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The Influence of Vacuum Electromagnetic Fluctuations on the motion of Charged Particles

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1. Introduction

Lorentz-Dirac equation (LDE) is the widely accepted classical equation to describe the motion of a scalar point charge acted by external electromagnetic fields and its own radiating fields. Because LDE is highly nonlinear, the ubiquitous electromagnetic fluctuating fields of the vacuum would produce a nonzero contribution to the motion of charged particles, which provides a promising way to understand the century-old problems and puzzles associated with uniformly accelerating motion of a point charge. The vacuum fluctuations also have an intimate relationship with the Unruh effect. In this chapter we will restrict our main stamina to investigate the influence of the vacuum electromagnetic fluctuating fields on the motion of a point charge. Because the discussion upon problems, such as the Unruh effect etc., would greatly digress from the motif of this book, we just disperse brief remarks on these problems at suitable places.

2. New reduction of order form of LDE

In 1938, Dirac for the first time systematically deduced the relativistic equation of motion for a radiating point charge in his classical paper [Dirac, 1938]. Being a singular third order differential equation, the controversy about the validity of LDE has never ceased due to its intrinsic pathological characteristics, such as violation of causality, nonphysical runaway solutions and anti-damping effect etc. [Wang et al., 2010]. All these difficulties of LDE can be traced to the fact that its order reduces from three to two as the Schott term is neglected. However, LDE derived by using the conservation laws of momentum and energy is quite elegant in mathematics and is of Lorentz invariance. Furthermore, many different methods used to derive the equation of motion for a radiating point charge lead to the same equation, and all pathological characteristics of LDE would disappear in its reduction of order form. Plass invented the backward integration method for scattering problems, and H. Kawaguchi et al. constructed a precise numerical integrator of LDE using Lorentz group Lie algebra property [Plass, 1961; Kawaguchi et al., 1997]. These methods are enough to numerically study the practical problems. On the other hand, Landau and Lifshitz obtained the reduction of order form of LDE [Landau & Lifshitz, 1962], which fully meets the requirements for dynamical equation of motion and is even recommended to substitute for the LDE. But one should keep in mind that Landau and Lifshitz equation (LLE) gaining the

advantages over LDE is at the price of losing the orthogonality of four-velocity and four-acceleration of point charges. This complexion means that LDE is still the most qualified equation of motion for a radiating point charge. To be clear, we make the assumption that LDE is the exact equation of motion for a radiating point charge.

2.1 Description of reduction of order form of LDE

For a point charge of mass m and charge e, LDE reads

$$\ddot{x}^{\mu} = \frac{e}{m} F^{\mu\nu} \dot{x}_{\nu} + \tau_0 (\dddot{x}^{\mu} + \ddot{x}^2 \dot{x}^{\mu}) ,$$ (1)

where $x^{\mu}(\tau)$ is the spacetime coordinates of the charge at proper time τ, $\tau_0 = 2e^2 / 3m$ the characteristic time of radiation reaction which approximately equals to the time for a light to transverse across the classical radius of a massive charge. The upper dots denote the derivative with respect to the proper time, Greek indices μ, ν etc. run over from 0 to 3. Repeated indices are summed tacitly, unless otherwise indicated. The diagonal metric of Minkowski spacetime is $(1, -1, -1, -1)$. For simplicity, we work in relativistic units, so that the speed of light is equal to unity. The second term of the right side of Eq. (1) is referred to as the radiation reaction force, and $\dddot{x}^{\mu}$ is the so called Schott term.

We have assumed in Eq. (1) that charged particles interact only with electromagnetic fields $F^{\mu\nu}$ which has the matrix expression:

$$\begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix} .$$

The Lorentz force is

$$eF^{\mu\nu} \dot{x}_{\nu} = e(\vec{E} \cdot \dot{\vec{x}}, \ \dot{x}_0 \vec{E} + \dot{\vec{x}} \times \vec{B}) = e\gamma(\vec{E} \cdot \frac{d\vec{x}}{dt}, \ \vec{E} + \frac{d\vec{x}}{dt} \times \vec{B}) ,$$ (2)

where $\gamma = (1 - \dot{x}^2)^{-1/2}$ is the relativistic factor.

By replacing the acceleration in the radiation reaction force with that produced only by external force, Landau and Lifshitz obtained the reduction of order form of LDE

$$\ddot{x}^{\mu} = f^{\mu}(\tau, x, \dot{x}) + \tau_0 [\frac{df^{\mu}}{d\tau} + f^2 \dot{x}^{\mu}] ,$$ (3)

where $f^{\mu}(\tau, x, \dot{x}) = eF^{\mu\nu} \dot{x}_{\nu} / m$. Eq. (3) is nonsingular and gets rid of most pathological characteristics of LDE, but the applicable scope is also slightly reduced. LLE is quite convenient to numerically study macroscopic motions of a point charge.

If one does not care about the complexity, there exists another more accurate reduction of order form of LDE than LLE, which also implies a corresponding reduction of order series form of LDE. In this section, we will present this reduction of order form of LDE. To do so, the most important step is using the acceleration produced only by external forces to approximate the Schott term, namely

$$\ddot{x}^{\mu} = f^{\mu}(\tau,\vec{x},\dot{\vec{x}}) + \tau_0[\dddot{x}^{\mu} + \ddot{x}^2\dot{x}^{\mu}] \approx F^{\mu} + \tau_0[\ddot{x}^{\nu}\frac{\partial f^{\mu}}{\partial \dot{x}^{\nu}} + \ddot{x}^2\dot{x}^{\mu}], \tag{4}$$

where

$$F^{\mu} = f^{\mu} + \tau_0(\frac{\partial f^{\mu}}{\partial \tau} + \dot{x}^{\nu}\frac{\partial f^{\mu}}{\partial x^{\nu}})$$

is the analytical function of proper time, spacetime coordinates and four-velocity. Letting $k = \ddot{x}^2$, Eq. (4) becomes

$$\ddot{x}^{\mu} - \tau_0\ddot{x}^{\nu}\partial f^{\mu}/\partial \dot{x}^{\nu} = F^{\mu} + \tau_0 k\dot{x}^{\mu},$$

which can be put into the matrix form $A\ddot{x} = F + \tau_0 k\dot{x}$, with $F = (F^0,\ F^1,\ F^2,\ F^3)^T$ and $\dot{x} = (\dot{x}^0,\ \dot{x}^1,\ \dot{x}^2,\ \dot{x}^3)^T$. The symbol "$T$" denotes transpose operation of a matrix. The explicit expression of matrix A is

$$\begin{bmatrix} 1 - \tau_0\partial f^0/\partial \dot{x}^0 & -\tau_0\partial f^0/\partial \dot{x}^1 & -\tau_0\partial f^0/\partial \dot{x}^2 & -\tau_0\partial f^0/\partial \dot{x}^3 \\ -\tau_0\partial f^1/\partial \dot{x}^0 & 1 - \tau_0\partial f^1/\partial \dot{x}^1 & -\tau_0\partial f^1/\partial \dot{x}^2 & -\tau_0\partial f^1/\partial \dot{x}^3 \\ -\tau_0\partial f^2/\partial \dot{x}^0 & -\tau_0\partial f^2/\partial \dot{x}^1 & 1 - \tau_0\partial f^2/\partial \dot{x}^2 & -\tau_0\partial f^2/\partial \dot{x}^3 \\ -\tau_0\partial f^3/\partial \dot{x}^0 & -\tau_0\partial f^3/\partial \dot{x}^1 & -\tau_0\partial f^3/\partial \dot{x}^2 & 1 - \tau_0\partial f^3/\partial \dot{x}^3 \end{bmatrix},$$

each element of matrix A is the analytical function of τ, x and $\dot{x}$. We define generalized four-velocity and four-acceleration vectors as

$$\dot{X}^{\mu} = (\frac{\Delta_0^{\dot{x}}}{\Delta},\frac{\Delta_1^{\dot{x}}}{\Delta},\frac{\Delta_2^{\dot{x}}}{\Delta},\frac{\Delta_3^{\dot{x}}}{\Delta}),\ D^{\mu} = (\frac{\Delta_0^{F}}{\Delta},\frac{\Delta_1^{F}}{\Delta},\frac{\Delta_2^{F}}{\Delta},\frac{\Delta_3^{F}}{\Delta}),$$

where Δ is the determinant of matrix A, and Δ_{μ}^{F} and $\Delta_{\mu}^{\dot{x}}$ are determinants of matrices obtained by replacing the μ-th column of A with column matrices F and $\dot{x}$ respectively. So four-acceleration can be expressed as

$$\ddot{x}^{\mu} = D^{\mu} + \tau_0 k\dot{X}^{\mu}. \tag{5}$$

Because the square of the four-acceleration k is involved, Eq. (5) is still not the explicit expression of acceleration. However, k can be expressed as

$$k = (D^{\mu} + \tau_0 k\dot{X}^{\mu})(D_{\mu} + \tau_0 k\dot{X}_{\mu}) = \tau_0^2\dot{X}^2 k^2 + 2\tau_0 kD^{\mu}\dot{X}_{\mu} + D^2, \tag{6}$$

which is a quadratic algebra equation of k. The physical solution of k is

$$k = \frac{2D^2}{(1 - 2\tau_0 D^{\mu}\dot{X}_{\mu}) + \sqrt{(1 - 2\tau_0 D^{\mu}\dot{X}_{\mu})^2 - 4\tau_0^2\dot{X}^2 D^2}}. \tag{7}$$

Thus we obtained the expression of acceleration, and the result is

$$\ddot{x}^{\mu} = D^{\mu} + \frac{2\tau_0 D^2\dot{X}^{\mu}}{(1 - 2\tau_0 D^{\mu}\dot{X}_{\mu}) + \sqrt{(1 - 2\tau_0 D^{\mu}\dot{X}_{\mu})^2 - 4\tau_0^2\dot{X}^2 D^2}}, \tag{8}$$

which is now the explicit function of proper time τ, spacetime coordinates x and velocity $\dot{x}$.

As a corollary, we can discuss the applicable scope of LDE from the existing condition of the solution of k, namely, the quantity under the square root appeared in Eq. (7) must be nonnegative. It is often taken for granted that LDE would be invalid at the scale of the Compton wavelength of the charge.

Following the above procedure, we can construct an iterative reduction of order form of LDE, which is a more accurate approximation to the original LDE. As the first step, we approximately expressed LDE as

$$\dddot{x}^\mu = f^\mu(\tau, x, \dot{x}) + \tau_0[\dddot{x}^\mu + \ddot{x}^2 \dot{x}^\mu] \approx G^\mu + \tau_0 k \dot{x}^\mu ,$$

where

$$G^\mu = f^\mu + \tau_0 \left(\frac{\partial \ddot{x}^\mu}{\partial \tau} + \dot{x}^\nu \frac{\partial \ddot{x}^\mu}{\partial x^\nu} + \ddot{x}^\nu_{known} \frac{\partial \ddot{x}^\mu}{\partial \dot{x}^\nu} \right)$$

is the function of τ, x and $\dot{x}$ owing to the four-acceleration $\ddot{x}^\nu_{known}$ being taken as that given by Eq. (8). From its definition, the square of four-acceleration k can be worked out

$$k = \frac{2G^2}{(1 - 2\tau_0 G \cdot \dot{x}) + \sqrt{(1 - 2\tau_0 G \cdot \dot{x})^2 - 4\tau_0^2 G^2}} ,$$

and the four-acceleration is

$$\ddot{x}^\mu = G^\mu + \frac{2\tau_0 G^2 \dot{x}^\mu}{(1 - 2\tau_0 G \cdot \dot{x}) + \sqrt{(1 - 2\tau_0 G \cdot \dot{x})^2 - 4\tau_0^2 G^2}} .$$

By repeating this procedure, we can obtain the *n-th* iterative expression of four-acceleration:

$$\ddot{x}^\mu_n = G^\mu_n + \frac{2\tau_0 G^2_n \dot{x}^\mu}{(1 - 2\tau_0 G_n \cdot \dot{x}) + \sqrt{(1 - 2\tau_0 G_n \cdot \dot{x})^2 - 4\tau_0^2 G^2_n}} , \tag{9}$$

where

$$G^\mu_n = f^\mu + \tau_0 \frac{d\ddot{x}^\mu_{n-1}}{d\tau} = f^\mu + \tau_0 \left(\frac{\partial \ddot{x}^\mu_{n-1}}{\partial \tau} + \dot{x}^\nu \frac{\partial \ddot{x}^\mu_{n-1}}{\partial x^\nu} + \ddot{x}^\nu_{n-1} \frac{\partial \ddot{x}^\mu_{n-1}}{\partial \dot{x}^\nu} \right) .$$

We emphasize again that $\ddot{x}^\mu_0$ is taken as that given by Eq. (8). Hereto we have obtained the iterative self-contained reduction of order form of LDE.

