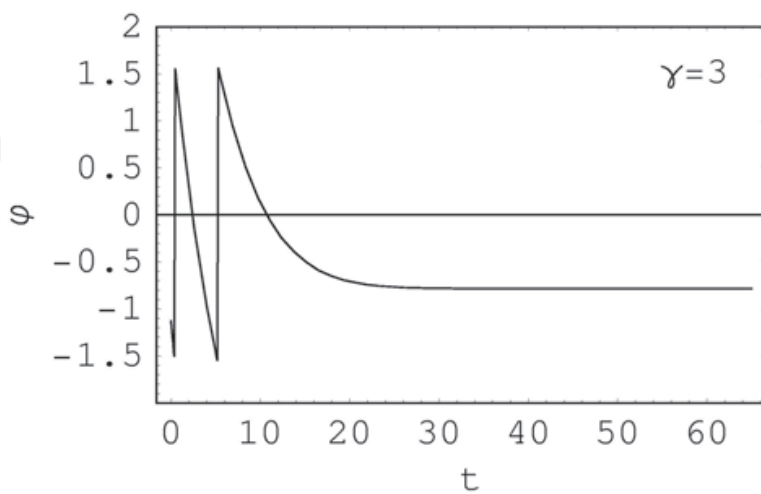$$\langle q|\alpha\rangle = \exp\left(-\frac{1}{2}\alpha_0^2\right) \sum_n \frac{\alpha^n}{\sqrt{n!}} \langle q|\phi_n(t)\rangle. \tag{54}$$
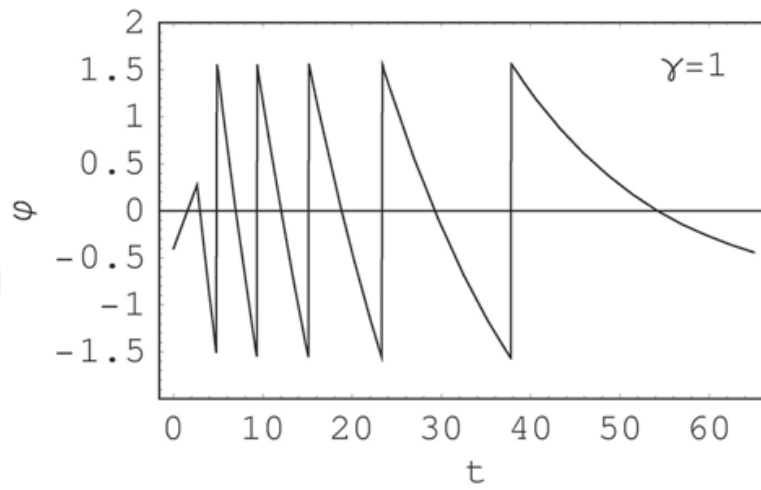
Fig. 2. The time evolution of φ for various values of γ. We used $\epsilon_0 = 1, \Omega = 1, c_1 = c_2 = 25,$ $\omega(0) = 1, h_1 = h_3 = \Omega/(2w), h_2 = 0,$ and $\rho_{1,0} = \rho_{2,0} = 0.5.$

We can easily show that the probability density $|\langle q|\alpha\rangle|^2$ is Gaussian from a fundamental evaluation. As mentioned earlier, the only pure states for which the WDF is positive everywhere are those that their corresponding probability density is Gaussian in cases like this. We can confirm from Fig. 3 that the trajectory of the peak of $|\langle q|\alpha\rangle|^2$ oscillates like a
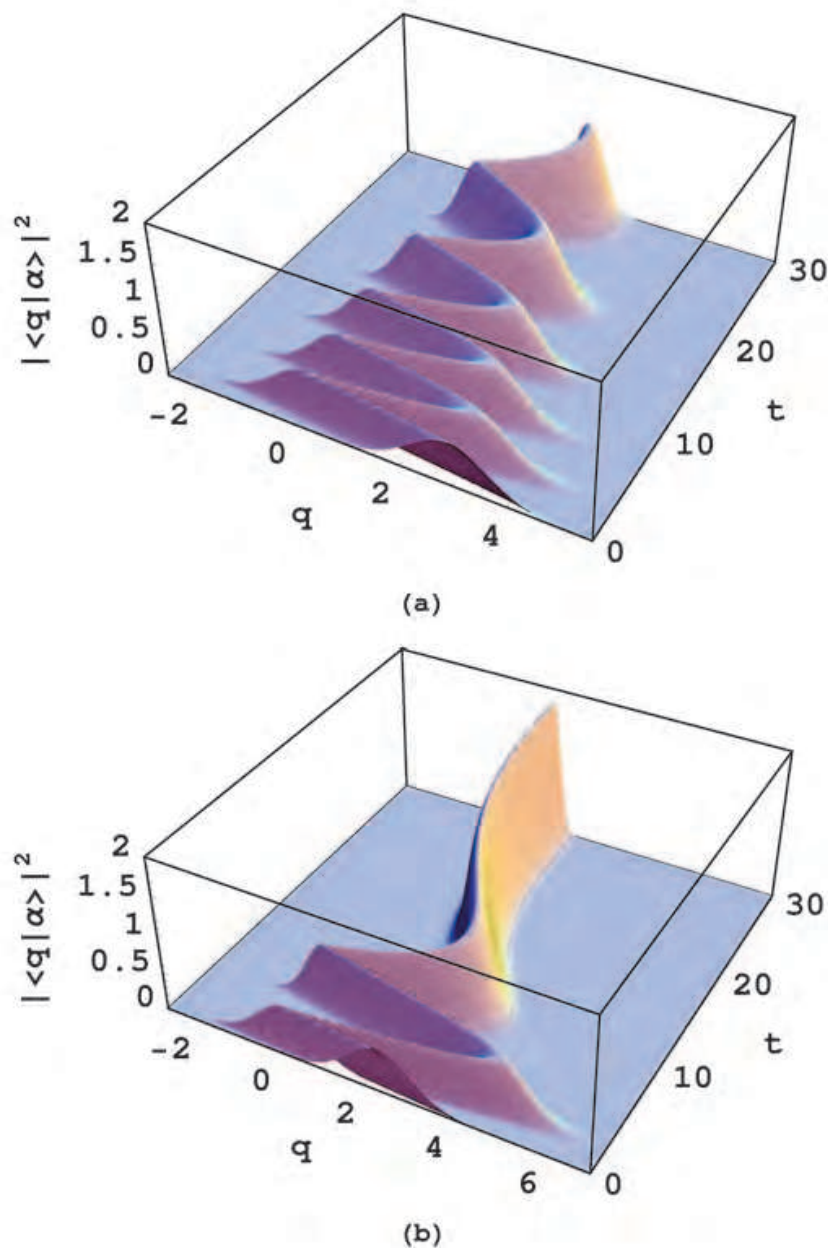

