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Output Feedback Control of Discrete-time LTI Systems: Scaling LMI Approaches

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1. Introduction

Most physical systems have only limited states to be measured and fed back for system controls. Although sometimes, a reduced-order observer can be designed to meet the requirements of full-state feedback, it does introduce extra dynamics, which increases the complexity of the design. This naturally motivates the employment of output feedback, which only use measurable output in its feedback design. From implementation point of view, static feedback is more cost effective, more reliable and easier to implement than dynamic feedback (Khalil, 2002; Kučera & Souza, 1995; Syrmos et al., 1997). Moreover, many other problems are reducible to some variation of it. Simply stated, the static output feedback problem is to find a static output feedback so that the closed-loop system has some desirable characteristics, or determine the nonexistence of such a feedback (Syrmos et al., 1997). This problem, however, still marked as one important open question even for LTI systems in control engineering.
Although this problem is also known NP-hard (Syrmos et al., 1997), the curious fact to note here is that these early negative results have not prevented researchers from studying output feedback problems. In fact, there are a lot of existing works addressing this problem using different approaches, say, for example, Riccati equation approach, rank-constrained conditions, approach based on structural properties, bilinear matrix inequality (BMI) approaches and min-max optimization techniques (e.g., Bara & Boutayeb (2005; 2006); Benton (Jr.); Gadewadikar et al. (2006); Geromel, de Oliveira & Hsu (1998); Geromel et al. (1996); Ghaoui et al. (2001); Henrion et al. (2005); Kučera & Souza (1995); Syrmos et al. (1997) and the references therein). Nevertheless, the LMI approaches for this problem remain popular (Bara & Boutayeb, 2005; 2006; Cao & Sun, 1998; Geromel, de Oliveira & Hsu, 1998; Geromel et al., 1996; Prempain & Postlethwaite, 2001; Yu, 2004; Zečević & Šiljak, 2004) due to simplicity and efficiency.
Motivated by the recent work (Bara & Boutayeb, 2005; 2006; Geromel et al., 1996; Xu & Xie, 2005a;b; 2006), this paper proposes several scaling linear matrix inequality (LMI) approaches to static output feedback control of discrete-time linear time invariant (LTI) plants. Based on whether a similarity matrix transformation is applied, we divide these approaches into two parts. Some approaches with similarity transformation are concerned with the dimension and rank of system input and output. Several different methods with respect to the system state dimension, output dimension and input dimension are given based on whether the distribution matrix of input B or the distribution matrix of output C is full-rank. The other

approaches apply Finsler's Lemma to deal with the Lyapunov matrix and controller gain directly without similarity transformation. Compared with the BMI approach (e.g., Henrion et al. (2005)) or VK-like iterative approach (e.g.,Yu (2004)), the scaling LMI approaches are much more efficient and convergence properties are generally guaranteed. Meanwhile, they can significantly reduce the conservatism of non-scaling method, (e.g.,Bara & Boutayeb (2005; 2006)). Hence, we show that our approaches actually can be treated as alternative and complemental methods for existing works.

The remainder of this paper is organized as follows. In Section 2, we state the system and problem. In Section 3, several approaches based on similarity transformation are given. In Subection 3.1, we present the methods for the case that B is full column rank. Based on the relationship between the system state dimension and input dimension, we discuss it in three parts. In Subsection 3.2, we consider the case that C is full row rank in the similar way. In Subsection 3.3, we propose another formulations based on the connection between state feedback and output feedback. In Section 4, we present the methods based on Finsler's lemma. In Section 5, we compare our methods with some existing works and give a brief statistical analysis. In Section 6, we extend the latter result to H_∞ control. Finally, a conclusion is given in the last section. The notation in this paper is standard. $\mathcal{R}^n$ denotes the n dimensional real space. Matrix $A > 0$ ($A \geq 0$) means A is positive definite (semi-definite).

2. Problem formulation

Consider the following discrete-time linear time-invariant (LTI) system:

$$x(t+1) = A_o x(t) + B_o u(t) \tag{1}$$

$$y(t) = C_o x(t) \tag{2}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$ and $y \in \mathcal{R}^l$. All the matrices mentioned in this paper are appropriately dimensioned. $m < n$ and $l < n$.

We want to stabilize the system (1)-(2) by static output feedback

$$u(t) = Ky(t) \tag{3}$$

The closed-loop system is

$$x(t+1) = \tilde{A}x(t) = (A_o + B_o K C_o)x(t) \tag{4}$$

The following lemma is well-known.

Lemma 1. *(Boyd et al., 1994) The closed-loop system (4) is (Schur) stable if and only if either one of the following conditions is satisfied:*

$$P > 0, \;\; \tilde{A}^T P \tilde{A} - P < 0 \tag{5}$$

$$Q > 0, \;\; \tilde{A} Q \tilde{A}^T - Q < 0 \tag{6}$$

3. Scaling LMIs with similarity transformation

This section is motivated by the recent LMI formulation of output feedback control (Bara & Boutayeb, 2005; 2006; Geromel, de Souze & Skelton, 1998) and dilated LMI formulation (de Oliveira et al., 1999; Xu et al., 2004).

3.1 B_o with full column-rank

We assume that B_o is of full column-rank, which means we can always find a non-singular matrix T_b such that $T_b B_o = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$. In fact, using singular value decomposition (SVD), we can obtain such T_b. Hence the new state-space representation of this system is given by

$$A = T_b A_o T_b^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = T_b B_o, \ C = C_o T_b^{-1} \tag{7}$$

The closed-loop system (4) is stable if and only if

$$\tilde{A}_b = A + BKC \text{ is stable}$$

In this case, we divide it into 3 situations: $m = n - m$, $m < n - m$, and $m > n - m$. Let

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \in \mathcal{R}^{n \times n}, P_{11} \in \mathcal{R}^{m \times m}, P_{12} \in \mathcal{R}^{m \times (n-m)} \tag{8}$$

For the third situation, let

$$P_{12} = [P_{12}^{(1)} \ P_{12}^{(2)}], \ P_{11} = \begin{bmatrix} P_{11}^{(1)} & P_{11}^{(2)} \\ P_{11}^{(2)T} & P_{11}^{(3)} \end{bmatrix} \tag{9}$$

where $P_{12}^{(1)} \in \mathcal{R}^{(n-m)\times(n-m)}$ and $P_{11}^{(1)} \in \mathcal{R}^{(n-m)\times(n-m)}$.

