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Multidimensional Dynamics: From Simple to Complicated

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1. Introduction

The most apparent look of a discrete-time dynamical system is that an orbit is composed of a collection of points in phase space, in contrast to a trajectory curve for a continuous-time system. A basic and prominent theoretical difference between discrete-time and continuous-time dynamical systems is that chaos occurs in one-dimensional discrete-time dynamical systems, but not for one-dimensional deterministic continuous-time dynamical systems; the logistic map and logistic equation are the most well-known example illustrating this difference. On the one hand, fundamental theories for discrete-time systems have also been developed in a parallel manner as for continuous-time dynamical systems, such as stable manifold theorem, center manifold theorem and global attractor theory etc. On the other hand, analytical theory on chaotic dynamics has been developed more thoroughly for discrete-time systems (maps) than for continuous-time systems. Li-Yorke's period-three-implies-chaos and Sarkovskii's ordering on periodic orbits for one-dimensional maps are ones of the most celebrated theorems on chaotic dynamics.
Regarding chaos theory for multidimensional maps, there are renowned Smale-Birkhoff homoclinic theorem and Moser theorem for diffeomorphisms. In addition, Marotto extended Li-Yorke's theorem from one-dimension to multi-dimension through introducing the notion of snapback repeller in 1978. This theory applies to maps which are not one-to-one (not diffeomorphism). But the existence of a repeller is a basic prerequisite for the theory. There have been extensive applications of this theorem to various applied problems. However, due to a technical flaw, Marotto fixed the definition of snapback repeller in 2005. While Marotto's theorem is valid under the new definition, its condition becomes more difficult to examine for practical applications. Accessible and computable criteria for applying this theorem hence remain to be developed. In Section 4, we shall introduce our recent works and related developments in the application of Marotto's theorem, which also provide an effective numerical computation method for justifying the condition of this theorem.
Multidimensional systems may also exhibit simple dynamics; for example, every orbit converges to a fixed point, as time tends to infinity. Such a scenario is referred to as convergence of dynamics or complete stability. Typical mathematical tools for justifying such dynamics include Lyapunov method and LaSalle invariant principle, a discrete-time version.

However, it is not always possible to construct a Lyapunov function to apply this principle, especially for multidimensional nonlinear systems. We shall illustrate other technique that was recently formulated for certain systems in Section 3.
As neural network models are presented in both continuous-time and discrete-time forms, and can exhibit both simple dynamics and complicated dynamics, we shall introduce some representative neural network models in Section 2.

2. Neural network models

In the past few decades, neural networks have received considerable attention and were successfully applied to many areas such as combinatorial optimization, signal processing and pattern recognition (Arik, 2000, Chua 1998). Discrete-time neural networks have been considered more important than their continuous-time counterparts in the implementations (Liu, 2008). The research interests in discrete-time neural networks include chaotic behaviors (Chen & Aihara, 1997; Chen & Shih, 2002), stability of fixed points (Forti & Tesi, 1995; Liang & Cao, 2004; Mak et al., 2007), and their applications (Chen & Aihara, 1999; Chen & Shih, 2008). We shall introduce some typical discrete-time neural networks in this section.
Cellular neural network (CNN) is a large aggregation of analogue circuits. It was first proposed by Chua and Yang in 1988. The assembly consists of arrays of identical elementary processing units called cells. The cells are only connected to their nearest neighbors. This local connectivity makes CNNs very suitable for VLSI implementation. The equations for two-dimension layout of CNNs are given by

$$C\frac{dx_{ij}}{dt} = -\frac{1}{R}x_{ij}(t) + \sum_{(k,\ell)\in N_{ij}} [a_{ij,k\ell}h(x_{k\ell}(t)) + b_{ij,k\ell}u_{k\ell}] + I, \tag{1}$$