As an example, we apply the new reduction of order iterative form of LDE to a special case, a point charge undergoing one-dimensional uniformly accelerating motion along x^1 direction acted by a constant electric field E. Assuming that the ratio of charge e to mass m is one, the equations of motion are

$$\ddot{x}^0 = E\dot{x}^1 + \tau_0[\dddot{x}^0 + \ddot{x}^2 \dot{x}^0] \approx E\dot{x}^1 + \tau_0[E\ddot{x}^1 + k\dot{x}^0]$$

$$\ddot{x}^1 = E\dot{x}^0 + \tau_0[\dddot{x}^1 + \ddot{x}^2 \dot{x}^1] \approx E\dot{x}^0 + \tau_0[E\ddot{x}^0 + k\dot{x}^1]$$

It is easy to obtain the expressions of $\dot{x}_0$ and $\dot{x}_1$, they are

$$\ddot{x}^0 = \frac{E\dot{x}^1 + \tau_0 E^2 \dot{x}^0 + \tau_0 k(\dot{x}^0 + \tau_0 E\dot{x}^1)}{1 - (\tau_0 E)^2},$$

$$\ddot{x}^1 = \frac{E\dot{x}^0 + \tau_0 E^2 \dot{x}^1 + \tau_0 k(\dot{x}^1 + \tau_0 E\dot{x}^0)}{1 - (\tau_0 E)^2}.$$

From its definition $k = \ddot{x}^2 = (\ddot{x}^0)^2 - (\ddot{x}^1)^2$, we obtain $k = -E^2$. The expression of four-acceleration is extremely simple, namely $\ddot{x}^0 = E\dot{x}^1$, $\ddot{x}^1 = E\dot{x}^0$. It is obvious that further iterative procedures will not bring any changes, which completely coincides with LDE for this motion. It is astonishing that a point charge undergoing one-dimensional uniformly accelerating motion emitting energy and momentum does not suffer radiation reaction force at all, which induces a series of puzzling problems lasted for over one hundred years since the radiation fields of this motion had been calculated [Born, 1909, as cited in Fulton & Rohrlich, 1960; Lyle, 2008].

We would like to make a brief remark on Eq. (8). Our approximate procedure contains all effects of the second term of radiation reaction force, so our result is more accurate than LLD. This conclusion is also embodied by its Taylor series form on τ_0 which includes infinite terms, while LLE only includes the linear term of τ_0.

It is easy for one to utilize the method of Landau and Lifshitz to construct a reduction of order iterative form of LDE [Aguirregabiria, 1997]. To compare two different reduction of order forms of LDE is the main content of the next subsection.

2.2 Reduction of order form of LDE up to τ_0^2 term

We know that the quantity τ_0 characterizing the radiation reaction effect is an extremely small time scale $(10^{-24}s)$, so every piece involved in Eq. (8) could be expanded as power series of the parameter τ_0. We are just interested in the first three terms of this series form of LDE, which is accurate enough to study practical problems and making the comparison between two series forms of LDE obtained respectively by Landau and Lifshitz's method and ours meaningful. To get this series form up to τ_0^2 term, we first expand matrix A to τ_0^2 term, and the result is

$$\Delta \approx 1 - \tau_0 \frac{\partial f^\mu}{\partial \dot{x}^\mu} + \tau_0^2 \left(\frac{\partial f^\mu}{\partial \dot{x}^\mu} \frac{\partial f^\nu}{\partial \dot{x}^\nu} - \frac{\partial f^\mu}{\partial \dot{x}^\nu} \frac{\partial f^\nu}{\partial \dot{x}^\mu} \right) + \cdots . \tag{10}$$

For the calculation of the four-vector D^μ, it is adequate to calculate its zero-th component and retain the result to τ_0^2 term

$$D^0 \approx F^0 + \tau_0 F^\mu \frac{\partial f^0}{\partial \dot{x}^\mu} + \tau_0^2 F^\mu \frac{\partial f^\nu}{\partial \dot{x}^\mu} \frac{\partial f^0}{\partial \dot{x}^\nu} + \cdots .$$

The space component expressions of four-vector D^μ can be obtained from the Lorentz covariance, and the result is

$$D^\mu \approx F^\mu + \tau_0 F^\alpha \frac{\partial f^\mu}{\partial \dot{x}^\alpha} + \tau_0^2 F^\alpha \frac{\partial f^\beta}{\partial \dot{x}^\alpha} \frac{\partial f^\mu}{\partial \dot{x}^\beta} + \cdots . \tag{11}$$

Due to the same footing as D^{μ}, the generalized four-velocity vector $\dot{X}^{\mu}$ in Eq. (8) can be immediately written out by changing F^{μ} in Eq. (11) to $\dot{x}^{\mu}$, namely

$$\dot{X}^{\mu} = \dot{x}^{\mu} + \tau_0 \dot{x}^{\alpha} \frac{\partial f^{\mu}}{\partial \dot{x}^{\alpha}} + \tau_0^2 \dot{x}^{\alpha} \frac{\partial f^{\beta}}{\partial \dot{x}^{\alpha}} \frac{\partial f^{\mu}}{\partial \dot{x}^{\beta}} + \cdots . \tag{12}$$

Then we need to calculate the square of four-acceleration k. Eqs. (8) and (9) show that it is enough for the expansion of k to retain the τ_0 term,

$$k = \frac{2D^2}{(1 - 2\tau_0 D^{\mu} \dot{X}_{\mu}) + \sqrt{(1 - 2\tau_0 D^{\mu} \dot{X}_{\mu})^2 - 4\tau_0^2 \dot{X}^2 D^2}}$$

$$\approx \frac{D^2}{(1 - 2\tau_0 D^{\mu} \dot{X}_{\mu})} \approx D^2 (1 + 2\tau_0 D^{\mu} \dot{X}_{\mu}) \tag{13}$$

$$= f^2 + 2\tau_0 [f^2 f^{\mu} \dot{x}_{\mu} + f_{\mu} (\frac{\partial f^{\mu}}{\partial \tau} + \dot{x}^{\nu} \frac{\partial f^{\mu}}{\partial x^{\nu}} + f^{\nu} \frac{\partial f^{\mu}}{\partial \dot{x}^{\nu}})]$$

So we obtian the four-acceleration accurate to τ_0 term is

$$\ddot{x}^{\mu} = D^{\mu} + \tau_0 k \dot{X}^{\mu} = f^{\mu} + \tau_0 (\frac{\partial f^{\mu}}{\partial \tau} + \dot{x}^{\nu} \frac{\partial f^{\mu}}{\partial x^{\nu}} + f^{\nu} \frac{\partial f^{\mu}}{\partial \dot{x}^{\nu}}) + \tau_0 f^2 \dot{x}^{\mu}$$

$$= f^{\mu} + \tau_0 (\frac{\partial f^{\mu}}{\partial \tau} + \dot{x}^{\nu} \frac{\partial f^{\mu}}{\partial x^{\nu}} + f^{\nu} \frac{\partial f^{\mu}}{\partial \dot{x}^{\nu}} + f^2 \dot{x}^{\mu}) + \cdots \tag{14}$$

which is the same as LLE.

We need to calculate the result of once iteration to obtain the τ_0^2 term of the series form of the acceleration. All involved calculation is straightforward but cumbersome. Retaining to the τ_0^2 term, the four-vector G^{μ} is

$$G^{\mu} = f^{\mu} + \tau_0 (\frac{\partial \ddot{x}^{\mu}}{\partial \tau} + \dot{x}^{\nu} \frac{\partial \ddot{x}^{\mu}}{\partial x^{\nu}} + \ddot{x}^{\nu} \frac{\partial \ddot{x}^{\mu}}{\partial \dot{x}^{\nu}})$$

$$= f^{\mu} + \tau_0 [\frac{\partial (f^{\mu} + \tau_0 \ddot{x}_1^{\mu})}{\partial \tau} + \dot{x}^{\nu} \frac{\partial (f^{\mu} + \tau_0 \ddot{x}_1^{\mu})}{\partial x^{\nu}} + (f^{\nu} + \tau_0 \ddot{x}_1^{\nu}) \frac{\partial (f^{\mu} + \tau_0 \ddot{x}_1^{\mu})}{\partial \dot{x}^{\nu}}] \tag{15}$$

$$= f^{\mu} + \tau_0 [\frac{\partial f^{\mu}}{\partial \tau} + \dot{x}^{\nu} \frac{\partial f^{\mu}}{\partial x^{\nu}} + f^{\nu} \frac{\partial f^{\mu}}{\partial \dot{x}^{\nu}}] + \tau_0^2 [\frac{\partial \ddot{x}_1^{\nu}}{\partial \tau} + \dot{x}^{\nu} \frac{\partial \ddot{x}_1^{\mu}}{\partial x^{\nu}} + f^{\nu} \frac{\partial \ddot{x}_1^{\mu}}{\partial \dot{x}^{\nu}} + \ddot{x}_1^{\nu} \frac{\partial f^{\mu}}{\partial \dot{x}^{\nu}}]$$

We then calculate the square of four-acceleration k and the result is

$$k = G^2 (1 + 2\tau_0 G \cdot \dot{x}) = (f^{\mu} + \tau_0 [\frac{\partial f^{\mu}}{\partial \tau} + \dot{x}^{\nu} \frac{\partial f^{\mu}}{\partial x^{\nu}} + \frac{\partial f^{\mu}}{\partial \dot{x}^{\nu}} f^{\nu}])^2 (1 + 2\tau_0 G \cdot \dot{x})$$

$$= f^2 + 2\tau_0 [f_{\mu} \frac{\partial f^{\mu}}{\partial \tau} + \dot{x}^{\nu} \frac{\partial f^{\mu}}{\partial x^{\nu}} f_{\mu} + f_{\mu} \frac{\partial f^{\mu}}{\partial \dot{x}^{\nu}} f^{\nu} + f^2 f \cdot \dot{x}] \tag{16}$$

At last, we arrive at the targeted expression of $\ddot{x}^{\mu}$, which can be expressed as

$$\ddot{x}^{\mu} = G^{\mu} + \tau_0 k \dot{x}^{\mu} = \ddot{x}_0^{\mu} + \tau_0 \ddot{x}_1^{\mu} + \tau_0^2 \ddot{x}_2^{\mu} , \tag{17}$$

and the quantities $\ddot{x}_0^\mu$, $\ddot{x}_1^\mu$ and $\ddot{x}_2^\mu$ are

$$\ddot{x}_0^\mu = f^\mu$$

$$\ddot{x}_1^\mu = \frac{\partial f^\mu}{\partial \tau} + \dot{x}^\nu \frac{\partial f^\mu}{\partial x^\nu} + f^\nu \frac{\partial f^\mu}{\partial \dot{x}^\nu} + f^2 \dot{x}^\mu$$

$$\ddot{x}_2^\mu = \frac{\partial \ddot{x}_1^\nu}{\partial \tau} + \dot{x}^\nu \frac{\partial \ddot{x}_1^\mu}{\partial x^\nu} + f^\nu \frac{\partial \ddot{x}_1^\mu}{\partial \dot{x}^\nu} + \ddot{x}_1^\nu \frac{\partial f^\mu}{\partial \dot{x}^\nu} \qquad , \tag{18}$$

$$+ 2(f_\alpha \frac{\partial f^\alpha}{\partial \tau} + \dot{x}^\nu \frac{\partial f^\alpha}{\partial x^\nu} f_\alpha + f^\nu \frac{\partial f^\alpha}{\partial \dot{x}^\nu} f_\alpha + f^2 f \cdot \dot{x}) \dot{x}^\mu$$

which is quite forbidding on the first face compared with the original LDE!
As stated before, by repeating the method of Landau and Lifshitz to reduce the order of LDE, one can also construct the series form of order reduced approximation of LDE, which is quite simple and the result is

$$\ddot{x}_0^\mu = f^\mu$$

$$\ddot{x}_i^\mu = f^\mu + \tau_0 (\frac{d\ddot{x}_{i-1}^\mu}{d\tau} + \ddot{x}_{i-1}^2 \dot{x}^\mu) \tag{19}$$

One needs to iterate two times to get the series expression up to τ_0^2 term using Eq. (19), and the result is

$$\ddot{x}_L^\mu = f^\mu + \tau_0 [\frac{df^\mu}{d\tau} + f^2 \dot{x}^\mu] + \tau_0^2 [\frac{d}{d\tau}(\frac{df^\mu}{d\tau} + f^2 \dot{x}^\mu) + \frac{df^2}{d\tau} \dot{x}^\mu] . \tag{20}$$

We just point out that Eqs. (17) and (20) obtained through two different ways are completely consistent with each other, which shows that the reduction of order form of LDE has unique expression so long as the external force is orthogonal to four-velocity and depends only on proper time, spacetime coordinate and four-velocity. This fact also supports that LDE is the correct equation describing the motion of charged particles.
The general reduction of order series form of LDE can be expressed as

$$\ddot{x}^\mu = \sum_{n=0}^{N} \tau_0^n \ddot{x}_n^\mu + R^\mu = \ddot{x}_N^\mu + R^\mu , \tag{21}$$

where R^μ is of the order τ_0^{N+1}. Apart from the first term $\ddot{x}_0^\mu$, all other single term is not orthogonal to $\dot{x}^\mu$ as the original LDE does, which can be seen by the calculation

$$(\frac{d\ddot{x}_N^\mu}{d\tau} + \ddot{x}_N^2 \dot{x}^\mu)\dot{x}_\mu = \dot{x}_\mu \frac{d\ddot{x}_N^\mu}{d\tau} + \ddot{x}_N^2 = \dot{x}_\mu \frac{d(\ddot{x}^\mu - R^\mu)}{d\tau} + \ddot{x}_N^2 = \dot{x}_\mu \frac{d\ddot{x}^\mu}{d\tau} + \ddot{x}_N^2 - \dot{x}_\mu \frac{dR^\mu}{d\tau}$$

$$= -\ddot{x}^\mu \frac{d\dot{x}_\mu}{d\tau} + \ddot{x}_N^2 - \dot{x}_\mu \frac{dR^\mu}{d\tau} = -\ddot{x}^2 + (\ddot{x} - R)^2 - \dot{x}_\mu \frac{dR^\mu}{d\tau} = R^2 - 2\ddot{x} \cdot R - \dot{x}_\mu \frac{dR^\mu}{d\tau} \qquad , \tag{22}$$

which is of order τ_0^{N+1}. To guarantee the orthogonal property between four-velocity and four-acceleration, the reduction of order series form of LDE must contain infinite terms. The

present method reducing the order of LDE contains infinite terms of τ_0 at each iterative process indicating that it is more accurate than that of Landau and Lifshitz.