Theorem 1. *The discrete-time system (1)-(2) is stabilized by (3) if there exist $P > 0$ defined in (8) and R, such that*

$$\begin{cases} \Phi(\Theta_1) < 0, \ m = n - m \\ \Phi(\Theta_2) < 0, \ m < n - m \\ \Phi(\Theta_3) < 0, \ m > n - m \end{cases} \tag{10}$$

where $\varepsilon \in \mathcal{R}$,

$$\Phi(\Theta_1) = \begin{bmatrix} A^T \Theta_1 A - P & * \\ RC + [P_{11} \ P_{12}]A & -P_{11} \end{bmatrix} < 0 \tag{11}$$

$$\Theta_1 = \begin{bmatrix} 0 & 0 \\ 0 & P_{22} + \varepsilon^2 P_{11} - \varepsilon P_{12} - \varepsilon P_{12}^T \end{bmatrix}, \tag{12}$$

$$\Theta_2 = \begin{bmatrix} 0 & 0 \\ 0 & P_{22} - \varepsilon \begin{bmatrix} P_{12} \\ 0 \end{bmatrix} - \varepsilon [P_{12}^T \ 0] + \varepsilon^2 \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix},$$

$$\Theta_3 = \begin{bmatrix} 0 & 0 \\ 0 & P_{22} - \varepsilon P_{12}^{(1)T} - \varepsilon P_{12}^{(1)} + \varepsilon^2 P_{11}^{(1)} \end{bmatrix}.$$

Furthermore, a static output feedback controller gain is given by

$$K = P_{11}^{-1} R \tag{13}$$

Proof: Noting that

$$(BKC)^T P (BKC) = C^T K^T P_{11} K C,$$

$$PBKC = \begin{bmatrix} P_{11} \\ P_{12}^T \end{bmatrix} KC$$

(5) is equivalent to

$$\begin{array}{l} ([P_{11}\ P_{12}]A + P_{11}KC)^T P_{11}^{-1}([P_{11}\ P_{12}]A + P_{11}KC) \\ -A^T \begin{bmatrix} P_{11} \\ P_{12}^T \end{bmatrix} P_{11}^{-1}[P_{11}\ P_{12}]A + A^T PA - P < 0 \end{array} \tag{14}$$

Considering that

$$P - \begin{bmatrix} P_{11} \\ P_{12}^T \end{bmatrix} P_{11}^{-1}[P_{11}\ P_{12}] = \begin{bmatrix} 0 & 0 \\ 0 & P_{22} - P_{12}^T P_{11}^{-1} P_{12} \end{bmatrix}$$

For the first situation $m = n - m$, consider the following inequality:

$$(P_{12} - \varepsilon P_{11})^T P_{11}^{-1}(P_{12} - \varepsilon P_{11}) \geq 0 \tag{15}$$

or equivalently

$$P_{12}^T P_{11}^{-1} P_{12} \geq \varepsilon P_{12}^T + \varepsilon P_{12} - \varepsilon^2 P_{11} \tag{16}$$

(14) is equivalent to

$$\begin{bmatrix} A^T \Theta_0 A - P & * \\ P_{11}KC + [P_{11}\ P_{12}]A & -P_{11} \end{bmatrix} < 0 \tag{17}$$

where

$$\Theta_0 = \begin{bmatrix} 0 & 0 \\ 0 & P_{22} - P_{12}^T P_{11}^{-1} P_{12} \end{bmatrix}$$

Using the fact (16), we have $\Theta_0 \leq \Theta_1$, and consequentially, $\Phi(\Theta_0) \leq \Phi(\Theta_1)$. Hence if (11) is satisfied, (5) is satisfied as well.

For the second situation, let the inequality

$$\left(\begin{bmatrix} P_{12} \\ 0 \end{bmatrix} - \varepsilon \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} \right)^T \begin{bmatrix} P_{11} & 0 \\ 0 & I \end{bmatrix}^{-1} \left(\begin{bmatrix} P_{12} \\ 0 \end{bmatrix} - \varepsilon \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} \right) \geq 0 \tag{18}$$

where $\begin{bmatrix} P_{12} \\ 0 \end{bmatrix} \in \mathcal{R}^{(n-m)\times(n-m)}$ and $\begin{bmatrix} P_{11} & 0 \\ 0 & I \end{bmatrix} \in \mathcal{R}^{(n-m)\times(n-m)}$. Note that (18) is equivalent to

$$P_{12}^T P_{11}^{-1} P_{12} \geq \varepsilon \begin{bmatrix} P_{12} \\ 0 \end{bmatrix}^T + \varepsilon \begin{bmatrix} P_{12} \\ 0 \end{bmatrix} - \varepsilon^2 \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} \tag{19}$$

For the third situation, noting that

$$(\begin{bmatrix} P_{12} & 0 \end{bmatrix} - \varepsilon P_{11})^T P_{11}^{-1}(\begin{bmatrix} P_{12} & 0 \end{bmatrix} - \varepsilon P_{11}) \geq 0 \tag{20}$$

we have

$$\begin{bmatrix} P_{12}^T \\ 0 \end{bmatrix} P_{11}^{-1} \begin{bmatrix} P_{12} & 0 \end{bmatrix} \geq \varepsilon \begin{bmatrix} P_{12}^T \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} P_{12} & 0 \end{bmatrix} - \varepsilon^2 P_{11} \tag{21}$$

(21) implies

$$P_{12}^T P_{11}^{-1} P_{12} \geq \varepsilon P_{12}^{(1)T} + \varepsilon P_{12}^{(1)} - \varepsilon^2 P_{11}^{(1)} \tag{22}$$

Hence we complete the proof.

Remark 1. *If $\varepsilon \equiv 0$ is set , then Theorem 1 recovers the result stated in (Bara & Boutayeb, 2006). We shall note that ε actually plays an important role in the scaling LMI formulation in Theorem 1. If $\varepsilon \equiv 0$, Theorem 1 implies $A_{22}^T P_{22} A_{22} - P_{22} < 0$ and $P_{22} > 0$, i.e., the system matrix A_{22} must be Schur stable, which obviously is an unnecessary condition and limits the application of this LMI formulation. However, with the aid of ε, we relax this constraint. A searching routine, such as fminsearch (simplex search method) in Matlab ©, can be applied to the following optimization problem (for a fixed ε, we have an LMI problem):*

$$\min_{\varepsilon,P,R} \lambda I, \; s.t. \; \Phi(\Theta) < \lambda I \tag{23}$$

The conservatism of Theorem 1 lies in these relaxations (15) or (16) on (5). To further relax the conservatism, we may choose a diagonal matrix $\triangle = diag\{\varepsilon_1, ..., \varepsilon_m\}, \varepsilon_i \geq 0$, instead of the single scalar ε. For example,

$$P_{12}^T P_{11}^{-1} P_{12} \geq P_{12}^T \triangle + \triangle P_{12} - \triangle P_{11} \triangle \tag{24}$$

Then we shall search the optimal value over multiple scalars for (23).

