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Cosmology: The Noncommutative Quantum and Classical Cosmology

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1. Introduction

During the early days of quantum mechanics and quantum field theory, continuous space-time and Lorentz symmetry was considered inappropriate to describe the small scale structure of the universe. It was also argued that one should introduce a fundamental length scale, limiting the precision of position measurements. The introduction of fundamental length is suggested to cure the ultraviolet divergencies occurring in quantum field theory. H. Snyder was the first to formulate these ideas mathematically (1), introducing noncommutative coordinates brings an uncertainty in the position. The success of the renormalisation made people forget about these ideas for some time. But when the quantization of gravity was considered thoroughly, it became clear that the usual concepts of space-time are inadequate and that spacetime has to be quantised or noncommutative, in some way. Quantum cosmology, is a simplified approach to the study of the very early universe, which means that the gravitational and matter variables have been reduced to a finite number of degrees of freedom (these models were extensively studied by means of Hamiltonian methods in the 1970's, (for reviews see (2; 3)); for homogenous cosmological models the metric depends only on time, this permits to integrate the space dependence and obtain a model with a finite dimensional configuration space, minisuperspace, whose variables are the 3-metric components. One way to extract useful dynamical information is through a WKB type method. The semiclassical or WKB approximations are usually discussed in text books on nonrelativistic quantum mechanics in the context of stationary states, i.e., determination of the energy eigenvalues and eigenfunctions (4). This approximation can also be used to obtain approximate and in some cases exact solutions of the dynamical problem, i.e., full Schroedinger equation, so the utility of the semiclassical approximation in obtaining exact solutions of the Schroedinger equation has not yet fully explored.

The same seems to be the case for the relativistic quantum mechanics. The importance of the semiclassical approximation in the relativistic case is probably best appreciated in quantum cosmology (5), specifically, in the analysis of the Wheeler-DeWitt equation, which is essentially a Klein-Gordon equation on the minisuperspace (6). In the last few years there have been several attempts to study the possible effects of noncommutativity in the classical cosmological scenario (7–9). The proposal of authors in Ref 9 introduces the effects

of noncommutativity at the quantum level, namely quantum cosmology, by deforming the minisuperspace through a Moyal deformation of the Wheeler-DeWitt (WDW) equation, similar to noncommutative quantum mechanics (10). The aim of this chapter is to introduce a deformation in the minisuperscpace variables through the Moyal product of the Wheeler DeWitt equation and apply a WKB type method to noncommutative quantum cosmology, and find the noncommutative classical solutions, avoiding in this way the difficult task to solve this cosmological models in the complicated framework of noncommutative gravity (11). We know how to introduce noncommutativity at a quantum level, by taking into account the changes that the Moyal product of functions induces on the quantum equation, and from there calculate the effects of noncommutativity at the classical level. This also has the advantage that for some noncommutative models for which the quantum solutions can not be found, the noncommutative classical solutions arise very easily from this formulation. This procedure is presented through many examples: first the Kanstowski-Sachs (KS) cosmological model is presented and the formalism developed in this model, is the applied to the Friedmann-Robertson-Walker (FRW) universe coupled to a scalar field and cosmological constant, besides we show the noncommutative proposal applied to a stringy model and the Bianchi I with Baratropic perfect fluid and Λ cosmological.

2. Cosmology

We start by reviewing the quantum cosmological models in which we are interested, and find the classical evolution through the WKB-type approximation.

2.1 Kantowski-Sachs (KS) Cosmology

The first example that we are interested because the simplest anisotropic, is the KS universe, part of the interest in this universe model is due to the wide set of analitycal solutions it admits, even if particular types of matter are coupled to gravity.

The Kanstowski-Sachs line element (12) is

$$ds^2 = -N^2 dt^2 + e^{2\sqrt{3}\beta} dr^2 + e^{-2\sqrt{3}\beta} e^{-2\sqrt{3}\Omega} \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right). \tag{1}$$

from the general relativity lagrangian we can construct the canonical momenta,

$$\Pi_\Omega = -\frac{12}{N} e^{-\sqrt{3}\beta - 2\sqrt{3}\Omega} \dot{\Omega}$$

$$\Pi_\beta = \frac{12}{N} e^{-\sqrt{3}\beta - 2\sqrt{3}\Omega} \dot{\beta}, \tag{2}$$

and the corresponding Hamiltonian

$$H = \frac{N}{24} e^{-\sqrt{3}\beta - 2\sqrt{3}\Omega} \left[-\Pi_\Omega^2 + \Pi_\beta^2 + 48 e^{-2\sqrt{3}\Omega} \right], \tag{3}$$

for this model we can use canonical quantization and obtain the Wheeler-DeWitt (WDW) equation. Using the usual identifications $\Pi_\Omega = -i\frac{\partial}{\partial\Omega}$ and $\Pi_\beta = -i\frac{\partial}{\partial\beta}$ we get

$$\left[\frac{\partial^2}{\partial\Omega^2} - \frac{\partial^2}{\partial\beta^2} - 48 e^{-2\sqrt{3}\Omega} \right] \psi(\Omega, \beta) = 0, \tag{4}$$

in this parametrization the WDW equation has very simple form; the solutions to this equation are given by

$$\psi = e^{\pm i\nu\sqrt{3}\beta} K_{i\nu}\left(4e^{-\sqrt{3}\Omega}\right), \tag{5}$$

where ν is the separation constant and $K_{i\nu}$ are the modified Bessel functions. We now proceed to apply the WKB method. For this we propose the wave function

$$\Psi(\beta, \Omega) = e^{i(S_1(\beta)+S_2(\Omega))}, \tag{6}$$

the WKB approximation is reached in the limit

$$\left|\frac{\partial S_1^2(\beta)}{\partial \beta^2}\right| << \left(\frac{\partial S_1(\beta)}{\partial \beta}\right)^2 \qquad \left|\frac{\partial S_2^2(\Omega)}{\partial \Omega^2}\right| << \left(\frac{\partial S_2(\Omega)}{\partial \Omega}\right)^2 \tag{7}$$

this gives the Einstein-Hamilton-Jacobi (EHJ) equation

$$-\left(\frac{\partial S_2(\Omega)}{\partial \Omega}\right)^2 + \left(\frac{\partial S_1(\beta)}{\partial \beta}\right)^2 - 48e^{-2\sqrt{3}\Omega} = 0, \tag{8}$$

solving (equation 8) gives the functions S_1, S_2 and using the definition for the momenta

$$\Pi_\beta = \frac{dS_1(\beta)}{d\beta}, \qquad \Pi_\Omega = \frac{dS_2(\Omega)}{d\Omega}, \tag{9}$$

which combined with (equation 2) and fixing the value of $N(t) = 24e^{-\sqrt{3}\beta - 2\sqrt{3}\Omega}$ we find

$$S_1(\beta) = P_{\beta_0}\beta,$$

$$S_2(\omega) = -\frac{1}{\sqrt{3}}\sqrt{P_{\beta_0}^2 - 48e^{-2\sqrt{3}\Omega}} + \frac{P_{\beta_0}}{\sqrt{3}}\mathrm{arctanh}\left[\frac{\sqrt{P_{\beta_0}^2 - 48e^{-2\sqrt{3}\Omega}}}{P_{\beta_0}}\right], \tag{10}$$

from this solutions and using (equation 2) and (equation 9) we obtain the classical solutions

$$\Omega(t) = \frac{1}{2\sqrt{3}}\ln\left[\frac{48}{P_{\beta_0}^2}\cosh^2\left(2\sqrt{3}P_{\beta_0}(t-t_0)\right)\right],$$

$$\beta(t) = \beta_0 + 2P_{\beta_0}(t-t_0), \tag{11}$$

this solutions are the same that solving the field equations of General Relativity.