In subsection 1, we have known that the uniformly accelerating motion of a point charge is quite special. According to the Maxwell's electromagnetic theory, the accelerated charge would for certain radiate electromagnetic radiation which would dissipate the charge's energy and momentum. W. Pauli observed that at $t = 0$ when the charge is instantaneously at rest [Pauli, 1920, as cited in Fulton & Rohrlich, 1960], the magnetic field of its radiating field is zero everywhere in the corresponding inertial frame shown by Born's original solution which means that the Poynting vector is zero everywhere in the rest frame of the charge at that instant of time and came to the conclusion that the uniformly accelerated charge does not radiate at all. Whether or not a charge undergoing uniformly accelerating motion emits electromagnetic radiation is still an open question. Nowadays, most authors of this area think that it does emit radiation. But one is faced another difficult question, namely what physical processes taking place near the neighborhood of the charge are able to give precise zero radiation reaction force. These problems have been extensively studied for a long time, but the situation still remains in a controversial status [Ginzburg, 1970; Boulware, 1980].

If one measures the macroscopic external field at the vicinity of the charge performing one-dimensional motion, what conclusions would he/she obtain? Just as one can not distinguish the gravitation field from a uniform acceleration by local experiments done in a tiny box, which is called Einstein equivalence principle, any macroscopic external fields felt by the charge are almost constant and the radiation reaction force would vanish according to LDE for charge's one-dimensional macroscopic motions! What is the mechanism of radiation reaction? There are various points of view for charges' radiation reaction. Lorentz regarded charges as rigid spheres of finite size, and the radiation reaction force comes from the interactions of the retarded radiation fields of all parts of the charge; Dirac regarded electrons as point charges and regarded the radiation reaction field as half the difference of its retarded radiation field minus its advanced field; Teitelboim obtained the radiation reaction force just using retarded radiating field of accelerated charges [Teitelboim , 1970; Teitelboim & Villarroel, 1980]; while Feynman and Wheeler thought that the radiation reaction of accelerated charges comes from the advanced radiating fields of all other charges in the Universe coherently superimposed at the location of the radiating charge [Wheeler & Feynman, 1945; 1949]. The common point of these different viewpoints is that the radiation reaction is represented by the variation of external force field with time, which gives zero radiation reaction for one-dimensional uniformly accelerating motion of charges showing that the general mechanism of radiation reaction is not complete. It has been proved that the electron of a hydrogen atom would never collapse using the nonrelativistic version of LDE which is quite contrary to the conventional idea of classical electrodynamics that the atoms would fall into the origin within rather a short time interval, which is called the theorem [Eliezer, 1947; Carati,2001]. Cole and Zou studied the stability of hydrogen atoms using LLE and obtained similar results with that of quantum mechanics [Cole & Zou, 2003].

It seems that the problems associated with the radiation reaction have little possibility to be resolved without introducing new factors. According to quantum field theory, the vacuum is not empty but full of all kinds of fluctuating fields. To investigate how the electromagnetic fluctuating fields of the vacuum influence the motion of a charge is the main content of this chapter.

3. Electromagnetic fluctuating fields of the vacuum

The development of quantum field theory (QFT) shows that the vacuum is not an empty space but an extremely complicated system. All kinds of field quanta are created and then annihilated or vice versa. The effects of electromagnetic fluctuating fields of the vacuum have been verified by experiments，such as Lamb shift of hydrogen atoms and Casimir force etc.. Unruh effect, which has been a very active area of physics and has not been verified by experiments as yet, shows that the vacuum state is dependent on the motion of observers. The quantized free fields of QFT are operator expressions, which can not be used directly in classical calculation. T. W. Marshall proposed a Lorentz invariant random classical radiation to model the corresponding fluctuating fields of the vacuum which was carried forward by T. H. Boyer [Boyer, 1980]. A number of phenomena associated with the vacuum of quantum electrodynamics can be understood in purely classical electrodynamics, provided we change the homogeneous boundary conditions on Maxwell's equations to include random classical radiation with a Lorentz invariant spectrum. This section briefly introduces this classical model of the vacuum fluctuating fields, which is borrowed heavily from T. H. Boyer's paper.

3.1 Random classical radiation fields for massless scalar cases

The introduced random radiation is not connected with temperature radiation but exist in the vacuum at the absolute zero of temperature; hence it is termed classical zero-point radiation, which is treated just as fluctuations of classical thermal radiation. The only special aspect of zero-point radiation is Lorentz invariant indicating there is no preferred frame. Thermal radiation involves radiation above the zero-point spectrum and involves a finite amount of energy and singles out a preferred frame of reference.

A spectrum of random classical radiation can be written as a sum over plane waves of various frequencies and wave vectors with random phases. For the massless scalar field, the spatially homogeneous and isotropic distribution in empty space can be written as an expansion in plane waves with random phases

$$\begin{aligned}
\varphi(\vec{r},t) &= \int d\vec{k}\ f(\omega)\cos[\vec{k}\cdot\vec{r} - \omega t - \theta(\vec{k})] \\
&= \int d\vec{k}\ \frac{1}{2} f(\omega)[a(\vec{k})\exp(-ik\cdot x) + a^*(\vec{k})\exp(ik\cdot x)]'
\end{aligned} \tag{23}$$

where the $\theta(\vec{k})$ is the random phase distributed uniformly on the interval $(0, 2\pi)$ and independently for each wave vector $\vec{k}$. The Lorentz invariance of Eq. (23) requires that $f(\omega)$ must be proportional to $1/\sqrt{\omega}$. The average over the random phases are

$$\begin{aligned}
&< \cos[\theta(\vec{k})]\cos[\theta(\vec{k}')] >=< \sin[\theta(\vec{k})]\sin[\theta(\vec{k}')] >= \frac{1}{2}\delta(\vec{k}-\vec{k}') \\
&< \cos[\theta(\vec{k})]\sin[\theta(\vec{k}')] >= 0
\end{aligned} \tag{24}$$

Eqs. (23) and (24) have immediate connection with the counterparts of QFT, which are free field

$$\phi(x) = \int \frac{d\vec{k}}{2\pi}\sqrt{\frac{\hbar}{\omega}}[\hat{a}(\vec{k})\exp(-ik\cdot x) + \hat{a}^\dagger(\vec{k})\exp(-ik\cdot x)],$$

and the commutators for creation and annihilation operators $\hat{a}^{\dagger}(\vec{k})$ and $\hat{a}(\vec{k})$ satisfying

$$[\hat{a}(\vec{k}),\hat{a}^{\dagger}(\vec{k}')] = \delta(\vec{k}-\vec{k}')$$
$$[\hat{a}(\vec{k}),\hat{a}(\vec{k}')] = [\hat{a}^{\dagger}(\vec{k}),\hat{a}^{\dagger}(\vec{k}')] = 0$$

Compared with the free field of QFT one can immediately get the Lorentz invariant spectral function, that is

$$f(\omega) = \frac{1}{\pi}\sqrt{\frac{\hbar}{2\omega}} .$$

These similarities make the quantitative calculations using random classical radiation fields comparable to the results given by full quantum theory.

As an example, we will deduce the Unruh effect using random classical radiation field. For a charge undergoing one-dimensional uniformly accelerating motion, the world line is

$$x = \frac{1}{a}\cosh(a\tau), \quad t = \frac{1}{a}\sinh(a\tau) , \tag{25}$$

where a is the constant acceleration seen at charge's rest frame. The uniform accelerating motion of a charge will be investigated in section 4.

We would like to evaluate the average value

$$< \varphi(\vec{r},s-t/2)\varphi(\vec{r},s+t/2) > ,$$

which characterizes the random classical field. By a Lorentz transformation, the point $\vec{r}$ can be changed to the origin of another inertial frame, which will be taken as the laboratory frame. Thus we just need to calculate

$$< \varphi(0,\sigma - \tau/2)\varphi(0,\sigma + \tau/2) > , \tag{26}$$

where $\sigma \pm \tau/2$ have two different interpretations; the results of Lorentz transformation or just two different proper times. The charge undergoes the uniformly accelerating motion along the x direction of the laboratory frame, and we calculate Eq. (26) at two positions of the charge at proper times $\sigma \pm \tau/2$, namely

$$\varphi(0,\sigma \pm \tau/2) = \varphi(\frac{\cosh(a(\sigma \pm \tau/2))}{a},\frac{\sinh(a(\sigma \pm \tau/2))}{a}) . \tag{27}$$

By the expression of field φ , the correlation function Eq. (26) becomes

$$< \varphi(0,\sigma - \tau/2)\varphi(0,\sigma + \tau/2) > = \int d^3k \frac{\hbar}{4\pi^2\omega}\cos\{\frac{k_x}{a}[\cosh(a(\sigma - \tau/2))$$
$$-\cosh(a(\sigma + \tau/2))] - \frac{\omega}{a}[\sinh(a(\sigma - \tau/2)) - \sinh(a(\sigma + \tau/2))]\} \tag{28}$$

The stationary character of the correlation function is not exhibited since the free parameter σ is included. However, the physical argument that there is no preferred time for uniformly

accelerating motions indicates that Eq. (28) must be independent of σ. In fact, by performing the Lorentz transformation

$$\omega' = \omega \cosh(a\sigma) - k_x \sinh(a\sigma),$$
$$k_x' = k_x \cosh(a\sigma) - \omega \sinh(a\sigma)$$

from unprimed laboratory frame to the primed frame in which the charge is instantaneously at rest at proper time σ, the correlation function becomes

$$< \varphi(0, \sigma - \tau/2)\varphi(0, \sigma + \tau/2) >$$
$$= \int \frac{d^3k'}{4\pi^2} \frac{\hbar}{\omega'} \cos[\frac{2\omega'}{a}[\sinh(\frac{a\tau}{2})] = -\frac{\hbar}{\pi}(\frac{a}{2})^2 \csc h^2(\frac{a\tau}{2}) \quad (29)$$

which is the function of the proper time τ.

The result Eq. (29) will be compared with the correlation function

$$< \varphi_T(0, s - t/2)\varphi_T(0, s + t/2) > \quad (30)$$

for a charge at the origin of an inertial frame where the random thermal radiation is

$$\pi^2 f_T^2(\omega) = \frac{\hbar}{\omega}(\frac{1}{2} + \frac{1}{\exp(\hbar\omega/kT) - 1}) = \frac{\hbar c^2}{2\omega} \coth(\frac{\hbar\omega}{2kT}),$$

where k is the Boltzmann constant, T the temperature. The calculation of correlation function Eq. (30) is straightforward and the result is

$$< \phi_T(0, s - t/2)\phi_T(0, s + t/2) >$$
$$= \frac{\hbar}{\pi c} \int_0^\infty d\omega \ \omega \coth(\frac{\hbar\omega}{2kT}) \cos \omega t = -\frac{\hbar}{\pi}(\frac{\pi k_B T}{\hbar})^2 \csc h^2(\frac{\pi k_B T t}{\hbar}) \quad (31)$$

The time parameter s is automatically counteracted without employing the Lorentz transformation.

If we compare the correlation function of the charge accelerated through the vacuum Eq. (29) with the correlation function Eq. (31) for a stationary charge in random classical scalar zero point radiation plus a Plank thermal spectrum, we find that they are identical in functional form provided that the acceleration and the temperature are related by

$$T = \frac{\hbar a}{2\pi k}. \quad (32)$$

It is in this sense that one speaks of an observer accelerated through the inertial vacuum as finding himself in a thermal bath which is called the Unruh effect [Unruh, 1976].

3.2 Random classical radiation fields for electromagnetic fields case
We just list the expressions used in the following sections of the classical model of the vacuum electromagnetic fluctuating fields, which can be written as

$$\vec{e}(\vec{r},t) = \sum_{\lambda=1,2} \int d^3k \sqrt{\frac{\hbar\omega}{2\pi^2}} \hat{\varepsilon}(\vec{k},\lambda) \cos[\vec{k}\cdot\vec{r}-\omega t-\theta(\vec{k},\lambda)]$$
$$\vec{b}(\vec{r},t) = \sum_{\lambda=1,2} \int d^3k \sqrt{\frac{\hbar\omega}{2\pi^2}} \hat{k}\times\hat{\varepsilon}(\vec{k},\lambda) \cos[\vec{k}\cdot\vec{r}-\omega t-\theta(\vec{k},\lambda)]$$
(33)

where λ denotes the polarization degree of freedom and $\hat{\varepsilon}(\vec{k},\lambda)$ the polarization vector which satisfies the sum rule

$$\sum_{\lambda=1}^{2} \varepsilon_i(\vec{k},\lambda)\varepsilon_j(\vec{k},\lambda) = \delta_{ij} - k_i k_j / k^2 .$$
(34)

The average over random phases satisfies the rules:

$$< \sin[\theta(\vec{k},\lambda)]\sin[\theta(\vec{k}',\lambda')] > = < \cos[\theta(\vec{k},\lambda)]\cos[\theta(\vec{k}',\lambda')] > = \frac{1}{2}\delta_{\lambda\lambda'}\delta(\vec{k}-\vec{k}')$$
$$< \cos[\theta(\vec{k},\lambda)]\sin[\theta(\vec{k}',\lambda')] > = 0$$
(35)

The average values of two components of random classical electromagnetic radiation fields can be obtained by using Eqs. (33)-(35). For example, the average value of two electric components is

$$e_i e_j = \sum_{\lambda=1,2} \int d^3k \sqrt{\frac{\hbar\omega}{2\pi^2}} \varepsilon_i(\vec{k},\lambda) \sum_{\lambda'=1,2} \int d^3k' \sqrt{\frac{\hbar\omega'}{2\pi^2}} \hat{\varepsilon}_j(\vec{k}',\lambda')$$
$$\times < \cos[k\cdot x-\theta(\vec{k},\lambda)]\cos[k'\cdot x-\theta(\vec{k}',\lambda')] >$$
$$= \sum_{\lambda,\lambda'=1,2} \int d^3k \sqrt{\frac{\hbar\omega}{2\pi^2}} \varepsilon_i(\vec{k},\lambda) \sum_{\lambda'=1,2} \int d^3k' \sqrt{\frac{\hbar\omega'}{2\pi^2}} \hat{\varepsilon}_j(\vec{k}',\lambda')\frac{1}{2}\delta_{\lambda\lambda'}\delta(\vec{k}-\vec{k}') ,$$
$$= \sum_{\lambda=1,2} \int d^3k \frac{\hbar\omega}{4\pi^2} \hat{\varepsilon}_i(\vec{k},\lambda)\hat{\varepsilon}_j(\vec{k},\lambda) = \frac{\hbar}{4\pi^2}\delta_{ij}\int d^3k\omega - \frac{\hbar}{4\pi^2}\int d^3k\frac{k_i k_j}{k}$$
$$= \delta_{ij}\frac{\hbar}{6\pi^2}\int d^3k\cdot k = \delta_{ij}\frac{\hbar}{6\pi}k_{cutoff}^4$$
(36)

where k_{cutoff} is the cutoff of wave vector and is usually taken as that corresponding to the Compton wavelength of massive charges. The other two average values are

$$< b_i b_j > = < e_i e_j > = \delta_{ij}\frac{\hbar}{6\pi}k_{cutoff}^4 , \qquad < e_i b_j > = 0 .$$
(37)

The preparation of the vacuum electromagnetic fluctuating fields is finished. However, we strongly recommend serious-minded readers to read the original papers of T. H. Boyer. In the next section, we will investigate the possible effects of electromagnetic fluctuating fields of the vacuum on the radiation reaction of a radiating charge using the reduction of order series form of LDE obtained in section 2.