where $u_{k\ell}$, x_{ij}, $h(x_{ij})$ are the controlling input, state and output voltage of the specified CNN cell, respectively. CNNs are characterized by the bias I and the template set A and B which consist of $a_{ij,k\ell}$ and $b_{ij,k\ell}$, respectively. $a_{ij,k\ell}$ represents the linear feedback, and $b_{ij,k\ell}$ the linear control. The standard output h is a piecewise-linear function defined by $h(\xi) = \frac{1}{2}(|\xi+1| - |\xi-1|)$. C is the linear capacitor and R is the linear resistor. For completeness of the model, boundary conditions need to be imposed for the cells on the boundary of the assembly, cf. (Shih, 2000). The discrete-time cellular neural network (DT-CNN) counterpart can be described by the following difference equation.

$$x_{ij}(t+1) = \mu x_{ij}(t) + \sum_{(k,\ell)\in N_{ij}} [\widetilde{a}_{ij,k\ell}h(x_{k\ell}(t)) + \widetilde{b}_{ij,k\ell}u_{k\ell}] + z_i, \tag{2}$$

where t is an integer. System (2) can be derived from a delta operator based CNNs. If one collects from a continuous-time signal $x(t)$ a discrete-time sequence $x[k] = x(kT)$, the delta operator

$$\delta x[k] = \frac{x[k+1] - x[k]}{T}$$

is an approximation of the derivative of $x(t)$. Indeed, $\lim_{T\to 0} \delta x[k] = \dot{x}(t)|_{t=kT}$. In this case, $\mu = 1 - \frac{T}{\tau}$, where T is the sampling period, and $\tau = RC$. The parameters $\widetilde{a}_{ij,k\ell}, \widetilde{b}_{ij,k\ell}$ in (2) correspond to $a_{ij,k\ell}$, $b_{ij,k\ell}$ in (1) under sampling, cf. (Hänggi et al., 1999). If (2) is considered in conjunction with (1), then T is required to satisfy $\tau \geq T$ to avoid aliasing effects. Under this situation, $0 \leq \mu \leq 1$. Thus CT-CNN is the limiting case of delta operator based CNNs with $T \to 0$. If the delta operator based CNNs is considered by itself, then there is no restriction on T, and thus no restrictions on μ in (2). On the other hand, a sampled-data based CNN has been introduced in (Harrer & Nossek, 1992). Such a network corresponds to the limiting case of delta operator based CNNs as $T \to 1$. For an account of unifying results on the above-mentioned models, see (Hänggi et al., 1999) and the references therein. In addition, Euler's difference scheme for (1) takes the form

$$x_{ij}(t+1) = (1 - \frac{\Delta t}{RC})x_{ij}(t) + \frac{\Delta t}{C}\left(\sum_{k \in N_{ij}} a_{ij,k\ell}h(x_{k\ell}(t)) + b_{ij,k\ell}u_{k\ell} + I\right). \tag{3}$$

Note that CNN of any dimension can be reformulated into a one-dimensional setting, cf. (Shih & Weng, 2002). We rewrite (2) into a one-dimensional form as

$$x_i(t+1) = \mu x_i(t) + \sum_{k=1}^{n} \omega_{ik}h(x_k(t)) + z_i. \tag{4}$$

The complete stability using LaSalle invariant principle has been studied in (Chen & Shih, 2004a). We shall review this result in Section 3.1.

Transiently chaotic neural network (TCNN) has been shown powerful in solving combinatorial optimization problems (Peterson & Söderberg, 1993; Chen & Aihara, 1995, 1997, 1999). The system is represented by

$$x_i(t+1) = \mu x_i(t) + w_{ii}(t)[y_i(t) - a_{0i}] + \Sigma_{k\neq i}^{n} w_{ik}y_k(t) + a_i \tag{5}$$