2.2 Friedmann-Robertson-Walker (FRW) Cosmology with scalar field and Λ

The next set of examples correspond to homogeneous and isotropic Universes, the so called Friedmann-Robertson-Walker (FRW) universe coupled to a scalar field and cosmological constant. The FRW metric is given by

$$ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}\left[\frac{dr^2}{1-kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right], \tag{12}$$

where $a(t) = e^{\alpha(t)}$ is the scale factor, $N(t)$ is the lapse function, and k is the curvature constant that takes the values $0, +1, -1$, which correspond to a flat, closed and open universes,

respectively. The Lagrangian we are to work on, is composed by the gravity sector and the matter sector, which for the FRW universe endowed with a scalar field and cosmological constant Λ is

$$\mathcal{L}_{tot} = \mathcal{L}_g + \mathcal{L}_\phi = e^{3\alpha} \left[6\frac{\dot{\alpha}^2}{N} - \frac{1}{2}\frac{\dot{\phi}^2}{N} - N\left(2\Lambda + 6ke^{-2\alpha}\right) \right], \tag{13}$$

the corresponding canonical momenta are

$$\Pi_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = 12e^{3\alpha}\frac{\dot{\alpha}}{N}, \qquad \Pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - e^{3\alpha}\frac{\dot{\phi}}{N}, \tag{14}$$

proceeding as before the WDW equation is obtained from the classical Hamiltonian. By the variation of (equation 13) with respect to N, $\partial \mathcal{L}/\partial N = 0$, implies the well-known result $\mathcal{H} = 0$.

$$e^{-3\alpha}N\left[-\frac{1}{24}\frac{\partial^2}{\partial \alpha^2} + \frac{1}{2}\frac{\partial^2}{\partial \phi^2} + e^{6\alpha}\left(2\Lambda + 6ke^{-2\alpha}\right) \right]\Psi(\alpha, \phi) = 0. \tag{15}$$

Now that we have the complete framework and the corresponding WDW equation, we can proceed to study different cases.

In table 1 we can see the different cases that we solved[1], all of them are calculated by using the WKB type procedure, the classical solutions are the same we would get by solving Einstein's field equations. We can expect that this approximation includes all the gravitational degrees of freedom of the particular cosmological model under study. This almost trivial observation is central to the ideas we are presenting in the next section.

2.3 Stringy Quantum Cosmology

In the case of strings, this example is related to the graceful exit of pre-big bang cosmology (13), this model is based on the gravi-dilaton effective action in 3+1 dimensions

$$S = -\frac{\lambda_s}{2}\int d^4x\sqrt{-g}e^{-\phi}(R + \partial_\mu\phi\partial^\nu\phi + V), \tag{16}$$

in this expression λ_s is the fundamental string length, ϕ is the dilaton field with V the possible dilaton potential. Working with an isotropic background, and setting $a(t) = e^{\beta(t)/\sqrt{3}}$, after integrating by parts, we get

$$S = -\frac{\lambda_s}{2}\int d\tau\left(\bar{\phi}'^2 - \beta'^2 + Ve^{-2\bar{\phi}}\right), \tag{17}$$

we have used the time parametrization[2] $dt = e^{-\bar{\phi}}d\tau$, the gauge $g_{00} = 1$, and defined $\bar{\phi} = \phi - \ln\int\left(\frac{d^3x}{\lambda_s^3}\right) - \sqrt{3}\beta$. From this action we calculate the canonical momenta, $\Pi_\beta = \lambda_s\beta'$ and $\Pi_{\bar{\phi}} = -\lambda_s\bar{\phi}'$. From the classical hamiltonian we find the WDW equation

$$\left[\frac{\partial^2}{\partial\bar{\phi}^2} - \frac{\partial^2}{\partial\beta^2} + \lambda_s^2 V(\bar{\phi}, \beta)e^{-2\bar{\phi}}\right]\Psi(\bar{\phi}, \beta) = 0, \tag{18}$$

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[1] The case $k \neq 0$, $\Lambda \neq 0$ does not have a closed analytical solution to the WDW equation.

[2] The prime denotes differentiation respect to τ.

case	Quantum Solution	Classical Solution
$k=0, \Lambda \neq 0$	$\psi = e^{\pm i\nu\frac{\sqrt{3}}{2}\phi}K_{i\nu}\left(4\sqrt{\frac{\Lambda}{3}}e^{3\alpha}\right)$ and J_ν for $\Lambda < 0$	$\phi(t) = \phi_0 - P_{\phi_0}t,$ $\alpha(t) = \frac{1}{6}\ln\left(\frac{P_{\phi_0}^2}{4\Lambda}\right)$ $+\frac{1}{3}\ln\left(\text{sech}\left[\frac{\sqrt{3}}{2}P_{\phi_0}(t-t_0)\right]\right),$
$k\neq0, \Lambda = 0$	$\psi^{(1)} = e^{\pm i\frac{\nu}{\sqrt{3}}\phi}K_{i\nu}\left(6e^{2\alpha}\right)$ for $k=1$, $\psi^{(2)} = e^{\pm i\frac{\nu}{\sqrt{3}}\phi}J_\nu\left(6e^{2\alpha}\right)$ for $k=-1$	$\phi(t) = \phi_0 - P_{\phi_0}(t-t_0),$ $\alpha(t) = \frac{1}{4}\ln\left[\frac{P_{\phi_0}^2}{12k}\right]$ $+\frac{1}{2}\ln\left(\text{sech}\left[\frac{1}{\sqrt{3}}P_{\phi_0}(t-t_0)\right]\right),$
$k\neq0, \Lambda \neq 0$	Unknown	$\phi(t) = \phi_0 - P_{\phi_0}(t-t_0),$ $\int \frac{d\alpha(t)}{\sqrt{P_{\phi_0}-2e^{6\alpha}(2\Lambda+6ke^{-2\alpha})}} = \frac{1}{\sqrt{12}}(t-t_0),$

Table 1. Classical and quantum solutions for the FRW universe coupled to a scalar field ϕ. For the case $\Lambda \neq 0$ $k \neq 0$, the classical solution for the scale factor is given in an implicit expression. We have fixed the lapse function to $N(t) = e^{3\alpha}$.

in particular for a potential of the form $V(\bar{\phi}) = -V_0e^{m\bar{\phi}}$, the quantum solution is

$$\Psi(\bar{\phi},\beta) = e^{\pm -i\frac{m-2}{2}\nu\beta}K_{i\nu}\left[\frac{2\lambda_s\sqrt{V_0}}{m-2}e^{(\frac{m-2}{2})\bar{\phi}}\right]. \tag{19}$$

The classical solutions for the scale factor and the dilaton are

$$\bar{\phi}(\tau) = \frac{1}{m-2}\ln\left[\frac{P_{\beta_0}^2}{V_0\lambda_s^2}\text{sech}^2\left(\frac{P_{\beta_0}}{2\lambda_s}(m-2)(\tau-\tau_0)\right)\right],$$

$$\beta(\tau) = \beta_0 + \frac{P_\beta}{\lambda_s}(\tau-\tau_0), \tag{20}$$

for $m=0$ and $m=4$, the solutions have been obtained in (13), and are used in connection to the graceful exit from pre-big bang cosmology in quantum string.

