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# Loop Quantum Cosmology: Effective Theory and Related Applications

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## 1. Introduction

The effort toward making quantum mechanics and general relativity compatible (quantum gravity) has lasted more than a century. The string theory and loop quantum gravity both stand out as strong candidates and currently they both make popular research subjects in quantum gravity. As we know, Loop quantum gravity (LQG)(Ashtekar & Lewandowski, 2004; Rovelli, 1998; 2004; Thiemann, 2007) is a background independent and non-perturbative canonical quantum gravity theory. LQG has made many breakthroughs in recent years: the establishment of the quantum Einstein equations Ashtekar & Tate (1994); Ashtekar et al. (1995a); Corichi & Zapata (1997); Lewandowshi & Thiemann (1999); Rovelli & Smolin (1994); Thiemann (1996; 1998a;b;c; 2001), the proof that the Riemannian operators have discrete eigenvalues Ashtekar et al. (1995b); Lewandowski (1997); Loll (1995a;b; 1997a;b); Rovilli & Smolin (1995); Thiemann (1998d;e), results concerning the entropy of the black hole horizon and cosmological horizon entropy with statistical mechanics Ashtekar et al. (1998; 1999; 2000; 2001; 2002; 2003a;b); Berreira et al. (1996); Rovelli (1996a;b); Smolin (1995), and so on. As an application of loop quantum gravity to cosmology, loop quantum cosmology (LQC) Bojowald (2005a; 2008); Date (2002) also presents itself as a possible path toward answers to the cosmological and astrophysical riddles.

As a symmetry reduced model of LQG, LQC inherits the quantum schemes originated from LQG that dealt with the isotropic and homogeneous universe firstly and then extended to the inhomogeneous and anisotropic model Bojowald (2002a). It plays an important role in connecting the LQG theory and the measurable world. On one hand, it is used to test the full theory, which, in its own form, is extremely complex and difficult to directly apply. On the other hand, making connections to the real world sheds light on further improvement of the LQG theory. These reasons make LQC a promising and enlightening subject to study.

In LQC, the collapsing and expanding phases are connected by the cyclic or oscillatory models Lidsey et al. (2004), and the universe is automatically born with a small scale factor at the fixed point near the Planck phase Bojowald (2005); Mulryne et al. (2005a). Unlike in the emergent universe model, this fixed point here is stable and allows the universe to start in an initial phase of oscillation. Then an inflationary phase is entered, which is the relevant regime for structure formation. In LQC, there are many different inflationary scenarios Artymowski et al. (2009); Bojowald et al. (2004); Mulryne et al. (2005b); Zhang & Ling (2007), among which the one without inflation is the most attractive, mainly because it can explain the inflationary phase directly from LQG. But, unfortunately, it is difficult to study the structure formation.

To know how and why this is true requires a study of the mathematical structure and the physical meaning of LQC.

As is well known, LQC is based on the connection dynamics. So far, the successful quantization of the cosmological model is still confined to the homogeneous Bianchi A class because one can refer to the diagonal technique Bojowald (2003). With the Ashtekar's new variables, the Hamilton constraint can be written as a difference equation. One of the major successes in LQC is that the big bang singularity can be replaced by the big bounce Ashtekar et al. (2006a;b; 2008). (The robust demonstration of bounces in LQC is confined to the cases where quantum back reaction can be safely ignored.). For the general case, the quantization is more complex and the research is still going on. For the general inhomogeneous cosmology, there have been three different approaches. The first one is to introduce the inhomogeneous matter, but still based on the isotropic quantum geometry and its effective theory. The second approach starts with the full constraint, splits it into the homogeneous part and the inhomogeneous part, and then obtains the effective theory Bojowald et al. (2006; 2007; 2008; 2009) (it has achieved a series of successes so far). The third approach is to deal with the inhomogeneous symmetric model explicitly, and to shed light on the full theory. At the time of writing, we still do not know whether the general solution is a difference equation or not.

As we all know, the difference equation of state is difficult to analyze even in the homogeneous and isotropic model. Thus we need a new tool to extract physical information out of the theory. That is why the effective theory comes in. The effective theory shares the form of the classical theory, but contains correction terms from the quantum theory (LQC). The commonly considered ones include the inverse volume correction, the holonomy correction and the back reaction correction. The inverse volume correction Bojowald (2002b;c) is used to solve the quantization problem  $p^{-\frac{3}{2}}$  in the matter Hamiltonian which cannot be quantized directly. Instead, we write the equivalent form of  $p^{-\frac{3}{2}}$  in the classical form, which can be promoted to the well-defined operator in the quantum theory. This brings a correction term to the classical theory. And the matter Hamiltonian derived in this way will behave differently from the classical one on small scales. The matter Hamiltonian shows a repulsive behavior. The holonomy correction originates from the fact that there does not exist an operator in the quantum theory corresponding to the connection  $c$  Banerjee & Date (2005); Date & Hossain (2004). When quantized,  $c$  should be expressed as  $\sin(\mu c)/\mu$ . It is obvious that the classical expression recovers only at the small value of  $c$ . This provides a new correction to the classical equation. The back reaction correction is the main source of correction in genuine quantum systems Bojowald et al. (2007); Chiou (2008). If brought together, they may counteract each other. Therefore, it is important to bring all possible quantum corrections together in a consistent manner and study the corresponding physics. Based on these theories, many interesting cosmological riddles have been studied, such as the big bang nucleosynthesis, the already mentioned inflationary scenario, the anisotropy of CMB, and the gravitational wave and so on. At the same time, many issues still need to be clarified in LQC, such as the ambiguity problem Bojowald et al. (2004) and the different quantum schemes Chiou & Li (2009a;b); Mielczarek & Szyd (2008); Yang et al. (2009).

Very recently, an integral formulation of loop quantum cosmology with the Feynman procedure has been discussed, which again shows that the loop quantum cosmology is different from the Wheeler-DeWitt theory Ashtekar (2010). The spin foam model of LQC has also been constructed (for recent progress see Ashtekar et al. (2009)). These theories add to the appeal of LQC from different perspectives.

In this Chapter, we will focus our attention on the effective theory from which we can easily extract the physical result, and analyzing the explicit model can in turn shed light on the full theory. The effective LQC theory is a semiclassical theory, and can be derived in different ways, such as the WKB approximation, the coherent state, and so on. And they are consistent with each other in the leading term. The effective theory is valid in the semiclassical region, in between the quantum one and the classical one. And this theory will go back to the classical theory in the classical region. So we can use effective loop quantum cosmology for both the semiclassical and the classical region. On the other hand, along with the development of modern space technology and high-precision measurement techniques, we have accumulated a large amount of experimental data, probably more than that can be explained well by the current theory, such as the Pioneer anomaly, dark matter, and the accelerating expansion of the universe, to name a few. Close connection between the theoretical results and real experiments is crucial, and motivates us to apply LQC to explicit physical models, and then to compare theoretical results with real or gedanken experiments. The works to be summarized in this chapter can be divided into the following three parts.

(I) We discuss the stability properties of an autonomous system in the effective LQC Xiao & Zhu (2010). The system is described by a self-interacting scalar field  $\phi$  with positive potential  $V$ , coupled with a barotropic fluid in the Universe. With  $\Gamma = VV''/V'^2$  considered as a function of  $\lambda = V'/V$ , the autonomous system is extended from three dimensions to four dimensions. We find that the dynamic behaviors of a subset, not all, of the fixed points are independent of the form of the potential. Considering the higher-order derivatives of the potential, we get an infinite-dimensional autonomous system which can describe the dynamical behavior of the scalar field with more general potential. We find that there is just one scalar-field-dominated scaling solution in the loop quantum cosmology scenario.

(II) We discuss the null energy condition in the effective LQC Li & Zhu (2009). Wormhole and time machine are objects of great interest in general relativity. However, it takes exotic matters which are impossible on the classical level to support them. But if we introduce the quantum effects of gravity into the stress-energy tensor, these peculiar objects can be constructed self-consistently. LQC, with the potential to bridge the classical theory and quantum gravity, provides a simple way to study quantum effect in the semiclassical case. We investigate the averaged null energy condition in LQC in the framework of effective Hamiltonian, and find out that LQC do violate the averaged null energy condition in the massless scalar field coupled model.

(III) We consider the covariant entropy bound conjecture in the effective LQC Li & Zhu (2010a). The covariant entropy bound conjecture is an important hint for the quantum gravity, with several versions available in the literature. For cosmology, Ashtekar and Wilson-Ewing showed the consistence between the loop gravity theory and one version of this conjecture. Recently, S. He and H. Zhang proposed a version for the dynamical horizon of the universe, which validates the entropy bound conjecture for the cosmology filled with perfect fluid in the classical scenario when the universe is far away from the big bang singularity. But their conjecture breaks down near the big bang region. We examine this conjecture in the context of LQC. With the example of photon gas, this conjecture is protected by the quantum geometry effects.

## 2. Effective Theory of Loop Quantum Cosmology

LQC is a symmetry-reduced sector of LQG. It is a direct application of the quantization technique that originated from LQG. This section serves two purposes: First, we show

that one can indeed find semiclassical solution which is an approximation to the classical Einstein equations at late times. Second, we derive the effective equations incorporating the dominating quantum corrections within the framework of geometric quantum mechanics. The derivation of the effective equation utilizes two main tools: the geometric quantum mechanics and the "shadow state framework".

### 2.1 Classical framework

The classical phase space  $\Gamma$  is constructed by the Ashtekar variables  $(A_a^i, E_i^a)$ , where  $A_a^i$  is an  $SU(n)$  connection and  $E_i^a$  is the corresponding canonically conjugate variable. With the Ashtekar variables, the classical constraint of the gravitational part can be expressed as

$$C_{grav} = -\gamma^{-2} \int_{\mathcal{V}} d^3x \epsilon_{ijk} e^{-1} E^{ai} E^{bj} F_{ab}^i. \quad (1)$$

In the following, we will see that the Ashtekar variables can describe the classical theory very well, as the ordinary ADM variables do. Considering isotropic and homogeneous universe, the pair,  $(A_a^i, E_i^a)$ , is equivalent to the following form:

$$A_a^i = c^0 V^{-\frac{1}{3}} {}^0\omega_a^i, \quad (2)$$

$$E_i^a = p \sqrt{{}^0q} V^{-\frac{2}{3}} {}^0e_i^a, \quad (3)$$

where  ${}^0e$  is a fiducial background triad,  ${}^0\omega_a^i$  is the connection, and  ${}^0V$  is the volume of the fiducial cell. From the above set of equations, we see that all the information about  $A_a^i$  and  $E_i^a$  are contained in the pair of new variables  $(c, p)$ . The classical Hamiltonian constraint for a spatially flat FRW universe with a free massless scalar field is

$$C = -\frac{3}{\kappa\gamma^2} c^2 p^{\frac{1}{2}} + \frac{1}{2} \frac{p_\phi^2}{p^{\frac{3}{2}}} = 0. \quad (4)$$

For convenience, we replace the pair  $(c, p)$  with  $(\beta, V)$  through the following relationship

$$\beta = \frac{c}{\sqrt{p}}, \quad (5)$$

$$V = p^{\frac{3}{2}}. \quad (6)$$

Then the Hamiltonian constraint can be expressed as

$$C = -\frac{3}{\kappa\gamma^2} \beta^2 V + \frac{1}{2} \frac{p_\phi^2}{V} = 0, \quad (7)$$

with the symplectic structure  $\Omega = \frac{2}{\kappa\gamma} d\beta \wedge dV + d\phi \wedge dp_\phi$ . The phase space  $\Gamma$  consists of all possible points  $(\beta, V, \phi, p_\phi)$ . The Poisson bracket on the phase space is given by

$$\{f, g\} = \frac{\kappa\gamma}{2} \left( \frac{\partial f}{\partial \beta} \frac{\partial g}{\partial V} - \frac{\partial g}{\partial \beta} \frac{\partial f}{\partial V} \right) + \frac{\partial f}{\partial \phi} \frac{\partial g}{\partial p_\phi} - \frac{\partial g}{\partial \phi} \frac{\partial f}{\partial p_\phi}. \quad (8)$$



The evolution of the canonical variable depends on the Poisson equations. Therefore, the classical equations of motion are given by

$$\dot{\beta} = \{\beta, C\} = -\frac{3}{2} \frac{\beta^2}{\gamma} - \frac{\kappa\gamma}{4} \frac{p_\phi^2}{V^2}, \tag{9}$$

$$\dot{V} = \{p, C\} = 3 \frac{\beta}{\gamma} V, \tag{10}$$

$$\dot{\phi} = \{\phi, C\} = \frac{p_\phi}{V}, \tag{11}$$

$$\dot{p}_\phi = \{p_\phi, C\} = 0. \tag{12}$$

We can verify that the above equation set is equivalent to the Friedmann equation for a free scalar field when it is written in terms of the ordinary ADM variables. The new canonically conjugate variables are related to the old geometrodynamics variables via  $\beta = \gamma \frac{\dot{a}}{a}$  and  $V = a^3$ .

Then, with the definition of the density  $\rho = \frac{1}{2} \frac{p_\phi^2}{p^3}$ , we can get the classical Friedmann equation and the Raychauduri equation as follows

$$H^2 = \frac{\kappa}{3} \rho, \tag{13}$$

$$3 \frac{\ddot{a}}{a} = -2\kappa\rho. \tag{14}$$

### 2.2 Quantum framework

LQC is the symmetry reduced model of LQG, and it inherits the quantization schemes of LQG. LQC is essentially different from the WDW theory. In LQC, the kinematical Hilbert space is in the “polymer representation” for  $p$ , instead of the standard Schrödinger representation. There is no operator corresponding to  $c$ . In addition, it is not densely defined if the inverse volume function  $|p|^{-3/2}$  is naively quantized as the operator with eigenvalues equal to the inverse of the volume eigenvalues. Thus, to construct the Hamiltonian constraint operator, we have to express the classical constraint in terms of the triad variable  $p$  and the holonomy  $h_k^{(\bar{\mu})}$ , both of which have direct quantum analogs.

In the development of LQC, there exist two different quantum schemes: the  $\mu_0$  scheme and the  $\bar{\mu}$  scheme. In the  $\mu_0$  scheme,  $\mu_0$  is a constant on the phase space, and the difference equation is in uniform step size. The greatest success of this scheme so far is that it can replace the big bang singularity with the big bounce, which reflects the nature of the quantum geometry effect. Unfortunately, it suffers from serious physical problems. For example, the critical value,  $\rho_{crit}$ , of the matter density at which the bounce occurs can be made arbitrarily small by increasing the initial momentum  $p_\phi$  of the scalar field. In other words, large values of  $p_\phi$  are permissible in the late universe, which leads to bounce at low matter density. This is a serious drawback because we do not expect the quantum effect to modify the evolution of the universe in the classical region. In the  $\bar{\mu}$  scheme,  $\bar{\mu}$  is a function on the phase space, unlike the constant  $\mu_0$  in the  $\mu_0$  scheme. This difference turns out to be enough to remove the major weakness of the  $\mu_0$  scheme, while keeping the desirable features of the original scheme. In this section, we only review the quantization procedure in the  $\bar{\mu}$ -scheme.