4. Nonzero contribution of vacuum fluctuations to radiation reaction

In section 2, we have obtained the reduction of order series form of LDE up to τ_0^2 term, and here we present it again

$$\ddot{x}^{\mu} = f^{\mu} + \tau_0 [\frac{\partial f^{\mu}}{\partial \tau} + \dot{x}^{\nu} \frac{\partial f^{\mu}}{\partial x^{\nu}} + f^{\nu} \frac{\partial f^{\mu}}{\partial \dot{x}^{\nu}} + f^2 \dot{x}^{\mu}] + \tau_0^2 [\frac{\partial \ddot{x}_1^{\mu}}{\partial \tau} + \dot{x}^{\nu} \frac{\partial \ddot{x}_1^{\mu}}{\partial x^{\nu}}$$
$$+ f^{\nu} \frac{\partial \ddot{x}_1^{\mu}}{\partial \dot{x}^{\nu}} + \ddot{x}_1^{\nu} \frac{\partial f^{\mu}}{\partial \dot{x}^{\nu}} + 2(f_{\alpha} \frac{\partial f^{\alpha}}{\partial \tau} + \dot{x}^{\nu} \frac{\partial f^{\alpha}}{\partial x^{\nu}} f_{\alpha} + f^{\nu} \frac{\partial f^{\alpha}}{\partial \dot{x}^{\nu}} f_{\alpha} + f^2 f \cdot \dot{x}) \dot{x}^{\mu}] \tag{38}$$

where

$$\ddot{x}_1^{\mu} = f^{\mu} + \tau_0 (\frac{\partial f^{\mu}}{\partial \tau} + \dot{x}^{\nu} \frac{\partial f^{\mu}}{\partial x^{\nu}} + f^{\nu} \frac{\partial f^{\mu}}{\partial \dot{x}^{\nu}} + f^2 \dot{x}^{\mu}) .$$

In getting this equation, we have assumed that the external fields are electromagnetic fields which can be expressed by the functions of the proper time and spacetime coordinates. Therefore the external force has the Lorentz-force form:

$$f^{\mu} = \frac{e}{m} F^{\mu\nu} \dot{x}_{\nu} = [(\vec{E} + \vec{e}) \cdot \dot{\vec{x}}, \ \dot{x}_0 (\vec{E} + \vec{e}) + \dot{\vec{x}} \times (\vec{B} + \vec{b})] , \tag{39}$$

where $\vec{E}(\tau, x)$ and $\vec{B}(\tau, x)$ are external electromagnetic fields, $\vec{e}(x)$ and $\vec{b}(x)$ the vacuum electromagnetic fluctuating fields expressed in Eq. (33). For the sake of simplicity, we have these electromagnetic fields absorb the factor e / m. The component expressions of the force field are

$$\begin{aligned} f^0 &= \dot{\vec{x}} \cdot (\vec{E} + \vec{e}) = f_E^0 + f_e^0 \\ f^1 &= \dot{x}_0 (E_1 + e_1) + [\dot{x}_2 (B_3 + b_3) - \dot{x}_3 (B_2 + b_2)] = f_E^1 + f_e^1 \\ f^2 &= \dot{x}_0 (E_2 + e_2) + [\dot{x}_3 (B_1 + b_1) - \dot{x}_1 (B_3 + b_3)] = f_E^2 + f_e^2 \\ f^3 &= \dot{x}_0 (E_3 + e_3) + [\dot{x}_1 (B_2 + b_2) - \dot{x}_2 (B_1 + b_1)] = f_E^3 + f_e^3 \end{aligned} \tag{40}$$

We would like to emphasize again that Eq. (38) is accurate enough for any macroscopic motions of a charge due to τ_0 ($\sim 10^{-24} s$) being an extremely small time scale. Because the atomic nucleus is of finite size, Eq. (38) could even be used to study the electron's motion of a hydrogen atom.

It is impossible or meaningless to trace the effects of vacuum electromagnetic fluctuations by directly solve the equation of motion of charged particles due to its stochastic property. Therefore, we should perform the average calculation for each term of the acceleration over the random phases of vacuum fluctuating fields, which is a coarse grained process. The nonlinear terms of Eq. (38) will produce nonzero contribution of vacuum electromagnetic fluctuating fields to the radiation reaction. According to the rules of average manipulation over random phases Eq. (35), The result of the average for the first term of charge's acceleration is just the external force f_E^{μ}. We will focus on the calculation of the average results for the next two terms of the acceleration, which is quite cumbersome.

4.1 Effects of the vacuum fluctuations on the radiation reaction in the order of τ_0

The part of acceleration linear with τ_0 is

$$\ddot{x}_1^{\mu} = \frac{\partial f^{\mu}}{\partial \tau} + \dot{x}^{\nu} \frac{\partial f^{\mu}}{\partial x^{\nu}} + f^{\nu} \frac{\partial f^{\mu}}{\partial \dot{x}^{\nu}} + f^2 \dot{x}^{\mu} . \tag{41}$$

Because the average value of the product of odd number vacuum electromagnetic fluctuating components is zero, the average results of the first two terms of Eq. (41) are just

$$\frac{\partial f_E^\mu}{\partial \tau} + \dot{x}^\nu \frac{\partial f_E^\mu}{\partial x^\nu} \; .$$

The third term

$$f^\nu \frac{\partial f^\mu}{\partial \dot{x}^\nu}$$

is possible to produce nonzero contribution to radiation reaction. Firstly, we calculate the average value of its zero component

$$
\begin{aligned}
< f^\nu \frac{\partial f^0}{\partial \dot{x}^\nu} > &=< (f_E^\nu + f_e^\nu)\frac{\partial (f_E^0 + f_e^0)}{\partial \dot{x}^\nu} >= f_E^\nu \frac{\partial f_E^0}{\partial \dot{x}^\nu} + < f_e^\nu \frac{\partial f_e^0}{\partial \dot{x}^\nu} > \\
&= f_E^\nu \frac{\partial f_E^0}{\partial \dot{x}^\nu} + < f_e^0 \frac{\partial f_e^0}{\partial \dot{x}^0} + f_e^1 \frac{\partial f_e^0}{\partial \dot{x}^1} + f_e^2 \frac{\partial f_e^0}{\partial \dot{x}^2} + f_e^3 \frac{\partial f_e^0}{\partial \dot{x}^3} > \\
&= f_E^\nu \frac{\partial f_E^0}{\partial \dot{x}^\nu} + < (\dot{x}_0 e_1 + \dot{x}_2 b_3 - \dot{x}_3 b_2)e_1 + (\dot{x}_0 e_2 + \dot{x}_3 b_1 - \dot{x}_1 b_3)e_2 + (\dot{x}_0 e_3 + \dot{x}_1 b_2 - \dot{x}_2 b_1)e_3 > \\
&= f_E^\nu \frac{\partial f_E^0}{\partial \dot{x}^\nu} + 3\dot{x}_0 < e_1 e_1 >
\end{aligned}
$$

Then we calculate the average value of the first space component and the result is

$$
\begin{aligned}
< f^\nu \frac{\partial f^1}{\partial \dot{x}^\nu} > &=< (f_E^\nu + f_e^\nu)\frac{\partial (f_E^1 + f_e^1)}{\partial \dot{x}^\nu} >= f_E^\nu \frac{\partial f_E^1}{\partial \dot{x}^\nu} + < f_e^\nu \frac{\partial f_e^1}{\partial \dot{x}^\nu} > \\
&= f_E^\nu \frac{\partial f_E^1}{\partial \dot{x}^\nu} + < f_e^0 \frac{\partial f^1}{\partial \dot{x}^0} + f_e^1 \frac{\partial f^1}{\partial \dot{x}^1} + f_e^2 \frac{\partial f^1}{\partial \dot{x}^2} + f_e^3 \frac{\partial f^1}{\partial \dot{x}^3} > \\
&= f_E^\nu \frac{\partial f_E^1}{\partial \dot{x}^\nu} + < \dot{x}_i e_i e_1 + (\dot{x}_0 e_2 + \dot{x}_3 b_1 - \dot{x}_1 b_3)b_3 - (\dot{x}_0 e_3 + \dot{x}_1 b_2 - \dot{x}_2 b_1)b_2 > \\
&= f_E^\nu \frac{\partial f_E^1}{\partial \dot{x}^\nu} + < \dot{x}_1 e_1^2 - \dot{x}_1 b_3^2 - \dot{x}_1 b_2^2 >= f_E^\nu \frac{\partial f_E^1}{\partial \dot{x}^\nu} - \dot{x}_1 < e_1 e_1 >
\end{aligned}
$$

Because all three space components are equivalent, the final result is

$$< f^\nu \frac{\partial f^\mu}{\partial \dot{x}^\nu} >= f_E^\nu \frac{\partial f_E^\mu}{\partial \dot{x}^\nu} + [3\dot{x}_0, \; -\dot{\vec{x}}] < e_1 e_1 > \; . \tag{42}$$

The average value of the fourth term of Eq. (41) can be directly calculated and the result is

$$
\begin{aligned}
< f^2 > \dot{x}^\mu &=< (f_E^\mu + f_e^\mu)(f_{E,\mu} + f_{e,\mu}) > \dot{x}^\mu = f_E^2 \dot{x}^\mu + < f_e^2 > \dot{x}^\mu \\
&= f_E^2 \dot{x}^\mu + < \dot{x}_i^2 e_i^2 - (\dot{x}_0 e_1 + \dot{x}_2 b_3 - \dot{x}_3 b_2)^2 - (\dot{x}_0 e_2 + \dot{x}_3 b_1 - \dot{x}_1 b_3)^2 \\
&\quad -(\dot{x}_0 e_3 + \dot{x}_1 b_2 - \dot{x}_2 b_1)^2 > \dot{x}^\mu \\
&= f_E^2 \dot{x}^\mu + < \dot{x}_i^2 e_i^2 - (\dot{x}_0^2 e_1^2 + \dot{x}_2^2 b_3^2 + \dot{x}_3^2 b_2^2) - (\dot{x}_0^2 e_2^2 + \dot{x}_3^2 b_1^2 + \dot{x}_1^2 b_3^2) \\
&\quad -(\dot{x}_0^2 e_3^2 + \dot{x}_1^2 b_2^2 + \dot{x}_2^2 b_1^2) > \dot{x}^\mu \\
&= f_E^2 \dot{x}^\mu - (3 + 4\dot{\vec{x}}^2)\dot{x}^\mu < e_1 e_1 >
\end{aligned}
$$

Therefore the complete expression linear with τ_0 of the contribution of vacuum fluctuations to the radiation reaction is

$$-4\tau_0(\dot{\ddot{x}}^2\dot{x}_0, \ \dot{x}_0^2\dot{\ddot{x}}) < e_1e_1 > .\tag{43}$$

4.2 Effects of the vacuum fluctuations on the radiation reaction in the order of τ_0^2

The part of acceleration that is proportional to τ_0^2 is

$$\ddot{x}_2^\mu = \frac{\partial \ddot{x}_1^\mu}{\partial \tau} + \dot{x}^\nu \frac{\partial \ddot{x}_1^\mu}{\partial x^\nu} + f^\nu \frac{\partial \ddot{x}_1^\mu}{\partial \dot{x}^\nu} + \ddot{x}_1^\nu \frac{\partial f^\mu}{\partial \dot{x}^\nu}$$
$$+2(f_\alpha \frac{\partial f^\alpha}{\partial \tau} + \dot{x}^\nu \frac{\partial f^\alpha}{\partial x^\nu} f_\alpha + f^\nu \frac{\partial f^\alpha}{\partial \dot{x}^\nu} f_\alpha + f^2 f \cdot \dot{x})\dot{x}^\mu \tag{44}$$

We had better expand it into explicit expression of external electromagnetic fields and vacuum fluctuating fields and then collect the same terms together. The expansion of the first term of Eq. (44) is

$$\frac{\partial \ddot{x}_1^\mu}{\partial \tau} = \frac{\partial}{\partial \tau}(\frac{\partial f^\mu}{\partial \tau} + \dot{x}^\nu \frac{\partial f^\mu}{\partial x^\nu} + f^\nu \frac{\partial f^\mu}{\partial \dot{x}^\nu} + f^2 \dot{x}^\mu)$$
$$= \frac{\partial^2 f^\mu}{\partial^2 \tau} + \dot{x}^\nu \frac{\partial^2 f^\mu}{\partial \tau \partial x^\nu} + f^\nu \frac{\partial f^\mu}{\partial x^\nu} + \frac{\partial f^\nu}{\partial \tau} \frac{\partial f^\mu}{\partial \dot{x}^\nu} + f^\nu \frac{\partial^2 f^\mu}{\partial \tau \partial \dot{x}^\nu} + 2f^\nu \frac{\partial f_\nu}{\partial \tau} \dot{x}^\mu + f^2 f^\mu$$