Remark 2. *In (Bara & Boutayeb, 2006), a different variable replacement is given:*

$$P_2 = P_{22} - P_{12}^T P_{11}^{-1} P_{12} \tag{25}$$

in (8). However, it is easily proved that these two transformations actually are equivalent. In fact, in (8), we have $P_{11} > 0$ and $P_2 > 0$ since $P > 0$. Based on (17), we have

$$\begin{bmatrix} A^T \begin{bmatrix} 0 & 0 \\ 0 & P_2 \end{bmatrix} A - \Lambda_0 & * \\ P_{11} K C + [P_{11} \; P_{12}] A & -P_{11} \end{bmatrix} < 0 \tag{26}$$

where

$$\Lambda_0 = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_2 + P_{12}^T P_{11}^{-1} P_{12} \end{bmatrix} = P \tag{27}$$

Hence, for the above three situations, we have an alternative condition, which is stated in the following lemma.

Theorem 2. *The discrete-time system (1)-(2) is stabilized by (3) if there exist $P_{11} > 0$, $P_2 > 0$, P_{12} and R with P defined in (27), such that*

$$\begin{cases} \Upsilon(\Lambda_1) < 0, \; m = n - m \\ \Upsilon(\Lambda_2) < 0, \; m < n - m \\ \Upsilon(\Lambda_3) < 0, \; m > n - m \end{cases} \tag{28}$$

where $\varepsilon \in \mathcal{R}$,

$$\Upsilon(\Lambda_i) = \begin{bmatrix} A^T \begin{bmatrix} 0 & 0 \\ 0 & P_2 \end{bmatrix} A - \Lambda_i & * \\ RC + [P_{11} \; P_{12}] A & -P_{11} \end{bmatrix},$$

$$\Lambda_1 = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_2 - \varepsilon^2 P_{11} + \varepsilon P_{12} + \varepsilon P_{12}^T \end{bmatrix},$$

$$\Lambda_2 = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_2 + \varepsilon \begin{bmatrix} P_{12} \\ 0 \end{bmatrix} + \varepsilon [P_{12}^T \; 0] - \varepsilon^2 \begin{bmatrix} P_{11} & 0 \\ 0 & I \end{bmatrix} \end{bmatrix},$$

$$\Lambda_3 = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_2 + \varepsilon P_{12}^{(1)T} + \varepsilon P_{12}^{(1)} - \varepsilon^2 P_{11}^{(1)} \end{bmatrix}.$$

Furthermore, a static output controller gain is given by (13).

Proof: We only consider the first case. Replacing P_2 and R by P_{22} and K using (25) and (13), we can derive that (28) is a sufficient condition for (5) with the P defined in (8).

3.2 C_o with full row-rank

When C_o is full row rank, there exists a nonsingular matrix T_c such that $C_o T_o^{-1} = [I_l \; 0]$. Applying a similarity transformation to the system (1)-(2), the closed-loop system (4) is stable if and only if

$$\tilde{A}_c = A + BKC \text{ is stable}$$

where $A = T_c A_o T_c^{-1}$, $B = T_c B_o$ and $C = C_o T_c^{-1} = [I_l \; 0]$.
Similarly to Section 3.1, we can also divide this problem into three situations: $l = n - l$, $l < n - l$ and $l > n - l$. We use the condition (6) here and partition Q as $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}$, where $Q_{11} \in \mathcal{R}^{l \times l}$.

Theorem 3. *The discrete-time system (1)-(2) is stabilized by (3) if there exist $Q > 0$ and R, such that*

$$\begin{cases} \Gamma(\bar{\Theta}_1) < 0, \; l = n - l \\ \Gamma(\bar{\Theta}_2) < 0, \; l < n - l \\ \Gamma(\bar{\Theta}_3) < 0, \; l > n - l \end{cases} \quad (29)$$

where $\varepsilon \in \mathcal{R}$,

$$\Gamma(\bar{\Theta}_i) = \begin{bmatrix} A\bar{\Theta}_i A^T - Q & * \\ (A[Q_{11} \; Q_{12}]^T + BR)^T & -Q_{11} \end{bmatrix},$$

$$\bar{\Theta}_1 = \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} + \varepsilon^2 Q_{11} - \varepsilon Q_{12} - \varepsilon Q_{12}^T \end{bmatrix},$$

$$\bar{\Theta}_2 = \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} + \varepsilon^2 \begin{bmatrix} Q_{11} & 0 \\ 0 & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} Q_{12} \\ 0 \end{bmatrix} - \varepsilon [Q_{12}^T \; 0] \end{bmatrix},$$

$$\bar{\Theta}_3 = \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} + \varepsilon^2 Q_{11}^{(1)} - \varepsilon Q_{12}^{(1)T} - \varepsilon Q_{12}^{(1)} \end{bmatrix},$$

$Q_{11}^{(1)}$ and $Q_{12}^{(1)}$ are properly dimensioned partitions of Q_{11} and Q_{12}. Furthermore, a static output feedback controller gain is given by

$$K = RQ_{11}^{-1} \quad (30)$$

Proof: We only prove the first case $l = n - l$, since the others are similar. Noting that $(BKC)Q(BKC)^T = BKQ_{11}K^TB$ and $BKCQ = BK[Q_{11}\ Q_{12}]$, (6) is equivalent to

$$\begin{aligned}(A[Q_{11}\ Q_{12}]^T + BKQ_{11})Q_{11}^{-1}(A[Q_{11}\ Q_{12}]^T + BKQ_{11})^T \\ -A\begin{bmatrix} Q_{11} \\ Q_{12}^T \end{bmatrix} Q_{11}^{-1}[Q_{11}\ Q_{12}]A^T + AQA^T - Q < 0\end{aligned} \tag{31}$$

Using the fact that

$$Q - \begin{bmatrix} Q_{11} \\ Q_{12}^T \end{bmatrix} Q_{11}^{-1}[Q_{11}\ Q_{12}] = \begin{bmatrix} 0 & 0 \\ 0 & Q_{12}^T Q_{11}^{-1} Q_{12} \end{bmatrix}$$

we infer that stability of the close-loop system is equivalent to the existing of a $Q > 0$ such that

$$\begin{bmatrix} A\bar{\Theta}_0 A^T - Q & * \\ (A[Q_{11}\ Q_{12}] + BKQ_{11})^T & -Q_{11} \end{bmatrix} < 0 \tag{32}$$

where

$$\bar{\Theta}_0 = \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} - Q_{12}^T Q_{11}^{-1} Q_{12} \end{bmatrix}$$

Since

$$(Q_{12} - \varepsilon Q_{11})^T Q_{11}^{-1}(Q_{12} - \varepsilon Q_{11}) \geq 0 \tag{33}$$

or equivalently,

$$Q_{12}^T Q_{11}^{-1} Q_{12} \geq \varepsilon Q_{12}^T + \varepsilon Q_{12} - \varepsilon^2 Q_{11} \tag{34}$$

It follows that (29) implies (32). Hence we complete the proof.