$$y_i(t) = (1 + e^{\frac{-x_i(t)}{\varepsilon}})^{-1} \tag{6}$$

$$w_{ii}(t+1) = (1-\gamma)w_{ii}(t), \tag{7}$$

where $i = 1, \cdots, n$, $t \in \mathbb{N}$ (positive integers), ε, γ are fixed numbers with $\varepsilon > 0$, $0 < \gamma < 1$. The main feature of TCNN contains chaotic dynamics temporarily generated for global searching and self-organizing. As certain variables (corresponding to temperature in the annealing process) decrease, the network gradually approaches a dynamical structure which is similar to classical neural networks. The system then settles at stationary states and provides a solution to the optimization problem. Equations (5)-(6) with constant self-feedback connection weights, that is, $w_{ii}(t) = w_{ii} =$ constant, has been studied in (Chen & Aihara, 1995, 1997); therein, it was shown that snapback repellers exist if $|w_{ii}|$ are large enough. The result hence implicates certain chaotic dynamics for the system. More complete analytical arguments by applying Marotto's theorem through the formulation of upper and lower dynamics to conclude the chaotic dynamics have been performed in (Chen & Shih, 2002, 2008, 2009). As the system evolves, w_{ii} decreases, and the chaotic behavior vanishes. In (Chen & Shih, 2004), they derived sufficient conditions under which evolutions for the system converge to fixed points of the system. Moreover, attracting sets and uniqueness of fixed point for the system were also addressed.

Time delays are unavoidable in a neural network because of the finite signals switching and transmission speeds. The implementation of artificial neural networks incorporating delays has been an important focus in neural systems studies (Buric & Todorovic, 2003; Campbell, 2006; Roska & Chua, 1992; Wu, 2001). Time delays can cause oscillations or alter the stability of a stationary solution of a system. For certain discrete-time neural networks with delays, the stability of stationary solution has been intensively studied in (Chen et al., 2006; Wu et al., 2009; Yua et al., 2010), and the convergence of dynamics has been analyzed in (Wang, 2008; Yuan, 2009). Among these studies, a typically investigated model is the one of Hopfield-type:

$$u_i(t+1) = a_i(t)u_i(t) + \sum_{j=1}^{m} b_{ij}(t)g_j(u_j(t-r_{ij}(t))) + J_i,\ i = 1, 2, \cdots, m. \tag{8}$$

Notably, system (8) represents an autonomous system if $a_i(t) \equiv a_i$, and $b_{ij}(t) \equiv b_{ij}$ (Chen et al., 2006), otherwise, a non-autonomous system (Yuan, 2009).
The class of Z-matrices consists of those matrices whose off-diagonal entries are less than or equal to zero. A M-matrix is a Z-matrix satisfying that all eigenvalues have positive real parts. For instance, one characterization of a nonsingular square matrix P to be a M-matrix is that P has non-positive off-diagonal entries, positive diagonal entries, and non-negative row sums. There exist several equivalent conditions for a Z-matrix P to be M-matrix, such as the one where there exists a positive diagonal matrix D such that PD is a diagonally dominant matrix, or all principal minors of P are positive (Plemmons, 1977). A common approach to conclude the stability of an equilibrium for a discrete-time neural network is through constructing Lyapunov-Krasovskii function/functional for the system. In (Chen, 2006), based on M-matrix theory, they constructed a Lyapunov function to derive the delay-independent and delay-dependent exponential stability results.
Synchronization is a common and elementary phenomenon in many biological and physical systems. Although the real network architecture can be extremely complicated, rich dynamics arising from the interaction of simple network motifs are believed to provide similar sources of activities as in real-life systems. Coupled map networks introduced by Kaneko (Kaneko, 1984) have become one of the standard models in synchronization studies. Synchronization in diffusively coupled map networks without delays is well understood, and the synchronizability of the network depends on the underlying network topology and the dynamical behaviour of the individual units (Jost & Joy, 2001; Lu & Chen, 2004). The synchronization in discrete-time networks with non-diffusively and delayed coupling is investigated in a series of works of Bauer and coworkers (Bauer et al., 2009; Bauer et al., 2010).