(a)

(b)

Fig. 3. Probability density in coherent state as a function of q and t. The value of γ is 0.1 for (a) and 0.3 for (b). We used $\hbar = 1$, $\epsilon_0 = 1$, $\mu(0) = 1$, $\Omega = 1$, $c_1 = c_2 = 25$, $\omega(0) = 1$, $h_1 = h_3 = \Omega/(2\mathbf{w})$, $h_2 = 0$, and $\rho_{1,0} = \rho_{2,0} = 0.5$.

classical state and converges near to origin as time goes by due to the influence of damping factor γ. Although coherent state is a pure quantum state, its properties lie on a borderline between those of classical state and quantum states.

From the early days of quantum mechanics, there have been great efforts for the problem of generating arbitrary quantum states including nonclassical states of an optical field mode. In particular, the superposition of coherent state (which is the main topic in this section) and the superposition of squeezed state (that will be treated in the next section) attracted much attention in the literature. A widely accepted criterion that a state to be classified as nonclassical one is exist: A quantum state has nonclassicality when the Glauber-Sudarshan P-function (Glauber, 1963; Sudarshan, 1963) fails to show the properties of a classical probability density. However, in many cases, this definition may hardly be applied to investigate the nonclassicality for a direct interpretation of experiments due to quite singular characteristics of P-function and the difficulty in determining P-function from given measurements. In fact, even for the simple harmonic oscillator, the exact characterization of the nonclassicality of a quantum state in terms of measurable quantities is somewhat ambiguous. A hierarchy of observable conditions for nonclassical quantum states, which allows one to verify whether the P-function for a specific state shows the properties of a classical probability density or not, has been reported (Richter & Vogel, 2002), while the global criteria for nonclassicality of states are yet the subject of researches.

Meanwhile, a method to reconstruct characteristic functions of a quantum state, such as the density matrix, the WDF, and the P-function, from experimentally accessible data is established, which is known as optical homodyne tomography (Smithey et al., 1993; Kiesel et al., 2008). It is possible to reconstruct the P-function up to sufficiently large thermal photon number whereas other criteria for nonclassicality, such as the Klyshko criterion (Klyshko, 1996), negativities of the WDF, and the entanglement potential (Asbóth, 2005), start to fail as the number of thermal photon increases (Kiesel et al., 2008). Though both definitions of nonclassicality in terms of P-function and in terms of WDF are sufficient but not necessary and leave some families of nonclassical quantum states outside their scope (Lvovsky & Shapiro, 2002). Of course, a satisfaction of the requirements of either definition does not automatically grantee satisfaction of the other. While the condition for nonclassicality based on P-function is more general than that based on WDF and covers more broad range of nonclassical quantum states, the negativity of the WDF, that is our main concern in this Chapter, is recognized as very strong indication for nonclassical character of quantum states.

The nonclassical properties of quantized light is highlighted by superposing two distinct states. Let us consider a superposition of two coherent states, that the corresponding wave function is represented in the form

$$\langle q|\psi(t)\rangle = \frac{1}{\sqrt{N}}(\langle q|\alpha_0 e^{i\varphi}\rangle + e^{i\phi}\langle q|\alpha_0 e^{-i\varphi}\rangle), \tag{55}$$

where

$$N = 2\{1 + \exp(-2\alpha_0^2 \sin^2 \varphi) \cos[\alpha_0^2 \sin(2\varphi) - \phi]\}. \tag{56}$$

Here, the total phase difference between two constituent states in superposition is 2φ, and the relative phase between the two components of the superposition is ϕ. Strictly speaking, this definition of cat state is somewhat different from that of Tara et al (Tara et al., 1993) [or that of Schleich et al. (Schleich et al., 1991), for $\phi = 0$]: The cat state of Tara et al. is defined in terms of $|\alpha e^{i\vartheta}\rangle$ and $|\alpha e^{-i\vartheta}\rangle$ instead of $|\alpha_0 e^{i\varphi}\rangle$ and $|\alpha_0 e^{-i\varphi}\rangle$, where ϑ is an arbitrary real constant. The interaction of coherent states with nonlinear medium can be a source for generating superposed coherent states (Tara et al, 1993). Not only the quadrature squeezing but also

the Sub-Poissonian and oscillatory photon statistics are typical consequences of nonclassical effects of quantum interference produced by superposition.

While the coherent states among all pure quantum states have the most properties of classicality, their superposition represented in Eq. (55) reveals remarkable features of nonclassicality. By substituting Eq. (53) into Eq. (55), we easily get the wave function in configuration space:

$$
\langle q|\psi(t)\rangle = \sqrt[4]{\frac{\Omega}{2\rho^2(t)\hbar\pi}}\frac{2}{\sqrt{N}}\exp\left[-\frac{1}{4\rho(t)\hbar}\left(\frac{\Omega}{\rho(t)}-2i\epsilon_0 e^{\Lambda(t)}\dot{\rho}(t)\right)q^2-\frac{1}{2}\alpha_0^2+\frac{i\phi}{2}\right]
$$

$$
\times \exp\left(\alpha_0\sqrt{\frac{\Omega}{\rho^2(t)\hbar}}q\cos\varphi-\frac{1}{2}\alpha_0^2\cos(2\varphi)\right)
$$

$$
\times \cos\left(\alpha_0\sqrt{\frac{\Omega}{\rho^2(t)\hbar}}q\sin\varphi-\frac{1}{2}\alpha_0^2\sin(2\varphi)-\frac{\phi}{2}\right). \tag{57}
$$

Though the illustration for the probability density $|\langle q|\psi(t)\rangle|^2$ given in Fig. 4 is somewhat complicate, the principal trajectory of $|\langle q|\psi(t)\rangle|^2$ is very similar to that of $|\langle q|\alpha\rangle|^2$ given in Fig. 3. The superposition in a case like this become a family of Schrödinger cat states only when the amplitude of the electromagnetic field is sufficiently large. Theoretical results of several previous researches (Kis et al., 2001; Varada &. Agarwal, 1993) show that certain quantum states can be approximated by superposing macroscopically distinguishable coherent states.