Remark 3. *How to compare the conditions in Theorem 3 and Theorem 1 remains a difficult problem. In the next section, we only give some experiential results based on numerical simulations, which give some suggestions on the dependence of the results with respect to m and l.*

3.3 Transformation-dependent LMIs

The result in this subsection builds a connection between the sets $\mathcal{L}$, $\mathcal{K}_c$, $\mathcal{K}_o$, $\tilde{\mathcal{K}}_c$ and $\tilde{\mathcal{K}}_o$, which are defined as follows. Without causing confusion, we omit the subscript $_o$ for A_o, B_o and C_o in this subsection.

$$\mathcal{L} = \{K \in \mathcal{R}^{m \times l} : \bar{A}\ stable\} \tag{35}$$

i.e., the set of all admissible output feedback matrix gains;

$$\mathcal{K}_c = \{K_c \in \mathcal{R}^{m \times n} : A + BK_c\ \ stable\} \tag{36}$$

i.e., the set of all admissible state feedback matrix gains;

$$\mathcal{K}_o = \{K_o \in \mathcal{R}^{n \times l} : A + K_oC\ \ stable\} \tag{37}$$

i.e., the set of all admissible observer matrix gains. Based on Lemma 1, we can easily formulate the LMI solution for sets $\mathcal{K}_c$ and $\mathcal{K}_o$. In fact, they are equivalent to following two sets respectively:

$$\tilde{\mathcal{K}}_c = \{K_c = W_{c2}W_{c1}^{-1} \in \mathcal{R}^{m \times n} : (W_{c1}, W_{c2}) \in \mathcal{W}_c\} \tag{38}$$

and

$$\mathcal{W}_c = \{W_{c1} \in \mathcal{R}^{n\times n}, W_{c2} \in \mathcal{R}^{m\times n} : W_{c1} > 0, \Psi_c < 0\} \tag{39}$$

where $\Psi_c = \begin{bmatrix} -W_{c1} & AW_{c1} + BW_{c2} \\ W_{c1}A^T + W_{c2}^T B^T & -W_{c1} \end{bmatrix}$.

$$\tilde{\mathcal{K}_o} = \{K_o = W_{o1}^{-1} W_{o2} \in \mathcal{R}^{n\times l} : (W_{o1}, W_{o2}) \in \mathcal{W}_o\} \tag{40}$$

and

$$\mathcal{W}_o = \{W_{o1} \in \mathcal{R}^{n\times n}, W_{o2} \in \mathcal{R}^{n\times l} : W_{o1} > 0, \Psi_o < 0\} \tag{41}$$

where $\Psi_o = \begin{bmatrix} -W_{1o} & W_{o1}A + W_{o2}C \\ A^T W_{o1} + C^T W_{o2}^T & -W_{1o} \end{bmatrix}$.

Lemma 2. *$\mathcal{L} \neq \emptyset$ if and only if*

1. *$\bar{\mathcal{K}}_c = \mathcal{K}_c \bigcap \{K_c : K_c Y_c = 0,\ Y_c = \mathcal{N}(C)\} \neq \emptyset$; or*
2. *$\bar{\mathcal{K}}_o = \mathcal{K}_o \bigcap \{K_c : Y_o K_o = 0,\ Y_o = \mathcal{N}(B')\} \neq \emptyset$.*

In the affirmative case, any $K \in \mathcal{L}$ can be rewritten as

1. *$K = K_c Q C^T (CQC^T)^{-1}$; or*
2. *$K = (B^T PB)^{-1} B^T PK_o$.*

where $Q > 0$ and $P > 0$ are arbitrarily chosen.

Proof: The first statement has been proved in Geromel et al. (1996). For complement, we give the proof of the second statement. The necessity is obvious since $K_o = BK$. Now we prove the sufficiency, i.e., given $K_o \in \bar{\mathcal{K}}_o$, there exists a K, such that the constraint $K_o = BK$ is solvable. Note that for $\forall P > 0$, $\Theta_o = \begin{bmatrix} B^T P \\ Y_o^T \end{bmatrix}$ is full rank, where $Y_o = \mathcal{N}(B^T)$. In fact, $rank(\Theta_o Y_o) = rank(\begin{bmatrix} B^T PY_o \\ I_{n-m} \end{bmatrix}) \geq n - m$. Multiplying Θ_o at the both side of $K_o = BK$ we have

$$\begin{bmatrix} B^T PK_o \\ Y_o^T K_o \end{bmatrix} = \begin{bmatrix} B^T PBL \\ 0 \end{bmatrix}$$

Since $B^T PB$ is invertible, we have $K = (B^T PB)^{-1} B^T PK_0$. Hence, we can derive the result.

Lemma 3. *$\mathcal{L} \neq \emptyset$ if and only if there exists $E_c \in \mathcal{R}^{n\times(n-l)}$ or $E_o \in \mathcal{R}^{n\times(n-m)}$, such that one of the following conditions holds:*

1. *$rank(T_c = \begin{bmatrix} C \\ E_c^T \end{bmatrix}) = n$ and $\mathcal{C}(E_c) \neq \emptyset$; or*
2. *$rank(T_o = \begin{bmatrix} B & E_o \end{bmatrix}) = n$ and $\mathcal{O}(E_o) \neq \emptyset$.*

where

$$\mathcal{C}(E_c) = \mathcal{W}_c \bigcap \{(W_{c1}, W_{c2}) : CW_{c1}E_c = 0, W_{c2}E_c = 0\}$$

$$\mathcal{O}(E_o) = \mathcal{W}_o \bigcap \{(W_{o1}, W_{o2}) : B^T W_{o1} E_o = 0, E_o^T W_{o2} = 0\}$$

In the affirmative case, any $K \in \mathcal{L}$ can be rewritten as

1. *$K = W_{c2} C^T (CW_{c1}C^T)^{-1}$; or*
2. *$K = (B^T W_{o1} B)^{-1} B^T W_{o2}$.*

Proof: We only prove the statement 2, since the statement 1 is similar. For the necessity, if there exist $K \in \mathcal{L}$, then it shall satisfy Lemma 1. Now we let

$$W_{o1} = P,\ W_{o2} = PBK$$

Choose $E_o = P^{-1}Y_o$, $Y_o = \mathcal{N}(B^T)$. It is known that $\begin{bmatrix} B & E_o \end{bmatrix}$ is full rank. Then we have

$$B^T W_{o1} E = B^T Y_o = 0, E^T W_{o2} = Y_o^T BK = 0$$

For sufficiency, we assume there exists E_o such that the statement 2) is satisfied. Notice that $W_{o1} > 0$ and the item W_{o2} in Ψ_o can be rewritten as $W_{o1}W_{o1}^{-1}W_{o2}$.

$$W_{o1}^{-1}W_{o2} = T_o(T_o^T W_{o1} T_o)^{-1} T_o^T W_{o2} = B(B^T W_{o1} B)^{-1} B^T W_{o2} \tag{42}$$

since T_o is invertible and $B^T W_{o1} E = 0, E^T W_{o2} = 0$. Hence, $W_{o1}^{-1}W_{o2}$ can be factorized as BK, where $K = (B^T W_{o1} B)^{-1} B^T W_{o2}$. Now we can derive (5) from the fact $\Psi_o < 0$. Thus we complete the proof.