3. Simple dynamics

Orbits of discrete-time dynamical system can jump around wildly. However, there are situations that the dynamics are organized in a simple manner; for example, every solution converges to a fixed point as time tends to infinity. Such a notion is referred to as *convergence of dynamics* or *complete stability*. Moreover, the simplest situation is that all orbits converge to a unique fixed point. We shall review some theories and results addressing such simple dynamics. In Subsection 3.1, we introduce LaSalle invariant principle and illustrate its application in discrete-time neural networks. In Subsection 3.2, we review the component-competing technique and its application in concluding global consensus for a discrete-time competing system.

3.1 Lyapunov method and LaSalle invariant principle

Let us recall LaSalle invariant principle for difference equations. We consider the difference equation

$$\mathbf{x}(t+1) = F(\mathbf{x}(t)), \tag{9}$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function. Let U be a subset of $\mathbb{R}^n$. For a function $V : U \longrightarrow \mathbb{R}$, define $\dot{V}(\mathbf{x}) = V(F(\mathbf{x})) - V(\mathbf{x})$. V is said to be a *Lyapunov function* of (9) on U if (i) V is continuous, and (ii) $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in U$. Set

$$S_0 := \{\mathbf{x} \in \overline{U} | \dot{V}(\mathbf{x}) = 0\}.$$

LaSalle Invariant Principle (LaSalle, 1976). Let F be a continuous mapping on $\mathbb{R}^n$, and let V be a Lyapunov function for F on a set $U \subseteq \mathbb{R}^n$. If orbit $\gamma := \{F^n(\mathbf{x}) | n \in \mathbb{N}\}$ is contained in a compact set in U, then its ω-limit set $\omega(\gamma) \subset S_0 \bigcap V^{-1}(c)$ for some $c = c(\mathbf{x})$.

This principle has been applied to the discrete-time cellular neural network (4) in (Chen & Shih, 2004a), where the Lyapunov function is constructed as

$$V(\mathbf{x}) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\omega_{ik}h(x_i)h(x_k) - \sum_{i=1}^{n} z_i h(x_i) + \frac{1}{2}(1-\mu)\sum_{i=1}^{n} h(x_i)^2,$$

and $h(\xi) = \frac{1}{2}(|\xi+1| - |\xi-1|)$ and $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Let us quote the main results therein.

Proposition (Chen & Shih, 2004a). Let W be a positive-definite symmetric matrix and $0 \leq \mu \leq 1$. Then V is a Lyapunov function for (4) on $\mathbb{R}^n$.

Consider the condition

$$\text{(H)} \quad \frac{1}{1-\mu}\left[\omega_{ii} - \sum_{k,j_k=\text{“m”}} |\omega_{ik}| + \sum_{j_k \neq \text{“m”},\, k \neq i} \delta(j_i, j_k)\omega_{ik} + z(i)\right] > -1.$$

Theorem (Chen & Shih, 2004a). Let W be a positive-definite symmetric matrix. If $0 < \mu < 1$ and condition (H) holds, then the DT-CNN with regular parameters is completely stabile.

Next, let us outline LaSalle invariant principle for non-autonomous difference equations. In addition to the classical result by LaSalle there is a modified version for the theorem reported in (Chen & Shih, 2004b). The alternative conditions derived therein is considered more applicable and has been applied to study the convergence of the TCNN.

Let $\mathbb{N}$ be the set of positive integers. For a given continuous function $\mathbf{F} : \mathbb{N} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, we consider the non-autonomous difference equation