The second term of Eq. (44) is

$$\dot{x}^\nu \frac{\partial \ddot{x}_1^\mu}{\partial x^\nu} = \dot{x}^\nu \frac{\partial}{\partial x^\nu}(\frac{\partial f^\mu}{\partial \tau} + \dot{x}^\alpha \frac{\partial f^\mu}{\partial x^\alpha} + f^\alpha \frac{\partial f^\mu}{\partial \dot{x}^\alpha} + f^2 \dot{x}^\mu)$$
$$= \dot{x}^\nu(\frac{\partial^2 f^\mu}{\partial \tau \partial x^\nu} + \dot{x}^\alpha \frac{\partial^2 f^\mu}{\partial x^\nu \partial x^\alpha} + \frac{\partial f^\alpha}{\partial x^\nu} \frac{\partial f^\mu}{\partial \dot{x}^\alpha} + f^\alpha \frac{\partial^2 f^\mu}{\partial x^\nu \partial \dot{x}^\alpha} + 2f_\alpha \frac{\partial f^\alpha}{\partial x^\nu} \dot{x}^\mu)$$

The third term of Eq. (44) is

$$f^\nu \frac{\partial \ddot{x}_1^\mu}{\partial \dot{x}^\nu} = f^\nu \frac{\partial}{\partial \dot{x}^\nu}(\frac{\partial f^\mu}{\partial \tau} + \dot{x}^\alpha \frac{\partial f^\mu}{\partial x^\alpha} + f^\alpha \frac{\partial f^\mu}{\partial \dot{x}^\alpha} + f^2 \dot{x}^\mu)$$
$$= f^\nu(\frac{\partial^2 f^\mu}{\partial \tau \partial \dot{x}^\nu} + \delta_\nu^\alpha \frac{\partial f^\mu}{\partial x^\alpha} + \dot{x}^\alpha \frac{\partial^2 f^\mu}{\partial x^\alpha \partial \dot{x}^\nu} + \frac{\partial f^\alpha}{\partial \dot{x}^\nu} \frac{\partial f^\mu}{\partial \dot{x}^\alpha} + f^\alpha \frac{\partial^2 f^\mu}{\partial \dot{x}^\alpha \partial \dot{x}^\nu} + 2f_\alpha \frac{\partial f^\alpha}{\partial \dot{x}^\nu} \dot{x}^\mu + f^2 \delta_\nu^\mu)$$
$$= f^\nu \frac{\partial^2 f^\mu}{\partial \tau \partial \dot{x}^\nu} + f^\alpha \frac{\partial f^\mu}{\partial x^\alpha} + \dot{x}^\alpha \frac{\partial^2 f^\mu}{\partial x^\alpha \partial \dot{x}^\nu} f^\nu + f^\nu \frac{\partial f^\alpha}{\partial \dot{x}^\nu} \frac{\partial f^\mu}{\partial \dot{x}^\alpha} + f^\alpha \frac{\partial^2 f^\mu}{\partial \dot{x}^\alpha \partial \dot{x}^\nu} f^\nu + 2f_\alpha \frac{\partial f^\alpha}{\partial \dot{x}^\nu} f^\nu \dot{x}^\mu$$
$$+ f^2 f^\mu$$

and the fourth term is

$$\ddot{x}_1^\nu \frac{\partial f^\mu}{\partial \dot{x}^\nu} = \frac{\partial f^\nu}{\partial \tau} \frac{\partial f^\mu}{\partial \dot{x}^\nu} + \dot{x}^\alpha \frac{\partial f^\nu}{\partial x^\alpha} \frac{\partial f^\mu}{\partial \dot{x}^\nu} + f^\alpha \frac{\partial f^\nu}{\partial \dot{x}^\alpha} \frac{\partial f^\mu}{\partial \dot{x}^\nu} + \frac{\partial f^\mu}{\partial \dot{x}^\nu} f^2 \dot{x}^\nu .$$

Now we collected all these terms together and found that there are three terms contained in $\ddot{x}_2^\mu$ which are linear with force field:

$$\frac{\partial^2 f^\mu}{\partial^2 \tau} + 2\dot{x}^\nu \frac{\partial^2 f^\mu}{\partial \varpi \partial x^\nu} + \dot{x}^\nu \dot{x}^\alpha \frac{\partial^2 f^\mu}{\partial x^\nu \partial x^\alpha} \,.$$

The average of these three terms over the random phases of vacuum fluctuating fields just leaves the contribution of the external force field, and the result is

$$< \frac{\partial^2 f^\mu}{\partial^2 \tau} + 2\dot{x}^\nu \frac{\partial^2 f^\mu}{\partial \varpi \partial x^\nu} + \dot{x}^\nu \dot{x}^\alpha \frac{\partial^2 f^\mu}{\partial x^\nu \partial x^\alpha} >= \frac{\partial^2 f_E^\mu}{\partial^2 \tau} + 2\dot{x}^\nu \frac{\partial^2 f_E^\mu}{\partial \varpi \partial x^\nu} + \dot{x}^\nu \dot{x}^\alpha \frac{\partial^2 f_E^\mu}{\partial x^\nu \partial x^\alpha} \,.$$

There are seven terms contained in $\ddot{x}_2^\mu$ which are quadratic in force fields. Because the vacuum fluctuating field just the function of spacetime coordinates, the derivatives with respect to proper time τ or four-velocity $\dot{x}^\mu$ equal to zero. Furthermore, the derivative with respect to spacetime coordinates would turn the cosine function into sine function, which combining with another cosine function of a vacuum fluctuating field would make the average value zero. Therefore, the average value of all quadratic terms of force fields contained in $\ddot{x}_2^\mu$ over the random phases of vacuum fluctuating fields is

$$\begin{aligned}
< & 2f^\nu \frac{\partial f^\mu}{\partial x^\nu} + 2\frac{\partial f^\nu}{\partial \tau}\frac{\partial f^\mu}{\partial \dot{x}^\nu} + 2f^\nu \frac{\partial^2 f^\mu}{\partial \varpi \partial \dot{x}^\nu} + 4f^\nu \frac{\partial f_\nu}{\partial \tau}\dot{x}^\mu \\
& + 2\dot{x}^\nu \frac{\partial f^\alpha}{\partial x^\nu}\frac{\partial f^\mu}{\partial \dot{x}^\alpha} + 2\dot{x}^\nu f^\alpha \frac{\partial^2 f^\mu}{\partial x^\nu \partial \dot{x}^\alpha} + 4\dot{x}^\nu f_\alpha \frac{\partial f^\alpha}{\partial x^\nu}\dot{x}^\mu > \\
= & 2f_E^\nu \frac{\partial f_E^\mu}{\partial x^\nu} + 2\frac{\partial f_E^\nu}{\partial \tau}\frac{\partial f_E^\mu}{\partial \dot{x}^\nu} + 2f_E^\nu \frac{\partial^2 f_E^\mu}{\partial \varpi \partial \dot{x}^\nu} + 4f_E^\nu \frac{\partial f_E^\nu}{\partial \tau}\dot{x}^\mu \\
& + 2\dot{x}^\nu \frac{\partial f_E^\alpha}{\partial x^\nu}\frac{\partial f_E^\mu}{\partial \dot{x}^\alpha} + 2\dot{x}^\nu f_E^\alpha \frac{\partial^2 f_E^\mu}{\partial x^\nu \partial \dot{x}^\alpha} + 4\dot{x}^\nu f_{E,\alpha}\frac{\partial f_E^\alpha}{\partial x^\nu}\dot{x}^\mu
\end{aligned}\,.$$

There are six terms contained in $\ddot{x}_2^\mu$ which are trinomial of force fields

$$2f^2 f^\mu + 2f^\nu \frac{\partial f^\alpha}{\partial \dot{x}^\nu}\frac{\partial f^\mu}{\partial \dot{x}^\alpha} + f^\alpha \frac{\partial^2 f^\mu}{\partial \dot{x}^\alpha \partial \dot{x}^\nu} f^\nu + 4f_\alpha \frac{\partial f^\alpha}{\partial \dot{x}^\nu} f^\nu \dot{x}^\mu + \frac{\partial f^\mu}{\partial \dot{x}^\nu} f^2 \dot{x}^\nu + 2\dot{x}^\mu f^2 f \cdot \dot{x} \,.$$

These terms are sequently denoted as $(a), (b), (c), (d), (e)$ and (f). We just present the calculation process for the first term, only final results of other five terms are given.
We first expand term (a) into

$$\begin{aligned}
2 < f^2 f^\mu > &= 2 < (f_E^\nu + f_e^\nu)(f_{E,\nu} + f_{e,\nu})(f_E^\mu + f_e^\mu) > \\
&= 2 < (f_E^\nu + f_e^\nu)(f_{E,\nu} + f_{e,\nu})f_E^\mu + (f_E^\nu + f_e^\nu)(f_{E,\nu} + f_{e,\nu})f_e^\mu > \\
&= 2 < (f_E^2 + f_e^2)f_E^\mu + 2f_E^\nu f_{e,\nu} f_e^\mu > \\
&= 2f_E^2 f_E^\mu + 2 < f_e^2 > f_E^\mu + 4f_E^\nu < f_{e,\nu} f_e^\mu >
\end{aligned}\,,$$

and last two terms are denoted as $(a.2)$ an $(a.3)$. We will calculate the average values of these two terms separately.
For the average value of term $(a.2)$, the result is

$$2f_E^\mu < f_e^2 >= 2f_E^\mu < f_{e,0}^2 - f_{e,1}^2 - f_{e,2}^2 - f_{e,3}^2 >$$

$$
\begin{aligned}
&= 2f_E^\mu < \dot{x}_i e_i \dot{x}_j e_j - (\dot{x}_0 e_1 + \dot{x}_2 b_3 - \dot{x}_3 b_2)^2 - (\dot{x}_0 e_2 + \dot{x}_3 b_1 - \dot{x}_1 b_3)^2 - (\dot{x}_0 e_3 + \dot{x}_1 b_2 - \dot{x}_2 b_1)^2 > \\
&= 2f_E^\mu < \dot{x}_i e_i \dot{x}_i e_i - (\dot{x}_0^2 + \dot{x}_2^2 + \dot{x}_3^2) e_1 e_1 - (\dot{x}_0^2 + \dot{x}_1^2 + \dot{x}_3^2) e_1 e_1 - (\dot{x}_0^2 + \dot{x}_2^2 + \dot{x}_1^2) e_1 e_1 > \\
&= 2f_E^\mu [\dot{x}_i \dot{x}_i - (\dot{x}_0^2 + \dot{x}_2^2 + \dot{x}_3^2) - (\dot{x}_0^2 + \dot{x}_1^2 + \dot{x}_3^2) - (\dot{x}_0^2 + \dot{x}_2^2 + \dot{x}_1^2)] < e_1 e_1 > \\
&= -2(4\dot{x}_0^2 - 1)f_E^\mu < e_1 e_1 >
\end{aligned}
$$

As for the term $(a.3)$, we first calculate its zero-th component and the result is

$$
\begin{aligned}
4f_E^\nu < f_{e,\nu} f_e^0 >&= 4f_E^0 < f_{e,0} f_e^0 > -4f_E^1 < f_{e,1} f_e^0 > -4f_E^2 < f_{e,2} f_e^0 > -4f_E^3 < f_{e,3} f_e^0 > \\
&= 4f_E^0 < \dot{x}_i \dot{x}_i e_1 e_1 > -4f_E^1 < \dot{x}_0 e_1 \dot{x}_i e_i > -4f_E^2 < \dot{x}_0 e_2 \dot{x}_i e_i > -4f_E^3 < \dot{x}_0 e_3 \dot{x}_i e_i > \\
&= 4f_E^0 \dot{x}_i \dot{x}_i < e_1 e_1 > -4f_E^1 \dot{x}_0 \dot{x}_1 < e_1 e_1 > -4f_E^2 \dot{x}_0 \dot{x}_2 < e_1 e_1 > -4f_E^3 \dot{x}_0 \dot{x}_3 < e_1 e_1 > \\
&= 4(f_E^0 \dot{\vec{x}}^2 - \dot{x}_0 \vec{f}_E \cdot \dot{\vec{x}}) < e_1 e_1 >
\end{aligned}
$$
,

then we calculate its first space component, and the result is

$$
\begin{aligned}
4f_E^\nu < f_{e,\nu} f_e^1 >&= 4f_E^0 < f_{e,0} f_e^1 > -4f_E^1 < f_{e,1} f_e^1 > -4f_E^2 < f_{e,2} f_e^1 > -4f_E^3 < f_{e,3} f_e^1 > \\
&= 4f_E^0 < \dot{x}_i \dot{x}_0 e_i e_1 > -4f_E^1 < (\dot{x}_0^2 + \dot{x}_2^2 + \dot{x}_3^2) e_1 e_1 > +4f_E^2 < \dot{x}_1 \dot{x}_2 e_1 e_1 > +4f_E^3 < \dot{x}_1 \dot{x}_3 e_1 e_1 > \\
&= [4f_E^0 \dot{x}_0 \dot{x}_1 - 4f_E^1 (2\dot{x}_0^2 - 1) + 4f_E^1 \dot{x}_1 \dot{x}_1 + 4f_E^2 \dot{x}_1 \dot{x}_2 + 4f_E^3 \dot{x}_1 \dot{x}_3] < e_1 e_1 > \\
&= [4(f_E^0 \dot{x}_0 + \vec{f}_E \cdot \dot{\vec{x}}) \dot{x}_1 - 4f_E^1 (2\dot{x}_0^2 - 1)] < e_1 e_1 >
\end{aligned}
$$
.