Remark 4. *For a given T_o, since $T_o^{-1}T_o = I_n$, $T_o^{-1}B = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$ and $T_o^{-1}E = \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix}$. Similarly, For a given T_c, $CT_c^{-1} = \begin{bmatrix} I_l & 0 \end{bmatrix}$.*

Theorem 4. *$\mathcal{L} \neq \emptyset$ if and only if there exists T_c or T_o, such that one of the following conditions holds:*

1.

$$\tilde{\mathcal{W}}_c \neq \emptyset,\ \tilde{\mathcal{W}}_c = \{\hat{W}_{c1} \in \mathcal{R}^{n\times n}, \hat{W}_{c2} \in \mathcal{R}^{m\times n} : \hat{W}_{c1} > 0, \Phi_c < 0\} \tag{43}$$

where

$$\hat{A} = T_c A T_c^{-1},\ \hat{B} = T_c B, \hat{W}_{c1} = \begin{bmatrix} W_{c11} & 0 \\ 0 & W_{c22} \end{bmatrix},$$

and

$$\hat{W}_{c2} = \begin{bmatrix} W_{c21} & 0 \end{bmatrix}, \quad W_{c11} \in \mathcal{R}^{l\times l}, \quad W_{c22} \in \mathcal{R}^{(n-l)\times(n-l)}, \quad W_{c21} \in \mathcal{R}^{m\times l},$$

$$\Phi_c = \begin{bmatrix} -\hat{W}_{c1} & \hat{A}\hat{W}_{c1} + \hat{B}\hat{W}_{c2} \\ \hat{W}_{c1}\hat{A}^T + \hat{W}_{c2}^T \hat{B}^T & -\hat{W}_{c1} \end{bmatrix};$$

2.

$$\check{\mathcal{W}}_o \neq \emptyset,\ \check{\mathcal{W}}_o = \{\check{W}_{o1} \in \mathcal{R}^{n\times n}, \check{W}_{o2} \in \mathcal{R}^{n\times r} : \check{W}_{o1} > 0, \Phi_o < 0\} \tag{44}$$

where

$$\check{A} = T_o^{-1} A T_o,\ \check{C} = CT_o, \quad \check{W}_{o1} = \begin{bmatrix} W_{o11} & 0 \\ 0 & W_{o22} \end{bmatrix},$$

and

$$\check{W}_{o2} = \begin{bmatrix} W_{o21} \\ 0 \end{bmatrix},$$

$$W_{o11} \in \mathcal{R}^{m\times m}, \quad W_{o22} \in \mathcal{R}^{(n-m)\times(n-m)}, \quad W_{o21} \in \mathcal{R}^{m\times r},$$

$$\Phi_o = \begin{bmatrix} -\check{W}_{o1} & \check{W}_{o1}\check{A} + \check{W}_{o2}\check{C} \\ \check{A}^T\check{W}_{o1} + \check{C}^T\check{W}_{o2}^T & -\check{W}_{o1} \end{bmatrix}.$$

In the affirmative case, any $K \in \mathcal{L}$ can be rewritten as

1. $K = W_{c21}W_{c11}^{-1}$; or

2. $K = W_{o11}^{-1} W_{o21}$.

Proof: We also only consider the statement 2) here. The sufficiency is obvious according to Lemma 3, hence, we only prove the necessity.
Note that

$$\begin{bmatrix} -\check{W}_{o1} & \check{W}_{o1}\check{A} + \check{W}_{o2}\check{C} \\ \check{A}^T\check{W}_{o1} + \check{C}^T\check{W}_{o2}^T & -\check{W}_{o1} \end{bmatrix} = \mathcal{T}_o^T \begin{bmatrix} -W_{o1} & W_{o1}A + W_{o2}C \\ A^T W_{o1} + C^T W_{o2}^T & -W_{o1} \end{bmatrix} \mathcal{T}_o$$

where $\mathcal{T}_o = \begin{bmatrix} T_o & 0 \\ 0 & T_o \end{bmatrix}$. Hence, we can conclude that

$$\check{W}_{o1} = T_o^T W_{o1} T_o,\ \check{W}_{02} = T_o^T W_{o2}$$

Since the system matrices also satisfy

$$B^T W_{o1} E = 0, E^T W_{o2} = 0$$

which implies

$$B^T T_o^{-T} \check{W}_{o1} T_o^{-1} E = 0,\ E^T T_o^{-T} \check{W}_{o2} = 0 \tag{45}$$

Let

$$\check{W}_{o1} = \begin{bmatrix} W_{o11} & W_{o12} \\ W_{o12}^T & W_{o22} \end{bmatrix},\ \check{W}_{o2} = \begin{bmatrix} W_{o21} \\ W_{o23} \end{bmatrix}$$

With the conclusion from Remark 4, (45) implies

$$W_{o12} = 0,\ \ W_{o23} = 0$$

Hence we have the structural constraints on $\check{W}_{o1}$ and $\check{W}_{o2}$. Using the results of Lemma 3, we can easily get the controller L. Thus we complete the proof.

Remark 5. *The first statements of Lemma 3 and Theorem 4 are corollaries of the results in Geromel, de Souze & Skelton (1998); Geromel et al. (1996). Based on Theorem 4, we actually obtain a useful LMI algorithm for output feedback control design of general LTI systems with fixed E_c and/or E_o. For these LTI systems, we can first make a similarity transformation that makes $C = [I\ 0]$ (or $B^T = [I\ 0]$). Then we force the W_{c1} and W_{c2} (or W_{o1} and W_{o2}) to be constrained structure shown in Theorem 4. If the corresponding LMIs have solution, we may conclude that the output feedback gain exists; otherwise, we cannot make a conclusion, as the choice of E_c or E_o is simply a special case. Thus we can choose a scaled E_c or E_o, i.e., ϵE_c or ϵE_o to perform a one-dimensional search, which converts the LMI condition in Theorem 4 a scaling LMI. For example, Φ_c in (43) should be changed as $\Phi_c = \begin{bmatrix} -\hat{W}_{1c} & A\hat{W}_{c1} + \varepsilon B\hat{W}_{c2} \\ \hat{W}_{c1}A^T + \varepsilon\hat{W}_{c2}^T B^T & -\hat{W}_{1c} \end{bmatrix}$.*

All the approaches in this section require similarity transformation, which can be done by some techniques, such as the singular value decomposition (SVD). However, those transformations often bring numerical errors, which sometimes leads to some problems for the marginal solutions. Hence in the next section, using Finsler's lemma, we introduce some methods without the pretreatment on system matrices.