2.4 Isotropization in Bianchi I with barotropic perfect fluid and Λ Cosmological
In our final example let us begin by recalling canonical formulation of the ADM formalism to the diagonal Bianchi Class A cosmological models. The metrics have the form

$$ds^2 = -(N^2 - N^jN_j)dt^2 + e^{2\Omega(t)}e^{2\beta_{ij}(t)}\omega^i\omega^j, \tag{21}$$

where N and N_i are the lapse and shift functions, respectively, $\Omega(t)$ is a scalar and $\beta_{ij}(t)$ a 3x3 diagonal matrix, $\beta_{ij} = diag(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$, ω^i are one-forms that characterize each cosmological Bianchi type model, and that obey $d\omega^i = \frac{1}{2}C^i_{jk}\omega^j \wedge \omega^k$, C^i_{jk} the

structure constants of the corresponding invariance group (8). The metric for the Bianchi type I, takes the form

$$ds_I^2 = -N^2 dt^2 + e^{2\Omega} e^{2\beta_+ + 2\sqrt{3}\beta_-} dx^2 + e^{2\Omega} e^{2\beta_+ - 2\sqrt{3}\beta_-} dy^2 + e^{2\Omega} e^{-4\beta_+} dz^2, \qquad (22)$$

The corresponding lagrangian density is

$$L_{Total} = \sqrt{-g}\,(R - 2\Lambda) + L_{matter}, \qquad (23)$$

and using (equation 22), this have the following form

$$L = 6e^{3\Omega} \left[-\frac{\dot{\Omega}^2}{N} + \frac{\dot{\beta}_+^2}{N} + \frac{\dot{\beta}_-^2}{N} - \frac{\Lambda}{3}N + \frac{8}{3}\pi G N \rho \right]. \qquad (24)$$

where the overdot denotes time derivatives. The canonical momentas to coordinate fields are defined in the usual way

$$P_\Omega = \frac{\partial L}{\partial \dot{\Omega}} = -12e^{3\Omega}\frac{\dot{\Omega}}{N}, \quad P_+ = \frac{\partial L}{\partial \dot{\beta}_+} = 12e^{3\Omega}\frac{\dot{\beta}_+}{N}, \quad P_- = \frac{\partial L}{\partial \dot{\beta}_-} = 12e^{3\Omega}\frac{\dot{\beta}_-}{N}, \qquad (25)$$

and the correspondent Hamiltonian function is

$$H = \frac{Ne^{-3\Omega}}{24}\left[-P_\Omega^2 + P_+^2 + P_-^2 - 48\Lambda e^{6\Omega} + 384\pi G M_\gamma e^{-3(\gamma-1)\Omega} \right] = 0, \qquad (26)$$

together with barotropic state equation $p = \gamma\rho$, the Hamilton-Jacobi equation is obtained when we substitute $P_{q^\mu} \to \frac{dS_i}{dq^\mu}$ into (equation 26). In what follows, we should consider the gauge $N = 1$.

2.4.1 Classical Solutions á la WKB

The quantum Wheeler-DeWitt (WDW) equation for these models is obtained by making the canonical quantization P_{q^μ} by $-i\partial_{q^\mu}$ in (equation 26) with $q^\mu = (\Omega, \beta_+, \beta_-)$ is

$$\frac{e^{-3\Omega}}{24}\left[\frac{\partial^2}{\partial\Omega^2} - \frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2} - \lambda e^{6\Omega} + b_\gamma e^{-3(\gamma-1)\Omega} \right]\Psi = 0. \qquad (27)$$

where $\lambda = 48\Lambda$, $b_\gamma = 384\pi G M_\gamma$. We now proceed to apply the WKB semiclassical approximation using the ansatz

$$\Psi(\Omega, \beta_\pm) = e^{i[S_1(\Omega) + S_2(\beta_+) + S_3(\beta_-)]}, \qquad (28)$$

into (equation 27), and without any loss of generality, one can consider the condition $\frac{d^2 S_i}{dq_i^2}$ be small i.e.,

$$\left(\frac{dS_1}{d\Omega}\right)^2 >> \frac{d^2 S_1}{d\Omega^2}, \qquad \left(\frac{dS_2}{d\beta_+^2}\right)^2 >> \frac{d^2 S_2}{d\beta_+^2}, \qquad \left(\frac{dS_2}{d\beta_-^2}\right)^2 >> \frac{d^2 S_2}{d\beta_-^2}, \qquad (29)$$

to get the classical Einstein-Hamilton-Jacobi equation

$$-\left(\frac{dS_1}{d\Omega}\right)^2 + \left(\frac{dS_2}{d\beta_+}\right)^2 + \left(\frac{dS_3}{d\beta_-}\right)^2 - \lambda e^{6\Omega} + be^{-3(\gamma-1)\Omega} = 0, \qquad (30)$$

which can be separate in a set of differential equations

$$-\left(\frac{dS_1}{d\Omega}\right)^2 + a_1^2 - \lambda e^{6\Omega} + be^{-3(\gamma-1)\Omega} = 0, \tag{31}$$

$$\left(\frac{dS_2}{d\beta_+}\right)^2 = n_1^2, \tag{32}$$

$$\left(\frac{dS_3}{d\beta_-}\right)^2 = p_1^2, \tag{33}$$

where a_1^2, n_1^2 and p_1^2 are the separation constants and their relations is $a_1^2 = n_1^2 + p_1^2$. Therefore using the relations between (equation 25), (equation 31), (equation 32) and (equation 33) we have the following equations of motion

$$\pm\sqrt{a_1^2 - \lambda e^{6\Omega} + b_\gamma e^{-3(\gamma-1)\Omega}} \equiv -12e^{3\Omega}\frac{\dot{\Omega}}{N}, \tag{34}$$

$$\pm n_1 \equiv 12e^{3\Omega}\frac{\dot{\beta}_+}{N}, \tag{35}$$

$$\pm p_1 \equiv 12e^{3\Omega}\frac{\dot{\beta}_-}{N}. \tag{36}$$

The main master equation to solved in the gauge $N = 1$, is

$$\frac{dt}{12} = \frac{d\Omega}{\sqrt{a_1^2 e^{-6\Omega} + b_\gamma e^{-3(\gamma+1)\Omega} - \lambda}}, \tag{37}$$

the other two equations (equation 35) and (equation 36) are trivially integrable. For particular stadium of the universe evolution, given by the γ parameter, we present these classical solutions in table 2.

2.4.2 Classical solutions via Hamiltonian formalism

In order to find the commutative equation of motion, we use the classical phase space variables $(\Omega, \beta_\pm)$, where the Poisson algebra for these minisuperspace variables are

$$\{\Omega, \beta_\pm\} = \{\beta_+, \beta_-\} = \{P_\Omega, P_\pm\} = \{P_+, P_-\} = 0, \qquad \{q^\mu, P_{q^\mu}\} = 1, \tag{38}$$

and recalling the Hamiltonian (equation 26), we obtain the classical solutions with the following procedure.

The classical equations of motion for the phase variables Ω, $\beta_\pm$, $P_\pm$, and P_Ω are

$$\dot{\Omega} = \{\Omega, H\} = -\frac{1}{12}e^{-3\Omega}P_\Omega, \tag{39}$$

$$\dot{\beta}_- = \{\beta_-, H\} = \frac{1}{12}e^{-3\Omega}P_-, \tag{40}$$

$$\dot{\beta}_+ = \{\beta_+, H\} = \frac{1}{12}e^{-3\Omega}P_+, \tag{41}$$

$$\dot{P}_\Omega = \{P_\Omega, H\} = \frac{1}{8}e^{-3\Omega}\left[-P_\Omega^2 + P_-^2 + P_+^2 + \lambda e^{6\Omega} + \gamma b_\gamma e^{-3(\gamma-1)\Omega}\right], \tag{42}$$

$$\dot{P}_- = \{P_-, H\} = 0, \qquad \rightarrow \qquad P_- = \pm p_1 = const. \tag{43}$$

$$\dot{P}_+ = \{P_+, H\} = 0, \qquad \rightarrow \qquad P_+ = \pm n_1 = const. \tag{44}$$