As is well known, WDF provides a possible method to describing a quantum system in terms of a quasi-distribution in phase space. It enables us to analyze the interference between two component states involved in the superposition. The WDF for the superposed coherent state is obtained from

$$
W(q,p,t)=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\langle\psi(t)|q+x\rangle\langle q-x|\psi(t)\rangle e^{2ipx/\hbar}dx. \tag{58}
$$

Performing the integration after inserting Eq. (57) into the above equation gives

$$
W(q,p,t)=\frac{2}{\pi\hbar N}\exp\left(2\sqrt{\frac{\Omega}{\hbar\rho^2(t)}}q\alpha_0\cos\varphi\right)\exp\left(-\frac{2}{\hbar\Omega}K(q,p,t)\right)
$$

$$
\times[\exp(-2\alpha_0^2)\cosh\Theta_1+\exp(-2\alpha_0^2\cos^2\varphi)\cos(\Theta_2-\phi)], \tag{59}
$$

where

$$
K(q,p,t)=\frac{\Omega^2}{4\rho^2(t)}q^2+\left[\rho(t)p-\epsilon_0 e^{\Lambda(t)}\dot{\rho}(t)q\right]^2, \tag{60}
$$

$$
\Theta_1=\frac{4\rho(t)\alpha_0\sin\varphi}{\sqrt{\Omega\hbar}}\left(p-\epsilon_0 e^{\Lambda(t)}\frac{\dot{\rho}(t)}{\rho(t)}q\right), \tag{61}
$$

$$
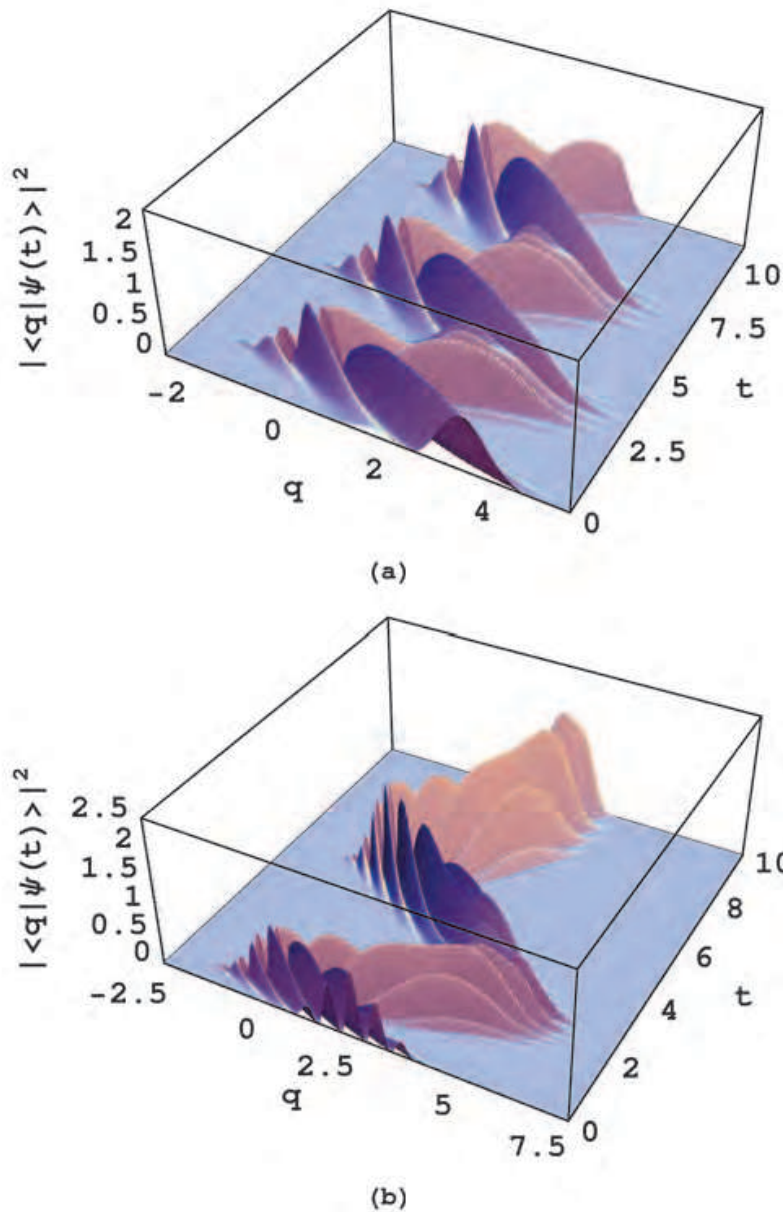\Theta_2=2\sqrt{\frac{\Omega}{\hbar\rho^2(t)}}q\alpha_0\sin\varphi-\alpha_0^2\sin(2\varphi). \tag{62}
$$

Fig. 4. Probability density for superposition of coherent states as a function of q and t. The value of γ is 0.1 for (a) and 0.3 for (b). We used $\hbar = 1$, $\epsilon_0 = 1$, $\mu(0) = 1$, $\phi = 1$, $\Omega = 1$, $c_1 = c_2 = 25$, $\omega(0) = 1$, $h_1 = h_3 = \Omega/(2\mathbf{w})$, $h_2 = 0$, and $\rho_{1,0} = \rho_{2,0} = 0.5$.