The other two space components of term $(a.3)$ can be obtained by the isotropy of the vacuum, and the final average result of term $(a.3)$ is

$$
\begin{aligned}
2 < f^2 f^\mu >&= 2f_E^2 f_E^\mu + 2 < f_e^2 > f_E^\mu + 4f_E^\nu < f_{e,\nu} f_e^\mu > \\
&= 2f_E^2 f_E^\mu - 2(4\dot{x}_0^2 - 1)f_E^\mu < e_1 e_1 > +[4(f_E^0 \dot{\vec{x}}^2 - \dot{x}_0 \vec{f}_E \cdot \dot{\vec{x}}), \ 4(f_E^0 \dot{x}_0 + \vec{f}_E \cdot \dot{\vec{x}})\dot{\vec{x}} - 4\vec{f}_E (2\dot{x}_0^2 - 1)] < e_1 e_1 > \\
&= 2f_E^2 f_E^\mu - [(4\dot{x}_0^2 + 2)f_E^0 + 4\dot{x}_0 \vec{f}_E \cdot \dot{\vec{x}}), \ -4(f_E^0 \dot{x}_0 + \vec{f}_E \cdot \dot{\vec{x}})\dot{\vec{x}} + (16\dot{x}_0^2 - 6)\vec{f}_E] < e_1 e_1 > \\
&= 2f_E^2 f_E^\mu - [(8\dot{x}_0^2 + 2)f_E^0, \ -8f_E^0 \dot{x}_0 \dot{\vec{x}} + (16\dot{x}_0^2 - 6)\vec{f}_E] < e_1 e_1 >
\end{aligned}
$$
.

In the following, we just list the final results for terms $(b) - (f)$, which would facilitate the check processes for serious-minded readers.
The average result for term (b) is

$$
2 < f^\alpha \frac{\partial f^\nu}{\partial \dot{x}^\alpha} \frac{\partial f^\mu}{\partial \dot{x}^\nu} >= 2f_E^\alpha \frac{\partial f_E^\nu}{\partial \dot{x}^\alpha} \frac{\partial f_E^\mu}{\partial \dot{x}^\nu} + 6[f_E^0, \ 2\dot{x}_0 \vec{E} - \vec{f}_E] < e_1 e_1 > ,
$$

and the result for term (c) is

$$
< f^\alpha \frac{\partial^2 f^\mu}{\partial \dot{x}^\alpha \partial \dot{x}^\nu} f^\nu >= 0 ,
$$

which is the result of the facts that external electromagnetic fields are linear function of four-velocity and vacuum fluctuating fields are the functions of spacetime coordinates.
The average result of term (d) is

$$
4 < f_\alpha \frac{\partial f^\alpha}{\partial \dot{x}^\nu} f^\nu > \dot{x}^\mu = 0 ,
$$

which is closely dependent on the structure of the Lorentz force , and the average result for term (e) is

$$< \dot{x}^\nu \frac{\partial f^\mu}{\partial \dot{x}^\nu} f^2 >= \dot{x}^\nu \frac{\partial f_E^\mu}{\partial \dot{x}^\nu} f_E^2 + [-f_E^0(4\dot{x}_0^2 + 1),\ 4\dot{x}_0 f_{E,0}\dot{\vec{x}} - \vec{f}_E(5\dot{x}_0^2 + 3\dot{\vec{x}}^2)] < e_1 e_1 > .$$

The last term (f) vanishes due to the fact that the Lorentz force is orthogonal to four-velocity vector. Collecting these pieces together, we arrive at the final expression proportional to τ_0^2 of the contribution of vacuum fluctuating fields to the radiation reaction of a radiating charge:

$$\tau_0^2[(-12\dot{x}_0^2 + 3)f_E^0,\ 12 f_{E,0}\dot{x}_0\dot{\vec{x}} + 12\dot{x}_0\vec{E} - (24\dot{x}_0^2 - 3)\vec{f}_E)] < e_1 e_1 > . \tag{45}$$

It is easy to check that the contribution of vacuum fluctuating fields to the radiation reaction is orthogonal to the four-velocity of the radiating charge. The only undetermined quantity contained in Eqs. (43) and (45) is the average value of two electric components of vacuum fluctuating fields, which can not be determined precisely. However, the cutoff of wave vector is often taken as that corresponding to the Compton wavelength of the massive charge. For electrons, this quantity is $< e_1 e_1 >= \hbar k_C^4 / 6\pi = \Omega = 9.3378 \times 10^{-5} T^2$, here "T" denotes the unit of magnetic field-Tesla.

It is necessary to illustrate the justification of our procedure, first performing the average calculation before solving the equation of motion, used to treat Eq. (38). It seems that the opposite procedure is more in line with general ideas. However, if one obtained the solution by directly solving Eq. (38) without performing the average calculation over random phases beforehand, we are justified to do the Taylor expansion about the vacuum fluctuations and then do average calculation for each term of this series. We can expect that two Taylor series obtained from two opposite procedures would coincide with each other at least for the first several terms.

In the following two sections, we will study the one-dimensional uniformly accelerating motion and the planar motion using the equation obtained in this section, which includes the radiation reaction up to τ_0^2 term with the contribution of the external field and the vacuum fluctuating fields.

5. One-dimensional motion of a charge acted by a constant electric field

In section 1, we have obtained the equation of motion of the one-dimensional uniformly accelerating motion of a point charge in a constant electric field. Although the radiation is emitted out during the charge undergoing this motion, the radiation reaction is bewilderingly zero! Much effort has been devoted to this problem, but there is no reasonable interpretation of this problem as yet. We propose a viewpoint that the radiation reaction of one-dimensional macroscopic motions of charged particles mainly comes from vacuum electromagnetic fluctuations expressed by Eqs. (43) and (45). Otherwise, it is very hard to understand the zero radiation reaction for the uniformly accelerating motion of a charge.

Assuming the constant electric field E is along x-axis, the radiation reaction of external field is zero, and the equation of motion of this case can be easily written out

$$\ddot{x}^\mu = f_E^\mu - \tau_0 \Omega [4\dot{\vec{x}}^2 \dot{x}_0, 4\dot{x}_0^2 \dot{\vec{x}}]$$
$$- \tau_0^2 \Omega [(12\dot{x}_0^2 - 3) f_E^0, -12\dot{x}_0 \vec{E} - 12 f_E^0 \dot{x}_0 \dot{\vec{x}} + (24\dot{x}_0^2 - 3) \vec{f}_E]' \tag{46}$$

where all the derivatives are with respect to proper time τ, and Ω denotes $< e_1 e_1 >$. In this case, the Lorentz force is $f_E^\mu = (\dot{x}E, \dot{x}_0 E, 0, 0)$. Eq. (46) shows that the magnitude of transverse motion exponentially decreases with proper time, namely $\dot{x}_{2,3}^2 \sim \exp(-8\tau_0 \Omega \dot{x}_0^2 \tau)$, which is reasonable because the mass of the charge increased with its velocity making it harder and harder for vacuum fluctuations to jolt it. However, the decay time $\sim (8\tau_0 \Omega \dot{x}_0^2)^{-1}$ is very large showing that transverse motion induced by vacuum fluctuations could remain considerable magnitude for a long time and triggers extra radiation which is interpreted as Unruh radiation by some authors [Chen & Tajima, 1999; Thirolf et al., 2009; Iso et al., 2010]. The transverse motion of the charge is neglected in the following calculation.

The four-velocity and acceleration are $\dot{x}^\mu = (\dot{x}_0, \dot{x})$, $\ddot{x}^\mu = (\ddot{x}_0, \ddot{x})$ respectively, and the component form of equation of motion can be expressed as

$$\ddot{x}_0 = \dot{x}E - 4\tau_0 \Omega \dot{x}^2 \dot{x}_0 - \tau_0^2 \Omega E (12\dot{x}_0^2 - 3)\dot{x}$$
$$\ddot{x} = \dot{x}_0 E - 4\tau_0 \Omega \dot{x}_0^2 \dot{x} - \tau_0^2 \Omega E (12\dot{x}_0^2 - 3)\dot{x}_0 \tag{47}$$

Letting $\dot{x}_0 = \cosh\beta(\tau)$, $\dot{x} = \sinh\beta(\tau)$, Eq. (47) is transformed into

$$\dot{\beta} = E(1 - 3\tau_0^2 \Omega) - 2\tau_0 \Omega \sinh(2\beta) - 6\tau_0^2 \Omega E \cosh(2\beta), \tag{48}$$

which can be further expressed as

$$\tau - \tau_o = \int_{\beta_0}^{\beta} \frac{d\beta}{E(1 - 3\tau_0^2 \Omega) - 2\tau_0 \Omega \sinh(2\beta) - 6\tau_0^2 \Omega E \cosh(2\beta)}$$

$$= \int_{\beta_0}^{\beta} \frac{exp(2\beta)d\beta}{E(1 - 3\tau_0^2 \Omega)exp(2\beta) - \tau_0 \Omega [exp(4\beta) - 1] - 3\tau_0^2 \Omega E [exp(4\beta) + 1]},$$

$$= \int_{\beta_0}^{\beta} \frac{ydy}{-(\tau_0 \Omega + 3\tau_0^2 \Omega E)y^4 + E(1 - 3\tau_0^2 \Omega)y^2 + \tau_0 \Omega - 3\tau_0^2 \Omega E}$$

$$= \int_{\beta_0}^{\beta} \frac{ydy}{-My^4 + Ny^2 + L} \tag{49}$$

where $y = exp(\beta) > 1$, and $M = \tau_0 \Omega + 3\tau_0^2 \Omega E$, $N = E(1 - 3\tau_0^2 \Omega)$, $L = \tau_0 \Omega - 3\tau_0^2 \Omega E$. The initial value of proper time τ_o should not be confused with the characteristic time of radiation reaction τ_0.

The time coordinate $x_0(\tau)$ can be obtained from $\dot{x}_0 = \cosh\beta$, namely

$$x_0(\tau) - x_0(0) = \int_0^\tau \cosh\beta d\tau = \int_0^\tau \frac{\cosh\beta}{\dot{\beta}} d\beta$$

$$= \int_0^\tau \frac{\cosh\beta d\beta}{E(1 - 3\tau_0^2 \Omega) - 2\tau_0 \Omega \sinh(2\beta) - 6\tau_0^2 \Omega E \cosh(2\beta)}$$

$$\begin{aligned} &= \frac{1}{2} \int_0^\tau \frac{[\exp(3\beta) + \exp(\beta)]d\beta}{E(1 - 3\tau_0^2\Omega)\exp[2\beta] - \tau_0\Omega[\exp(4\beta) - 1] - 3\tau_0^2\Omega E[\exp(4\beta) + 1]} \\ &= \frac{1}{2} \int_0^\tau \frac{y^2 + 1}{-My^4 + Ny^2 + L} dy \end{aligned} \tag{50}$$

Similarly, the space coordinate $x(\tau)$ has the expression:

$$x(\tau) - x(0) = \int_0^\tau \sinh\beta \, d\tau = \int_0^\tau \frac{\sinh\beta}{\dot{\beta}} d\beta = \frac{1}{2} \int_0^\tau \frac{y^2 - 1}{-My^4 + Ny^2 + L} dy \; . \tag{51}$$

Now all interesting quantities are ascribed to the calculation of three integrals:

$$\int_{y0}^y \frac{y \, dy}{-My^4 + Ny^2 + L}, \quad \int_{y0}^y \frac{y^2 + 1}{-My^4 + Ny^2 + L} dy, \quad \int_{y0}^y \frac{y^2 - 1}{-My^4 + Ny^2 + L} dy \; .$$

Before we cope with these complicated integrals, let us first see two simple cases: (a) the vacuum fluctuating fields are neglected, namely the conventional uniformly accelerating motion, thus $M = L = 0$, $N = E$, and Eq. (49) becomes

$$\tau - \tau_o = \frac{1}{E} \int_{y0}^y \frac{dy}{y} = \frac{1}{E} \ln(\frac{y}{y_0}) \, ,$$

which leads to the result $\beta - \beta_0 = E(\tau - \tau_o)$. The expressions of spacetime coordinates are

$$x_0(\tau) - x_0(0) = \frac{1}{2E} \int_{y0}^y \frac{y^2 + 1}{y^2} dy = \frac{1}{E}[\sinh\beta - \sinh\beta_0]$$

$$x_1(\tau) - x_1(0) = \frac{1}{2E} \int_{y0}^y \frac{y^2 - 1}{Ey^2} dy = \frac{1}{E}[\cosh\beta - \cosh\beta_0] \, ,$$

which are extensively used in the study of Unruh effect.

Case (b): the external field is absent, so $M = L = \tau_0\Omega$ and $N = E = 0$. The solutions are

$$\tau - \tau_o = \frac{1}{2M} \int_{y0}^y \frac{dy^2}{1 - y^4} = \frac{1}{2M} \int_{y_0^2}^{y^2} \frac{dy}{1 - y^2} = \frac{1}{2M}[\tanh^{-1}(y^{-2}) - \tanh^{-1}(y_0^{-2})]$$

$$x_0(\tau) - x_0(0) = \frac{1}{2M} \int_{y0}^y \frac{y^2 + 1}{1 - y^4} dy = \frac{1}{2M} \int_{y0}^y \frac{1}{1 - y^2} dy = \frac{1}{2M}[\tanh^{-1}(y^{-1}) - \tanh^{-1}(y_0^{-1})] \, ,$$

$$x(\tau) - x(0) = \frac{1}{2M} \int_{y0}^y \frac{y^2 - 1}{1 - y^4} dy = \frac{-1}{2M} \int_{y0}^y \frac{1}{1 + y^2} dy = \frac{1}{2M}[\tan^{-1}(y_0) - \tan^{-1}(y)]$$

in getting these results, we used the integral formula

$$\int \frac{1}{1 - x^2} dx = \begin{cases} \tanh^{-1}(x) & (0 \le |x| < 1) \\ \tanh^{-1}(x^{-1}) & (|x| > 1) \end{cases} \, ,$$

where functions $\tanh^{-1}(y)$ and $\tan^{-1}(y)$ are inverse functions of $\tanh(x)$ and $\tan(x)$. From the expression of the proper time, we can see that $\beta(\infty) \to 0$ as $\tau \to \infty$ showing that a free charge moving in vacuum would exhaust its energy ending up with a quivering state.