In the  $\bar{\mu}$  scheme, one can shrink the loop  $\square_{ij}$  until the area of the loop approaches the area gap  $\Delta$ , measured by the physical metric  $q_{ab}$ . The physical area of the elementary cell is  $|p|$ . Each

side of  $\square_{ij}$  is  $\bar{\mu}(p)$  times the edge of the elementary cell, which leads to

$$\bar{\mu}^2 |p| = \Delta \equiv (2\sqrt{3}\pi\gamma)l_{pl}^2. \quad (15)$$

Therefore,  $\bar{\mu}$  is a non-trivial function on the phase space and can be rewritten as

$$\bar{\mu} = \sqrt{\frac{\Delta}{p}}, \quad (16)$$

where  $\gamma$  is the Immirzi parameter<sup>1</sup>.

Following Dirac, in the quantum theory, we should first construct a kinematical description. The Hilbert space  $H_{kin}^{grav}$  is the space  $L^2(\mathbb{R}_{Bohr}, d\mu_{Bohr})$  consisting of square integrable functions on the Bohr compactification of the real line. To specify states concretely, we work with the representation of the operator  $\hat{p}$  in which the operator  $\hat{p}$  is diagonal. Eigenstates of  $\hat{p}$  are labeled by a real number  $\mu$  and satisfy the orthonormality relation:

$$\langle \mu_1 | \mu_2 \rangle = \delta_{\mu_1, \mu_2}. \quad (17)$$

The right-hand side of the above equation is the Kronecker delta rather than the Dirac delta distribution. A general state in  $H_{kin}^{grav}$  can be expressed as a countable sum  $\langle \Psi | = \sum_n c^{(n)} \langle \mu_n |$ , where  $c^{(n)}$  are complex coefficients and the inner product is given by

$$\langle \Psi_1 | \Psi_2 \rangle = \sum_n \bar{c}_1^{(n)} c_2^{(n)}.$$

The fundamental operators are  $\hat{p}$  and  $\exp(i\widehat{\bar{\mu}c}/2)$ . The action of the operator  $\hat{p}$  on its eigenvalue is

$$\hat{p} |\mu\rangle = \frac{8\pi\gamma l p^2}{6} \mu |\mu\rangle \quad (18)$$

and the action of the operator  $\exp(i\widehat{\bar{\mu}c}/2)$  on  $|\mu\rangle$  will be given later.

In order to achieve quantization, we should represent the Hamiltonian constraint operator  $\widehat{C}_{grav}$  in terms of the above well-defined operators. Following the full theory, with the Thiemann trick, we can rewrite the term that involves the inverse triad  $e^{-1}$  as

$$e^{-1} \epsilon_i^{jk} E_j^a E_k^b = \sum_l \frac{{}^o q^{1/3}}{2\pi G \gamma \bar{\mu} L} \epsilon^{jkl0} e_j^{a0} e_k^{b0} \text{Tr}(h_l^{(\bar{\mu})} \{ (h_l^{(\bar{\mu})})^{-1}, V \} \tau_i), \quad (19)$$

where the holonomy

$$h_l^{(\bar{\mu})} = \mathcal{P} \exp \int_0^{\bar{\mu}L} \tau_i A_a^i dx^a = \exp(\bar{\mu}c\tau_l) \quad (20)$$

<sup>1</sup> The general form of  $\bar{\mu} = (\frac{\Delta}{p})^x$ ,  $0 < x < 1$ . Here we choose  $x = \frac{1}{2}$  according to the suggestions of Ashtekar. However, we still cannot determine it for the following two reasons. (1) The coordinate area is more natural than an invariant geometrical area when we consider the quantization of the single curvature components. (2) The quantization requires the area operator which is not well understood in the full theory.

is along the edge of coordinate length  $\bar{\mu}L$  aligned with the direction of  ${}^o e_i^a \partial_a$ .  $\tau_i$  is the generator of the  $SU(2)$  and satisfies the relationship  $[\tau_i, \tau_j] = \epsilon_{ij}^k \tau_k$ , where  $\tau_i = \frac{1}{2i} \sigma_i$  and  $\sigma_i$  are the Pauli matrices in the standard convention.

According to the techniques in the gauge theory, the field strength  $F_{ab}^i$  can be expressed as

$$F_{ab}^k = -2 \lim_{Ar_{\square} \rightarrow 0} \text{Tr} \left( \frac{h_{\square ij}^{(\bar{\mu})} - 1}{\bar{\mu}^2 V_0^{2/3}} \right) \tau^{ko} \omega_a^{io} \omega_b^j \tag{21}$$

where

$$h_{\square jk}^{(\bar{\mu})} := h_j^{(\bar{\mu})} h_k^{(\bar{\mu})} (h_j^{(\bar{\mu})})^{-1} (h_k^{(\bar{\mu})})^{-1} \tag{22}$$

is the holonomy along the the four edges of  $\square_{ij}$ .

Combining Eq.(19) with Eq.(21), the classical Hamiltonian constraint for the gravitational sector can be rewritten as

$$\begin{aligned} C_{grav} &= -\frac{4 \text{sgn}(p)}{8\pi\gamma^3 \bar{\mu}^3} \sum_{ijk} \epsilon^{ijk} \text{Tr} (h_j^{(\bar{\mu})} h_k^{(\bar{\mu})} (h_j^{(\bar{\mu})})^{-1} (h_k^{(\bar{\mu})})^{-1} h_i^{(\bar{\mu})} \{ (h_i^{(\bar{\mu})})^{-1}, V \}) \\ &= \sin(\bar{\mu}c) \left[ -\frac{4}{8\pi\gamma^3} \frac{\text{sgn}(p)}{\mu^3} \sum_k \text{Tr} \tau_k h_i^{(\bar{\mu})} \{ (h_i^{(\bar{\mu})})^{-1}, V \} \right] \sin(\bar{\mu}c), \end{aligned} \tag{23}$$

where in the last step we have used a symmetric ordering of the three terms for later convenience.

Now, we consider the action of the operator  $e^{\widehat{\frac{i\bar{\mu}c}{2}}}$  on the state  $|\mu\rangle$ . Although the geometrical meaning of this action of  $\exp i(\bar{\mu}c/2)$  is simple, its expression in the  $|\mu\rangle$  representation is complicated because  $\mu$  is not an affine parameter along the integral curve of the vector  $\bar{\mu} \frac{d}{d\mu}$ . After calculation, we get

$$e^{\widehat{\frac{i\bar{\mu}c}{2}}} \tilde{\Psi}(\mu) = \tilde{\Psi}(\text{sgn}(\bar{\mu})|\bar{\mu}|^{\frac{2}{3}}), \text{ where } \bar{\mu} = \text{sgn}(\mu)|\mu|^{\frac{3}{2}} + \frac{1}{K}. \tag{24}$$

Next, we will change to the  $v$  representation for simplicity. In the  $v$  representation, the action of  $e^{\widehat{\frac{i\bar{\mu}c}{2}}}$  on  $|v\rangle$  is extremely simple

$$e^{\widehat{\frac{i\bar{\mu}c}{2}}} \Psi(v) = \Psi(v + 1), \tag{25}$$

and the action of the volume operator on it is

$$\hat{V}|v\rangle = \left(\frac{8\pi\gamma}{6}\right)^{\frac{3}{2}} \frac{|v|}{K} |v\rangle, \tag{26}$$

where  $v = K \text{sgn}(\mu)|\mu|^{\frac{3}{2}}$  and  $K = \frac{2\sqrt{2}}{3\sqrt{3\sqrt{3}}}$ . Therefore, we will use the  $v$  representation in the following.



We can straightforwardly get the action of the operators  $\widehat{\sin(\frac{\bar{\mu}c}{2})}$  and  $\widehat{\cos(\frac{\bar{\mu}c}{2})}$  on  $v$  representation,

$$\widehat{\sin(\frac{\bar{\mu}c}{2})}|v\rangle = \frac{1}{2i}[|v+1\rangle - |v-1\rangle], \quad (27)$$

$$\widehat{\cos(\frac{\bar{\mu}c}{2})}|v\rangle = \frac{1}{2}[|v+1\rangle + |v-1\rangle]. \quad (28)$$

Then we promote the corresponding physical quantities  $\sin(\frac{\bar{\mu}c}{2}), \cos(\frac{\bar{\mu}c}{2})$  and  $V$  to the operators. We have

$$\begin{aligned} \widehat{C}_{grav} &= \widehat{\sin(\frac{\bar{\mu}c}{2})} \left[ \frac{24i \operatorname{sgn}(\mu)}{8\pi\gamma^3\bar{\mu}^3} (\widehat{\sin(\frac{\bar{\mu}c}{2})} \widehat{V} \widehat{\cos(\frac{\bar{\mu}c}{2})} - \widehat{\cos(\frac{\bar{\mu}c}{2})} \widehat{V} \widehat{\sin(\frac{\bar{\mu}c}{2})}) \right] \widehat{\sin(\frac{\bar{\mu}c}{2})} \\ &= : \widehat{\sin(\bar{\mu}c)} \widehat{A} \widehat{\sin(\bar{\mu}c)}. \end{aligned} \quad (29)$$

The action of the operator  $\widehat{A}$  on the state  $\Psi(v)$  can be written as

$$\widehat{A}\Psi(v) = -\frac{27K}{4} \sqrt{\frac{8\pi}{6\gamma^{3/2}}} |v||v-1| - |v+1| \Psi(v). \quad (30)$$

One needs to use caution when deriving this expression.  $\operatorname{sgn}(v)$  is unambiguously defined only on states other than the point  $v=0$ . Since the right-hand side vanishes at  $v=0$ , it is just the domain of  $\operatorname{sgn}(\mu)$ , where  $\widehat{A}$  is well defined. Therefore, the operator on the right-hand side is densely defined.

The action of the gravitational constraint on  $\Psi(v)$  is given by

$$\widehat{C}_{grav}\Psi(v) = f_+(v)\Psi(v+4) + f_0(v)\Psi(v) + f_-(v)\Psi(v-4) \quad (31)$$

with

$$f_+(v) = \frac{27}{16} \sqrt{\frac{8\pi}{6}} \frac{Kl_{pl}}{\gamma^{3/2}} |v+2||v+1| - |v+3| \quad (32)$$

$$f_-(v) = f_+(v-4) \quad (33)$$

$$f_0(v) = -f_+(v) - f_-(v). \quad (34)$$

From the above, we can see that the gravitational constraint is again a difference operator. Compared with the  $\mu_0$ -scheme, the new constraint involves steps which are constant in the eigenvalues of the volume operator  $\widehat{V}$ , not in the eigenvalues of  $\widehat{p}$ .

Next, we continue to quantize the matter part of the constraint

$$C_{matt} = 8\pi|p|^{-\frac{3}{2}} p_\phi^2. \quad (35)$$

It turns out that despite the existence of an inverse operator  $\widehat{p}^{-\frac{3}{2}}$ , one can quantize it successfully. With the Thiemann trick, one can always write the inverse operator in an equivalent way, which contains the quantities that can be easily promoted to the operators. We express the inverse triad as

$$\begin{aligned}
 |p|^{-n} &= \left( \left| \frac{3}{\gamma\kappa l \bar{\mu} j(j+1)(2j+1)} \sum_i \text{Tr}(^j \tau_i^j h_i^{\bar{\mu}} \{j(h_i^{\bar{\mu}})^{-1}, V^{2l/3}\}) \right| \right)^{\frac{n}{1-l}} \\
 &= \left( \left| \frac{9}{\gamma\kappa l \bar{\mu} j(j+1)(2j+1)} \text{Tr}(^j \tau_3^j h_3^{\bar{\mu}} \{j(h_3^{\bar{\mu}})^{-1}, V^{2l/3}\}) \right| \right)^{\frac{n}{1-l}}.
 \end{aligned}
 \tag{36}$$

In this quantization, there are two ambiguities Bojowald (2002b; 2005), labeled by a half integer  $j$  and a real number  $l$  in the range  $0 < l < 1$ . Following the considerations in Perez (2006); Vandersloot (2005), we will set  $j = 1/2$ , and the general case for  $j$  can be found in Chiou & Li (2009a;b). For  $l$ , there is no universally accepted concept, and  $l = 1/2$  and  $l = 3/4$  have been used in the literature. Fortunately, the results do not change qualitatively with the exact choice. Here, we choose  $j = 1/2$  and  $l = 3/4$ . Then

$$\widehat{|p|^{-\frac{3}{2}} \Psi(v)} = \left( \frac{6}{8\pi\gamma} \right)^{3/2} B(v) \Psi(v)
 \tag{37}$$

where

$$B(v) = \left( \frac{3}{2} \right)^3 K |v| \left| |v+1|^{1/3} - |v-1|^{1/3} \right|^3.
 \tag{38}$$

Combining all the results above, we can write down the full constraint

$$\hat{C} \Psi(v) = (\hat{C}_{\text{grav}} + \hat{C}_{\text{matt}}) \Psi(v) = 0
 \tag{39}$$

as follows:

$$\begin{aligned}
 p_\phi^2 \Psi(v, \phi) &= [B(v)]^{-1} (C^+(v) \Psi(v+4, \phi) + C^0(v) \Psi(v, \phi) + C^-(v) \Psi(v-4, \phi)) \\
 &=: -\Theta \Psi(v, \phi)
 \end{aligned}
 \tag{40}$$

where the coefficients  $C^\pm(v)$  and  $C^0(v)$  are given by:

$$\begin{aligned}
 C^+(v) &= \frac{3\pi KG}{8} |v+2| \left| |v+1| - |v+3| \right| \\
 C^-(v) &= C^+(v-4) \\
 C^0(v) &= -C^+(v) - C^-(v).
 \end{aligned}
 \tag{41}$$

### 2.3 Effective theory

The effective theory can be derived through the geometric quantum mechanics method. Because of the fiber bundle structure, any horizontal section can be identified with the classical phase space. If we can find such a section, then the quantum dynamics on it can be expressed in terms of effective Hamiltonian, which is simply the expectation value of the quantum Hamiltonian constraint operator. The expectation value yields the classical term as the leading term and has quantum correction in the subleading terms. This is the key idea for deriving the effective equation. Here we can look for a natural section that is approximately preserved by the flow of the Hamiltonian constraint in a precise sense.