4. Scaling LMIs without similarity transformation

Finsler's Lemma has been applied in many LMI formulations, e.g., (Boyd et al., 1994; Xu et al., 2004). With the aid of Finsler's lemma, we can obtain scaling LMIs without similarity transformation.

Lemma 4. *(Boyd et al., 1994) The following expressions are equivalent:*

1. $x^T Ax > 0$ *for* $\forall x \neq 0$, *subject to* $Bx = 0$;
2. $B^{\perp T} AB^{\perp} > 0$, *where* $B^{\perp}$ *is the kernel of* B^T, *i.e.,* $B^{\perp} B^T = 0$;
3. $A + \sigma B^T B > 0$, *for some scale* $\sigma \in R$;
4. $A + XB + B^T X^T > 0$, *for some matrix* X.

In order to apply Finsler's lemma, several manipulation on the Lyapunov inequalities should be done first. Note that the condition (5) actually states $V(x(t)) = x^T(t)Px(t) > 0$ and $\Delta V(x) = V(x(t+1)) - V(x(t)) < 0$. The latter can be rewritten as

$$\xi^T \mathcal{P} \xi < 0,\ \xi = [x^T(t)\ x^T(t+1)]^T, \mathcal{P} = \begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix} \tag{46}$$

Define $\zeta = [x^T\ u^T]^T$. It is easy to verify:

$$\xi = M_p \zeta \tag{47}$$

$$[K\ \ -1] N_p \zeta = 0 \tag{48}$$

where

$$M_p = \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}, N_p = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \tag{49}$$

That is

$$(46)\text{ s.t. }(47)\text{-}(48) \tag{50}$$

Now based on the statements 1) and 4) of Finsler's Lemma, we can conclude that (50) is equivalent to

$$M_p^T \mathcal{P} M_p + N_p^T \begin{bmatrix} K^T \\ -I \end{bmatrix} \mathcal{X}^T + \mathcal{X} \begin{bmatrix} K & -I \end{bmatrix} N_p < 0 \tag{51}$$

for some $\mathcal{X}$. Now we let

$$\mathcal{X}^T = [\varepsilon \tilde{Z}^T\ Z^T] \tag{52}$$

where ε is a given real scalar, $Z = [z_1^T, z_2^T, \cdots, z_m^T]^T \in \mathcal{R}^{m\times m}$ and $\tilde{Z} \in \mathcal{R}^{n\times m}$. Note that $\tilde{Z}$ is constructed from Z with n rows drawing from Z, i.e., $\tilde{Z} = [z_{\tilde{1}}^T, z_{\tilde{2}}^T, \cdots, z_{\tilde{n}}^T]^T$, where $z_{\tilde{i}}^T$, $1 \leq \tilde{i} \leq m$ is a vector from Z. Since $n \geq m$, there are some same vectors in $\tilde{Z}$. Now we define

$$W = ZK = [w_1^T, w_2^T, \cdots, w_m^T]^T \tag{53}$$

and

$$\tilde{W} = \tilde{Z}K = [w_{\tilde{1}}^T, w_{\tilde{2}}^T, \cdots, w_{\tilde{n}}^T]^T \tag{54}$$

where $w_{\tilde{i}}^T$, $1 \leq \tilde{i} \leq m$ is a vector from W. Then (51) can be transferred into following LMI:

$$M_p^T \mathcal{P} M_p + \begin{bmatrix} \varepsilon(C^T \tilde{W}^T + \tilde{W}C) & * \\ WC - \varepsilon \tilde{Z}^T & -(Z^T + Z) \end{bmatrix} < 0 \tag{55}$$

Since $Z^T + Z > B^T P B \geq 0$, Z is invertible, $K = Z^{-1}W$.

Theorem 5. *The discrete-time system (1)-(2) is stabilized by (3) if there exist $P > 0$ and Z, W, such that (55) is satisfied for some scalar ε. Furthermore, the controller is given by $K = Z^{-1}W$.*

The conservatism lies in the construction of $\tilde{Z}$, which has to be a special structure. $\tilde{Z}$ can be further relaxed using a transformation $\tilde{Z} = \varepsilon\hat{Z}Z$, where $\hat{Z} \in \mathcal{R}^{n\times m}$ is a given matrix. In Theorem 5, the condition (5) is applied. Based on the condition (6), we have the following Lemma.

Theorem 6. *The discrete-time system (1)-(2) is stabilized by (3) if there exist $Q > 0$ and Z, W, such that*

$$M_q \mathcal{Q} M_q^T + \begin{bmatrix} \varepsilon(\tilde{W}^T B^T + B\tilde{W}) & * \\ W^T B^T - \varepsilon\tilde{Z} & -(Z^T + Z) \end{bmatrix} < 0 \tag{56}$$

where

$$M_q = \begin{bmatrix} I & A \\ 0 & C \end{bmatrix}, \mathcal{Q} = \begin{bmatrix} -Q & 0 \\ 0 & Q \end{bmatrix} \tag{57}$$

is satisfied for some scalar ε. Furthermore, the controller is given by $K = Z^{-1}W$.

Proof: The condition (6) can be rewritten as

$$\begin{bmatrix} I \\ (BK)^T \end{bmatrix}^T M_q \mathcal{Q} M_q^T \begin{bmatrix} I \\ (BK)^T \end{bmatrix} < 0 \tag{58}$$

Since $\begin{bmatrix} I \\ (BK)^T \end{bmatrix}^T \begin{bmatrix} (BK) \\ -I \end{bmatrix} = 0$, (58) can be rewritten as

$$\begin{bmatrix} (BK) \\ -I \end{bmatrix}^{\perp} M_q \mathcal{Q} M_q^T \begin{bmatrix} (BK) \\ -I \end{bmatrix}^{\perp T} < 0 \tag{59}$$

Now applying Finsler's lemma, we have

$$M_q \mathcal{Q} M_q^T + \begin{bmatrix} (BK) \\ -I \end{bmatrix} \mathcal{X} + \mathcal{X}^T \begin{bmatrix} (BK) \\ -I \end{bmatrix}^T < 0 \tag{60}$$

for some $\mathcal{X} = [\varepsilon\tilde{Z}\ Z]$. Similar to (52), we construct $\tilde{Z}$ from Z with its columns. Hence we have (56), which is a sufficient condition for (6). Thus we complete the proof.