At last we deal with the general case (c), namely Eqs. (49)-(51) which are expressed by three integrals. We just give the results of these integrals, the calculating process is not difficult. The results of these integrals are

$$\int \frac{ydy}{-My^4 + Ny^2 + L} = \frac{1}{\sqrt{4ML + N^2}} \tanh^{-1}[(\frac{2My^2 - N}{\sqrt{4ML + N^2}})^{\pm}]$$

$$\int \frac{y^2}{-My^4 + Ny^2 + L} dy = \sqrt{\frac{\sqrt{4ML + N^2} + N}{2M\sqrt{4ML + N^2}}} \tanh^{-1}[(y\sqrt{\frac{2M}{\sqrt{4ML + N^2} + N}})^{\pm}]$$

$$- \sqrt{\frac{\sqrt{4ML + N^2} - N}{2M\sqrt{4ML + N^2}}} \tan^{-1}(y\sqrt{\frac{2M}{\sqrt{4ML + N^2} - N}})$$

$$\int \frac{1}{-My^4 + Ny^2 + L} dy = \sqrt{\frac{\sqrt{4ML + N^2} + N}{2L(4ML + N^2)}} \tan^{-1}[\sqrt{\frac{2M}{\sqrt{4ML + N^2} - N}}y]$$

$$+ \sqrt{\frac{\sqrt{4ML + N^2} - N}{2L(4ML + N^2)}} \tanh^{-1}[(\sqrt{\frac{2M}{\sqrt{4ML + N^2} + N}}y)^{\pm}]$$

where the symbol " $\pm$ " is involved. When the argument of function $\tanh^{-1}(x)$ is larger than one, we take minus symbol, otherwise we take plus symbol.

It is not necessary to write out the expressions of proper time and spacetime coordinates as the functions of $y = \exp(\beta)$. We have already gained some insight into this kind of motion from cases (a) and (b). But the general case (c) could provide further insight into the effects of vacuum on the charge's motion. Let us investigate some information contained in Eq. (40). Its explicit expression is

$$\tau - \tau_o = \frac{1}{\sqrt{4ML + N^2}} \tanh^{-1}[(\frac{2My^2 - N}{\sqrt{4ML + N^2}})^{\pm}]\Big|_{y_0}^{y}$$

$$\sim \frac{1}{E} \tanh^{-1}[(\frac{2\tau_0\Omega y^2 - E}{E})^{\pm}]\Big|_{y_0}^{y} \tag{52}$$

in the second line we have used the approximation $M = L \sim \tau_0\Omega$, $N \sim E$, and we also assumed that external field $E \ll \tau_0^{-1}$ (complete expression is $E \ll mc / e\tau_0$). If we choose the " $+$ " as the superscript of the argument of function $\tanh^{-1}(x)$, we tacitly use the condition $y < \sqrt{E / (\tau_0\Omega)}$ and arrive at the expression

$$y^2 \sim \frac{E}{2\tau_0\Omega}[1 + \tanh(\tau E)],$$

with the initial value $y_0^2 = y^2(0) = E / 2\tau_0\Omega$, which corresponds to an extremely large energy of the charge. when $\tau \to \infty$, we have $y \to \sqrt{E / (\tau_0\Omega)}$ and the velocity $\tanh\beta$ of charge observed in laboratory frame approaches to the value $1 - 2\tau_0\Omega / E$. All these results and conditions are self-consistent.

When the initial value y_0^2 is small and does not meet the condition $y_0^2 = E / 2\tau_0\Omega$, we have approximate expression

$$y^2 = \frac{E(1+Y)}{2\tau_0\Omega} \frac{1+\tanh(E\tau)}{1+Y\tanh(E\tau)},$$

where $Y = (2\tau_0\Omega y_0^2 - E) / E$. As $\tau \to \infty$, the value of y increases to $\sqrt{E / (\tau_0\Omega)}$ which is close to what happens in linear accelerators. The vacuum fluctuations and the external electric field E together put an upper limit to its final energy.

Similarly we can discuss the opposite situation $y > \sqrt{E / (\tau_0\Omega)}$, Eq. (52) becomes

$$E(\tau - \tau_o) \sim \tanh^{-1}[\frac{E}{2\tau_0\Omega y^2 - E}]\Big|_{y_0}^{y}.$$

For initial value $y_0 = y(0) = \infty$, the solution can be expressed as

$$y^2 = \frac{E}{2\tau_0\Omega}[\frac{1}{\tanh(E\tau)} + 1].$$

when $\tau \to \infty$, y^2 decreases to $\sqrt{E / (\tau_0\Omega)}$.

If the initial value $y_0 > \sqrt{E / (\tau_0\Omega)}$ is finite, the solution becomes

$$y^2 = \frac{E(1+Y)}{2\tau_0\Omega} \frac{1+\tanh(E\tau)}{Y+\tanh(E\tau)},$$

where $Y = E / (2\tau_0\Omega y_0^2 - E) < 1$. As $\tau \to \infty$, y^2 also decreases to $\sqrt{E / (\tau_0\Omega)}$. This situation can be used to estimate the energy of charged cosmic rays and the electric field of the universe. In fact the limit value $\sqrt{E / (\tau_0\Omega)}$ of y^2 can be obtained directly from Eq. (47) by using the condition that the total force of the charge equals to zero.

We would like to briefly discuss about what we obtian in this section. If our starting point Eq. (38) is right, then the existence of the limit value of y^2 shows that the conventional uniformly accelerating motion can not be strictly realized in the real world, and the radiation reaction originated from the vacuum fluctuations could solve the puzzles associated with this motion, such as whether or not charges uniformly accelerated radiate with respect to inertial observers and the problems associated with a freely falling charge in a gravitation field etc. [Dewitt & Brehme, 1960; Singal, 1995, 1997; Lyle, 2008; Spallicci, 2010]. As seen in section 3, uniformly accelerating motion has an intimate relationship with the Unruh effect. Although B. L. Hu based on the model for a spinless charge interacting with a scalar field objects to relating the radiation reaction to vacuum fluctuation [Johnson, & Hu, 2005], we still insist on the viewpoint that the radiation reaction of the charge's one-dimensional macroscopic motions mainly comes from the vacuum electromagnetic fluctuating fields, which is another source besides external fields responsible for the radiation reaction of charges. As expressed by Eqs. (34) and (36), the radiation reaction produced by vacuum fluctuations exists in all motions of charges, but only for macroscopic linear motions it preponderates over that of external fields.

6. Planar motion of a charge in a constant magnetic field

Since the LDE is nonlinear, it is hard to solve it even in simple external field configurations. The planar motion of a charge in a constant magnetic field is another extremely important case except for the uniformly accelerating motion. D. J. Endres has proved that in this situation LDE admits a unique physical solution [Endres, 1993; Wang, 2010]. Many authors studied this problem and obtained various approximate solutions under certain conditions [Lubart, 1974]. To our knowledge no precise solution of LDE for this case has been found yet. J. S. Bell and J. M. Leinaas suggested that the residue polarization of electrons in storage rings could be served as a measure of Unruh temperature [Bell, & Leinaas, 1983, 1987; Unruh, 1998].

Classically electrons are regarded as scalar point charges, and all effects associated with spin are neglected [Jackson, 1976]. A charge moving in the plane with a constant magnetic field along its normal direction should execute an inward spiral motion radiating away energy and momentum according to Larmor law. Using the equation obtained in section 3, which contains radiation reaction effects produced by external field and vacuum fluctuations up to τ_0^2 term and is accurate enough for any macroscopic motions of charges. The solution of electrons' planar motion also means a way to experimentally detect the contribution of vacuum electromagnetic fluctuating fields to the radiation reaction.

The component form of the equation of motion is

$$
\begin{aligned}
\ddot{x}_0 &= -\tau_0(B^2 + 4\Omega)(\dot{x}_0^2 - 1)\dot{x}_0 \\
\ddot{x}_1 &= \dot{x}_2 B - \tau_0(B^2 + 4\Omega)\dot{x}_0^2\dot{x}_1 - \tau_0^2(B^3 + B\Omega)(26\dot{x}_0^2 - 3)\dot{x}_2 \quad , \\
\ddot{x}_2 &= -\dot{x}_1 B - \tau_0(B^2 + 4\Omega)\dot{x}_0^2\dot{x}_2 + \tau_0^2(B^3 + B\Omega)(26\dot{x}_0^2 - 3)\dot{x}_1
\end{aligned}
\tag{53}
$$

which contains the radiation reaction up to τ_0^2 term. The external magnetic field $\vec{B}$ is along z-axis, and the electron moves in x-y plane.

The first expresion of Eq. (53) describes the time component of four-velocity changing with the proper time, which can be written as

$$
\int \frac{d\dot{x}_0^2}{(\dot{x}_0^2 - 1)\dot{x}_0^2} = \int -2\tau_0(B^2 + 4\Omega)d\tau \ .
$$

This expression can be integrated out immediately, and the result is

$$
\dot{x}_0^2(\tau) = \frac{\dot{x}_0^2(0)}{\dot{x}_0^2(0) - [\dot{x}_0^2(0) - 1]exp[-2\tau_0(B^2 + 4\Omega)\tau]} \ .
\tag{54}
$$

When $\tau \to \infty$, we can see that $\dot{x}_0^2(\tau) \to 1$ indicating that the electron's kinetic energy was exhausted by radiation reaction force after some time. The total energy of the electron changing with proper time τ is described by

$$
m\dot{x}_0 = \frac{m\dot{x}_0(0)}{\sqrt{\dot{x}_0^2(0) - [\dot{x}_0^2(0) - 1]exp[-2\tau_0(B^2 + 4\Omega)\tau]}} \ .
$$

To solve the equations of $\dot{x}_1$ and $\dot{x}_2$, we introduce the complex variable $\dot{Z} = \dot{x}_1 + i\dot{x}_2$, which satisfies the equation:

$$\ddot{Z} = -[iB + \tau_0(B^2 + 4\Omega)\dot{x}_0^2 - i\tau_0^2(B^3 + B\Omega)(26\dot{x}_0^2 - 3)]\dot{Z} . \qquad (55)$$

This equation can be easily solved and the result is

$$\dot{Z} = \dot{Z}_0 exp[-\int_0^\tau [iB + \tau_0(B^2 + 4\Omega)\dot{x}_0^2 - i\tau_0^2(B^3 + B\Omega)(26\dot{x}_0^2 - 3)]d\tau]$$

$$= \dot{Z}_0 exp[-iB\tau - 3i\tau_0^2(B^3 + B\Omega)\tau]exp\{[-\tau_0(B^2 + 4\Omega) + 26i\tau_0^2(B^3 + B\Omega)]\int_0^\tau \dot{x}_0^2(\tau)d\tau\}$$

Using Eq. (54), the integral of $\dot{x}_0^2(\tau)$ can be carried out

$$\int_0^\tau \dot{x}_0^2(\tau)d\tau = \int_0^\tau \frac{\dot{x}_0^2(0)}{\dot{x}_0^2(0) - [\dot{x}_0^2(0) - 1]exp[-2\tau_0(B^2 + 4\Omega)\tau]}d\tau$$

$$= \int_0^\tau \frac{exp[2\tau_0(B^2 + 4\Omega)\tau]}{exp[2\tau_0(B^2 + 4\Omega)\tau] - [1 - 1/\dot{x}_0^2(0)]}d\tau \qquad .$$

$$= \frac{Ln[\dot{x}_0^2(0)exp[2\tau_0(B^2 + B_v^2)\tau] - \dot{x}_0^2(0) + 1]}{2\tau_0(B^2 + 4\Omega)}$$

Now, the expression of $\dot{Z}$ becomes

$$\dot{Z} = \dot{Z}_0 exp[-iB\tau - 3i\tau_0^2(B^3 + B\Omega)\tau]exp\{[-\tau_0(B^2 + 4\Omega) + 26i\tau_0^2(B^3 + B\Omega)]$$

$$\times \frac{Ln[\dot{x}_0^2(0)exp[2\tau_0(B^2 + 4\Omega)\tau] - \dot{x}_0^2(0) + 1]}{2\tau_0(B^2 + 4\Omega)}\} \qquad , \quad (56)$$

$$= \dot{Z}_0\dot{x}_0^{2\alpha}(0)exp[(-iB - \tau_0(B^2 + 4\Omega) + 23i\tau_0^2(B^3 + B\Omega))\tau][1 - |\dot{Z}_0|^2 exp[-2\tau_0(B^2 + 4\Omega)\tau]/\dot{x}_0^2(0)]^\alpha$$

where

$$\alpha = -\frac{1}{2} + 13i\tau_0\frac{B^3 + B\Omega}{B^2 + 4\Omega} ,$$

and $\dot{x}_0^2(0) = 1 + |\dot{Z}_0|^2$. It is very hard to find the exact expressions for $x_{1,2}(\tau)$, or they do not exist at all. However, Eq. (56) is the product of two parts; the first part is

$$\dot{Z}_0 exp[-iB\tau - 3i\tau_0^2(B^3 + B\Omega)\tau + 2\alpha\tau_0(B^2 + 4\Omega)\tau] ,$$

which varies quickly with the proper time compared with the second part

$$[\dot{x}_0^2(0) - |\dot{Z}_0|^2 exp[-2\tau_0(B^2 + 4\Omega)\tau]]^\alpha .$$

Therefore, $Z(\tau)$ can be expressed as the series in the quantity

$$\frac{2\tau_0(B^2 + 4\Omega)}{-iB - \tau_0(B^2 + 4\Omega) + 23i\tau_0^2(B^3 + B\Omega)} .$$

By the method of integration by parts, $Z(\tau)$ can be expressed as

$$
Z(\tau) = Z_0 + \frac{\dot{Z}_0 \dot{x}_0^{2\alpha}(0)}{-iB - \tau_0(B^2 + 4\Omega) + 23i\tau_0^2(B^3 + B\Omega)}
$$
$$
\times \{ \exp[(-iB - \tau_0(B^2 + 4\Omega) + 23i\tau_0^2(B^3 + B\Omega))\tau][1 - |\dot{Z}_0|^2 \exp[-2\tau_0(B^2 + 4\Omega)\tau]/\dot{x}_0^2(0)]^{\alpha} - 1 \} + \cdots
$$
, (57)

where only the first term is presented. This expression is of high accuracy for any proper time and any initial value of velocity. It is easy to prove that the complete series of Eq. (57) is absolutely convergent.