Case	Commutative solutions
$\gamma = -1, \ \Lambda \neq 0, \ \rho_{-1} = M_{-1}$	$\Omega = \frac{1}{3} Ln \left[\frac{e^{2qt} - 4a_1^2}{16qe^{qt}} \right], \qquad q^2 = 24\pi G M_{-1} - 3\Lambda,$ $\beta_+ = \pm \frac{2}{3} \frac{n_1}{a_1} \ arctanh \left[\frac{e^{qt}}{2a_1} \right], \quad a_1^2 = n_1^2 + p_1^2,$ $\beta_- = \pm \frac{2}{3} \frac{p_1}{a_1} \ arctanh \left[\frac{e^{qt}}{2a_1} \right].$
$\gamma = 1, \ \Lambda < 0, \ \rho_1 = M_1 e^{-6\Omega}$	$\Omega = \frac{1}{3} Ln \left[\frac{e^{2qt} - 4a_1^2}{16qe^{qt}} \right], \qquad q = \sqrt{3|\Lambda|},$ $\beta_+ = \pm \frac{2}{3} \frac{n_1}{a_1} \ arctanh \left[\frac{e^{qt}}{2a_1} \right], \quad a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1,$ $\beta_- = \pm \frac{2}{3} \frac{p_1}{a_1} \ arctanh \left[\frac{e^{qt}}{2a_1} \right].$
$\gamma = 1, \ \Lambda = 0, \ \rho_1 = M_1 e^{-6\Omega}$	$\Omega = \frac{1}{3} Ln \left[\frac{a_1}{4} t \right], \qquad a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1,$ $\beta_+ = \pm Ln \left[t^{-\frac{n_1}{3a_1}} \right],$ $\beta_- = \pm Ln \left[t^{-\frac{p_1}{3a_1}} \right].$
$\gamma = 0, \ \Lambda = 0, \ \rho_0 = M_0 e^{-3\Omega}$	$\Omega = \frac{1}{3} Ln \left[\frac{b_0 t^2}{64} + \frac{a_1 t}{4} \right], \qquad b_0 = 384\pi G M_0,$ $\beta_+ = \pm \frac{n_1}{3a_1} Ln \left[\frac{16a_1 + b_0 t}{t} \right], \qquad a_1^2 = n_1^2 + p_1^2,$ $\beta_- = \pm \frac{p_1}{3a_1} Ln \left[\frac{16a_1 + b_0 t}{t} \right].$

Table 2. Classical Solutions for $\gamma = -1, 1, 0$, and constraints q, a_1 and b_0.

Introducing (equation 26) into (equation 42), we have

$$8e^{-3\Omega} \dot{P}_\Omega = 2\lambda + (\gamma - 1) b_\gamma e^{-3(\gamma+1)\Omega}, \tag{45}$$

which can be integrate to obtain the relation for P_Ω

$$P_\Omega = \pm \sqrt{a_1^2 - \lambda e^{6\Omega} + b_\gamma e^{-3(\gamma-1)\Omega}}, \tag{46}$$

where $a_1^2 = n_1^2 + p_1^2$.

The set of equations (equation 39), (equation 40) and (equation 41) are equivalents to the set of equations (equation 34), (equation 35) and (equation 36), equations used to obtain the classical solutions.

Just to remark, the solutions obtained with the Hamiltonian formalism and the WKB-like procedure are equivalent to solving GR field equations.

3. Noncommutative Quantum Cosmology

There is a huge interest to noncommutative theories to explain the appropriate modification of Classical General Relativity, and hence of spacetime symmetries at short-distance scales, that implies modifications at large scales. General Quantum Mechanics arguments indicate that, it is not possible to measure a classical background spacetime at the Planck scale, due to the effects of gravitational backreaction (14). It is therefore tempting to incorporate the dynamical features of spacetime at deeper kinematical level using the standard techniques of noncommutative classical field theory based in the so called Moyal product in which for all calculations purposes (differentiation, integration, etc.) the space time coordinates are treated as ordinary (commutative) variables and noncommutativity enters into play in the way in which fields are multiplied (16). Using a deformation in the minisuperspace of this space variables in the Hamilton approach, as we are trying with the idea of noncommutative space

time, we propose that the minisuperspace variables do not commute, for that purpose we will modified the Poisson structure, this approach does not modify the hamiltonian structure in the noncommutative fields.

Finding the classical cosmological solutions for any cosmological model in noncommutative gravity (5) is a very difficult task, this is a consequence of the highly nonlinear character of the theory. To avoid these difficulties, we will follow the original proposals of noncommutative quantum cosmology that was developed in (12). We start by presenting, in quite a general form the construction of noncommutative quantum cosmology and the WKB type method to calculate the classical evolution.

Let us start with a generic form for the commutative WDW equation, this is defined in the minisuperspace variables x, y. As mentioned in (12) a noncommutative deformation of the minisuperspace variables is assumed

$$[x, y] = i\theta, \tag{47}$$

this noncommutativity can be formulated in terms of noncommutative minisuperspace functions with the Moyal product of functions

$$f(x, y) \star g(x, y) = f(x, y) e^{i\frac{\theta}{2}\left(\overleftarrow{\partial_x}\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\overrightarrow{\partial_x}\right)} g(x, y). \tag{48}$$

Then the noncommutative WDW equation can be written as

$$\left(-\Pi_x^2 + \Pi_y^2 - V(x, y)\right) \star \Psi(x, y) = 0, \tag{49}$$

we know from noncommutative quantum mechanics, that the symplectic structure is modified changing the commutator algebra. It is possible to return to the original commutative variables and usual commutation relations if we introduce the following change of variables

$$x \to x + \frac{\theta}{2}\Pi_y \qquad \text{and} \qquad y \to y - \frac{\theta}{2}\Pi_x, \tag{50}$$

the efects of the Moyal star product are reflected in the WDW equation, only through the potential

$$V(x, y) \star \Psi(x, y) = V(x + \frac{\theta}{2}\Pi_y, y - \frac{\theta}{2}\Pi_x), \tag{51}$$

taking this into account and using the usual substitutions $\Pi_{q^\mu} = -i\partial_{q^\mu}$ we arrive to

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - V\left(x - i\frac{\theta}{2}\frac{\partial}{\partial y}, y + i\frac{\theta}{2}\frac{\partial}{\partial x}\right)\right]\Psi(x, y) = 0, \tag{52}$$

this is the Noncommutative WDW equation (NCWDW) and its solutions give the quantum description of the noncommutative universe. We can use the NCWDW to find the temporal evolution of our noncommutative cosmology by a WKB type procedure.

3.1 Noncommutative Kantowski-Sachs Cosmology

Using the method outlined and using (equation 4) we find the NCWDW equation

$$\left[\frac{\partial^2}{\partial \Omega^2} - \frac{\partial^2}{\partial \beta^2} - 48e^{-2\sqrt{3}\left(\Omega - i\frac{\theta}{2}\frac{\partial}{\partial \beta}\right)}\right]\psi(\Omega, \beta) = 0, \tag{53}$$

assuming that we can write $\Psi(\Omega, \beta) = e^{\sqrt{3}\nu\beta} X(\Omega)$ the equation for $X(\Omega)$ is

$$\left[-\frac{d^2}{d\Omega^2} + 48 e^{-3i\nu\theta} e^{-2\sqrt{3}\Omega} + 3\nu^2 \right] X(\Omega) = 0, \tag{54}$$

then the solutions of the NCWDW equation are

$$\Psi(\Omega, \beta) = e^{\pm i\sqrt{3}\nu\beta} K_{i\nu}\left(4e^{-\sqrt{3} \pm \frac{3}{2}\nu\theta} \right), \tag{55}$$

as already mentioned the NCWDW equation has the same problems as its commutative counterpart, it has no time dependence and unfortunately it can not be normalized. Usually the next step is to construct a " Gaussian" wave packet that can be normalized and do the physics with the new wave function. This is not needed for our purposes, as we will be applying the WKB method as in the previous section. Using equations (equation 6) and (equation 7) we arrive at

$$S_1(\beta) = P_{\beta_0}\beta,$$

$$S_2(\beta) = -\frac{1}{\sqrt{3}}\sqrt{P_{\beta_0}^2 - 48 e^{-\sqrt{3}\theta P_{\beta_0}} e^{-2\sqrt{3}\Omega}}$$

$$+ \frac{P_{\beta_0}}{\sqrt{3}}\operatorname{arctanh}\left[\frac{\sqrt{P_{\beta_0}^2 - 48 e^{-\sqrt{3}\theta P_{\beta_0}} e^{-2\sqrt{3}\Omega}}}{P_{\beta_0}} \right], \tag{56}$$

using (equation 50) we get

$$\dot{\beta}_C = \frac{1}{2}\beta - \frac{1}{2}\dot{P}_\Omega, \qquad \dot{\Omega}_C = \frac{1}{2}\Omega + \frac{1}{2}\dot{P}_\beta \tag{57}$$

and the fact that the momenta are not modified we arrive to

$$\Omega(t) = \frac{1}{2\sqrt{3}}\ln\left[\frac{48}{P_{\beta_0}^2}\cosh^2\left(2\sqrt{3}P_{\beta_0}(t - t_0) \right) \right] - \frac{1}{2}\theta P_{\beta_0},$$

$$\beta(t) = \beta_0 + 2P_{\beta_0}(t - t_0)$$

$$- \frac{\theta}{2}P_{\beta_0}\tanh^2\left[2\sqrt{3}P_{\beta_0}(t - t_0) \right], \tag{58}$$

this solutions have already been obtained in (15), in that paper they do a deformation of the simplectic structure at a classical level, modifying the Poisson brackets, to include noncommutativity.