In order to obtain the effective theory, we should choose a coherent state. Here we use a Gaussian coherent state mostly for the reason that it is the simplest for getting the effective equation with a late-time, large-volume approximation. We can also choose a more general form to do this. The Gaussian coherent state is as follows:

$$\begin{aligned} \langle \psi_{\beta', V'; \phi', p'_\phi} | &= \int dp_\phi \sum_v e^{-\frac{1}{2}\epsilon^2(v-v')^2} e^{\frac{i}{2}\sqrt{\Delta}\beta'(v-v')} \\ &\times e^{-\frac{1}{2}\epsilon_\phi^2(p_\phi-p'_\phi)^2} e^{-i\phi'(p_\phi-p'_\phi)} \langle v; p_\phi | \\ &=: \int dp_\phi \sum_v \overline{\Psi}_n(p_\phi) \langle v; p_\phi |, \end{aligned} \quad (42)$$

where  $v$  and  $v'$  are defined as  $V = (\frac{8\pi\gamma}{6})^{\frac{3}{2}} \frac{l_p^3}{K} v$  and  $V' = (\frac{8\pi\gamma}{6})^{\frac{3}{2}} \frac{l_p^3}{K} v'$ , here  $K = \frac{2\sqrt{2}}{3\sqrt{3}\sqrt{3}}$ .

Additionally, we should put on it three constraints:

- (1)  $v' \gg 1$ ,  $\sqrt{\delta}\beta' \ll 1$ . This pair of conditions means that the scalar factor is much larger than the Planck length and demands that the rate of change of the scale factor is much smaller than the speed of light, which holds even in the early universe.
- (2)  $v'\epsilon \gg 1$  and  $\epsilon \ll \sqrt{\delta}\beta'$ . This pair means that the spreads of operator  $\hat{v}$  and  $\hat{\beta}$  must be small.
- (3)  $\phi \gg \epsilon_\phi$  and  $p_\phi \epsilon_\phi \gg 1$ . The last pair of restrictions on parameters demands that the spreads of  $\phi$  and  $p_\phi$  are small.

We need to show that the semiclassical state is sharply peaked at the classical point  $(\beta', V', \phi', p'_\phi)$ . Here, we face two difficulties. First, the operator corresponding to  $\hat{\beta}$  in the Schrödinger representation is not defined in the polymer framework. So we need to define a fundamental operator in  $\mathcal{H}_{poly}$  which is approximated by  $\hat{\beta}$  of the Schrödinger representation. We define this operator as follows:

$$\widehat{\beta}_\Delta = \frac{1}{i\sqrt{\Delta}} (\widehat{e^{\frac{i}{2}\sqrt{\Delta}\beta}} - \widehat{e^{-\frac{i}{2}\sqrt{\Delta}\beta}}). \quad (43)$$

The operator  $\widehat{\beta}_\Delta$  agrees approximately with the classical  $\beta$  in the regime  $\sqrt{\Delta}\beta \ll 1$ . And its action on the basis kets  $|v; p_\phi\rangle$  is

$$\widehat{\beta}|v; p_\phi\rangle = \frac{1}{i\sqrt{\Delta}} (|v+1; p_\phi\rangle - |v-1; p_\phi\rangle). \quad (44)$$

The second difficulty is that the coherent state defined in Eq.(42) lies on  $Cyl^*$ . But there is no inner product on the  $Cyl^*$ . In other words, the solutions to the constraints do not reside in the kinematical Hilbert space, but rather in its algebraic dual space, therefore the required expectation values cannot be defined on it. Fortunately, we can carry out calculation within the "shadow state framework". Each shadow captures only a part of the information contained in our state, but the collection of shadows can be used to determine the full properties of the state in  $Cyl^*$ . We can indeed prove this Gaussian coherent state is sharply peaked at some classical values  $(\beta', V', \phi', p'_\phi)$ . Here we do not give the detailed proof, and refer interested readers to Taveras (2008).

The required projection  $\hat{P}_\gamma$  from  $Cyl^*$  to  $Cyl_\gamma$  can be defined as

$$\langle \Psi | \hat{P}_\gamma := \sum_{x_j \in \gamma} \Psi(x_j) |x_j\rangle \equiv |\Psi_\gamma^{shad}\rangle. \quad (45)$$

The corresponding shadow state in our case is

$$|\Psi_\gamma^{shad}\rangle = \int dp_\phi \sum_n e^{-\frac{1}{2}\epsilon^2(n-N)^2} e^{-\frac{i}{2}\sqrt{\Delta}\beta'(n-N)} \times e^{-\frac{1}{2}\epsilon_\phi^2(p_\phi-p'_\phi)^2} e^{i\phi'(p_\phi-p'_\phi)} |n; p_\phi\rangle. \tag{46}$$

Then one can compute the expectation value of the constraint operator directly

$$\langle \hat{C} \rangle = -\frac{3}{16\pi G\gamma^2 \bar{\mu}'^2} p^{\frac{1}{2}} \left[ 1 + e^{-4\epsilon^2} \left( 2\sin^2(\sqrt{\Delta}\beta') - 1 \right) \right] + \frac{1}{2} \left( p'^2_\phi + \frac{1}{2\epsilon_\phi^2} \right) \left( \frac{6}{8\pi\gamma l_p^2} \right)^{\frac{3}{2}} K \left[ \frac{1}{v'} + O(v'^{-3}, v'^{-3}\epsilon^{-2}) \right]. \tag{47}$$

Furthermore, we want to know that the equations of motion for the other physical quantities  $O$ . We will use the commutator between  $O$  and the Hamiltonian. The corresponding results are as follows:

$$\begin{aligned} \langle \dot{\beta} \rangle &\simeq -\frac{1}{16\pi} \frac{27}{16} \left( \frac{8\pi\gamma}{6} \right)^{\frac{1}{2}} \frac{K}{\gamma^2 \sqrt{\Delta}} \left[ 4e^{-\frac{25}{4}\epsilon^2} \cos\left(\frac{5}{2}\sqrt{\Delta}\beta'\right) + 4e^{-\frac{9}{4}\epsilon^2} \cos\left(\frac{3}{2}\sqrt{\Delta}\beta'\right) \right. \\ &\quad \left. - 8e^{-\frac{1}{4}\epsilon^2} \cos\left(\frac{1}{2}\sqrt{\Delta}\beta'\right) \right] - \left( p'^2_\phi + \frac{1}{2\epsilon_\phi^2} \right) \frac{1}{2V'^2}, \\ \langle \dot{V} \rangle &\simeq \frac{3V'}{\gamma} e^{-4\epsilon^2} \frac{\sin(2\sqrt{\Delta}\beta')}{2\sqrt{\Delta}}, \\ \langle \dot{\phi} \rangle &\simeq \frac{p'_\phi}{V'} + O\left(\frac{1}{V'^3}\right), \\ \langle \dot{p}_\phi \rangle &= 0. \end{aligned}$$

### 2.4 Ordinary formalism

As mentioned above, in the geometric quantization picture, we take the expectation values as our basic observables, and try to obtain an effective description in terms of these variables. We denote the expectation of  $\beta, V, \phi, p_\phi$  as  $\bar{\beta}, \bar{V}, \bar{\phi}, \bar{p}_\phi$ , respectively. Then we obtain the effective equations of motion to the first order as follows

$$\begin{aligned} \bar{C} &= -\frac{3}{\kappa\gamma^2} \bar{V} \bar{\beta}^2 \left( 1 - \frac{1}{4} \Delta \bar{\beta}^2 \right) - \frac{6\epsilon^2}{\kappa\gamma^2} \frac{\bar{V}}{\Delta} \\ &\quad + \frac{\bar{p}_\phi^2}{2\bar{V}} \left[ 1 + O(\bar{V}^{-2}, \bar{V}^{-2}\epsilon^{-2}) \right], \end{aligned} \tag{48}$$

$$\begin{aligned} \dot{\bar{\beta}} &= \frac{3}{4\gamma} \sqrt{1 - \frac{1}{4} \Delta \bar{\beta}^2} \left[ -2\bar{\beta}^2 + \Delta \bar{\beta}^4 \right] \\ &\quad - \frac{\kappa}{4} \sqrt{1 - \frac{1}{4} \Delta \bar{\beta}^2} \frac{\bar{p}_\phi^2}{V'^2} \left[ 1 + O(\bar{V}^{-2}, \bar{V}^{-2}\epsilon^{-2}) \right], \end{aligned} \tag{49}$$

$$\dot{\bar{V}} = 3 \frac{\bar{\beta}}{\gamma} \bar{V} \sqrt{1 - \frac{\Delta \bar{\beta}^2}{4}} \left( 1 - \frac{1}{2} \Delta \bar{\beta}^2 \right), \tag{50}$$

$$\dot{\phi} = \frac{\bar{p}_\phi}{\bar{V}} + O(\bar{V}^{-3}), \quad (51)$$

$$\dot{\bar{p}}_\phi = 0. \quad (52)$$

Combining Eq.(48) with Eq.(49), we get the effective Friedmann equation

$$H^2 = \frac{\kappa\rho}{3} \left(1 - \frac{\rho}{\rho_{crit}}\right) + O(\epsilon^2), \quad (53)$$

where  $H = \frac{\dot{a}}{a}$  and  $\rho_{crit} = \frac{3}{\kappa\gamma^2\Delta}$  originated from the quantum effect.

From the effective theory, we can easily see that the semiclassical states follow the classical trajectory until the scalar field density is on the order of 1% of the Planck density where deviations from the classical trajectory emerge. Then there can be major deviations from the classical theory. The existence of the correction  $\rho_{crit}$  can allow  $H = 0$ , meaning a bounce is possible.

Also, from the equation of motion, we can get the conservation equation and the corrected Raychaudhuri equation for this effective theory. With the Poisson bracket, we can calculate  $\dot{\rho} = \{\rho, \bar{C}\}$ , and get the conservation equation

$$\dot{\rho} + 6\frac{\dot{a}}{a}\rho = 0, \quad (54)$$

which is the same as the classical conservation equation.

Similarly, we calculate  $\dot{\bar{V}} = \{\bar{V}, \bar{C}\}$ , and express it in terms of the scale factor  $a$ . Then we get

$$3\frac{\ddot{a}}{a} = -2\kappa\rho \left(1 - \frac{5}{2}\frac{\rho}{\rho_c}\right) + o(\epsilon^2). \quad (55)$$

Compared with the classical Raychaudhuri equation, it also gets a quantum correction term  $\frac{\rho^2}{\rho_c}$ . Classically, this equation is always negative, but in the effective framework, there is a bounce at the  $\frac{\rho}{\rho_c} = 1$ , which makes  $\ddot{a}$  positive when  $\rho > \frac{2}{5}\rho_c$ .

In the end, there are still two points that need to be clarified. First, theoretically, Eq.(53) cannot describe the correct dynamics near the bounce point because the bounce point  $\frac{\rho}{\rho_c} = 1$  is outside of the regime of our approximation. The effective framework is applicable only to a late-time, large-volume universe because  $\epsilon \ll \sqrt{\Delta\beta'}$  is violated badly at the point of  $\frac{\rho}{\rho_c} = 1$ . However, numerical work Ashtekar et al. (2006a;b) has shown that the dynamics derived by the above describes the evolution of the universe very well even at the bounce, and hence the results obtained in the effective framework continue to be reliable even beyond their expected regime. Second, we omit the high-order correction term  $o(\epsilon^2)$  in the modified Friedmann equation Eq.(53). We should note that near the bounce point, the term in parentheses in the modified Friedmann equation is approaching 0. It is not known if  $O(\epsilon^2)$  should be omitted there because  $\epsilon \ll \sqrt{\Delta\beta'}$  is violated. But the numerical results show that the modified Friedmann equation holds with negligible  $O(\epsilon^2)$  corrections. Therefore, the effective theory remains valid beyond the domain for which it is constructed.

**2.5 Phenomenological analysis**

In the above subsection, we obtain the effective framework of LQC systematically. In this one, we analyze the holonomy correction and the inverse volume correction phenomenologically, which is easy to handle and can describe the evolution of the universe correctly. Meanwhile, it can lead us to new physics heuristically.

**2.5.1 The holonomy correction**

As mentioned above, the classical Hamiltonian with a free scalar field for the  $k = 0$  FRW model is given by

$$H_{cl} = -\frac{3N}{8\pi G\gamma^2} c^2 \sqrt{|p|} + \frac{Np_\phi^2}{2|p|^{3/2}} \tag{56}$$

in terms of the Ashtekar variables  $c$  and  $p$ .  $p_\phi$  is the conjugate momentum of  $\phi$ .  $N$  is the lapse function and  $\gamma$  is the Barbero-Immirzi parameter. At the heuristic level, we can impose the loop quantum corrections of LQC phenomenologically. We simply replace  $c$  with

$$c \longrightarrow \frac{\sin(\bar{\mu}c)}{\bar{\mu}}. \tag{57}$$

Then we get the description of "holonomization". This effective dynamics is solved as if the dynamics was classical but governed by the new "holonomized" Hamiltonian, which reads as

$$H_{eff} = -\frac{3N}{8\pi G\gamma^2} \frac{\sin^2 \bar{\mu}c}{\bar{\mu}^2} \sqrt{|p|} + \frac{Np_\phi^2}{2|p|^{3/2}}. \tag{58}$$

As to be expected, the bouncing scenario can be easily obtained at the level of heuristic effective dynamics without invoking the sophisticated features of LQC. In particular, with the "improved" scheme imposed for  $\bar{\mu}$ , the modified Hamiltonian constraint  $C_{\bar{\mu}} = 0$  immediately sets an upper bound for the matter density:

$$\rho_\phi := \frac{p_\phi^2}{2|p|^3} = \frac{3}{8\pi G\gamma^2\Delta} \sin^2 \bar{\mu}c \leq 3\rho_{Pl}, \tag{59}$$

where the Planckian density is defined as

$$\rho_{Pl} := (8\pi G\gamma^2\Delta)^{-1}. \tag{60}$$

With this effective Hamiltonian, we have the canonical equation

$$\dot{p} = \left\{ p, H_{eff} \right\} = -\frac{8\pi\gamma}{3} \frac{\partial H_{eff}}{\partial c}, \tag{61}$$

or,

$$\dot{a} = \frac{\sin(\bar{\mu}c) \cos(\bar{\mu}c)}{\gamma\bar{\mu}}. \tag{62}$$

Combining that with the constraint on Hamiltonian,  $H_{eff} = 0$ , we obtain the modified Friedmann equation

$$H^2 = \frac{8\pi}{3} \rho \left( 1 - \frac{\rho}{\rho_c} \right), \tag{63}$$



where  $H \equiv \frac{\dot{a}}{a}$  denotes the Hubble rate, and  $\rho_c \equiv \frac{3}{8\pi\gamma^2\mu^2 p}$  is the quantum critical density. Compared with the standard Friedmann equation, we can define the effective density

$$\rho_{eff} = \rho \left(1 - \frac{\rho}{\rho_c}\right). \quad (64)$$

Taking derivative of Eq.(63) and also using the conservation equation of matter,  $\dot{\rho} + 3H(\rho + P) = 0$ , we obtain the modified Raychaudhuri equation

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi}{3} \left\{ \rho \left(1 - \frac{\rho}{\rho_c}\right) + 3 \left[ P \left(1 - \frac{2\rho}{\rho_c}\right) - \frac{\rho^2}{\rho_c} \right] \right\}. \quad (65)$$

Comparing that with the standard Raychaudhuri equation, we can define the effective pressure,

$$P_{eff} = P \left(1 - \frac{2\rho}{\rho_c}\right) - \frac{\rho^2}{\rho_c}. \quad (66)$$

In terms of the effective density and the effective pressure, the modified Friedmann, Raychaudhuri and conservation equations take the following forms,

$$H^2 = \frac{8\pi}{3} \rho_{eff}, \quad (67)$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi}{3} (\rho_{eff} + 3P_{eff}), \quad (68)$$

$$\dot{\rho}_{eff} + 3H(\rho_{eff} + P_{eff}) = 0. \quad (69)$$

Therefore, we can get the important features of LQC without going into the detailed construction of LQC at all. This can help us to extract the physics easily and provide insight into the full theory.