From Fig. 5, we can find the nonclassical characteristics of the superposed coherent state. The two bells that are Gaussian type correspond to the two constituent coherent states, and the ripple given in the middle between them is taken place from quantum interference. Interference occurs when the two bells do not overlap, but the WDF should have non-zero values along their common intervals in q and/or p coordinates (Dragoman, 2001). As you can see, some parts of the ripple take on negative value. This is a clear signal for the existence of nonclassical features of the system. The number of peaks in the structure of the ripple becomes large as the distance between the two bells increases. We can confirm from this aspect that the wavelength of interference fringe is inversely proportional to the value of α_0. The wavelength
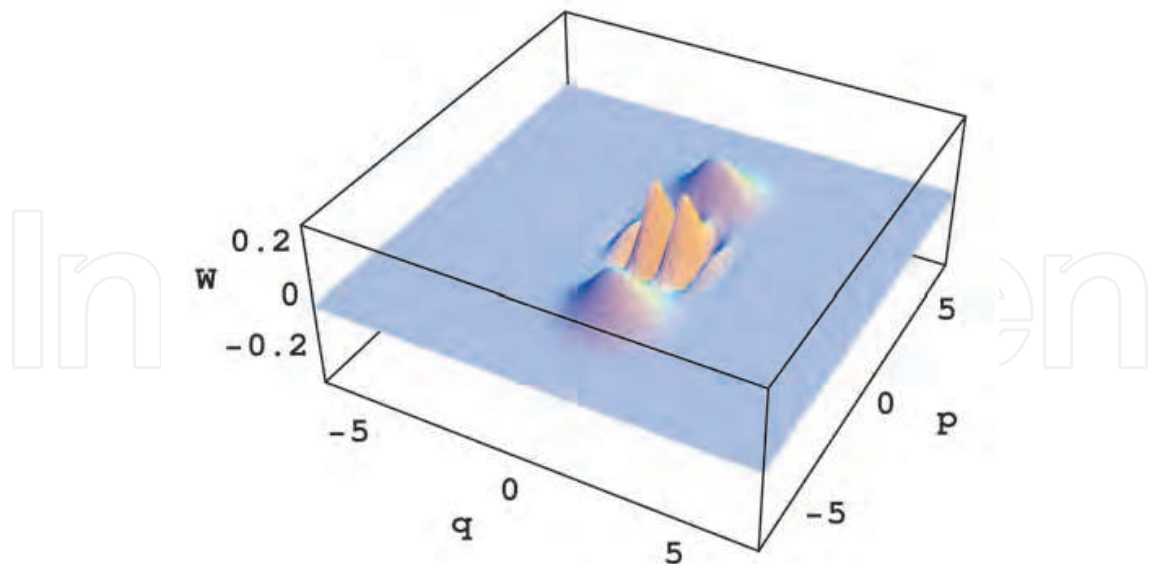
Fig. 5. Quadrature plot of WDF for superposition of coherent states. We used $\hbar = 1$, $\gamma = 0.1$, $\epsilon_0 = 1$, $\mu(0) = 1$, $\phi = 1$, $\Omega = 1$, $c_1 = c_2 = 25$, $\omega(0) = 1$, $t = 1$, $h_1 = h_3 = \Omega/(2\mathbf{w})$, $h_2 = 0$, and $\rho_{1,0} = \rho_{2,0} = 0.5$.

of interference fringe is a major factor that determine the shape of interference in probability density displayed in Fig. 4 (and Figs. 6 and 7 for the superposed squeezed state). Recall that the interference pattern in probability density for an arbitrary superposition state is connected with its WDF via Eqs. (38) and (39). If we consider, in the context of classical mechanics, that the physical attributes of a system exist objectively even when it is unknown, classical mechanics fails to give a reasonable explanation for the negative values in the superposition states.

If we consider that the last term in Eq. (59) involving $\cos(\Theta_2 - \phi)$ is the interference term, the structure of interference varies according to the value of ϕ. The superposition with $\phi = 0$ for simple harmonic oscillator and the corresponding WDF are studied in detail by several researchers (Bužek et al., 1992; Raimond et al., 1997; Schleich et al., 1991; Varada &. Agarwal, 1993). In particular, some researchers (Simon et al., 1997; Yurke & Stoler, 1986) are interested in superpositions with $\phi = \pi/2$ for a little different aspect than here, thanks to their experimental realizability, for this family of states, through the evolution of a coherent state in a Kerr medium. For $\phi = \pi/2$, Eq. (59) becomes

$$W(\phi = \pi/2; q, p, t) = \frac{2}{\pi\hbar N_{\pi/2}} \exp\left[-2\sqrt{\frac{\Omega}{\hbar\rho^2(t)}}q\alpha_0\cos\varphi\right]\exp\left(-\frac{2}{\hbar\Omega}K(q, p, t)\right)$$
$$\times[\exp(-2\alpha_0^2)\cosh\Theta_1 + \exp(-2\alpha_0^2\cos^2\varphi)\sin\Theta_2], \qquad (63)$$

where
$$N_{\pi/2} = 2\{1 + \exp(-2\alpha_0^2\sin^2\varphi)\sin[\alpha_0^2\sin(2\varphi)]\}. \qquad (64)$$

Nonclassical features for this family of states are studied extensively in the literature (Ahmad et al., 2011).