3.2 Noncommutative FRW Cosmology with scalar field and Λ

We can use the NCWKB type method to FRW universe coupled to a scalar field. Proceeding as before the corresponding NCWDW equation is

$$\left[-\frac{1}{24}\frac{\partial^2}{\partial\alpha^2} + \frac{1}{2}\frac{\partial^2}{\partial\phi^2} + e^{6(\alpha - i\frac{\theta}{2}\frac{\partial}{\partial\phi})}\left(2\Lambda + 6k e^{-2(\alpha - i\frac{\theta}{2}\frac{\partial}{\partial\phi})} \right) \right]\Psi = 0. \tag{59}$$

From the NCWDW equation, we use the method developed in the previous sections and calculate the classical evolution by appliying the NCWKB type method. These results are presented in the next table

case	NC Quantum Solution	NC Classical Solution
$k=0, \Lambda \neq 0$	$\psi = e^{\pm i \nu \frac{\sqrt{3}}{2}\phi} K_{i\nu}\left[4\sqrt{\frac{\Lambda}{3}}e^{3\left(\alpha-\frac{3}{2}\nu\theta\right)}\right]$ and J_ν for $\Lambda < 0$	$\phi(t) = \phi_0 - P_{\phi_0}t$ $-\sqrt{3}\theta P_{\phi_0}\tanh\left(\frac{\sqrt{3}}{2}P_{\phi_0}(t-t_0)\right),$ $\alpha(t) = \frac{\theta}{2}P_{\phi_0} + \frac{1}{6}\ln\left(\frac{P_{\phi_0}^2}{4\Lambda}\right)$ $+\frac{1}{3}\ln\left(\text{sech}\left[\frac{\sqrt{3}}{2}P_{\phi_0}(t-t_0)\right]\right),$
$k \neq 0, \Lambda = 0$	$\psi^{(1)} = e^{\pm i \frac{\nu}{\sqrt{3}}\phi} K_{i\nu}\left[6e^{2\left(\alpha-\frac{\theta}{2}\nu\right)}\right]$ for $k=1$ $\psi^{(2)} = e^{\pm i\nu/\sqrt{3}\phi}J_\nu\left[6e^{2\left(\alpha-\frac{\theta}{2}\nu\right)}\right]$, for $k=-1$	$\phi(t) = \phi_0 - P_{\phi_0}(t-t_0)$ $-\sqrt{3}\theta P_{\phi_0}\tanh\left(\frac{P_{\phi_0}}{\sqrt{3}}(t-t_0)\right),$ $\alpha(t) = \frac{\theta}{2}P_{\phi_0} + \frac{1}{4}\ln\left[\frac{P_{\phi_0}^2}{12k}\right]$ $+\frac{1}{2}\ln\left(\text{sech}\left[\frac{1}{\sqrt{3}}P_{\phi_0}(t-t_0)\right]\right)$
$k \neq 0, \Lambda \neq 0$	Unknown	$\phi(t) = \phi_0 - P_{\phi_0}(t-t_0)$ $+6\theta\int e^{6\alpha}\left(\Lambda + 2e^{-2\alpha}\right)dt,$ $\int \dfrac{d\alpha(t)}{\sqrt{P_{\phi_0}-2e^{6\alpha+3\theta P_{\phi_0}}\left(2\Lambda+6ke^{-2\alpha-\theta P_{\phi_0}}\right)}}$ $= \frac{1}{\sqrt{12}}(t-t_0),$

Table 3. Classical and quantum solutions for noncommutative FRW universe coupled to a scalar field. For these models noncommutativity is introduced in the gravitational and matter sectors. As in the commutative scenario, for $\Lambda \neq 0$ and $k \neq 0$ the noncommutative classical solution is given in an implicit form, and there is not a closed analytical quantum solutions. As in the commutative case we have fixed the value of the lapse function $N(t) = e^{3\alpha}$.

3.3 Stringy Noncommutative Quantum Cosmology

As in the previous examples we introduce the noncommutative relation $[\bar{\phi}, \beta] = i\theta$, and from the classical hamiltonian we find the NCWDW equation

$$\left[\frac{\partial^2}{\partial\bar{\phi}^2} - \frac{\partial^2}{\partial\beta^2} - \lambda_s^2 V(\bar{\phi}, \beta)e^{(m-2)\left(\bar{\phi}-i\frac{\theta}{2}\frac{\partial}{\partial\beta}\right)}\right]\Psi(\bar{\phi}, \beta) = 0. \tag{60}$$

The noncommutative wave function is

$$\Psi(\bar{\phi}, \beta) = e^{\pm -i\frac{m-2}{2}\nu\beta}K_{i\nu}\left[\frac{2\lambda_s\sqrt{V_0}}{m-2}e^{(m-2)\left(\bar{\phi}\mp\frac{m-2}{4}\theta\nu\right)}\right], \tag{61}$$

using the NCWKB type method the classical solutions for the noncommutative stringy cosmology are

$$\bar{\phi}(\tau) = \frac{1}{m-2}\ln\left[\frac{P_{\beta_0}^2}{V_0\lambda_s^2}\text{sech}^2\left(\frac{P_{\beta_0}}{2\lambda_s}(m-2)(\tau-\tau_0)\right)\right]$$

$$-\frac{\theta}{2}P_{\beta_0},$$

$$\beta(\tau) = \beta_0 + \frac{P_\beta}{\lambda_s}(\tau-\tau_0)$$

$$+\theta\frac{P_{\beta_0}}{2}\tanh\left[\frac{P_{\beta_0}}{2\lambda_s}(m-2)(\tau-\tau_0)\right], \tag{62}$$

the classical evolution for string cosmology can be calculated for $m = 0$ and $m = 4$. An interesting issue concerns the B field that is turned off in the string cosmology model (13) and does not contribute to the effective action. In open string theory, however noncommutativity arises precisely in the low energy limit of string theory in the presence of a constant B field. The θ parameter we have introduced in the minisuperspace could then be understood as a kind of B-field related with the Neveu-Schwarz B-field.