For the inward spiral motion of electrons, the magnitude r of the radius vector $\vec{r}(\tau) = x_1(\tau)\vec{e}_x + x_2(\tau)\vec{e}_y$ changing with proper time is an interested quantity. Due to $dr = \sqrt{dx_1^2 + dx_2^2} = |dZ|$ and with Eq. (56), we have

$$
-\frac{dr}{d\tau} = |\dot{Z}_0[\dot{x}_0^2(0)\exp[2\tau_0(B^2 + 4\Omega)\tau] - \dot{x}_0^2(0) + 1]^{\alpha}| = \frac{|\dot{Z}_0|}{\sqrt{\dot{x}_0^2(0)\exp[2\tau_0(B^2 + 4\Omega)\tau] - |\dot{Z}_0|^2}} , \quad (58)
$$

which can be further expressed as

$$
r(\tau) = r(0) + \frac{|\dot{Z}_0|}{\tau_0(B^2 + 4\Omega)} \int_0^{\tau} \frac{d\exp[-\tau_0(B^2 + 4\Omega)\tau]}{\sqrt{\dot{x}_0^2(0) - |\dot{Z}_0|^2 \exp[-2\tau_0(B^2 + 4\Omega)\tau]}} , \quad (59)
$$

where $r(0)$ is the initial distance of the electron away from the origin.

Using the integral formulas

$$
\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1}(x), \quad \int \frac{-1}{\sqrt{1 - x^2}} dx = \cos^{-1}(x) ,
$$

the calculation of Eq. (59) can be carried out and the result is

$$
\int_0^{\tau} \frac{d\exp[-\tau_0(B^2 + 4\Omega)\tau]}{\sqrt{\dot{x}_0^2(0) - |\dot{Z}_0|^2 \exp[-2\tau_0(B^2 + 4\Omega)\tau]}}
$$
$$
= \frac{1}{|\dot{Z}_0|} \{ \sin^{-1}[\frac{|\dot{Z}_0|}{\dot{x}_0(0)} \exp(-\tau_0(B^2 + 4\Omega)\tau)] - \sin^{-1}[\frac{|\dot{Z}_0|}{\dot{x}_0(0)}] \}
$$

Thus we obtain the expression of $r(\tau)$:

$$
r(\tau) = r(0) + \frac{1}{\tau_0(B^2 + 4\Omega)} \{ \sin^{-1}[\frac{|\dot{Z}_0|}{\dot{x}_0(0)} \exp(-\tau_0(B^2 + 4\Omega)\tau)] - \sin^{-1}[\frac{|\dot{Z}_0|}{\dot{x}_0(0)}] \} , \quad (60)
$$

which shows that the electron's distance away from the origin slowly decreases with the proper time. The electron really undergoes an inward spiral motion.

We prefer to express physical quantities by laboratory time x_0 rather than proper time τ. To do so, we first calculate

$$\frac{dt}{d\tau} = \frac{\dot{x}_0(0)}{\sqrt{\dot{x}_0^2(0) - |\dot{Z}_0|^2 \, exp[-2\tau_0(B^2 + 4\Omega)\tau]}}$$

$$= \frac{\dot{x}_0(0)exp[\tau_0(B^2 + 4\Omega)\tau]}{\sqrt{\dot{x}_0^2(0)exp[2\tau_0(B^2 + 4\Omega)\tau] - |\dot{Z}_0|^2]}} \tag{61}$$

from Eq. (54). Using integral formula $\int (x^2 - 1)^{-1/2} \, dx = \cosh^{-1}(x)$, we obtain

$$t = \int_0^\tau \frac{\dot{x}_0(0)exp[\tau_0(B^2 + 4\Omega)\tau]d\tau}{\sqrt{\dot{x}_0^2(0)exp[2\tau_0(B^2 + 4\Omega)\tau] - |\dot{Z}_0|^2]}}$$

$$= \frac{\dot{x}_0(0)}{\tau_0(B^2 + 4\Omega)|\dot{Z}_0|}\{\cosh^{-1}[\frac{\dot{x}_0(0)}{|\dot{Z}_0|}exp[\tau_0(B^2 + 4\Omega)\tau]] - \cosh^{-1}[\frac{\dot{x}_0(0)}{|\dot{Z}_0|}]\} \tag{62}$$

in getting this result, we have assumed that initial laboratory time $t = 0$ corresponds to initial proper time $\tau = 0$.

The distance of electron away from the origin varies with the laboratory time t can be written out according to Eqs. (54) and (58) and the result is

$$-\frac{dr}{dt} = -\frac{dr/d\tau}{dt/d\tau} = \frac{|\dot{Z}_0|}{\dot{x}_0(0)}exp[-\tau_0(B^2 + 4\Omega)\tau] . \tag{63}$$

If we can express the right hand side of this equation by the expression of t, then we get the equation describing r changing with laboratory time t. From Eq. (62), after some calculation we can obtain a quadratic equation of $exp[\tau_0(B^2 + 4\Omega)\tau]$, which is

$$exp[2\tau_0(B^2 + 4\Omega)\tau] - 2exp[\tau_0(B^2 + 4\Omega)\tau]\cosh[\tau_0(B^2 + 4\Omega)t]$$

$$+ \frac{1}{\dot{x}_0^2(0)} + \frac{|\dot{Z}_0|^2}{\dot{x}_0^2(0)}\cosh^2[\tau_0(B^2 + 4\Omega)t] = 0 .$$

It is easy to solve this equation and the result is

$$exp[\tau_0(B^2 + 4\Omega)\tau] = \cosh[\tau_0(B^2 + 4\Omega)t] \pm \frac{1}{\dot{x}_0(0)}\sinh[\tau_0(B^2 + 4\Omega)t] . \tag{64}$$

We know that when the electron is stationary, $\dot{x}_0(0) = 1$ and $t = \tau$. So we should take the " + " in Eq. (64), and Eq. (63) becomes

$$-\frac{dr}{dt} = \frac{|\dot{Z}_0|}{\dot{x}_0(0)\cosh[\tau_0(B^2 + 4\Omega)t] + \sinh[\tau_0(B^2 + 4\Omega)t]} .$$

This equation can be easily solved and the result is

$$r(0) - r(t) = \int_0^t \frac{\left| \dot{Z}_0 \right| dt}{\dot{x}_0(0) \cosh[\tau_0(B^2 + 4\Omega)t] + \sinh[\tau_0(B^2 + 4\Omega)t]}$$

$$= \frac{2 \left| \dot{Z}_0 \right|}{\tau_0(B^2 + 4\Omega)} \int_0^t \frac{d exp(y)}{[\dot{x}_0(0) + 1] exp(2y) + [\dot{x}_0(0) - 1]}$$

$$= \frac{2}{\tau_0(B^2 + 4\Omega)} \{ \tan^{-1}[\sqrt{\frac{\dot{x}_0(0)+1}{\dot{x}_0(0)-1}} exp[\tau_0(B^2 + 4\Omega)t]] - \tan^{-1}[\sqrt{\frac{\dot{x}_0(0)+1}{\dot{x}_0(0)-1}}] \}$$

So the final expression of the electron's distance away from the origin expressed by the laboratory time is

$$r(t) = r(0) - \frac{2}{\tau_0(B^2 + 4\Omega)} \{ \tan^{-1}[\sqrt{\frac{\dot{x}_0(0)+1}{\dot{x}_0(0)-1}} exp[\tau_0(B^2 + 4\Omega)t]] - \tan^{-1}(\sqrt{\frac{\dot{x}_0(0)+1}{\dot{x}_0(0)-1}}) \}, \quad (65)$$

which explicitly shows the inward spiral characteristic of the electron's planar motion with a constant magnetic field along its normal direction. Classically the final destiny of electrons performing such a motion is falling into the origin.

For ultrarelativistic electrons, $\dot{x}_0(0)$ is very large and $\tan^{-1}[\sqrt{(\dot{x}_0(0)+1)/(\dot{x}_0(0)-1)}] \doteq \pi/4$. Using Eq. (65), We can estimate the laboratory time needed for an ultrarelativistic electron to decrease its the distance from $r(0)$ to $r(0)/2$, and the result is

$$t_{half} = \frac{1}{\tau_0(B^2 + 4\Omega)} \ln \frac{1 + \tan[\tau_0(B^2 + 4\Omega)r(0)/4]}{1 - \tan[\tau_0(B^2 + 4\Omega)r(0)/4]}.$$

The initial value $r(0)$ can be approximated by that of the situation without considering the radiation reaction effects, namely $r(0) \approx \dot{x}_0/B$, so t_{haff} can be written as

$$t_{half} = \frac{1}{\tau_0(B^2 + 4\Omega)} \ln \frac{1 + \tan[\tau_0(B^2 + 4\Omega)\dot{x}_0(0)/4B]}{1 - \tan[\tau_0(B^2 + 4\Omega)\dot{x}_0(0)/4B]}. \quad (66)$$

However, due to τ_0 being a very small quantity, we are justified to further simplify this expression and the result is

$$t_{half} \sim \frac{\dot{x}_0(0)}{2} \frac{mc}{eB} \sim 10^{-3} \frac{\dot{x}_0(0)}{B}$$

seconds for electrons which shows that t_{half} has little relationship with vacuum fluctuations. For ultrarelativistic electrons, according to Eq. (64), we have approximate expression

$$exp[\tau_0(B^2 + 4\Omega)\tau] \doteq \cosh[\tau_0(B^2 + 4\Omega)t],$$

so the energy of electrons can be simplified as

$$\frac{\dot{x}_0(\tau)}{\dot{x}_0(0)} \sim \frac{\cosh[\tau_0(B^2 + 4\Omega)t]}{\sqrt{\dot{x}_0^2(0) \cosh^2[\tau_0(B^2 + 4\Omega)t] - \left| \dot{Z}_0 \right|^2}} \sim \frac{\cosh[\tau_0(B^2 + 4\Omega)t]}{\dot{x}_0(0) \sinh[\tau_0(B^2 + 4\Omega)t]}.$$

The ratio of two $\dot{x}_0(\tau)/\dot{x}_0(0)$'s respectively with and without the considerations of vacuum fluctuations is

$$\frac{[\dot{x}_0(\tau)/\dot{x}_0(0)]_{with}}{[\dot{x}_0(\tau)/\dot{x}_0(0)]_{without}} \sim 1 - 4\Omega/B^2 \;, \tag{67}$$

which shows that the effect of the vacuum fluctuations. We look forward to seeing that this result would be tested in future experiments. The nonzero effects of vacuum fluctuations had been recognized in microscopic world long time ago, such as the Lamb shift and the Casmir effect etc. However, whether or not the vacuum fluctuations has a relationship with the radiation reaction for the motion of charges is still an open question. The planar motion of high energy electrons with a constant magnetic field perpendicular to its moving plane provides a possible experimental scheme to test this viewpoint.

7. Conclusion

In this chapter, we presented a new reduction of order form of LDE, which coincides with that obtained by the method of Landau and Lifshitz in its Taylor series form. Using the classical version of zero-point electromagnetic fluctuating fields of the vacuum, we obtained the contributions of vacuum fluctuations to radiation reaction of a radiating charge up to the τ_0^2 term. Then we use the obtained reduction of order equation of LDE including the radiation reaction induced by external force and vacuum fluctuations up to the τ_0^2 term, which is accurate enough for any macroscopic motions of charges and even applicable to the electron's motion of a hydrogen atom due to τ_0 being extremely small, to study the one-dimensional uniformly accelerating motion produced by a constant electric field and the planar motion produced by a constant magnetic field. Our calculations show that for any one-dimensional uniformly accelerating motion the velocity of charges has a limit value and almost all puzzles associated with this special motion disappear; while the planar motion of electrons provides an experimental scheme to test the conjecture that the interaction between charged particles and the vacuum electromagnetic fluctuations is anther mechnisim for the charge's radiation reaction, which plays a dominant role only for one-dimensional macroscopic motions of charged particles.

8. Acknowledgment

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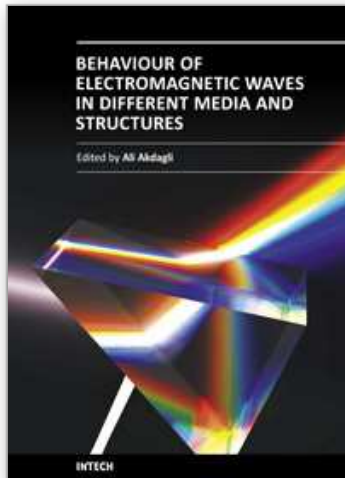
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