### 2.5.2 The inverse volume correction

The inverse volume  $|p|^{-\frac{3}{2}}$  of the matter Hamiltonian can get a quantum correction and the matter Hamiltonian obtained in this manner will behave differently at small  $p$ . This is called the inverse volume correction, and can be interpreted as providing a natural curvature cut-off. For a scalar field, the modified matter Hamiltonian is

$$\mathcal{H}_{matt} = \frac{1}{2} d(a) p_\varphi^2 + a^3 V(\varphi), \quad (70)$$

where  $p_\varphi$  is the momentum canonically conjugate to  $\varphi$ , and  $d(a)$ , which is classically  $1/a^3$ , encodes the quantum corrections. In the semi-classical regime, where spacetime may be treated as continuous, it is given by

$$d(a) = \frac{D(q)}{a^3}, \quad q = \frac{a^2}{a_*^2}, \quad a_* = \sqrt{\frac{\gamma j}{3}} \ell_{\text{Pl}}. \quad (71)$$

and

$$D(q) = q^{-3/2} \left\{ \frac{3}{2l} \left( \frac{1}{l+2} \left[ (q+1)^{l+2} - |q-1|^{l+2} \right] - \frac{q}{1+l} \left[ (q+1)^{l+1} - \text{sgn}(q-1) |q-1|^{l+1} \right] \right) \right\}^{3/(2-2l)}. \quad (72)$$

Here  $\gamma$  is the Barbero-Immirzi parameter. Since the expression for  $a^{-1}$  is rather complicated, the final quantization contains quantization ambiguities of different types. Here,  $j$  (a half integer) is resulted from the use of arbitrary representations, and  $0 < l < 1$  is another quantization parameter. The approximation to the eigenvalues becomes better for values of  $j$  larger than the minimal one,  $1/2$ .

The scale below which non-perturbative modifications become important is given by  $a_*$ . Typically, one chooses  $j \gg 1$ , so that  $a_* \gg \ell_{\text{Pl}}$ . The Planck scale marks the onset of discrete spacetime effects. For  $\ell_{\text{Pl}} < a \ll a_*$ , the universe is in the semiclassical regime. In this regime,  $q \ll 1$ . With the Taylor expansion, the geometrical density  $d(a)$  behaves as

$$d(a) \sim \left[ \left( \frac{3}{1+l} \right)^{3/(2-2l)} \left( \frac{a}{a_*} \right)^{3(2-l)/(1-l)} \right] \frac{1}{a^3}. \tag{73}$$

The Hamiltonian determines the dynamics completely. The equation of motion for the matter is

$$\dot{\varphi} = \{ \varphi, \mathcal{H} \} = d(a) p_\varphi. \tag{74}$$

Combined with the Hamiltonian equation of motion for  $p_\varphi$ , the above equation can be cast into a second order equation for  $\varphi$  Bojowald & Vandersloot (2002); Singh & Toporensky (2004); Vereshchagin (2004); Tsujikawa et al. (2004),

$$\ddot{\varphi} + \left( 3H - \frac{\dot{D}}{D} \right) \dot{\varphi} + D V_{,\varphi} = 0. \tag{75}$$

For  $\ell_{\text{Pl}} < a \ll a_*$ , we find that  $\dot{D}/D > 3H$ , which leads to the classical frictional term for an expanding universe. The case is the opposite if the universe is contracting.

The Friedman equation and the Raychaudhuri equation are as follows:

$$H^2 = \frac{8\pi G}{3} \left[ \frac{\dot{\varphi}^2}{2D(a)} + V(\varphi) \right], \tag{76}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3} \left[ \frac{\dot{\varphi}^2}{D} \left( 1 - \frac{\dot{D}}{4HD} \right) - V(\varphi) \right]. \tag{77}$$

The Friedman equation implies a bounce in the scale factor, i.e.,  $\dot{a} = 0$  and  $\ddot{a} > 0$ , which requires a negative potential. (In a closed model, the curvature term allows for a bounce with positive potential Singh & Toporensky (2004); Vereshchagin (2004).) Vanishing Hubble parameter at the bounce implies

$$\dot{\varphi}^2 = -2D(a)V(\varphi), \tag{78}$$

so that at the bounce,

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3} \left( 6 - \frac{d \ln D}{d \ln a} \right) V. \tag{79}$$

Classically, i.e., for  $D = 1$ , a bounce for a negative  $V(\varphi)$  is not allowed. With the modified  $D(a)$ , however,  $d \ln D / d \ln a > 6$  will hold for sufficiently small  $a$ , so  $\ddot{a} > 0$  is possible. Thus, the universe has to collapse sufficiently deep into the modified regime before it can bounce back.

So far, the two corrections appear independently in the discussion. We hope to find a consistent way to bring them together and study their impact on the universe. All the works discussed in the following are based on the effective theory.

### 3. Stability analysis of an autonomous system

Scalar field plays an important role in modern cosmology. Indeed, scalar field cosmological models have great importance in the study of the early universe, especially in the investigation of inflation. The dynamical properties of scalar fields also make an interesting research topic for modern cosmological studies Copeland et al. (2006); Coley (2003). The dynamical behavior of scalar field coupled with a barotropic fluid in spatially flat Friedmann-Robertson-Walker (FRW) universe has been studied by many authors (see Copeland et al. (2006); Coley (2003); Leon et al. (2010), and the first section of Copeland et al. (2009)).

The phase-plane analysis of the cosmological autonomous system is a useful method for studying the dynamical behavior of scalar field. One always considers the dynamical behavior of a scalar field with an exponential potential in the classical cosmology Copeland et al. (1998); Hao & Li (2003; 2004) or modified cosmology Li & Hao (2004); Samart & Gumjudpai (2007). And, if one considers the dynamical behavior of a scalar field coupled with a barotropic fluid, the exponential potential is also the first choice Billyard & Coley (2000); Ferreira & Joyce (1998); Hoogen et al. (1999); Yu & Wu (2008). The exponential potential  $V$  leads to the facts that the variables  $\Gamma = VV''/V'^2$  equals 1 and that  $\lambda = V'/V$  is also a constant. Then the autonomous system is always 2-dimensional in the classical cosmology Copeland et al. (1998), and 3-dimensional in LQC Samart & Gumjudpai (2007). Although one can also consider a more complex case with  $\lambda$  being a dynamically changing quantity Copeland et al. (2006); Macorra & Piccinelli (2000); Ng et al. (2001), the fixed point is not a real one, and this method is not exact. Recently, Zhou *et al* Fang et al. (2009); Zhou (2008) introduced a new method by which one can make  $\Gamma$  a general function of  $\lambda$ . Then the autonomous system is extended from 2-dimensional to 3-dimensional in the classical cosmology. They found that this method can help investigate many quintessence models with different potentials. One of our goals is to extend this method for studying the dynamical behavior of a scalar field with a general potential coupled with a barotropic fluid in LQC.

Based on the holonomy modification, the dynamical behavior of dark energy in LQC scenario has recently been investigated by many authors Fu et al. (2008); Lamon & Woehr (2010); Li & Ma (2010); Samart & Gumjudpai (2007); Wei & Zhang (2007); Xiao & Zhu (2010). The attractor behavior of scalar field in LQC has also been studied Copeland et al. (2008); Lidsey (2004). It was found that the dynamical properties of dark-energy models in LQC are significantly different from those in the classical cosmology. In this section, we examine the background dynamics of LQC dominated by a scalar field with a general positive potential coupled with a barotropic fluid. By considering  $\Gamma$  as a function of  $\lambda$ , we investigate scalar fields with different potentials. Since the Friedmann equation has been modified by the quantum effect, the dynamical system will be very different from the one in classical cosmology, e.g., the number of dimensions of autonomous system will change to four in LQC. It must be pointed out that this method cannot be used to describe the dynamical behavior of scalar field with arbitrary potential. To overcome this problem, therefore, we should consider an infinite-dimensional autonomous system.

This section is organized as follows. We present in Subsection 3.1 the basic equations and the 4-dimensional dynamical system, and discuss in Subsection 3.2 the properties of this system. In Subsection 3.3, we give more discussions on the autonomous system, and also on an infinite-dimensional autonomous system. We conclude the section in the last subsection.

### 3.1 Basic equations

We focus on the flat FRW cosmology. The modified Friedmann equation in the effective LQC with holonomy correction can be written as

$$H^2 = \frac{1}{3}\rho \left(1 - \frac{\rho}{\rho_c}\right), \tag{80}$$

in which  $\rho$  is the total energy density and the natural unit  $\kappa = 8\pi G = 1$  is adopted for simplicity. We consider a self-interacting scalar field  $\phi$  with a positive potential  $V(\phi)$  coupled with a barotropic fluid. Then the total energy density can be written as  $\rho = \rho_\phi + \rho_\gamma$ , with the energy density of scalar field  $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$  and the energy density of barotropic fluid  $\rho_\gamma$ . We consider that the energy momenta of this field to be covariant conserved. Then one has

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0, \tag{81}$$

$$\dot{\rho}_\gamma + 3\gamma H\rho_\gamma = 0, \tag{82}$$

where  $\gamma$  is an adiabatic index and satisfies  $p_\gamma = (\gamma - 1)\rho_\gamma$  with  $p_\gamma$  being the pressure of the barotropic fluid, and the prime denotes the differentiation w.r.t. the field  $\phi$ . Differentiating Eq. (80) and using Eqs. (81) and (82), one can obtain

$$\dot{H} = -\frac{1}{2} \left(\dot{\phi}^2 + \gamma\rho_\gamma\right) \left[1 - \frac{2(\rho_\gamma + \rho_\phi)}{\rho_c}\right]. \tag{83}$$

Eqs. (80)-(82) or Eqs. (81)-(83) characterize a closed system which can determine the cosmic behavior. To analyze the dynamical behavior of the universe, one can further introduce the following variables Copeland et al. (1998); Samart & Gumjudpai (2007):

$$x \equiv \frac{\dot{\phi}}{\sqrt{6}H}, \quad y \equiv \frac{\sqrt{V}}{\sqrt{3}H}, \quad z \equiv \frac{\rho}{\rho_c}, \quad \lambda \equiv \frac{V'}{V}, \tag{84}$$

where the  $z$  term is a special variable in LQC (see Eq. (80)). In the LQC scenario, the total energy density  $\rho$  should be less or equal to the critical energy density  $\rho_c$ , and thus  $0 \leq z \leq 1$ . Notice that, in the classical region,  $z = 0$  for  $\rho \ll \rho_c$ . Using these new variables, one can obtain

$$\frac{\rho_\gamma}{3H^2} = \frac{1}{1-z} - x^2 - y^2, \tag{85}$$

$$\frac{\dot{H}}{H^2} = - \left[3x^2 + \frac{3\gamma}{2} \left(\frac{1}{1-z} - x^2 - y^2\right)\right] (1 - 2z) \tag{86}$$

Using the new variables (84), and considering Eqs. (85) and (86), one can rewrite Eqs. (80)-(82) in the following forms,

$$\begin{aligned} \frac{dx}{dN} &= -3x - \frac{\sqrt{6}}{2}\lambda y^2 + x \left[3x^2 + \frac{3\gamma}{2} \left(\frac{1}{1-z} - x^2 - y^2\right)\right] \\ &\times (1 - 2z), \end{aligned} \tag{87}$$

$$\begin{aligned} \frac{dy}{dN} &= \frac{\sqrt{6}}{2}\lambda xy + y \left[3x^2 + \frac{3\gamma}{2} \left(\frac{1}{1-z} - x^2 - y^2\right)\right] \\ &\times (1 - 2z), \end{aligned} \tag{88}$$

$$\frac{dz}{dN} = -3\gamma z - 3z(1-z) \left(2x^2 - \gamma x^2 - \gamma y^2\right), \tag{89}$$

$$\frac{d\lambda}{dN} = \sqrt{6}\lambda^2 x (\Gamma - 1), \tag{90}$$

where  $N = \ln a$  and

$$\Gamma \equiv \frac{VV''}{V'^2}. \quad (91)$$

Note that the potential  $V(\phi)$  is positive in this section, but one can also discuss a negative potential. Just as Heard & Wands (2002) shown, the negative scalar potential could slow down the growth of the scale factor and cause the Universe to be in a collapsing phase. The dynamical behavior of scalar field with positive and negative potential in brane cosmology has been discussed by Copeland et al. (2009). In this section we are concerned only with an expanding universe, and both the Hubble parameter and the potential are positive. Differentiating  $\lambda$  w.r.t. the scalar field  $\phi$ , we obtain the relationship between  $\lambda$  and  $\Gamma$ ,

$$\frac{d\lambda^{-1}}{d\phi} = 1 - \Gamma. \quad (92)$$

If we only consider a special case of the potential, like exponential potential Billyard & Coley (2000); Copeland et al. (1998); Ferreira & Joyce (1998); Hao & Li (2003; 2004); Hoogen et al. (1999); Li & Hao (2004); Samart & Gumjudpai (2007); Yu & Wu (2008), then  $\lambda$  and  $\Gamma$  are both constants. In this case, the 4-dimensional dynamical system, Eqs. (87)-(90), reduces to a 3-dimensional one, since  $\lambda$  is a constant. (In the classical dynamical system, the  $z$  term does not exist, and then the dynamical system is reduced from 3-dimensional to 2-dimensional.) The cost of this simplification is that the potential of the field is restricted. Recently, Zhou *et al* Fang et al. (2009); Zhou (2008) considered the potential parameter  $\Gamma$  as a function of another potential parameter  $\lambda$ , which enables one to study the fixed points for a large number of potentials. We will follow this method in this and the next subsections to discuss the dynamical behavior of scalar field in the LQC scenario, and we have

$$\Gamma(\lambda) = f(\lambda) + 1. \quad (93)$$

In this case, Eq. (93) can cover many scalar potentials.