4. Superposition of squeezed states

The investigation of the properties of squeezed state and its generation is also a central topic in quantum optics since it enables us to utilize an optical field with reduced quadrature noise. We introduce a squeeze operator as

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^{\dagger}, \tag{65}$$

where μ and ν obey

$$|\mu|^2 - |\nu|^2 = 1. \tag{66}$$

As is the case of $\hat{a}$ and $\hat{a}^{\dagger}$, this operator and its Hermitian conjugate satisfy the boson commutation relation, $[\hat{b}, \hat{b}^{\dagger}] = 1$. Let us consider only real values for μ and ν on the purpose to simplify the problem. Squeezed state $|\beta\rangle$ is the eigenstate of $\hat{b}$:

$$\hat{b}|\beta\rangle = \beta|\beta\rangle. \tag{67}$$

For convenience in further study, we introduce a squeezing parameter as $d = \mu/\nu$. Then, the wave function of squeezed state in configuration space can be evaluated in terms of d using Eq. (67):

$$\langle q|\beta\rangle = \mathcal{N}_q \exp\left[-\frac{q^2}{2\rho(t)\hbar}\left(\frac{\Omega}{2\rho(t)}\frac{d+1}{d-1} - i\epsilon_0 e^{\Lambda(t)}\dot{\rho}(t) \right) + \frac{d\alpha + \alpha^*}{d-1}\sqrt{\frac{\Omega}{\hbar\rho^2(t)}}q \right], \tag{68}$$

where a normalization factor $\mathcal{N}_q$ is given by

$$\mathcal{N}_q = \left(\frac{\Omega}{2\rho^2(t)\hbar\pi}\frac{d+1}{d-1} \right)^{1/4} \exp\left(-\frac{d+1}{d-1}\alpha_0^2\cos^2\varphi + i\delta_{s,q}(\alpha, \alpha^*) \right), \tag{69}$$

with an arbitrary real phase $\delta_{s,q}(\alpha, \alpha^*)$. Considering Eqs. (66) and (67), it is easy to show that the eigenvalue β can be written in the form

$$\beta = \mu\alpha + \nu\alpha^*. \tag{70}$$

Now we represent β as

$$\beta = \beta_0 e^{i\varphi_\beta}, \tag{71}$$

where β_0 and φ_β are real. Execution of an algebra with the substitution of Eq. (46) into Eq. (70) yields

$$\beta_0 = \alpha_0\sqrt{\mu^2 + \nu^2 + 2\mu\nu\cos(2\varphi)}, \tag{72}$$

$$\varphi_\beta = \tan^{-1}\left(\frac{d-1}{d+1}\tan\varphi \right). \tag{73}$$

Let us take $\delta_{s,q}(\alpha, \alpha^*)$ in the form

$$\delta_{s,q}(\alpha, \alpha^*) = -\alpha_0^2\sin\varphi\cos\varphi. \tag{74}$$

Then, Eq. (68) reduces to a simple form which is

$$\langle q|\beta \rangle = \left(\frac{\Omega}{2\pi\hbar\rho^2(t)} \frac{d+1}{d-1} \right)^{1/4} \exp \left[-\frac{1}{2\rho(t)\hbar} \left(\frac{\Omega}{2\rho(t)} \frac{d+1}{d-1} - i\epsilon_0 e^{\Lambda(t)} \dot{\rho}(t) \right) q^2 \right.$$

$$\left. +\frac{d\alpha + \alpha^*}{d-1} \left(\sqrt{\frac{\Omega}{\hbar\rho^2(t)}} q - \alpha_0 \cos\varphi \right) \right]. \tag{75}$$

The squeezed state which have this wave function belongs to a nonclassical state. If we recall that any non-commuting observables in quantum mechanics can be determined simultaneously in classical mechanics with any order of precision, classical analogue for squeezing is unthinkable.

A superposition of squeezed states may also be useful in understanding nonclassical features of quantum states. Let us consider the superposition of $\langle q|\beta \rangle$ and $\langle q|\beta^* \rangle$ with an arbitrary relative phase ϕ. The wave function for this system is given by

$$\langle q|\Psi(t) \rangle = \frac{1}{\sqrt{\mathcal{N}}} (\langle q|\beta_0 e^{i\varphi_\beta} \rangle + e^{i\phi} \langle q|\beta_0 e^{-i\varphi_\beta} \rangle), \tag{76}$$

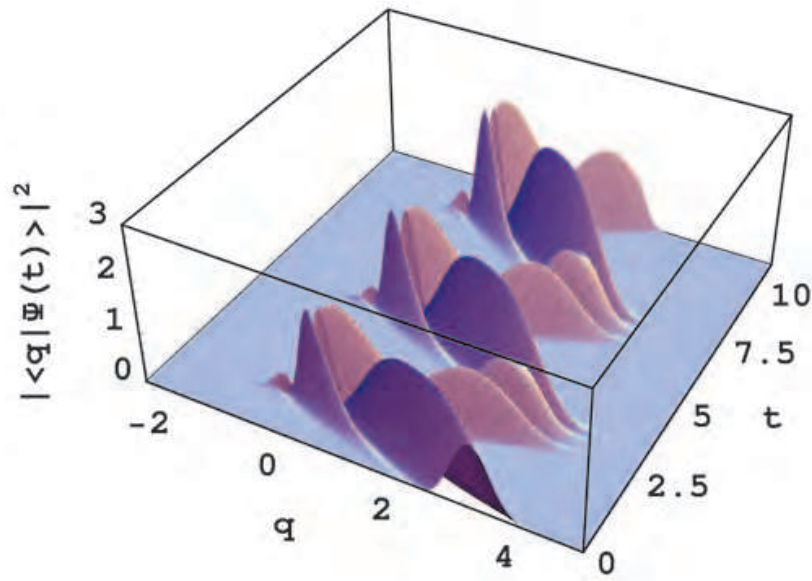where $\mathcal{N}$ is a normalization constant of the form

$$\mathcal{N} = 2 \left[1 + \exp \left(-\frac{2(d-1)}{d+1} \alpha_0^2 \sin^2\varphi \right) \cos[\alpha_0^2 \sin(2\varphi) - \phi] \right]. \tag{77}$$