3.4 Noncommutative solutions of the isotropization in Bianchi I with barotropic perfect fluid and Λ cosmological

Let us begin introducing the noncommutative deformation of the minisuperspace in the WDW equation, this time, between all the variables of the minisuperspace, assuming that Ω_{nc} and $\beta_{\pm nc}$ obey the commutation relation

$$[\Omega_{nc}, \beta_{-nc}] = i\theta_1, \qquad [\Omega_{nc}, \beta_{+nc}] = i\theta_2, \qquad [\beta_{-nc}, \beta_{+nc}] = i\theta_3. \tag{63}$$

Instead of working directly with the physical variables Ω and $\beta_{\pm}$ we may achieve all the above solutions by making use of the auxiliary canonical variables Ω_{nc} and $\beta_{\pm nc}$ defined as

$$\Omega_{nc} \equiv \Omega - \frac{\theta_1}{2}P_- - \frac{\theta_2}{2}P_+, \tag{64}$$

$$\beta_{-nc} \equiv \beta_- + \frac{\theta_1}{2}P_\Omega - \frac{\theta_3}{2}P_+, \tag{65}$$

$$\beta_{+nc} \equiv \beta_+ + \frac{\theta_2}{2}P_\Omega + \frac{\theta_3}{2}P_-. \tag{66}$$

maintaining the usual commutation relations between the fields, i.e., $[q^\mu, q^\nu] = 0$ and the identifications $P_\Omega = P_{\Omega nc}$ and $P_\pm = P_{\pm nc}$. With this shift and the usual canonical quantization $P_{q^\mu} \to -i\partial_{q^\mu}$, we arrive to the noncommutative WDW equation

$$\left[\frac{\partial^2}{\partial\Omega_{nc}^2} - \frac{\partial^2}{\partial\beta_{+nc}^2} - \frac{\partial^2}{\partial\beta_{-nc}^2} - \lambda e^{6\Omega_{nc}} + b_\gamma e^{-3(\gamma-1)\Omega_{nc}}\right]\Psi(\Omega, \beta_\pm) = 0, \tag{67}$$

where $\lambda = 48\Lambda$, $b_\gamma = 384\pi G M_\gamma$. At this point we have a noncommutative WDW equation and noncommutative hamiltonian. In what follows, we shall consider a wave function and apply the WKB procedure to obtain classical solutions.

3.4.1 Noncommutative classical solutions á la WKB

In order to find noncommutative classical solutions through the WKB approximation, we use the fact that $e^{i\theta\frac{\partial}{\partial x}}e^{\eta x} \equiv e^{i\eta\theta}e^{\eta x}$, and the ansatz for the wavefunction $\Psi(\Omega_{nc}, \beta_{\pm nc}) = e^{i[S_1(\Omega_{nc})\pm n_1\beta_{+nc}\pm p_1\beta_{-nc}]}$, where we use explicitly $S_2(\beta_{+nc}) = \pm n_1\beta_{+nc}$ and $S_3(\beta_{-nc}) = \pm p_1\beta_{-nc}$ to get the classical noncommutative Einstein-Hamilton-Jacobi (EHJ) equation

$$-\left(\frac{dS_1}{d\Omega_{nc}}\right)^2 + \left(\frac{dS_2}{d\beta_{+nc}}\right)^2 + \left(\frac{dS_3}{d\beta_{-nc}}\right)^2 - \lambda e^{6\Omega_{nc}} + b e^{-3(\gamma-1)\Omega_{nc}} = 0, \tag{68}$$

which can be separate in a set of differential equations with $m_1^2 = n_1^2 + p_1^2$. We have the

Case	Noncommutative Solutions
$\gamma = -1$, $\Lambda \neq 0$, $\rho_{-1} = M_{-1}$ $a_1^2 = n_1^2 + p_1^2$, $q^2 = 24\pi G M_{-1} - 3\Lambda$,	$\Omega_{nc} = \frac{1}{3} Ln \left[\frac{e^{2qt} - 4a_1^2}{16qe^{qt}} \right] - \frac{\theta_1}{2} p_1 - \frac{\theta_2}{2} n_1$, $\beta_{+nc} = \pm \frac{2}{3} \frac{n_1}{a_1} \, arctanh \left[\frac{e^{qt}}{2a_1} \right] + \frac{\theta_2}{8} \left(\frac{e^{qt}}{4} + a_1^2 e^{-qt} \right) - \frac{\theta_3}{2} p_1$, $\beta_{-nc} = \pm \frac{2}{3} \frac{p_1}{a_1} \, arctanh \left[\frac{e^{qt}}{2a_1} \right] + \frac{\theta_1}{8} \left(\frac{e^{qt}}{4} + a_1^2 e^{-qt} \right) + \frac{\theta_3}{2} n_1$,
$\gamma = 1$, $\Lambda < 0$, $\rho_1 = M_1 e^{-6\Omega}$ $a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1$,	$\Omega_{nc} = \frac{1}{3} Ln \left[\frac{e^{2qt} - 4a_1^2}{16qe^{qt}} \right] - \frac{\theta_1}{2} p_1 - \frac{\theta_2}{2} n_1$, $\quad q = \sqrt{3|\Lambda|}$, $\beta_{+nc} = \pm \frac{2}{3} \frac{n_1}{a_1} \, arctanh \left[\frac{e^{qt}}{2a_1} \right] + \frac{\theta_2}{8} \left(\frac{e^{qt}}{4} + a_1^2 e^{-qt} \right) - \frac{\theta_3}{2} p_1$, $\beta_{-nc} = \pm \frac{2}{3} \frac{p_1}{a_1} \, arctanh \left[\frac{e^{qt}}{2a_1} \right] + \frac{\theta_1}{8} \left(\frac{e^{qt}}{4} + a_1^2 e^{-qt} \right) + \frac{\theta_3}{2} n_1$,
$\gamma = 1$, $\Lambda = 0$, $\rho_1 = M_1 e^{-6\Omega}$ $a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1$,	$\Omega_{nc} = \frac{1}{3} Ln \left[\frac{a_1}{4} t \right] - \frac{\theta_1}{2} p_1 - \frac{\theta_2}{2} n_1$, $\beta_{+nc} = \pm Ln \left[t^{-\frac{n_1}{3a_1}} \right] + \frac{\theta_2}{2} a_1 - \frac{\theta_3}{2} p_1$, $\beta_{-nc} = \pm Ln \left[t^{-\frac{p_1}{3a_1}} \right] + \frac{\theta_1}{2} a_1 + \frac{\theta_3}{2} n_1$,
$\gamma = 0$, $\Lambda = 0$, $\rho_0 = M_0 e^{-3\Omega}$ $b_0 = 384\pi G M_0$, $a_1^2 = n_1^2 + p_1^2$,	$\Omega_{nc} = \frac{1}{3} Ln \left[\frac{b_0 t^2}{64} + \frac{a_1 t}{4} \right] - \frac{\theta_1}{2} p_1 - \frac{\theta_2}{2} n_1$, $\beta_{+nc} = \pm \frac{n_1}{3a_1} Ln \left[\frac{16a_1 + b_0 t}{t} \right] + \frac{\theta_2}{2} \sqrt{a_1^2 + \frac{b_0 t^2}{64} + \frac{a_1 t}{4}} - \frac{\theta_3}{2} p_1$, $\beta_{-nc} = \pm \frac{p_1}{3a_1} Ln \left[\frac{16a_1 + b_0 t}{t} \right] + \frac{\theta_1}{2} \sqrt{a_1^2 + \frac{b_0 t^2}{64} + \frac{a_1 t}{4}} + \frac{\theta_3}{2} n_1$.

Table 4. Noncommutative solutions for, $\gamma = -1, 1, 0$, and constraints q, a_1 and b_0.

following noncommutative equations of motion

$$\pm \sqrt{a_1^2 - \lambda e^{6\Omega_{nc}} + b_\gamma e^{-3(\gamma - 1)\Omega_{nc}}} \equiv -12 e^{3\Omega_{nc}} \frac{\dot{\Omega}_{nc}}{N}, \tag{69}$$

$$\pm n_1 \equiv 12 e^{3\Omega_{nc}} \frac{\dot{\beta}_{+nc}}{N}, \tag{70}$$

$$\pm p_1 \equiv 12 e^{3\Omega_{nc}} \frac{\dot{\beta}_{-nc}}{N}. \tag{71}$$

One just need to be careful in (equation 69), (equation 70) and (equation 71), and apply the chain rule to the variables (equation 64), (equation 65) and (equation 66), in order to get the right solution, $\dot{\beta}_{-nc} = \frac{\partial \beta_-}{\partial t} + \frac{\partial \beta_{-nc}}{\partial P_\Omega} \frac{\partial p_\Omega}{\partial t} + \frac{\partial \beta_{-nc}}{\partial P_+} \frac{\partial p_+}{\partial t} + \frac{\partial \beta_{-nc}}{\partial P_-} \frac{\partial p_-}{\partial t} = \dot{\beta}_- + \frac{\theta_1}{2} \dot{P}_\Omega$. In this sense, all solutions to find in the commutative case, remain for the noncommutative case with the corresponding shift, as we show in the table 4.