Remark 6. *The proof of Theorem 6 is based on the equivalence between 1 and 2 of Finsler's lemma. It also provides an alterative proof of Theorem 5 if we note that (5) is equivalent to*

$$\begin{bmatrix} I \\ KC \end{bmatrix}^T M_p^T \mathcal{P} M_p \begin{bmatrix} I \\ KC \end{bmatrix} < 0 \tag{61}$$

Remark 7. *Except for the case that $m = 1$ for Theorem 5 and $l = 1$ for Theorem 6, the construction of $\tilde{Z}$ is a problem to be considered. So far, we have no systematic method for this problem. However, based on our experience, the choose of different vectors and their sequence do affect the result.*

The following simple result is the consequence of the equivalence of 1 and 3 in Finsler's Lemma.

Theorem 7. *The discrete-time system (1)-(2) is stabilized by (3) if there exist $P > 0$ and K, such that*

$$\begin{bmatrix} -P - \varepsilon\bar{A} - \varepsilon\bar{A}^T + \varepsilon^2 I & \bar{A}^T \\ \bar{A} & P - I \end{bmatrix} < 0 \tag{62}$$

where $\varepsilon \in \mathcal{R}$.

Proof: It is obvious that inequality (42) holds subject to $[\bar{A} - I]\xi = 0$. Now we apply the equivalence between 1 and 3 of Finsler's lemma and obtain

$$\mathcal{P} - \sigma[\bar{A} - I]\begin{bmatrix} \bar{A} \\ -I \end{bmatrix} = \begin{bmatrix} -P - \sigma\bar{A}^T\bar{A} & \sigma\bar{A} \\ \sigma\bar{A} & P - \sigma I \end{bmatrix} < 0 \tag{63}$$

for some $\sigma > 0$. Note that $-\bar{A}^T\bar{A} < -\varepsilon\bar{A}^T - \varepsilon\bar{A} + \varepsilon^2 I$, (63) can be implied by

$$\begin{bmatrix} -P + \sigma(-\varepsilon\bar{A}^T - \varepsilon\bar{A} + \varepsilon^2 I) & \sigma\bar{A}^T \\ \sigma\bar{A} & P - \sigma I \end{bmatrix} < 0 \tag{64}$$

By redefining P as $\frac{1}{\sigma}P$, we can obtain the result.

Remark 8. *Inequality (51) is also equivalent to*

$$M_p^T \mathcal{P} M_p - \sigma N_p^T \begin{bmatrix} K^T \\ -I \end{bmatrix} \begin{bmatrix} K & -I \end{bmatrix} N_p < 0 \tag{65}$$

for some positive scalar σ. Hence, we have

$$M_p^T \tilde{\mathcal{P}} M_p - N_p^T \begin{bmatrix} K^T \\ -I \end{bmatrix} \begin{bmatrix} K & -I \end{bmatrix} N_p < 0 \tag{66}$$

where $\tilde{\mathcal{P}} = \begin{bmatrix} -\tilde{P} & 0 \\ 0 & \tilde{P} \end{bmatrix}$, $\tilde{P} = \sigma^{-1}P$. Using the fact that $(K - K_0)^T(K - K_0) \geq 0$, we may obtain an iterative solution from initial condition K_0, where K_0 may be gotten from Lemma 5.

5. Comparison and examples

We shall note that the comparisons of some existing methods (Bara & Boutayeb, 2005; Crusius & Trofino, 1999; Garcia et al., 2001) with the case of $\varepsilon = 0$ in Theorem 1 has been given in (Bara & Boutayeb, 2006), where it states that there are many numerical examples for which Theorem 1 with $\varepsilon = 0$ works successfully while the methods in (Bara & Boutayeb, 2005; Crusius & Trofino, 1999; Garcia et al., 2001) do not and vice-versa. It also stands for our conditions. Hence, in the section, we will only compare these methods introduced above. The LMI solvers used here are SeDuMi (v1.3) Sturm et al. (2006) and SDPT3 (v3.4) Toh et al. (2006) with YALMIP Löfberg (2004) as the interface.

In the first example, we will show the advantage of the scaling LMI with ε compared with the non-scaling ones. In the second example, we will show that different scaling LMI approaches have different performance for different situations. As a by-product, we will also illustrate the different solvability of the different solvers.

Example 1. *Consider the unstable system as follows.*

$$A_o = \begin{bmatrix} 0.82 & 0.0576 & 0.2212 & 0.8927 & 0.0678 \\ 0.0574 & 0.0634 & 0.6254 & 0.0926 & 0.9731 \\ 0.0901 & 0.7228 & 0.5133 & 0.2925 & 0.9228 \\ 0.6967 & 0.0337 & 0.5757 & 0.8219 & 0.9587 \\ 0.1471 & 0.6957 & 0.2872 & 0.994 & 0.5632 \end{bmatrix}$$

$$B_o = \begin{bmatrix} 0.9505 & 0.2924 \\ 0.3182 & 0.4025 \\ 0.2659 & 0.0341 \\ 0.0611 & 0.2875 \\ 0.3328 & 0.2196 \end{bmatrix}$$

$$C_o = \begin{bmatrix} 0.5659 & 0.255 & 0.5227 & 0.0038 & 0.3608 \\ 0.8701 & 0.5918 & 0.1291 & 0.3258 & 0.994 \end{bmatrix}$$

This example is borrowed from (Bara & Boutayeb, 2006), where output feedback controllers have been designed. For A_{22} from A, it has stable eigenvalue. In this paper, we compare the design problem with the maximum decay rate, i.e.,

$$\max \rho \quad s.t. \;\; \tilde{A}^T P \tilde{A} - P < -\rho P$$

Note that in this example, $m < n - m$. With $\varepsilon = 0$, i.e., using the method in (Bara & Boutayeb, 2006), we obtain the maximum $\rho = 0.16$, while Theorem 1 gives $\rho = 0.18$ with $\varepsilon = -0.09$. However, Theorem 5 only obtains a maximum $\rho = 0.03$ with a choice of $\hat{Z} = [I_2 \; I_2 \; 0]^T$. Note that the solvability heavily depends on the choice of ε. For example, when $\varepsilon = 0.09$ for Theorem 1, the LMI is not feasible.

Now we consider a case that A_{22} has an unstable eigenvalue. Consider the above example with slight changes on A_o

$$A_o = \begin{bmatrix} 0.9495 & 0.12048 & 0.14297 & 0.19192 & 0.019139 \\ 0.8656 & 0.28816 & 0.67152 & 0.01136 & 0.38651 \\ 0.5038 & 0.46371 & 0.9712 & 0.93839 & 0.42246 \\ 0.13009 & 0.76443 & 0.47657 & 0.54837 & 0.4089 \\ 0.34529 & 0.61187 & 0.15809 & 0.46639 & 0.53536 \end{bmatrix}$$

We can easily verify that A_{22} from A has one unstable eigenvalue 1.004. Hence, the method in (Bara & Boutayeb, 2006) cannot solve it. However, Theorem 1 generates a solution as $K = \begin{bmatrix} -0.233763 & -0.31506 \\ -3.61207 & 0.376493 \end{bmatrix}$. Meanwhile, Theorem 5 also can get a feasible solution for $\varepsilon = -0.1879$ and $K = \begin{bmatrix} 0.9373 & -0.4008 \\ 1.5244 & -0.7974 \end{bmatrix}$. Theorem 4 via a standard SVD without scaling can also obtain $K = \begin{bmatrix} -0.3914 & -0.3603 \\ -2.3604 & -1.1034 \end{bmatrix}$ using (43) or $K = \begin{bmatrix} 1.4813 & 0.5720 \\ -3.7203 & -1.8693 \end{bmatrix}$ using (44).