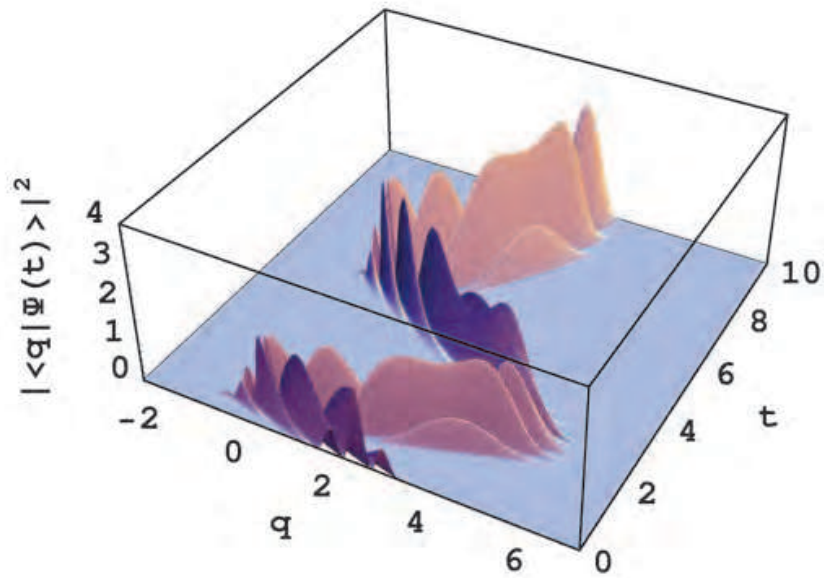Substituting Eq. (75) into Eq. (76) and, then, executing some algebra results in

$$\langle q|\Psi(t) \rangle = \left(\frac{\Omega}{2\rho^2(t)\hbar\pi} \frac{d+1}{d-1} \right)^{1/4} \frac{2}{\sqrt{\mathcal{N}}} \exp \left[-\frac{1}{2\rho(t)\hbar} \left(\frac{\Omega}{2\rho(t)} \frac{d+1}{d-1} - i\epsilon_0 e^{\Lambda(t)} \dot{\rho}(t) \right) q^2 \right.$$

$$\left. +\frac{(d+1)\alpha_0 \cos\varphi}{d-1} \left(\sqrt{\frac{\Omega}{\hbar\rho^2(t)}} q - \alpha_0 \cos\varphi \right) + \frac{i\phi}{2} \right]$$

$$\times \cos \left[\alpha_0 \sin\varphi \left(\sqrt{\frac{\Omega}{\hbar\rho^2(t)}} q - \alpha_0 \cos\varphi \right) - \frac{\phi}{2} \right]. \tag{78}$$

Figures 6 and 7 are probability densities, $|\langle q|\Psi(t) \rangle|^2$, with the squeezing for q-quadrature and for p-quadrature, respectively. By comparing these with Fig. 4, we see that the width of densities are narrowed for the squeezing for q-quadrature and broadened for p-quadrature. Thus the uncertainty of q is reduced for the case of the squeezing for q-quadrature, while increased for p-quadrature. Therefore, through application of squeezed states, we are able to reduce noise dispersion in one quadrature at the expense of increased noise in the complementary quadrature as compared with that of coherent state. For this reason, squeezed states of light can be applied to high precision interferometer that provides high resolutions in measurement beyond the standard limits.

All wave functions that are given in Eqs. (32)[or (33)], (53), (57), (75), and (78) satisfy Schrödinger equation when supplemented by some time-dependent phase factors suitable

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Fig. 6. Probability density for superposition of squeezed states as a function of q and t. The value of γ is 0.1 for (a) and 0.3 for (b). The same values as that of Fig. 4 are used except for $d = 2$.

for each. The phase (factor) for number state is given in Eq. (34) and that for other wave functions can also be derived from the same method as that of the number state (Choi, 2011).

The quantum wave functions have no reality of their own and are just associated with the probability to find photon in a certain domain as mentioned previously. The mathematical description in classical optics for the interference in phase space is very similar to that in quantum optics, but the classically represented optical waves have a real character.
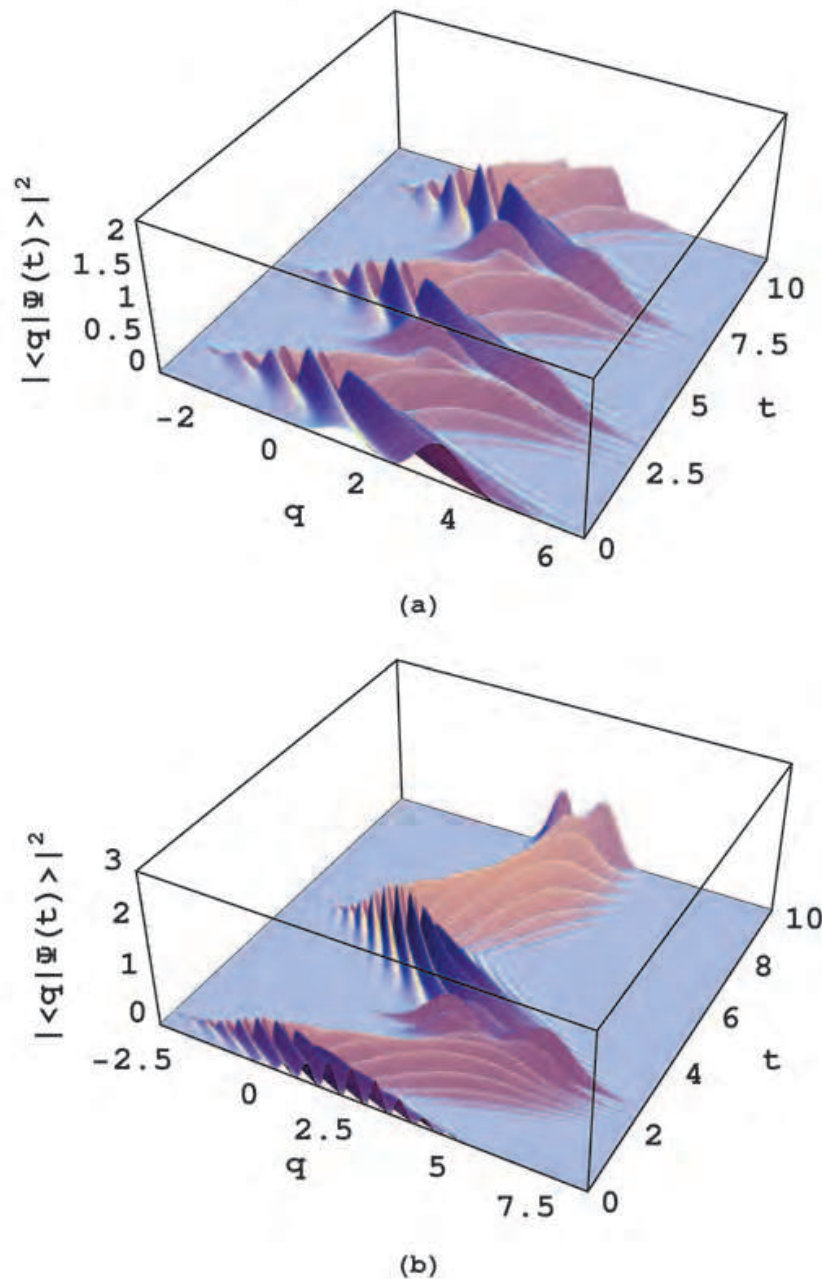
(a)