For completeness' sake, we briefly review the discussion of Fang et al. (2009) in the following. From Eq. (92), one can obtain

$$\frac{d\lambda}{\lambda f(\lambda)} = \frac{dV}{V}. \quad (94)$$

Integrating out  $\lambda = \lambda(V)$ , one has the following differential equation of potential

$$\frac{dV}{V\lambda(V)} = d\phi. \quad (95)$$

Then, Eqs. (94) and (95) give a route for obtaining the potential  $V = V(\phi)$ . If we consider a concrete form of the potential (e.g., an exponential potential), the dynamical system is specialized (e.g., the dynamical system is reduced to 3-dimensional if one considers the exponential potential for  $d\lambda/dN = 0$ ). These specialized dynamical systems are too special if one hopes to distinguish the fixed points that are the common properties of scalar field from those that are just related to the special potentials Fang et al. (2009). If we consider a more general  $\lambda$ , then we can get the more general stability properties of scalar field in the LQC



scenario. We will continue the discussion of this topic in Subsection 3.3. In this case, Eq. (90) becomes

$$\frac{d\lambda}{dN} = \sqrt{6}\lambda^2 x f(\lambda). \tag{96}$$

Hereafter, Eqs. (87)-(89) along with Eq. (96) are definitely describing a dynamical system. We will discuss the stability of this system in the following subsection.

### 3.2 Properties of the autonomous system

Obviously, the terms on the right-hand side of Eqs. (87)-(89) and (96) only depend on  $x, y, z, \lambda$ , but not on  $N$  or other variables. Such a dynamical system is usually called an autonomous system. For simplicity, we define  $\frac{dx}{dN} = F_1(x, y, z, \lambda) \equiv F_1$ ,  $\frac{dy}{dN} = F_2(x, y, z, \lambda) \equiv F_2$ ,  $\frac{dz}{dN} = F_3(x, y, z, \lambda) \equiv F_3$ , and  $\frac{d\lambda}{dN} = F_4(x, y, z, \lambda) \equiv F_4$ . The fixed points  $(x_c, y_c, z_c, \lambda_c)$  satisfy  $F_i = 0, i = 1, 2, 3, 4$ . From Eq. (96), it is straightforward to see that  $x = 0, \lambda = 0$  or  $f(\lambda) = 0$  can make  $F_4(x, y, z, \lambda) = 0$ . Also, we must consider  $\lambda^2 f(\lambda) = 0$ . Just as Fang et al. (2009) argued, it is possible that  $\lambda^2 f(\lambda) \neq 0$  and  $\frac{d\lambda}{dN} \neq 0$  when  $\lambda = 0$ . Thus the necessary condition for the existence of the fixed points with  $x \neq 0$  is  $\lambda^2 f(\lambda) = 0$ . Taking into account these factors, we can easily obtain all the fixed points of the autonomous system described by Eqs. (87)-(89) and (96), and they are shown in Tab. (1).

The properties of each fixed points are determined by the eigenvalues of the Jacobi matrix

$$\mathcal{M} = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial \lambda} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial \lambda} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} & \frac{\partial F_3}{\partial \lambda} \\ \frac{\partial F_4}{\partial x} & \frac{\partial F_4}{\partial y} & \frac{\partial F_4}{\partial z} & \frac{\partial F_4}{\partial \lambda} \end{pmatrix} \Bigg|_{(x_c, y_c, z_c, \lambda_c)}. \tag{97}$$

According to Lyapunov’s linearization method, the stability of a linearized system is determined by the eigenvalues of the matrix  $\mathcal{M}$  (see Chapter 3 of Slotine & Li (1991)). If all of the eigenvalues are strictly in the left-half complex plane, then the autonomous system is stable. If at least one eigenvalue is strictly in the right-half complex plane, then the system is unstable. If all of the eigenvalues are in the left-half complex plane, but at least one of them is on the  $i\omega$  axis, then one cannot conclude anything definite about the stability from the linear approximation. By examining the eigenvalues of the matrix  $\mathcal{M}$  for each fixed point shown in Tab. (1), we find that points  $P_{1,2,4-8,10}$  are unstable and point  $P_9$  is stable only under some conditions. We cannot determine the stability properties of  $P_3$  from the eigenvalues, and we have gave the full analysis of  $P_3$  in the appendix of Xiao & Zhu (2010).

Some remarks on Tab.(1):

1. Apparently, points  $P_2$  and  $P_6$  have the same eigenvalues, and the difference between these two points is just on the value of  $\lambda$ . As noted in the caption of Tab. (1),  $\lambda_*$  means that  $\lambda$  can be any value, and  $\lambda_a$  is just the value that makes  $f(\lambda) = 0$ . Obviously,  $\lambda_a$  is just a special value of  $\lambda_*$ , and point  $P_6$  is a special case of point  $P_2$ .  $P_6$  is connected with  $f(\lambda)$ , but  $P_2$  is not. From now on, we do not consider separately the special case of  $P_6$  when we discuss the property of  $P_2$ . Hence the value of  $\lambda_a$  is contained in our discussion of  $\lambda_*$ .
2. It is straightforward to check that, if  $x_c = \lambda_c = 0, y_c$  can be any value  $y_*$  when it is greater than or equal 1. But, if  $y_* > 1$ , then  $z_c = 1 - 1/y_*^2 < 1$ , and this means that the fixed



Fixed-points	$x_c$	$y_c$	$z_c$	$\lambda_c$	Eigenvalues	
$P_1$	0	0	0	0	$\mathbf{M}^T = (0, -3\gamma, \frac{3}{2}\gamma, -3 + \frac{3}{2}\gamma)$	
$P_2$	0	0	0	$\lambda_*$	$\mathbf{M}^T = (0, \frac{3}{2}\gamma, -3\gamma, -3 + \frac{3}{2}\gamma)$	
$P_3$	0	1	0	0	$\mathbf{M}^T = (-3, -3\gamma, 0, 0)$	S, U, S,
$P_4$	1	0	0	0	$\mathbf{M}^T = (0, -6, 0, 6 - 3\gamma)$	
$P_5$	-1	0	0	0	$\mathbf{M}^T = (0, -6, 0, 6 - 3\gamma)$	
$P_6$	0	0	0	$\lambda_a$	$\mathbf{M}^T = (0, \frac{3}{2}\gamma, -3\gamma, -3 + \frac{3}{2}\gamma)$	
$P_7$	1	0	0	$\lambda_a$	$\mathbf{M}^T = (-6, 6 - 3\gamma, \frac{1}{2}\sqrt{6}\lambda_a + 3, \sqrt{6}\lambda_a A)$	
$P_8$	-1	0	0	$\lambda_a$	$\mathbf{M}^T = (-6, 6 - 3\gamma, -\frac{1}{2}\sqrt{6}\lambda_a + 3, -\sqrt{6}\lambda_a A)$	
$P_9$	$-\frac{\sqrt{6}}{6}\lambda_a$	$\sqrt{1 - \frac{\lambda_a^2}{6}}$	0	$\lambda_a$	$\mathbf{M}^T = (-\lambda_a^2, -3 + \frac{1}{2}\lambda_a^2, \lambda_a^2 - 3\gamma, -\lambda_a^3 - f_1(\lambda_a))$	S, for U, for $f_1$
$P_{10}$	$-\sqrt{\frac{3}{2}}\frac{\gamma}{\lambda_a}$	$\sqrt{\frac{3}{2\lambda_a^2}\gamma(2 - \gamma)}$	0	$\lambda_a$	See the Eq. (98)	

Table 1. The stability analysis of an autonomous system in LQC. The system is described by a self-interacting positive potential  $V$  coupled with a barotropic fluid  $\rho_\gamma$ . Explanation of the symbols used in this table:  $P_i$  are fixed points located in the 4-dimensional phase space, which are earmarked by the coordinates  $(x_c, y_c, z_c, \lambda_c)$ .  $\lambda_*$  is the value that makes  $f(\lambda) = 0$ .  $\mathbf{M}^T$  means the inverted matrix of the eigenvalues of the fixed points.  $\Lambda = 0, \lambda_a$ .  $A = \left[ 2f(\lambda_a) + \lambda_a \left( \frac{df(\lambda)}{d\lambda} \Big|_{\lambda_a} \right) \right]$ . U stands for unstable, and S stands for stable.

point is located in the quantum-dominated regions. Although the stability of this point in the quantum regions may depend on  $f(\lambda)$ , it is not necessary to analyze its dynamical properties, since it does not have any physical meanings. The reason is the following. If the universe is stable, it will not evolve to today's pictures. If the Universe is unstable, it will always be unstable. We will just focus on point  $P_3$  staying in the classical regions. Then  $y_c = y_* = 1, z_c = 1 - 1/y_*^2 = 0$ , i.e., for the classical cosmology region,  $\rho \ll \rho_c$ .

3. Since the adiabatic index  $\gamma$  satisfies  $0 < \gamma < 2$  (in particular, for radiation  $\gamma = \frac{4}{3}$  and for dust  $\gamma = 1$ ), all the terms that contain  $\gamma$  should not change sign. A more general situation of  $\gamma$  is  $0 \leq \gamma \leq 2$  Billyard et al. (1998). If  $\gamma = 0$  or  $\gamma = 2$ , the eigenvalues corresponding to points  $P_{1,2,4,5,9}$  will have some zero elements and some negative ones. To analyze the stability of these points, we need to resort to other more complex methods, just as we did in the appendix of Xiao & Zhu (2010) for the dynamical properties of point  $P_3$ . In this subsection, we just consider the barotropic fluid which includes radiation and dust, and  $\gamma \neq 0, 2$ . Notice that if one considers  $\gamma = 0$ , the barotropic fluid describes the dark energy. This is an interesting topic, but will not be considered here for the sake of simplicity.
4.  $-\sqrt{6} < \lambda_a < \sqrt{6}, \lambda_a \neq 0$  should hold for point  $P_9$ , hence  $-3 + \frac{1}{2}\lambda_a^2 < 0$ .
5.  $\lambda_a > 0$  should hold, since  $y_c > 0$  for point  $P_{10}$ . The eigenvalue of this point is

$$\mathbf{M} = \begin{pmatrix} -3\gamma \\ -3\lambda_a\gamma f_1(\lambda_a) \\ -\frac{3}{2} + \frac{3}{4}\gamma + \frac{3}{4\lambda_a} \sqrt{(2-\gamma)(\lambda_a^2(2-\gamma) + 8\gamma + 24\gamma^2)} \\ -\frac{3}{2} + \frac{3}{4}\gamma - \frac{3}{4\lambda_a} \sqrt{(2-\gamma)(\lambda_a^2(2-\gamma) + 8\gamma + 24\gamma^2)} \end{pmatrix}. \tag{98}$$

Since we just consider  $0 < \gamma < 2$  in this subsection, it is easy to check that  $(2 - \gamma)(\lambda_a^2(2 - \gamma) + 8\gamma + 24\gamma^2) > 0$  is always satisfied. And this point is unstable with  $f_1(\lambda_a) = \left. \frac{df(\lambda)}{d\lambda} \right|_{\lambda=\lambda_a}$  being either negative or positive, since  $-\frac{3}{2} + \frac{3}{4}\gamma + \frac{3}{4\lambda_a} \sqrt{(2 - \gamma)(\lambda_a^2(2 - \gamma) + 8\gamma + 24\gamma^2)}$  is always positive.

Based on Tab. (1) and the related remarks above, we have the following conclusions.

1. Points  $P_{1,2}$ : The related critical values, eigenvalues and stability properties do not depend on the specific form of the potential, since  $\lambda_c = 0$  or  $\lambda$  can be any value  $\lambda_*$ .
2. Point  $P_3$ : The related stability properties depend on  $f_1(0) = \left. \frac{df(\lambda)}{d\lambda} \right|_{\lambda=0}$ .
3. Points  $P_{4,5}$ : The related eigenvalues and stability properties do not depend on the form of the potential, but the critical values of these points should satisfy  $\lambda^2 f(\lambda) = 0$  since  $x_c \neq 0$ .
4. Point  $P_6$ : It is a special case of  $P_2$ , but  $f(\lambda_a) = 0$  should be satisfied.
5. Points  $P_{7,8}$ : Same as  $P_6$ , they would not exist if  $f(\lambda_a) \neq 0$ .
6. Point  $P_{9,10}$ :  $f(\lambda_a) = 0$  should hold. The fixed values and the eigenvalues of these two points depend on  $f_1(\lambda_a) = \left. \frac{df(\lambda)}{d\lambda} \right|_{\lambda=\lambda_a}$ .

Thus, only points  $P_{1,2}$  are independent of  $f(\lambda)$ .

Comparing the fixed points in LQC and the ones in the classical cosmology (see the Table I of Fang et al. (2009)), we can see that, even though the values of the coordinates  $(x_c, y_c, \lambda_c)$  are

the same, the stability properties are very different. This is reasonable, because the quantum modification is considered, and the autonomous system in the LQC scenario is very different from the one in the classical scenario, e.g., the autonomous system is 4-dimensional in LQC but 3-dimensional in the classical scenario. Notice that all of the fixed points lie in the classical regions, and therefore the coordinates of fixed points remain the same from classical to LQC, which we also pointed out in Xiao & Zhu (2010).

Now we focus on the later time attractors: point  $P_3$  under the conditions of  $\gamma = 1, f_1(0) \geq 0$  and  $\gamma = 4/3, f_1(0) = 0$ , and point  $P_9$  under the conditions of  $\lambda_a^2 < 6, f_1(\lambda_a) > \lambda_a, \lambda_a < 3\gamma$ . Obviously, these points are scalar-field dominated, since  $\rho_\gamma = H^2(1/(1-z_c) - x_c^2 - y_c^2) = 0$ . For point  $P_3$ , the effective adiabatic index  $\gamma_\phi = (\rho_\phi + p_\phi)/\rho_\phi = 0$ , which means that the scalar field is an effective cosmological constant. For point  $P_9$ ,  $\gamma_\phi = \lambda_a^2/2$ . This describes a scaling solution that, as the universe evolves, the kinetic energy and the potential energy of the scalar field scale together. And we can see that there is not any barotropic fluid coupled with the scalar field dominated scaling solution. This is different from the dynamical behavior of scalar field with exponential potential  $V = V_0 \exp(-\lambda\kappa\phi)$  in the classical cosmology Billyard & Coley (2000); Copeland et al. (1998); Ferreira & Joyce (1998); Hao & Li (2003; 2004); Hoogen et al. (1999); Li & Hao (2004); Samart & Gumjudpai (2007); Yu & Wu (2008), and also is different from the properties of the scalar field in brane cosmology Copeland et al. (2009), in which  $\lambda = \text{const.}$  (notice that the definition of  $\lambda$  in Copeland et al. (2009) is different from the one in this section) and  $\Gamma$  is a function of  $L(\rho(a))$  and  $|V|$ . In these models, the universe may enter a stage dominated by scalar field coupled with fluid when  $\lambda, \gamma$  satisfy some conditions Copeland et al. (1998; 2009).

We discuss the dynamical behavior of the scalar field by considering  $\Gamma$  as a function of  $\lambda$  in this and the preceding subsections. But  $\Gamma$  can not always be treated as a function of  $\lambda$ . We need to consider a more general autonomous system, which we will introduce in the next subsection.