$$\mathbf{x}(t+1) = \mathbf{F}(t, \mathbf{x}(t)). \tag{10}$$

A sequence of points $\{\mathbf{x}(t)\}_1^\infty$ in $\mathbb{R}^n$ is a solution of (10) if $\mathbf{x}(t+1) = \mathbf{F}(t, \mathbf{x}(t))$, for all $t \in \mathbb{N}$. Let $\mathcal{O}_\mathbf{x} = \{\mathbf{x}(t) \mid t \in \mathbb{N}, \mathbf{x}(1) = \mathbf{x}\}$, be the orbit of $\mathbf{x}$. We say that $\mathbf{p}$ is a ω-limit point of $\mathcal{O}_\mathbf{x}$ if there exists a sequence of positive integers $\{t_k\}$ with $t_k \to \infty$ as $k \to \infty$, such that $\mathbf{p} = \lim_{k\to\infty} \mathbf{x}(t_k)$. Denote by $\omega(\mathbf{x})$ the set of all ω-limit points of $\mathcal{O}_\mathbf{x}$.

Let $\mathbb{N}_i$ represent the set of all positive integers larger than n_i, for some positive integer n_i. Let G be any set in $\mathbb{R}^n$ and $\overline{G}$ be its closure. For a function $V : \mathbb{N}_0 \times G \longrightarrow \mathbb{R}$, define $\dot{V}(t, \mathbf{x}) = V(t+1, \mathbf{F}(t, \mathbf{x})) - V(t, \mathbf{x})$. If $\{\mathbf{x}(t)\}$ is a solution of (10), then $\dot{V}(t, \mathbf{x}) = V(t+1, \mathbf{x}(t+1)) - V(t, \mathbf{x}(t))$. V is said to be a *Lyapunov function* for (10) if
(i) $\{V(t, \cdot) \mid t \in \mathbb{N}_0\}$ is equi-continuous, and
(ii) for each $p \in \overline{G}$, there exists a neighborhood U of p such that $V(t, \mathbf{x})$ is bounded below for $\mathbf{x} \in U \cap G$ and $t \in \mathbb{N}_1$, $n_1 \geq n_0$, and
(iii) there exists a continuous function $Q_0 : \overline{G} \to \mathbb{R}$ such that $\dot{V}(t, \mathbf{x}) \leq -Q_0(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in G$ and for all $t \in \mathbb{N}_2$, $n_2 \geq n_1$,
or
(iii)$'$ there exist a continuous function $Q_0 : \overline{G} \to \mathbb{R}$ and an equi-continuous family of functions $Q : \mathbb{N}_2 \times \overline{G} \to \mathbb{R}$ such that $\lim_{t\to\infty} |Q(t, \mathbf{x}) - Q_0(\mathbf{x})| = 0$ for all $\mathbf{x} \in G$ and $\dot{V}(t, \mathbf{x}) \leq -Q(t, \mathbf{x}) \leq 0$ for all $(t, \mathbf{x}) \in \mathbb{N}_2 \times G$, $n_2 \geq n_1$.
Define

$$S_0 = \{\mathbf{x} \in \overline{G} : Q_0(\mathbf{x}) = 0\}.$$

Theorem (Chen & Shih, 2004a). Let $V : \mathbb{N}_0 \times G \to \mathbb{R}$ be a Lyapunov function for (10) and let $\mathcal{O}_\mathbf{x}$ be an orbit of (10) lying in G for all $t \in \mathbb{N}_0$. Then $\lim_{t\to\infty} Q(t, \mathbf{x}(t)) = 0$, and $\omega(\mathbf{x}) \subset S_0$.

This theorem with conditions (i), (ii), and (iii) has been given in (LaSalle, 1976). We quote the proof for the second case reported in (Chen & Shih, 2004b). Let $\mathbf{p} \in \omega(\mathbf{x})$. That is, there exists a sequence $\{t_k\}_1^\infty$, $t_k \to \infty$ as $k \to \infty$ and $\mathbf{x}(t_k) \to \mathbf{p}$ as $k \to \infty$. Since $V(t_k, \mathbf{x}(t_k))$ is non-increasing and bounded below, $V(t_k, \mathbf{x}(t_k))$ approaches a real number as $k \to \infty$. Moreover, $V(t_{k+1}, \mathbf{x}(t_{k+1})) - V(t_1, \mathbf{x}(t_1)) \leq -\sum_{t=t_1}^{t_{k+1}-1} Q(t, \mathbf{x}(t))$, by (iii)$'$. Thus, $\sum_{t=t_1}^{\infty} Q(t, \mathbf{x}(t)) < \infty$. Hence, $Q(t, \mathbf{x}(t)) \to 0$ as $t \to \infty$, since $Q(t, \mathbf{x}(t)) \geq 0$. Notably, $Q(t_k, \mathbf{x}(t_k)) \to Q_0(\mathbf{x}(t_k))$ as $k \to \infty$. This can be justified by observing that