(b)

Fig. 7. Probability density for superposition of squeezed states as a function of q and t. The value of γ is 0.1 for (a) and 0.3 for (b). The same values as that of Fig. 4 are used except for $d = -2$.

Using the same method as that of previous section, the WDF that corresponds to Eq. (78) is evaluated to be

$$
\begin{aligned}
\mathcal{W}(q, p, t) = {} & \frac{2}{\pi \hbar \mathcal{N}} \exp \left(-\frac{2}{\hbar \Omega} K_s(q, p, t) \right) \\
& \times \exp \left[\frac{d+1}{d-1} \left(2 \sqrt{\frac{\Omega}{\hbar \rho^2(t)}} q \alpha_0 \cos \varphi - 2 \alpha_0^2 \cos^2 \varphi \right) \right] \\
& \times \left[\exp \left(-\frac{d-1}{d+1} 2 \alpha_0^2 \sin^2 \varphi \right) \cosh \left(\frac{d-1}{d+1} \Theta_1 \right) + \cos(\Theta_2 - \phi) \right],
\end{aligned} \tag{79}
$$

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where

$$K_s(q,p,t) = \frac{\Omega^2}{4\rho^2(t)}\frac{d+1}{d-1}q^2 + \frac{d-1}{d+1}[\rho(t)p - \epsilon_0 e^{\Lambda(t)}\dot{\rho}(t)q]^2. \tag{80}$$

The WDF with squeezing is plotted in Fig. 8: (a) corresponds to squeezing for q-quadrature
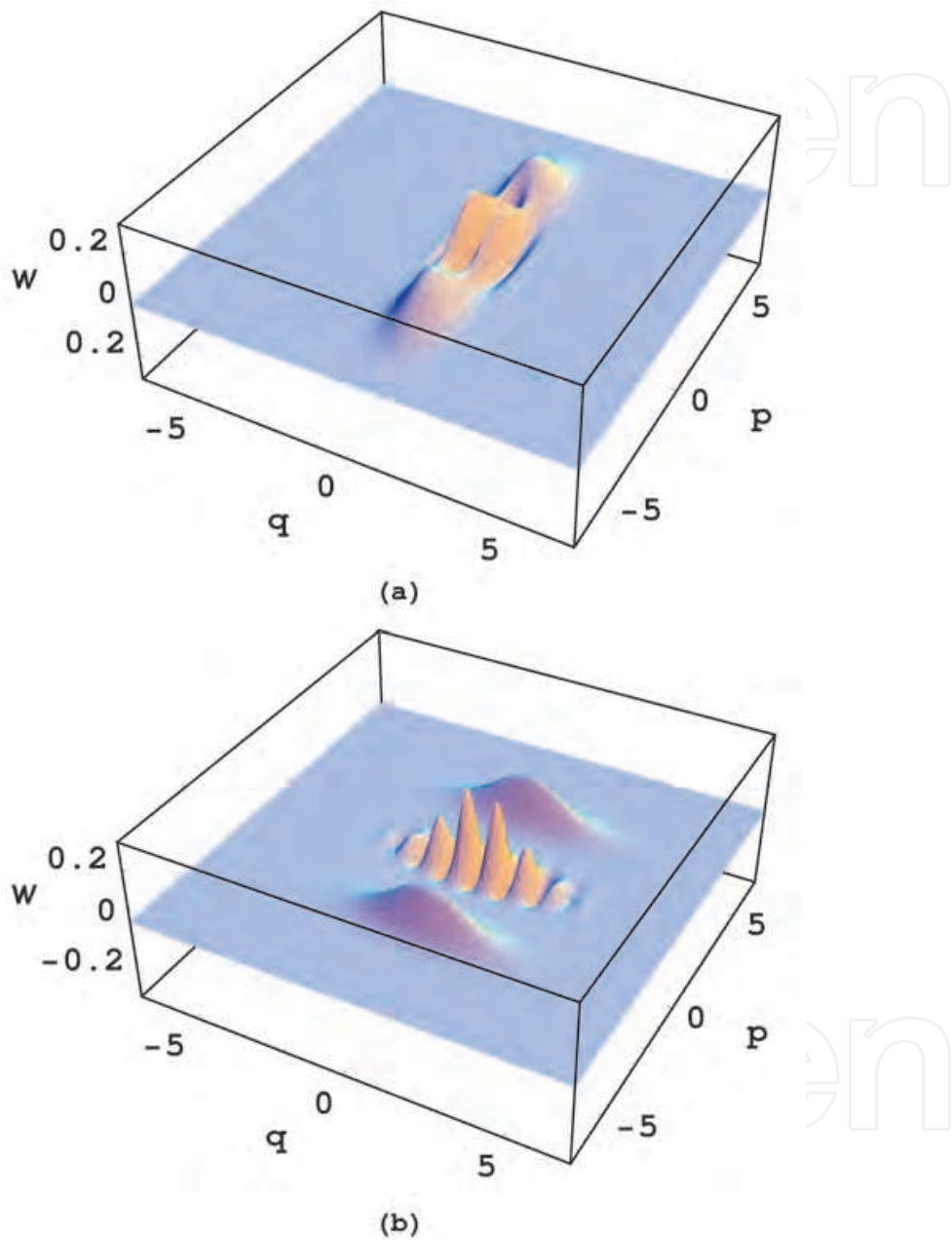

(a)

(b)

Fig. 8. Quadrature plot of WDF for superposition of squeezed states with $d = 3$ for (a) and $d = -2$ for (b). All values used here are the same as those in Fig. 5.

and (b) for p-quadrature. The width of two bells is shortened along the direction of q for (a) and shortened along the direction of p for (b) when they are compared to that of coherent state shown in Fig. 5.