### 3.3 More discussions on the autonomous system

The dynamical behavior of scalar field has been discussed by many authors (e.g., see Billyard & Coley (2000); Copeland et al. (1998; 2006); Coley (2003); Ferreira & Joyce (1998); Hao & Li (2003; 2004); Hoogen et al. (1999); Li & Hao (2004); Samart & Gumjudpai (2007); Yu & Wu (2008)). If one wants to get the potentials that yield the cosmological scaling solutions beyond the exponential potential, one can add a  $\frac{d\phi}{dN}$  term into the autonomous system Nunes & Mimoso (2000). All of these methods deal with special cases of the dynamical behavior of scalar fields in backgrounds of some specific forms. By considering  $\Gamma$  as a function of  $\lambda$ , one can treat potentials of more general forms and get the common fixed points of the general potential, as shown in Fang et al. (2009); Zhou (2008) and in the two preceding subsections. However, as is discussed in Fang et al. (2009), sometimes  $\Gamma$  is not a function of  $\lambda$ , and then the dynamical behaviors of scalar fields discussed above are still not general in the strict sense. For a more general discussion, we must consider the higher order derivatives of the potential. We define

$$\begin{aligned} (1)\Gamma &= \frac{VV_3}{V'^2}, & (2)\Gamma &= \frac{VV_4}{V'^2}, & (3)\Gamma &= \frac{VV_5}{V'^2}, \\ \dots & & (n)\Gamma &= \frac{VV_{n+2}}{V'^2}, & \dots & \end{aligned} \quad (99)$$

in which  $V_n = \frac{d^n V}{d\phi^n}$ ,  $n = 3, 4, 5, \dots$ . Then we can get

$$\frac{d\Gamma}{dN} = \sqrt{6}x \left[ \Gamma\lambda + {}^{(1)}\Gamma - 2\lambda\Gamma^2 \right], \tag{100}$$

$$\frac{d({}^{(1)}\Gamma)}{dN} = \sqrt{6}x \left[ {}^{(1)}\Gamma\lambda + {}^{(2)}\Gamma - 2\lambda\Gamma({}^{(1)}\Gamma) \right], \tag{101}$$

$$\frac{d({}^{(2)}\Gamma)}{dN} = \sqrt{6}x \left[ {}^{(2)}\Gamma\lambda + {}^{(3)}\Gamma - 2\lambda\Gamma({}^{(2)}\Gamma) \right], \tag{102}$$

$$\frac{d({}^{(3)}\Gamma)}{dN} = \sqrt{6}x \left[ {}^{(3)}\Gamma\lambda + {}^{(4)}\Gamma - 2\lambda\Gamma({}^{(3)}\Gamma) \right], \tag{103}$$

$$\dots\dots\dots$$

$$\frac{d({}^{(n)}\Gamma)}{dN} = \sqrt{6}x \left[ {}^{(n)}\Gamma\lambda + {}^{(n+1)}\Gamma - 2\lambda\Gamma({}^{(n)}\Gamma) \right], \tag{104}$$

$$\dots\dots\dots$$

To discuss the dynamical behavior of scalar field with more general potential, e.g., when neither  $\lambda$  nor  $\Gamma$  is constant, we need to consider a dynamical system described by Eqs. (87)-(90) coupled with Eqs. (100)-(104). It is easy to see that this dynamical system is also an autonomous one. We can discuss the values of the fixed points of this autonomous system. Considering Eq. (90), we can see that the values of fixed points should satisfy  $x_c = 0$ ,  $\lambda_c = 0$ , or  $\Gamma_c = 1$ . Then, we can get the fixed points of this infinite-dimensional autonomous system.

1. If  $x_c = 0$ , considering Eqs. (87)-(89), one can get  $(y_c, z_c, \lambda_c) = (0, 0, 0)$  or  $(y_c, z_c, \lambda_c) = (0, 0, \lambda_*)$ , and  $\Gamma_c, {}^{(n)}\Gamma_c$  can be any values.
2. If  $\lambda_c = 0$ , considering Eqs. (87)-(89), one can see that the fixed points of  $(x, y, z)$  are  $(x_c, y_c, z_c) = (0, y_*, 1 - 1/y_*^2)$  and  $(x_c, y_c, z_c) = (\pm 1, 0, 0)$ . If  $x_c = 0$ ,  $\Gamma_c$  and  ${}^{(n)}\Gamma_c$  can be any values, and if  $x_c = \pm 1$ ,  ${}^{(n)}\Gamma_c = 0$ .
3. If  $\Gamma_c = 1$ , considering Eqs. (87)-(89), one can get that the fixed points of  $(x, y, z, \lambda)$  are  $(x_c, y_c, z_c, \lambda_c) = (0, 0, 0, \lambda_*)$  and  $(x_c, y_c, z_c, \lambda_c) = (\pm 1, 0, 0, \lambda_*)$ . And  ${}^{(n)}\Gamma_c$  should satisfy  ${}^{(n)}\Gamma_c = \lambda_*^n$ . There are other fixed points, which will be discussed below.

Based on the above analysis and Tab. (1), one can find that points  $P_{1-10}$  are just special cases of the fixed points of an infinite-dimensional autonomous systems. Considering the definition of  $\Gamma$  (see Eq. (91)), the simplest potential is an exponential potential when  $\Gamma_c = 1$ . The properties of these fixed points have been discussed by many authors Billyard & Coley (2000); Copeland et al. (1998); Ferreira & Joyce (1998); Hao & Li (2003; 2004); Hoogen et al. (1999); Li & Hao (2004); Samart & Gumjudpai (2007); Yu & Wu (2008). If  $x_c = 0$  and  $y_c = 0$ , this corresponds to a fluid-dominated universe, which we do not consider here. If  $x_c = \pm 1$ ,  $\Gamma_c = 0$  and  ${}^{(n)}\Gamma_c = 0$ , we do not need to consider the  $\Gamma$  and the  ${}^{(n)}\Gamma$  terms. Then the stability properties of these points are the same as points  $P_{4,5}$  in Tab. (1), and there are unstable points. The last case is  $(x_c, y_c, z_c, \lambda_c) = (0, y_*, 1 - 1/y_*^2, 0)$  and  $\Gamma, {}^{(n)}\Gamma$  can be any value. To analyze the dynamical properties of this autonomous system, we need to consider the  ${}^{(n)}\Gamma_c$  terms. We will get an infinite series. In order to solve this infinite series, we must truncate it by setting a sufficiently high-order  ${}^{(M)}\Gamma$  to be a constant, for a positive integer  $M$ , so that  $d({}^{(M)}\Gamma)/dN = 0$ . Thus we can get an  $(M + 4)$ -dimensional autonomous

system. One example is the quadratic potential  $V = \frac{1}{2}m^2\phi^2$  with some positive constant  $m$  that gives a 5-dimensional autonomous system, and another example is the Polynomial (concave) potential  $V = M^{4-n}\phi^n$  Kallosh et al. (1991); Linde et al. (1991; 1994) that gives an  $(n + 3)$ -dimensional autonomous system. Following the method we used in the two preceding subsections, we can get the dynamical behavior of such finite-dimensional systems.

In the rest of this subsection, we discuss whether this autonomous system has scaling solution. If  $x_c = 0$ , then  $\Gamma_c \neq 0$ ,  $^{(n)}\Gamma_c \neq 0$ , and the stability of the fixed points may depend on the truncation. As an example, if we choose  $^{(2)}\Gamma = 0$ , then we can get a 6-dimensional autonomous system. The eigenvalues for the fixed point  $(x_c, y_c, z_c, \lambda_c, \Gamma_c, ^{(1)}\Gamma_c) = (0, 0, 0, \lambda_b, \Gamma_*, ^{(1)}\Gamma_*)$ , where  $\lambda_b = 0$  or  $\lambda_b = \lambda_*$ , is

$$\mathbf{M}^T = (0, 0, 0, \frac{3}{2}\gamma, -3\gamma, -3 + \frac{3}{2}\gamma).$$

Obviously, this is an unstable point, and it has no scaling solution. The eigenvalues for the fixed point  $(x_c, y_c, z_c, \lambda_c, \Gamma_c, ^{(1)}\Gamma_c) = (0, 1, 0, 0, \Gamma_*, ^{(1)}\Gamma_*)$  is

$$\mathbf{M}^T = (0, 0, 0, 0, -3\gamma, -3 - 3\gamma).$$

According to the center manifold theorem (see Chapter 8 of Khalil (1996)), there are two non-zero eigenvalues, and we need to reduce the dynamical system to 2-dimensional to get the stability properties of the autonomous system. This point may have scaling solution, but we need more complex mathematical method. But it is easy to find that this point is scalar field dominated if it has a scaling solution.

We discuss the last case. If  $\Gamma_c = 1$ , we can consider an exponential potential. Then the autonomous system is reduced to 3-dimensional. It is easy to check that the values  $(x_{ec}, y_{ec}, z_{ec})$  of the fixed points are just the values  $(x_c, y_c, z_c)$  of points  $P_{6-10}$  in Tab. (1). We focus on the two special fixed points:

$$F_1 : (x_{ec}, y_{ec}, z_{ec}) = (-\lambda/\sqrt{6}, \sqrt{1 - \lambda^2/6}, 0),$$

$$F_2 : (x_{ec}, y_{ec}, z_{ec}) = (-\sqrt{3/2}\gamma/\lambda, \sqrt{3\gamma(2 - \gamma)/(2\lambda^2)}, 0).$$

Using Lyapunov's linearization method, we can find that  $F_2$  is unstable and  $F_1$  is stable if  $\lambda < 3\gamma$ . It is easy to check that  $\rho_\gamma = H^2[1/(1 - z_{ec}) - x_{ec}^2 - y_{ec}^2] = 0$  when  $(x_{ec}, y_{ec}, z_{ec}) = (-\lambda/\sqrt{6}, \sqrt{1 - \lambda^2/6}, 0)$ . From the above analysis, we find that there is just the scalar-field-dominated scaling solution when we consider the autonomous system to be described by a self-interacting scalar field coupled with a barotropic fluid in the LQC scenario.

### 3.4 Conclusions

To discuss the dynamical properties of scalar field in the LQC scenario, we take  $\Gamma$  as a function of  $\lambda$ , and extend the autonomous system from 3-dimensional to 4-dimensional. We find this extended autonomous system has more fixed points than the 3-dimensional one does. And we find that for some fixed points, the function  $f(\lambda)$  affects either their values, e.g., for points  $P_{4-10}$ , or their stability properties, e.g., for points  $P_{3,9}$ . In other words, the dynamical properties of these points depend on the specific form of the potential. But some other fixed points, e.g., points  $P_{1,2}$ , are independent of the potential. The properties of these fixed points are satisfied by all scalar fields. We also find that there are two later time attractors, but the universe is scalar-field-dominated since  $\rho_\gamma = 0$  at these later time attractors.



The method developed by Fang et al. (2009); Zhou (2008) can describe the dynamical behavior of the scalar field with potential of a more general form than, for example, an exponential potential Billyard & Coley (2000); Copeland et al. (1998); Ferreira & Joyce (1998); Hao & Li (2003; 2004); Hoogen et al. (1999); Li & Hao (2004); Samart & Gumjudpai (2007); Yu & Wu (2008). But it is not all-encompassing. If one wants to discuss the dynamical properties of a scalar field with an arbitrary potential, one needs to consider the higher-order derivatives of the potential  $V(\phi)$ . Hence the dynamical system will extend from 4-dimensional to infinite-dimensional. This infinite-dimensional dynamical system is still autonomous, but it is impossible to get all of its dynamical behavior unless one considers  $\Gamma_c = 1$  which just gives an exponential potential. If one wants to study as much as possible the dynamical properties of this infinite-dimensional autonomous system, one has to consider a truncation that sets  ${}^{(M)}\Gamma = \text{Const.}$ , with  $M$  above a certain positive integer. Then the infinite-dimensional system can be reduced to  $(M + 4)$ -dimensional. And we find that there is just the scalar-field-dominated scaling solution for this autonomous system.

We only get the scalar-field-dominated scaling solutions, whether we consider  $\Gamma$  as a function of  $\lambda$  or consider the higher order derivatives of the potential. This conclusion is very different from the autonomous system which is just described by a scalar field with an exponential potential Samart & Gumjudpai (2007).

#### 4. Averaged null energy condition

Wormholes and time machines are attractive objects in general relativity, always among top reasons that draw young minds to the study of this subject Morris & Thorn (1988), and they continue to be active research fields in general relativity Lobo (2007). The stress-energy tensor components of these exotic spacetime violate all known pointwise energy conditions, which is forbidden in classical general relativity. In contrast, the energy condition violation can be easily met in the semiclassical case because of quantum fluctuations Epstein et al. (1965); Klinkhanmmer (1991); Pitaevsky & Zeldovich (1971). For example, the Casimir vacuum for the electromagnetic field between two perfectly conducting plates has a negative local energy density Casimir (1948); squeezed states of light can result in negative energy densities. Based on semiclassical gravitational analysis, many self-consistent wormhole solutions have been found Barcelo & Visser (1999); Garattini (2005); Garattini & Lobo (2007); Hochberg et al. (1997); Khusnutdinov (2003); Sushkov (1992). On the other hand, the topological censorship theorem proved by Friedman, Schleich, and Witt Friedman et al. (1993) implies that the existence of macroscopic traversable wormholes requires the violation of the averaged null energy condition (ANEC). ANEC can be stated as

$$\int_{\gamma} T_{\mu\nu} k^{\mu} k^{\nu} dl \geq 0, \quad (105)$$

where the integral is along any complete, achronal null geodesic  $\gamma$ ,  $k^{\mu}$  denotes the geodesic tangent, and  $l$  is an affine parameter. Unfortunately the quantum effects in semiclassical gravitational analysis are always confined to an extremely thin band Roman (2004). So it seems impossible to find a macroscopic traversable wormhole based on semiclassical gravitational analysis.

As a quantum gravitational theory, loop quantum gravity (LQG) Ashtekar & Lewandowski (2004); Rovelli (1998); Smolin (2004); Thiemann (2007) is a non-perturbative and background-independent quantization of gravity. Physically, the Einstein equation in LQG is modified, while the stress-energy tensor  $T_{\mu\nu}$  is unchanged. But mathematically, we can



move the terms modified by LQG to the side of the stress-energy tensor, and combine them with the stress-energy tensor to get an effective stress-energy tensor. This viewpoint allows to directly apply some previous analysis results in the considered situation. For instance, that proof of censorship theorem in Friedman et al. (1993) uses the Einstein equation extensively. But the proof places no restrictions on the form of matter. Instead, the geometric quantities are of paramount importance in the proof. Therefore with the concept of effective stress-energy tensor, the proof in Ref. Friedman et al. (1993) can be directly applied in the LQG-corrected spacetime.