3.4.2 Noncommutative classical solutions á la Hamilton

In the commutative model we know that the solutions to hamiltons equations are the same as in General Relativity. Now the natural extension is to consider the noncommutative version of our model, with the idea of noncommutative between the three variables $(\Omega_{nc}, \beta_{\pm nc})$, so we apply a deformation of the Poisson algebra. For this we start with the usual hamiltonian (equation 26), but the symplectic structure is modify as follow

$$\{P_\Omega, P_\pm\}_\star = \{P_+, P_-\}_\star = 0, \qquad \{q^\mu, P_{q^\mu}\}_\star = 1, \tag{72}$$

$$\{\Omega, \beta_-\}_\star = \theta_1, \quad \{\Omega, \beta_+\}_\star = \theta_2, \quad \{\beta_-, \beta_+\}_\star = \theta_3. \tag{73}$$

where the $\star$ is the Moyal product. In the second case, the hamiltonian is modify by the shift (equation 64),(equation 65) and (equation 66) resulting

$$H_{nc} = \frac{Ne^{-3\Omega_{nc}}}{24} \left[-P_\Omega^2 + P_+^2 + P_-^2 - \lambda e^{6\Omega_{nc}} + b_\gamma e^{-3(\gamma-1)\Omega_{nc}} \right] = 0, \qquad (74)$$

but the symplectic structure is the one that we know, the commutative one (equation 38).
The noncommutative equations of motion, for the first formalism that we exposed have the original variables, but with the variables modified,

$$\begin{aligned} \dot{q}_{nc}^\mu &= \{q^\mu, H\}_\star, \\ \dot{P}_{nc}^\mu &= \{P^\mu, H\}_\star, \end{aligned} \qquad (75)$$

and for the second formalism we use the shifted variables but with the original (commutative) symplectic structure

$$\begin{aligned} \dot{q}_{nc}^\mu &= \{q_{nc}^\mu, H_{nc}\}, \\ \dot{P}_{nc}^\mu &= \{P_{nc}^\mu, H_{nc}\}, \end{aligned} \qquad (76)$$

in both approaches we have the same result. Therefore the equations of motion take the form

$$\dot{\Omega}_{nc} = \{\Omega, H\}_\star = \{\Omega_{nc}, H_{nc}\} = -\frac{e^{-3\Omega_{nc}}}{12} P_\Omega, \qquad (77)$$

$$\dot{\beta}_{-nc} = \{\beta_-, H\}_\star = \{\beta_{-nc}, H_{nc}\} = \frac{e^{-3\Omega_{nc}}}{12} P_- + \frac{\theta_1}{2} \dot{P}_\Omega, \qquad (78)$$

$$\dot{\beta}_{+nc} = \{\beta_+, H\}_\star = \{\beta_{+nc}, H_{nc}\} = \frac{e^{-3\Omega_{nc}}}{12} P_+ + \frac{\theta_2}{2} \dot{P}_\Omega, \qquad (79)$$

$$\dot{P}_\Omega = \{P_\Omega, H\}_\star = \{P_\Omega, H_{nc}\} = \frac{e^{-3\Omega_{nc}}}{8} \left[6\lambda e^{6\Omega_{nc}} + 3(\gamma-1)b_\gamma e^{-3(\gamma-1)\Omega_{nc}} \right], \qquad (80)$$

$$\dot{P}_- = \{P_-, H\}_\star = \{P_-, H_{nc}\} = 0, \quad \rightarrow \quad P_- = p_1, \qquad (81)$$

$$\dot{P}_+ = \{P_+, H\}_\star = \{P_+, H_{nc}\} = 0, \quad \rightarrow \quad P_+ = n_1. \qquad (82)$$

if we proceed as in the commutative case we get the solutions showed in the table IVA.

4. Conclusions and outlook

In this chapter we have presented the NCWKB type method for noncommutative quantum cosmology and with this procedure, found the noncommutative classical solutions for several noncommutative quantum cosmological models.
Noncommutativity is incorporated in the minisuperspace variables, in a similar manner as it is a proposal that originally emerged at the quantum level, by this reason we considered it as in standard quantum mechanics. By means of the WKB approximation on the corresponding NCWDW equation, one gets the noncommutative generalized Einstein-Hamilton-Jacobi equation (NCEHJ), from which the classical evolution of the noncommutative model is obtained. The examples we studied were the Kantowski-Sachs cosmological model, the FRW universe with cosmological constant and coupled to a scalar field, a string quantum cosmological model and Bianchi I with Barotropic perfect fluid and Λ Cosmological.
In the commutative scenario, that the classical solutions found from the WKB-type method

are solutions to the corresponding Einsteins field equations. Due to the complexity of the noncommutative theories of gravity(5), classical solutions to the noncommutative field equations are almost impossible to find, but in the approach of noncommutative quantum cosmology and by means of the WKB-type procedure, they can be easily constructed. Also the quantum evolution of the system is not needed to find the classical behavior, from table 2 we can see that for the case $\Lambda \neq 0$ and $k \neq 0$ the wave function can not be analitacally calculated, but still the noncommutative effects can be incorporated and the classical evolution is found implicitly. This procedure gives a straightforward algorithm to incorporate noncommutative effects to cosmological models. In this approach the effects of noncommutativity are encoded in the potential through the Moyal product of functions (equation 51). We only need the NCWDW equation and the approximations (equation 7), to get the NCEHJ and from it, the noncommutative classical behavior can easily be constructed. As already mentioned, in(11) the effects of noncommutativity were studied in connection with inflation, but the noncommutative deformation was only done in the matter sector neglecting the gravity sector. For completeness to the section 2.4 and 3.4 we present the solutions in the gauge $N = 24e^{3\Omega}$ (see appendix A and B), one of the advantages of this gauge is that the solutions are very9 simple, this is something to take into account when we introduce a more complex form of matter, where in the gauge $N(t) = 1$ analytical solutions can not be found. The procedure developed here has the advantage that we can implement noncommutativity in both sectors in a straightforward way and find the classical solutions (i.e. inflationary models). The study on deformed phase space for all of this cosmologies should be constructed with this noncommutative proposal. These ideas are being explored and will be reported elsewhere.

5. Acknowledgments

6. Appendix

A Commutative classical solutions in the Gauge $N = 24e^{3\Omega}$

In this appendix we present the classical solutions in the gauge $N = 24e^{3\Omega}$, the equations are much more simpler to solve in this guage.

A.1 Commutative Classical Solutions á la WKB
The master equation becomes

$$2dt = \frac{d\Omega}{\sqrt{a_1^2 - \lambda e^{6\Omega} + b_\gamma e^{-3(\gamma-1)\Omega}}}, \tag{83}$$

and the other two equations are immediately integrable. Again for particular cases in the γ parameter, we present the classical solutions, table 5

A.2 Classical solutions via Hamiltonian formalism
With the gauge fixed to $N = 24e^{3\Omega}$ we can see that the hamiltonian takes the form

$$H = -P_\Omega^2 + P_+^2 + P_-^2 - \lambda e^{6\Omega} + b_\gamma e^{-3(\gamma-1)\Omega} = 0. \tag{84}$$