Example 2. *We randomly generate 5000 stabilizable and detectable systems of dimension* $n = 4(6,6,6,7,7)$, $m = 2(3,1,5,4,3)$ *and* $l = 2(3,5,1,3,4)$.

	T 1	T 3
SeDuMi	5000	4982
SDPT3	4975	5000

Table 1. Different solvability of different solvers

T 1[α]	T 3	4.2.2[β]	6.3.3	6.1.5	6.5.1	7.4.3	7.3.4
Y	Y	4999	4999	4994	4996	4998	4998
Y	N	1	0	2	3	1	1
N	Y	0	1	4	1	1	1
N	N	0	0	0	0	0	0

Superscript[γ]: Y (N) means that the problem can (not) be solved by the corresponding theorems. For example, the value 4 of third row and third column means that in the random 5000 examples, there are 4 cases that cannot be solved by Theorem 1 while can be solved by Theorem 3.

Table 2. Comparison of Theorem 1 and Theorem 3

Hence we can use Theorem 1 and Theorem 3 with $\varepsilon = 0$ to solve this problem. Note that different solvers may give different solvability. For example, given $n = 6$, $m = 3$ and $l = 3$, in a one-time simulation, the result is given in Table 1. Thus in order to partially eliminate the effect of the solvers, we choose the combined solvability result from two solvers in this section.

Table 2 shows the comparison of Theorem 1 and Theorem 3. Some phenomenons (the solvability of Theorem 1 and Theorem 3 depends on the l and m. When $m > l$, Theorem 1 tends to have a higher solvability than Theorem 3. And vise verse.) was observed from these results obtained using LMITOOLS provided by Matlab is not shown here.

6. Extension to H_∞ synthesis

The aforementioned results can contribute to other problems, such as robust control. In this section, we extend it to H_∞ output feedback control problem. Consider the following system:

$$x(t+1) = Ax(t) + B_2u(t) + B_1w \tag{67}$$

$$y(t) = Cx(t) + Dw \tag{68}$$

$$z(t) = Ex(t) + Fw \tag{69}$$

We only consider the case that B_2 is with full rank and assume that the system has been transferred into the form like (7). Using the controller as (3), the closed-loop system is

$$\begin{aligned} x(t+1) &= \hat{A}x(t) + \hat{B}w \\ &= (A + B_2KC)x(t) + (B_1 + B_2KD)w \end{aligned} \tag{70}$$

We attempt to design the controller, such that the L_2 gain $\sup \frac{\|z\|_2}{\|w\|_2} \leq \gamma$. It should be noted that all the aforementioned scaling LMI approaches can be applied here. However, we only choose one similar to Theorem 1.

Theorem 8. *The discrete-time system (67)-(69) is stabilized by (3) and satisfies H_∞, if there exist a matrix $P > 0$ defined in (8) and R, such that*

$$\begin{cases} \Re(\Theta_1) < 0, \; m = n - m \\ \Re(\Theta_2) < 0, \; m < n - m \\ \Re(\Theta_3) < 0, \; m > n - m \end{cases} \tag{71}$$

where $\varepsilon \in \mathcal{R}$, Θ_i is defined in Theorem 1,

$$\Re(\Theta_i) = \begin{bmatrix} -P_{11} & RC + [P_{11}\; P_{12}]A & RD + [P_{11}\; P_{12}]B_1 & 0 \\ * & A^T\Theta_i A - P & A^T\Theta_i B_1 & E^T \\ * & * & B_1^T\Theta_i B - \gamma I & F^T \\ * & * & * & -\gamma I \end{bmatrix} \tag{72}$$

Proof: Following the arguments in Theorem 1, we can see that (71) implies

$$\Re(\Theta_i) = \begin{bmatrix} \hat{A}^T P \hat{A} - P & \hat{A}^T P \hat{B} & E^T \\ * & \hat{B}^T P \hat{B} - \gamma I & F^T \\ * & * & -\gamma I \end{bmatrix} < 0 \tag{73}$$

Using bounded real lemma (Boyd et al., 1994), we can complete the proof.

7. Conclusion

In this paper, we have presented some sufficient conditions for static output feedback control of discrete-time LTI systems. Some approaches require a similarity transformation to convert B or C to a special form such that we can formulate the design problem into a scaling LMI problem with a conservative relaxation. Based on whether B or C is full rank, we consider several cases with respect to the system state dimension, output dimension and input dimension. These methods are better than these introduced in (Bara & Boutayeb, 2006) and might achieve statistical advantages over other existing results (Bara & Boutayeb, 2005; Crusius & Trofino, 1999; Garcia et al., 2001). The other approaches apply Finsler's lemma directly such that the Lyapunov matrix and the controller gain can be separated, and hence gain benefits for the design. All the presented approaches can be extended to some other problems. Note that we cannot conclude that the approaches presented in this paper is definitely superior to all the existing approaches, but introduce some alternative conditions which may achieve better performance than others in some circumstances.

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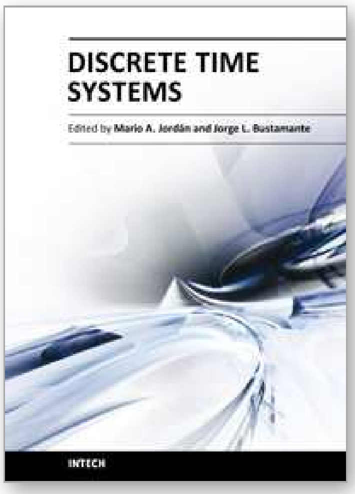

Discrete Time Systems

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Discrete-Time Systems comprehend an important and broad research field. The consolidation of digital-based computational means in the present, pushes a technological tool into the field with a tremendous impact in areas like Control, Signal Processing, Communications, System Modelling and related Applications. This book attempts to give a scope in the wide area of Discrete-Time Systems. Their contents are grouped conveniently in sections according to significant areas, namely Filtering, Fixed and Adaptive Control Systems, Stability Problems and Miscellaneous Applications. We think that the contribution of the book enlarges the field of the Discrete-Time Systems with signification in the present state-of-the-art. Despite the vertiginous advance in the field, we also believe that the topics described here allow us also to look through some main tendencies in the next years in the research area.