The application of the techniques of LQG to the cosmological sector is known as loop quantum cosmology (LQC). Some of the main features of LQG such as discreteness of spatial geometry are inherited in LQC. A major success of LQC is that it resolves the problem of classical singularities both in an isotropic model Bojowald (2001) and in a less symmetric model Bojowald (2003), replacing the big bang spacetime singularity of cosmology with a big bounce. This bouncing scenario depends crucially on the discreteness of the theory. It has also been shown that non-perturbative modification of the matter Hamiltonian leads to a generic phase of inflation Bojowald (2002c); Date & Hossain (2005); Xiong & Zhu (2007a). These inflation models are built by taking only certain modification terms which affect the stress-energy tensor while ignoring the discretized geometry effect. But these modifications are also negligible for semiclassical gravitational theory. The effective stress-energy tensor in the inflation models has already been found to violate several kinds of energy conditions. For example, in loop quantum cosmology, non-perturbative modification to a scalar matter field at short scales induces a violation of the strong energy condition Xiong & Zhu (2007b). The ANEC is different from the other energy conditions such as the strong energy condition mentioned above in that the ANEC is an integral along any complete null-like geodesic, instead of being confined to the neighborhood of a certain point of the space-time. For a system without symmetry, it is a very complicated issue, making it almost impossible to calculate. But in the context of isotropic LQC, we can get an exact result, which can provide a hint for studying the wormholes in LQG and testing the validity of LQG. In this paper, we adopt effective method to study the quantum effect of effective stress-energy in loop quantum cosmology. From our calculation, we find that LQC does violate the averaged null energy condition in the massless scalar field coupled model.

This section is organized as follows. We introduce an exactly solvable model containing a massless scalar field in Subsec. 4.1. Then in Subsec. 4.2, we investigate the averaged null energy condition in this exactly solvable model. Finally, Subsec. 4.3 contains the discussion of our results and their implications. In this section we adopt  $c = \hbar = G = 1$ .

#### 4.1 An exactly solvable model

The effective dynamics of LQC was formulated in Refs. Ashtekar et al. (2006a;b); Singh & Vandersloot (2005); Taveras (2008). Here we follow closely Mielczarek et al. (2008) to consider a universe containing a massless scalar field. Then the matter Hamiltonian in equation (58) can be written as

$$H_M(p, \phi) = \frac{1}{2} \frac{p_\phi^2}{p^{3/2}},$$

where  $p_\phi$  is the conjugate momentum for the scalar field  $\phi$ . The complete equations of motion for the universe containing a massless scalar field are

$$\begin{cases} \dot{c} = -\frac{1}{\gamma} \frac{\partial}{\partial p} \left( \sqrt{p} \left[ \frac{\sin(\bar{\mu}c)}{\bar{\mu}} \right]^2 \right) - \frac{\kappa\gamma}{4} \frac{p_\phi^2}{p^{5/2}}, \\ \dot{p} = \frac{2}{\gamma} \frac{\sqrt{p}}{\bar{\mu}} \sin(\bar{\mu}c) \cos(\bar{\mu}c), \end{cases} \tag{106}$$

and

$$\begin{cases} \dot{\phi} = p^{-3/2} p_\phi, \\ \dot{p}_\phi = 0, \end{cases} \tag{107}$$

where  $\kappa = 8\pi$ . In addition, the Hamiltonian constraint  $H_{eff} = 0$  becomes

$$\frac{3}{8\pi\gamma^2\bar{\mu}^2} \sqrt{p} \sin^2(\bar{\mu}c) = \frac{1}{2} \frac{p_\phi^2}{p^{3/2}}. \tag{108}$$

Combining equations (106) with (108), we obtain

$$\left( \frac{dp}{dt} \right)^2 = \Omega_1 p^{-1} - \Omega_3 p^{-4}, \tag{109}$$

with  $\Omega_1 = \frac{2}{3}\kappa p_\phi^2$  and  $\Omega_3 = \frac{1}{9}\kappa^2\gamma^2\bar{\mu}^2 p_\phi^4$ . Equation (107) implies that  $p_\phi$  is a constant which characterizes the scalar field in the system. To solve equation (109) we introduce a new dependent variable  $u$  in the form

$$u = p^3. \tag{110}$$

With this new variable  $u$ , Eq.(109) becomes

$$\left( \frac{du}{dt} \right)^2 = 9\Omega_1 u - 9\Omega_3, \tag{111}$$

and has a solution

$$u = \frac{\Omega_3}{\Omega_1} + \frac{9}{4}\Omega_1 t^2 - \frac{9}{2}\Omega_1 C_1 t + \frac{9}{4}\Omega_1 C_1^2, \tag{112}$$

where  $C_1$  is an integral constant. We can choose  $C_1 = 0$  through coordinate freedom. Then the solution for  $p$  is

$$p = \left[ \frac{\Omega_3}{\Omega_1} + \frac{9}{4}\Omega_1 t^2 \right]^{1/3}. \tag{113}$$

#### 4.2 The averaged null energy condition in LQC

Based on the above discussion, we calculate the averaged null energy condition in the context of LQC in this section. Because of the homogeneity of the universe, the null geodesic curves through different spatial points are the same. To investigate the ANEC, we only need to consider one of the null geodesic lines through any point in space. Due to the isotropy of the FRW metric, the null geodesic curves through the same point in different directions are also the same. Therefore, our problem is reduced to test any but one null geodesic line. Specifically, we consider a null geodesic line generated by vectors

$$\left( \frac{\partial}{\partial t} \right)^\mu + \frac{1}{a} \left( \frac{\partial}{\partial x} \right)^\mu.$$

According to the definition of affine parameter,  $(\frac{\partial}{\partial l})^\mu \nabla_\mu (\frac{\partial}{\partial l})^\nu = 0$ , we can reparameterize it with  $l$  to get

$$k^\mu = \left(\frac{\partial}{\partial l}\right)^\mu = \frac{1}{a} \left(\frac{\partial}{\partial t}\right)^\mu + \frac{1}{a^2} \left(\frac{\partial}{\partial x}\right)^\mu.$$

Then we can get the relationship between  $t$  and the affine parameter  $l$ ,

$$t = \frac{l}{a}.$$

For the considered universe containing a massless scalar field, the energy density and the pressure of matter can be expressed as

$$\rho = \frac{1}{2} \dot{\phi}^2, \quad (114)$$

$$P = \frac{1}{2} \dot{\phi}^2, \quad (115)$$

according to the definition of the density and pressure. The effective energy density and pressure of matter take the forms

$$\rho_{eff} = \frac{1}{2} \dot{\phi}^2 \left(1 - \frac{1}{2} \frac{\dot{\phi}^2}{\rho_c}\right), \quad (116)$$

$$P_{eff} = \frac{1}{2} \dot{\phi}^2 \left(1 - \frac{\dot{\phi}^2}{\rho_c}\right) - \frac{1}{4} \frac{\dot{\phi}^4}{\rho_c}. \quad (117)$$

Since the effective stress-energy tensor takes an ideal fluid form,

$$\begin{aligned} T_{\mu\nu}^{eff} &= \rho_{eff} (dt)_\mu (dt)_\nu + a^2 P_{eff} \\ &\quad \times \left[ (dx)_\mu (dx)_\nu + (dy)_\mu (dy)_\nu + (dz)_\mu (dz)_\nu \right], \end{aligned} \quad (118)$$

the average null energy condition (105) for the effective stress-energy tensor becomes

$$\begin{aligned} \int_\gamma T_{\mu\nu}^{eff} k^\mu k^\nu dl &= \int_{-\infty}^{\infty} \frac{1}{a} (\rho_{eff} + P_{eff}) dt \\ &= \int_{-\infty}^{\infty} \frac{1}{a} \left( \dot{\phi}^2 - \frac{\dot{\phi}^4}{\rho_c} \right) dt \\ &= p_\phi^2 \int_{-\infty}^{\infty} (p^{-7/2} - p^{-13/2} \frac{p_\phi^2}{\rho_c}) dt. \end{aligned}$$

In the last line we have used equation (107) and the relationship between  $p$  and  $a$ . Substituting the exact solution (113) into the above equation, we get

$$\int_{\gamma} T_{\mu\nu}^{eff} k^{\mu} k^{\nu} dl = -\frac{\Gamma(5/6)\Gamma(2/3)}{7\rho_c\Omega_I(\frac{\Omega_{III}}{\Omega_I})^{13/6}\sqrt{\frac{\pi}{\Omega_{III}}}} p_{\phi}^4, \quad (119)$$

where  $\Gamma$  is the gamma function. From the above result it is obvious that

$$\int_{\gamma} T_{\mu\nu}^{eff} k^{\mu} k^{\nu} dl < 0. \quad (120)$$

The above result shows that, in addition to the violation of some local energy conditions, the effective stress-energy tensor of loop quantum cosmology also violates the averaged null energy condition.

### 4.3 Conclusions and discussion

Given some kinds of local energy condition violation in loop quantum cosmology and motivated by the topological censorship theorem which rules out traversable wormholes in spacetime where the averaged null energy condition is satisfied, we investigate this kind of nonlocal energy condition in the context of LQC. Our analysis is based on a flat universe containing a massless scalar field. This model can be solved analytically. With the help of the analytical solution and taking advantage of the homogeneity and isotropy of the universe, we calculate the average of energy directly. Although the quantum correction is focused on the early universe around the Planck scale, the correction is so strong that it makes the universe violate the null averaged energy condition. Mathematically we have written the modified Einstein equation in LQC in the standard form but with effective stress-energy tensor. This form of equations allows to directly apply the original proof of Ref. Friedman et al. (1993) in the effective LQC. So the ANEC (for the original stress-energy tensor instead of the effective one) argument cannot forbid the existence of traversable wormhole once the Loop Quantum Gravity effects are taken into account. But we do not expect the existence of wormhole in LQC, because it is too symmetric to support wormholes. For less symmetric situations, the traversable wormholes might exist if quantum gravity effects make the effective stress-energy tensor violate the ANCE. On the other hand, LQC adopts the essence of LQG, so our result can shed some light on the ANEC of LQG. And we hope this result can give some hints on looking for wormhole solutions in the LQG theory. These interesting objects will provide some gedanken-experiments to test our quantum gravity theory.

### 5. Dynamical horizon entropy bound conjecture

The thermodynamical property of spacetime is an important hint for the quantization of gravity. Starting from Hawking's discovery of black hole's radiation Hawking (1975), a theory of thermodynamics of spacetime is being constructed gradually. Recently, the second law of this thermodynamics was generalized to the covariant entropy bound conjecture Bousso (1999). It states that the entropy flux  $S$  through any null hypersurface generated by geodesics with non-positive expansion, emanating orthogonally from a two-dimensional (2D) spacelike surface of area  $A$ , must satisfy

$$\frac{S}{A} \leq \frac{1}{4l_p^2}, \quad (121)$$

where  $l_p = \sqrt{\hbar}$  is the Planck length. Here and in what follows, we adopt the units  $c = G = k_B = 1$ . Soon, Flanagan, Marolf and Wald Flanagan et al. (2000) proposed a new version of the entropy bound conjecture. If one allows the geodesics generating the null hypersurface from a 2D spacelike surface of area  $A$  to terminate at another 2D spacelike surface of area  $A'$  before coming to a caustic, boundary or singularity of spacetime, one can replace the above conjecture with

$$\frac{S}{A' - A} \leq \frac{1}{4l_p^2}. \quad (122)$$

More recently, He and Zhang related these conjectures to dynamical horizon and proposed a covariant entropy bound conjecture on the cosmological dynamical horizon He & Zhang (2007): Let  $A(t)$  be the area of the cosmological dynamical horizon at cosmological time  $t$ , then the entropy flux  $S$  through the cosmological dynamical horizon between time  $t$  and  $t'$  ( $t' > t$ ) must satisfy

$$\frac{S}{A(t') - A(t)} \leq \frac{1}{4l_p^2}, \quad (123)$$

if the dominant energy condition holds for matter.

Since it has been suggested that the holographic principle is a powerful hint and should be used as an essential building block for any quantum gravity theory Bousso (2002), it is important and tempting to investigate the covariant entropy bound conjecture in the framework of the LQC, which is a successful application of the non-perturbative quantum gravity scheme—the LQG. The authors of Ashtekar & Wilson-Ewing (2008) investigated the Bousso's covariant entropy bound Bousso (1999; 2002) with a cosmology filled with photon gas and found that the conjecture is violated near the big bang in the classical scenario. But they found the LQC can protect this conjecture even in the deep quantum region. In Zhang & Ling (2007), He and Zhang proposed a new version of the entropy bound conjecture for the dynamical horizon in cosmology and validated it through a cosmology filled with adiabatic perfect fluid, governed by the classical Einstein equation when the universe is far away from the big bang singularity. But when the universe approaches the big bang singularity, the strong quantum fluctuation does break down their conjecture. In analogy to Ashtekar and Wilson-Ewing's result Ashtekar & Wilson-Ewing (2008), one may wonder if He and Zhang's conjecture can also be protected by the quantum geometry effect of the LQG. Following Ashtekar & Wilson-Ewing (2008), we use photon gas as an example to investigate this problem. As expected, we find that the loop quantum effects can indeed protect the conjecture. Besides the result of Ashtekar & Wilson-Ewing (2008), our result presents one more evidence for the consistence between the loop gravity and the covariant entropy conjecture.

This section is organized as follows. In Subsec. 5.1, we describe the covariant entropy bound conjecture proposed by He and Zhang He & Zhang (2007). Then in Subsec. 5.2, we test this conjecture with cosmology filled with photon gas, and show that the LQC is able to protect the conjecture in all. We conclude the paper in Subsec. 5.3 and discuss the implications.

### 5.1 The covariant entropy bound conjecture

According to Zhang & Ling (2007), the cosmological dynamical horizon Bousso (2002) is defined geometrically as a three-dimensional hypersurface foliated by spheres, where at least

one orthogonal null congruence with vanishing expansion exists. For a sphere characterized by any value of  $(t, r)$ , there are two future directed null directions

$$k_{\pm}^a = \frac{1}{a} \left( \frac{\partial}{\partial t} \right)^a \pm \frac{1}{a^2} \left( \frac{\partial}{\partial r} \right)^a, \tag{124}$$

satisfying geodesic equation  $k^b \nabla_b k^a = 0$ . The expansion of these null directions is

$$\theta := \nabla_a k_{\pm}^a = \frac{2}{a^2} \left( \dot{a} \pm \frac{1}{r} \right), \tag{125}$$

where the dot denotes differential with respect to  $t$ , and the sign  $+(-)$  represents the null direction pointing to larger (smaller) values of  $r$ . For an expanding universe, i.e.  $\dot{a} > 0$ ,  $\theta = 0$  determines the location of the dynamical horizon,  $r_H = 1/\dot{a}$ , by the definition of dynamical horizon given above. The LQC replaces the big bang with the big bounce, so the universe is symmetric with respect to the point of the bounce, expanding on one side of the bounce and contracting on the other side. The dynamical horizon in the contracting stage of the LQC corresponds to  $r_H = -1/\dot{a}$ , and all of the relations are similar to the ones given here. In this paper we only consider the expanding stage for the LQC, but note that the contracting stage is the same.