We can represent the position of two bells in polar coordinate as $(\alpha_0, \pm\varphi)$ (Schleich et al., 1991). For $\varphi = 0$ and $\varphi = \pi$, the two bells overlap: There are no interferences in this case, but are quantum transitions between two different quantum states overlapped (Dragoman, 2001). On the other hand, for $\varphi = \pi/2$, they are separated from each other with maximum distance. The microscopical pattern of the structure of interference is directly related to the value of ϕ. Of course, these rules mentioned in this paragraph are equally applied to the case of the superposition of coherent state.

The WDF plays a crucial role in analyzing nonclassical characteristics of quantum states. You can confirm the nonclassical features for the superposed squeezed states from negative values in interference structure displayed in Fig. 8, which are very similar to that of previous section. The methods for interpreting quantum superpositions are different from that for simple addition of probability distributions, because, in quantum mechanics, we deal with superpositions of probability amplitudes instead of those of probabilities themselves. This is closely related to the appearance of interference terms in the distribution functions of probability. The novel effects of nonclassical states that admits no analogue in classical mechanics have drawn special attention both in theoretical and experimental physics thanks to their applicability in modern technology employing optical and/or other dynamical systems (Ourjoumtsev et al, 2006).

The development of modern technology in experimental photon engineering have made it possible to produce Schrödinger cat states and/or kitten states (small Schrödinger cat states) on the basis of effective nonlinear operations that can be realized via projective measurements and post-selection. Projective measurements based on the Hilbert space formulation of quantum theory produce complete determinations of the post-measurement states through the projection-valued measures of a Hermitian operator (von Neumann, 1932). Kitten states can be produced by squeezing a single-photon. An interesting and useful way to obtain a squeezed single-photon is subtracting one photon from a single-mode squeezed vacuum beam generated by an optical nonlinear process, so-called degenerate optical parametric down-conversion (Ourjoumtsev et al., 2006). A sufficiently large Schrödinger cat states with a smaller overlap between two constituent states can be created by subtracting multiphoton from a squeezed vacuum beam (Neergaard-Nielsen et al., 2011). Other methods for preparation of superposition states include a squeezed Schrödinger cat state prepared by conditional homodyning of a two-photon Fock state (Ourjoumtsev et al., 2007), high-fidelity superposition states prepared using cavity QED technology (de Queirós et al., 2007), and preparation of entangled non-local superposition states (Ourjoumtsev et al., 2009).

5. Conclusion

Nonclassical features of superpositions of coherent states and squeezed states for electromagnetic field in linear media whose electromagnetic parameters vary with time are examined. The expansion of Maxwell equations in charge-source free medium gives second order differential equations for both position function $\mathbf{u}(\mathbf{r})$ and time function $q(t)$. The Hamiltonian associated with the classical equation of motion for $q(t)$ varies with time. Among several methods that are useful in managing time-dependent Hamiltonian systems, the quantum invariant operator method is used in order to solve quantum solutions of the system. The annihilation and the creation operators related to quantum invariant operator

satisfy boson commutation relation. The wave functions in number state are derived by taking advantage of the annihilation and the creation operators.

Coherent state is obtained from the expansion of the wave functions of number state. By solving the eigenvalue equation of squeeze operator, squeezed state is also obtained. We can confirm from Figs. 4, 6 and 7, that the detailed structure of probability densities for superposition states are somewhat complicated due to the interference between the two component states. We cannot observe nonclassical properties of the coherent state from WDF, because the value of WDF for a single coherent state is always positive. However, a minor nonclassicality of the coherent state has been reported (Johansen, 2004): A particular quantum distribution function for the coherent state of simple harmonic oscillator, the so-called Margenau-Hill distribution, can take negative values in some regions, but the negative values are relatively very small. This appearance reflects the nonclassicality demonstrated in weak measurements which are, in general, performed under the situation where the coupling of a measuring device to the measured system is very weak. The average values obtained from the weak measurement reveal a time-symmetric dependence on initial and final conditions (Shikano & Hosoya, 2010), providing a natural definition of conditional probabilities in quantum mechanics and, consequently, enabling a more complete description for quantum statistics. However, the interpretation of the results of weak measurements is somewhat controversial on account of its peculiar feature that the measured (weak) value can take strange ones which are outside the range of the eigenvalues of a target observable and may even be complex. The detailed analysis of the strange features of weak measurements may provide a better understanding for the essential differences between quantum and classical statistics.

From Figs. 5 and 8, we can see the interference structure produced between the two main bells. If we consider Eqs. (38) and (39), this determines the pattern of interference fringe in real space of q and p. Though two main bells are always positive, interference structure takes positive and negative values in turn, where the appearance of the negative values is an important signal for the nonclassicality of the system. In fact, nonclassical quantum states are general and ubiquitous. Not only any pure state of the harmonic oscillator can be represented in terms of nonclassical quantum states but also even the number state is a class of nonclassical quantum states (Kis et al., 2001). Nonclassical states in physical fields such as various optical systems, ion motion in a Paul trap, and quantum dot can be applied to fundamental problems in modern technology ranging from high-resolution spectroscopy to low-noise communication and quantum information processing (Kis et al., 2001). In particular, nonclassical properties of correlated quantum systems are expected to play the key role to overcome some limitations relevant to information processing in classical computer system and classical communication.

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