Case	Commutative solutions
$\gamma = -1, \ \Lambda \neq 0, \ \rho_{-1} = M_{-1}$	$\Omega = \frac{1}{6} Ln \left[-\frac{a_1^2}{384\pi G M_{-1} - 48\Lambda} Sech^2 (6a_1 t) \right], \quad a_1^2 = n_1^2 + p_1^2,$ $\beta_+ = \pm 2n_1 t,$ $\beta_- = \pm 2p_1 t.$
$\gamma = 1, \ \Lambda \neq 0, \ \rho_1 = M_1 e^{-6\Omega}$	$\Omega = \frac{1}{6} Ln \left[\frac{a_1^2}{48\Lambda} Sech^2 (6a_1 t) \right], \quad a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1,$ $\beta_+ = \pm 2n_1 t,$ $\beta_- = \pm 2p_1 t.$
$\gamma = 1, \ \Lambda = 0, \ \rho_1 = M_1 e^{-6\Omega}$	$\Omega = 2\sqrt{a_1^2 + b_1 t}, \qquad\qquad a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1,$ $\beta_+ = -2n_1 t,$ $\beta_- = -2p_1 t.$
$\gamma = 0, \ \Lambda = 0, \ \rho_0 = M_0 e^{-3\Omega}$	$\Omega = \frac{1}{3} Ln \left[-\frac{a_1^2}{b_0} sech^2 (3a_1 t) \right], \qquad b_0 = 384\pi G M_0,$ $\beta_+ = \pm 2n_1 t, \qquad a_1^2 = n_1^2 + p_1^2,$ $\beta_- = \pm 2p_1 t.$
$\gamma = \frac{1}{3}, \ \Lambda = 0, \ \rho_0 = M_{\frac{1}{3}} e^{-4\Omega}$	$\Omega = \frac{1}{2} Ln \left[-\frac{a_1^2}{b_{\frac{1}{3}}} sech^2 (2a_1 t) \right], \quad a_1^2 = n_1^2 + p_1^2,$ $\beta_+ = \pm 2n_1 t, \qquad\qquad b_{\frac{1}{3}} = 384\pi G M_{\frac{1}{3}},$ $\beta_- = \pm 2p_1 t.$

Table 5. Classical Solutions for $\gamma = -1, \frac{1}{3}, 1, 0$, and constraints a_1, b_0 and b_1.

The Poisson brackets structure yields to equations of motion

$$\dot{\Omega} = \{\Omega, H\} = -2P_\Omega, \tag{85}$$

$$\dot{\beta_-} = \{\beta_-, H\} = 2P_-, \qquad \rightarrow \qquad \beta_- = \pm 2p_1 t, \tag{86}$$

$$\dot{\beta_+} = \{\beta_+, H\} = 2P_+, \qquad \rightarrow \qquad \beta_+ = \pm 2n_1 t, \tag{87}$$

$$\dot{P}_\Omega = \{P_\Omega, H\} = \left[+6\lambda e^{6\Omega} + 3(\gamma - 1) b_\gamma e^{-3(\gamma-1)\Omega} \right], \tag{88}$$

$$\dot{P}_- = \{P_-, H\} = 0, \qquad \rightarrow \qquad P_- = \pm p_1 = const. \tag{89}$$

$$\dot{P}_+ = \{P_+, H\} = 0, \qquad \rightarrow \qquad P_+ = \pm n_1 = const. \tag{90}$$

Using (equation 84), introducing (equation 89) and (equation 90), we obtain the expression for P_Ω

$$P_\Omega = \sqrt{m_1^2 - \lambda e^{6\Omega} + b_\gamma e^{-3(\gamma-1)\Omega}}, \tag{91}$$

being self-consistent with equation (equation 88), where $a_1^2 = n_1^2 + p_1^2$. Introducing this equation into (equation 85) we get the master equation found to solve the Einstein field equation in this gauge, where the classical solutions are presented in table IIIA.

B Noncommutative classical solutions

B.1 Noncommutative classical solutions in the Gauge $N = 24e^{3\Omega}$ á la WKB and via Hamiltonian formalism

The noncommutative solutions in the space q^μ become

| Case	Noncommutative Solutions
$\gamma = -1, \ \Lambda \neq 0, \ \rho_{-1} = M_{-1}$ $a_1^2 = n_1^2 + p_1^2,$	$\Omega_{nc} = \frac{1}{6} Ln \left[-\frac{a_1^2}{384\pi G M_{-1} - 48\Lambda} \, Sech^2 \, (6a_1 t) \right] - \frac{\theta_1}{2} p_1 - \frac{\theta_2}{2} n_1,$ $\beta_{+nc} = \pm 2n_1 t + \frac{\theta_2 a_1}{2} \, tanh(6a_1 t) - \frac{\theta_3}{2} p_1,$ $\beta_{-nc} = \pm 2p_1 t + \frac{\theta_1 a_1}{2} \, tanh(6a_1 t) + \frac{\theta_3}{2} n_1,$
$\gamma = 1, \ \Lambda \neq 0, \rho_1 = M_1 e^{-6\Omega}$ $a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1,$	$\Omega_{nc} = \frac{1}{6} Ln \left[\frac{a_1^2}{48\Lambda} \, Sech^2(6a_1 t) \right] - \frac{\theta_1}{2} p_1 - \frac{\theta_2}{2} n_1,$ $\beta_{+nc} = \pm 2n_1 t + \frac{\theta_2 a_1}{2} \, tanh(6a_1 t) - \frac{\theta_3}{2} p_1,$ $\beta_{-nc} = \pm 2p_1 t + \frac{\theta_1 a_1}{2} \, tanh(6a_1 t) + \frac{\theta_3}{2} n_1,$
$\gamma = 1, \ \Lambda = 0, \ \rho_1 = M_1 e^{-6\Omega}$ $a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1,$	$\Omega_{nc} = 2a_1 t - \frac{\theta_1}{2} p_1 - \frac{\theta_2}{2} n_1,$ $\beta_{+nc} = -2n_1 t + \frac{\theta_2}{2} a_1 - \frac{\theta_3}{2} p_1,$ $\beta_{-nc} = -2p_1 t + \frac{\theta_1}{2} a_1 + \frac{\theta_3}{2} n_1,$
$\gamma = 0, \ \Lambda = 0, \ \rho_0 = M_0 e^{-3\Omega}$ $b_0 = 384\pi G M_0,$ $a_1^2 = n_1^2 + p_1^2,$	$\Omega_{nc} = \frac{1}{3} Ln \left[-\frac{a_1^2}{b_0} \, sech^2(3a_1 t) \right] - \frac{\theta_1}{2} p_1 - \frac{\theta_2}{2} n_1,$ $\beta_{+nc} = \pm 2n_1 t + \frac{\theta_2 a_1}{2} \, tanh(3a_1 t) - \frac{\theta_3}{2} p_1,$ $\beta_{nc} - = \pm 2p_1 t + \frac{\theta_1 a_1}{2} \, tanh(3a_1 t) + \frac{\theta_3}{2} n_1.$
$\gamma = \frac{1}{3}, \ \Lambda = 0, \ \rho_0 = M_{\frac{1}{3}} e^{-4\Omega}$ $a_1^2 = n_1^2 + p_1^2,$	$\Omega_{nc} = \frac{1}{2} Ln \left[-\frac{a_1^2}{b_{\frac{1}{3}}} \, sech^2(2a_1 t) \right] - \frac{\theta_1}{2} p_1 - \frac{\theta_2}{2} n_1,$ $\beta_{+nc} = \pm 2n_1 t + \frac{\theta_2 a_1}{2} \, tanh(2a_1 t) - \frac{\theta_3}{2} p_1$ $\beta_{-nc} = \pm 2p_1 t + \frac{\theta_1 a_1}{2} \, tanh(2a_1 t) + \frac{\theta_3}{2} n_1.$

Table 6. Noncommutative solutions for $\gamma = -1, \frac{1}{3}, 1, 0$, and constraints a_1, b_0 and b_1.

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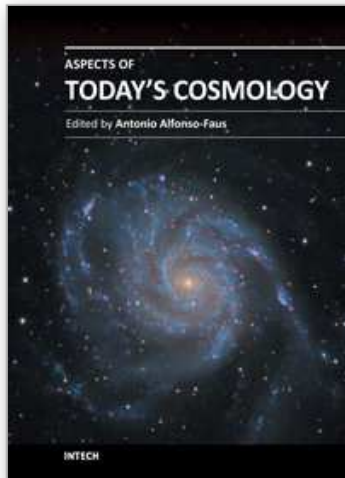
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