Since the area of the dynamical horizon is  $A = 4\pi a^2 r_H^2 = 4\pi H^{-2}$ , the covariant entropy bound conjecture in our question becomes

$$l_p^2 S \leq \pi \left[ H^{-2}(t') - H^{-2}(t) \right], \tag{126}$$

where  $S$  is the entropy flux through the dynamical horizon between cosmological time  $t$  and  $t'$  ( $t' > t$ ), and  $H$  is the Hubble parameter. Considering that the cosmology model discussed here is isotropic and homogeneous, we can write the entropy current vector as

$$s^a = \frac{s}{a^3} \left( \frac{\partial}{\partial t} \right)^a, \tag{127}$$

where  $s$  is the ordinary comoving entropy density, independent of space. If the entropy current of the perfect fluid is conserved, i.e.,  $\nabla_a s^a = 0$ ,  $s$  will be independent of  $t$  as well. For simplicity we restrict ourselves to this special case. The entropy flux through the dynamical horizon (shown in Fig.1) is given by

$$S = \int_{CDH} s^a \epsilon_{abcd} = \frac{4\pi s}{3} \left( r_H^3(t') - r_H^3(t) \right) \tag{128}$$

where  $\epsilon_{abcd} = a^3 r^2 \sin \theta (dt \wedge dr \wedge d\theta \wedge d\phi)_{abcd}$  is the spacetime volume 4-form. So the conjecture is reduced to

$$H^{-2}(t') - \frac{4}{3} l_p^2 s \dot{a}^{-3}(t') \geq H^{-2}(t) - \frac{4}{3} l_p^2 s \dot{a}^{-3}(t), \quad t' > t. \tag{129}$$



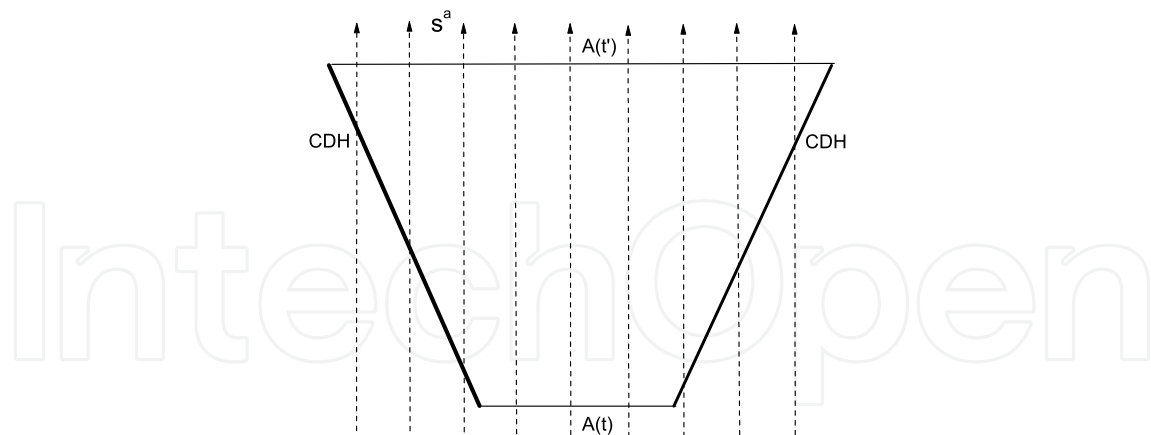


Fig. 1. A schematic of the entropy current flowing across the cosmological dynamical horizon. The thick solid line marked by "CDH" is the cosmological dynamical horizon. The thin solid line is the region enclosed by the CDH at time  $t$  and  $t'$  respectively. The dashed lines are the entropy current.

### 5.2 Conjecture test for a cosmology filled with perfect fluid

Given that the FRW universe is filled with photon gas, the energy momentum tensor can be expressed as

$$T_{ab} = \rho(t)(dt)_a(dt)_b + P(t)a^2(t) \left\{ (dr)_a(dr)_b + r^2[(d\theta)_a(d\theta)_b + \sin^2 \theta(d\phi)_a(d\phi)_b] \right\}. \quad (130)$$

The pressure  $P$  and the energy density  $\rho$  satisfy a fixed equation of state

$$P = \omega\rho, \quad (131)$$

where the constant  $\omega = \frac{1}{3}$ . From  $\nabla^a T_{ab} = 0$ , we have the conservation equation

$$\dot{\rho} + 3H(\rho + P) = 0. \quad (132)$$

The comoving entropy density  $s$  is given by

$$s = a^3 \frac{\rho + P}{T} = a^3 (1 + \omega) \frac{\rho}{T}, \quad (133)$$

and  $\rho$  depends only on the temperature  $T$ ,

$$\rho = K_0 l_p^{-2 - \frac{1+\omega}{\omega}} T^{\frac{1+\omega}{\omega}}, \quad (134)$$

where  $K_0$  is a dimensionless constant depending on the density of energy state of the perfect fluid. For photon gas  $K_0 = \frac{\pi^2}{15}$ . Plugging above thermodynamics relation into equation (133) we get  $s = (1 + \omega)K_0^{\frac{\omega}{1+\omega}} l_p^{-1 - \frac{2\omega}{1+\omega}} \rho^{\frac{1}{1+\omega}} a^3$ . Written the above conservation equation as

$$\dot{\rho} + 3(1 + \omega)\rho \frac{\dot{a}}{a} = 0, \quad (135)$$

we have an integration constant  $C = \rho^{\frac{1}{1+\omega}} a^3$ . Then  $s = (1 + \omega) K_0^{\frac{\omega}{1+\omega}} l_p^{-1-\frac{2\omega}{1+\omega}} C$ . Combining our equation of state Eq. (131) with the above conservation equation, we get the relationship between  $\rho$  and the Hubble parameter,

$$H = -\frac{1}{3(1 + \omega)} \frac{\dot{\rho}}{\rho}. \tag{136}$$

Substituting the above relation (136) into the modified Friedmann equation (67), we can get

$$\rho = \frac{1}{6\pi(t + C_1)^2(1 + \omega)^2 + \frac{1}{\rho_c}} \tag{137}$$

where  $C_1$  is an integration constant without direct physical significance, and we can always drop it by resetting the time coordinate. Setting  $C_1 = 0$  gives

$$H = \frac{4\pi t(1 + \omega)}{6\pi t^2(1 + \omega)^2 + \frac{1}{\rho_c}}. \tag{138}$$

With the definition of the Hubble parameter, we can integrate once again to get

$$a(t) = C^{1/3} \left[ 6\pi t^2(1 + \omega)^2 + \frac{1}{\rho_c} \right]^{\frac{1}{3(1+\omega)}}. \tag{139}$$

When  $\rho_c$  goes to infinity, all of the above solutions become the same as the classical ones<sup>2</sup> presented in Zhang & Ling (2007). In the classical scenario,

$$\begin{aligned} H^{-2} - \frac{4}{3} l_p^2 s \dot{a}^{-3} &= \frac{9}{4} t^2 (1 + \omega)^2 - \frac{9 K_0^{\frac{\omega}{1+\omega}} l_p^{1-\frac{2\omega}{1+\omega}}}{2(6\pi)^{1/(1+\omega)}} t^{3-\frac{2}{1+\omega}} (1 + \omega)^{4-\frac{2}{1+\omega}}. \end{aligned} \tag{140}$$

When  $t \ll 1$ ,  $H^{-2} - \frac{4}{3} l_p^2 s \dot{a}^{-3} \sim -t^{3-\frac{2}{1+\omega}} = -t^{3/2}$  which is a decreasing function of  $t$ , so the conjecture breaks down when the universe approaches the big bang singularity.

We introduce a new variable  $\tau = \sqrt{2\pi\rho_c(1 + \omega)}t$  for the LQC to simplify the above expressions to

$$H = \sqrt{2\pi\rho_c} \frac{2\tau}{3\tau^2 + 1}, \tag{141}$$

$$a = C^{1/3} \rho_c^{-\frac{1}{3(1+\omega)}} (3\tau^2 + 1)^{\frac{1}{3(1+\omega)}}, \tag{142}$$

$$\dot{a} = aH = 2\tau C^{1/3} \rho_c^{-\frac{1}{3(1+\omega)}} \sqrt{2\pi\rho_c} (3\tau^2 + 1)^{\frac{1}{3(1+\omega)}-1}. \tag{143}$$

<sup>2</sup> Note that the original result in Zhang & Ling (2007) used conformal time  $\eta$ , while we use universe time  $t$  in this paper.  $\eta$  can be negative which divides the discussion into two cases.  $t$  is always positive and makes the discussion simpler.

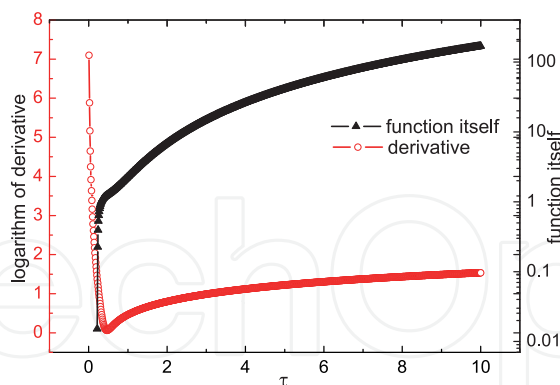


Fig. 2. Function  $H^{-2} - \frac{4}{3}l_p^2 s \dot{a}^{-3}$  and its derivative respect to  $\tau$  for photon gas.

Then

$$\begin{aligned}
 & H^{-2} - \frac{4}{3}l_p^2 s \dot{a}^{-3} \\
 &= \frac{1}{2\pi\rho_c} \left[ \left( \frac{3}{2}\tau + \frac{1}{2\tau} \right)^2 \right. \\
 & \quad \left. - \frac{(1+\omega)}{6\sqrt{2\pi}} K_0^{\frac{\omega}{1+\omega}} \left( \frac{\sqrt{3}}{16\pi^2\gamma^2} \right)^{\frac{1}{1+\omega} - \frac{1}{2}} \frac{(3\tau^2 + 1)^{3 - \frac{1}{1+\omega}}}{\tau^3} \right].
 \end{aligned}
 \tag{144}$$

It is obvious that the necessary and sufficient condition for meeting the covariant entropy bound conjecture is that the above expression increases with  $\tau$ . In order to investigate the monotone property of above function, we plot  $H^{-2} - \frac{4}{3}l_p^2 s \dot{a}^{-3}$  itself and its derivative respect to  $\tau$  in Fig.2. The minimal value of the derivative is about  $1.16 > 0$ . The covariant entropy bound conjecture for dynamical horizon in cosmology is fully protected by loop quantum effect.

### 5.3 Conclusions and discussion

The covariant entropy bound conjecture comes from the holographic principle and is an important hint for the quantum gravity theory. In the recent years we have witnessed more and more success of the loop quantum gravity, especially for the problem of the big bang singularity in cosmology. The entropy bound conjecture usually breaks down in the strong gravity region of spacetime where the quantum fluctuation is strong, and one would expect the loop quantum correction to protect the conjecture from the quantum fluctuation. And Ashtekar and Wilson-Ewing do find a result in Ashtekar & Wilson-Ewing (2008) which is consistent with above expectation. In this paper, we generalized the covariant entropy conjecture for the cosmological dynamical horizon proposed in Zhang & Ling (2007) to the loop quantum cosmology scenario. We found that the quantum geometry effects of the loop quantum gravity can also protect the conjecture. Our result gives out one more evidence for the consistence of covariant entropy conjecture and loop quantum gravity theory. This adds one more encouraging result of loop quantum gravity theory besides previous ones.

## 6. Summary

Based on quantum geometry, the mathematical structure of LQC has been strictly defined. LQC inherits the non-perturbative and background-independent quantization schemes of LQG. In the framework of LQC, the evolution of the universe is described by the difference equation. In the past years, LQC achieved great successes. The most outstanding result is replacing the big bang space singularity of cosmology with a big bounce. LQC also gives a quantum suppression of classical chaotic behavior near singularities in the Bianchi-IX models. Furthermore, it has been shown that the non-perturbative modification of the matter Hamiltonian leads to a generic phase of inflation. LQC gives some possible answers to cosmological riddles due to the discreteness of the quantum geometry.

With the geometric quantum mechanics and the shadow state framework, we can get the effective theory of LQC. In this effective theory, the classical Hamiltonian gets a quantum correction, and the classical Einstein equation is replaced by the equations of motion induced by the effective Hamiltonian. Our works are based on this effective theory.

Due to the space limitations, this Chapter includes only our recently following three works: (1) We discussed the dynamical behavior of a scalar field with a general potential coupled with a barotropic fluid in LQC. (2) We found that the averaged null energy condition is violated in LQC which provides the possibility for the traversable time machine. (3) We found that the dynamical horizon entropy bound conjecture breaks down in classical general relativity near the big bang region but is protected by the quantum geometry effects in LQC.

It is undeniable that LQC is developing rapidly and showing its power in solving cosmological riddles. However, it still faces two major challenges:

- So far, we have not proven that LQC is truly derived from the full theory;
- At the time of writing, the predictions made by LQC are yet to be compared with cosmological observations.

Here involved is only a big branch of LQC. There are other issues worth studying, such as the perturbation theory, the cosmological spin foam theory, and so on. There is still a long way to go before the arrival of the victory.

Acknowledgments: This work was supported by the National Natural Science Foundation of China under Grant No. 10875012 and the Fundamental Research Funds for the Central Universities.

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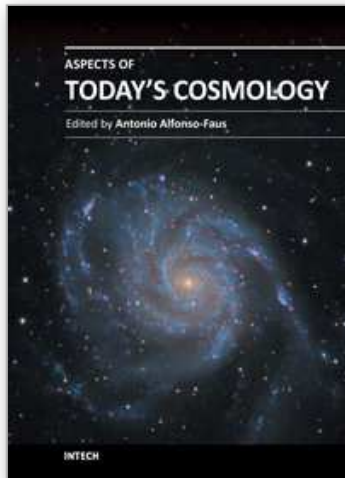
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## **Aspects of Today's Cosmology**

Edited by Prof. Antonio Alfonso-Faus

ISBN 978-953-307-626-3

Hard cover, 396 pages

**Publisher** InTech

**Published online** 09, September, 2011

**Published in print edition** September, 2011

This book presents some aspects of the cosmological scientific odyssey that started last century. The chapters vary with different particular works, giving a versatile picture. It is the result of the work of many scientists in the field of cosmology, in accordance with their expertise and particular interests. Is a collection of different research papers produced by important scientists in the field of cosmology. A sample of the great deal of efforts made by the scientific community, trying to understand our universe. And it has many challenging subjects, like the possible doomsday to be confirmed by the next decade of experimentation. May be we are now half way in the life of the universe. Many more challenging subjects are not present here: they will be the result of further future work. Among them, we have the possibility of cyclic universes, and the evidence for the existence of a previous universe.

### **How to reference**

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Li-Fang Li, Kui Xiao and Jian-Yang Zhu (2011). Loop Quantum Cosmology: Effective Theory and Related Applications, Aspects of Today's Cosmology, Prof. Antonio Alfonso-Faus (Ed.), ISBN: 978-953-307-626-3, InTech, Available from: <http://www.intechopen.com/books/aspects-of-today-s-cosmology/loop-quantum-cosmology-effective-theory-and-related-applications>

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