$$\begin{aligned} &|Q(t_k, \mathbf{x}(t_k)) - Q_0(\mathbf{x}(t_k))| \\ \leq\ &|Q(t_k, \mathbf{x}(t_k)) + Q(t_k, \mathbf{p}) - Q(t_k, \mathbf{p}) + Q_0(\mathbf{p}) - Q_0(\mathbf{p}) - Q_0(\mathbf{x}(t_k))|. \end{aligned}$$

In addition, $|Q_0(\mathbf{x}(t))| \leq |Q(t, \mathbf{x}(t))| + |Q(t, \mathbf{x}(t)) - Q_0(\mathbf{x}(t))|$. It follows from (iii)$'$ that $Q_0(\mathbf{x}(t_k)) \to 0$ as $k \to \infty$. Therefore, $Q_0(\mathbf{p}) = 0$, since Q_0 is continuous. Thus, $\mathbf{p} \in S_0$.
If we further assume that V is bounded, then it is obvious that the proof can be much simplified. In the investigations for the asymptotic behaviors of TCNN, condition (iii)$'$ is more achievable.
We are interested in knowing whether if an orbit of the system (10) approaches an equilibrium state or fixed point as time tends to infinity. The structure of ω-limit sets for the orbits provides an important information toward this investigation. In discrete-time dynamical systems, the ω-limit set of an orbit is not necessarily connected. However, the following proposition has been proved by Hale and Raugel in 1992.

Proposition (Hale & Raugel, 1992). Let T be a continuous map on a Banach space X. Suppose that the ω-limit set $\omega(\mathbf{x})$ is contained in the set of fixed points of T, and the closure of the orbit $\mathcal{O}_\mathbf{x}$ is compact. Then $\omega(\mathbf{x})$ is connected.

This proposition can be extended to non-autonomous systems for which there exist limiting maps. Namely,

$(\mathbf{A})$ There exists a continuous map $\overline{\mathbf{F}}:\mathbb{R}^n\to\mathbb{R}^n$ such that $\lim_{t\to\infty}\|\mathbf{F}(t,\mathbf{x})-\overline{\mathbf{F}}(\mathbf{x})\|=0$, for all $\mathbf{x}\in\mathbb{R}^n$.

Theorem (Chen & Shih, 2004b). Assume that (10) satisfies $(\mathbf{A})$, the orbit $\mathcal{O}_{\mathbf{x}}$ is bounded, and $\omega(\mathbf{x})$, the ω-limit set of $\mathbf{x}$, is contained in the set of fixed points of $\overline{\mathbf{F}}$. Then $\omega(\mathbf{x})$ is connected. Under this circumstances, if $\overline{\mathbf{F}}$ has only finitely many fixed points, then the orbit $\mathcal{O}_{\mathbf{x}}$ approaches some single fixed point of $\overline{\mathbf{F}}$, as t tends to infinity.

Let us represent the TCNN system (5)-(7) by the following time-dependent map

$$\mathbf{F}(t,\mathbf{x})=(F_1(t,\mathbf{x}),\cdots,F_n(t,\mathbf{x}))$$

where

$$F_i(t,\mathbf{x})=\alpha x_i+(1-\gamma)^t\omega_{ii}(0)(y_i-a_{0i})+\sum_{j\neq i}^{n}\omega_{ij}y_j+a_i,$$

where $y_i=h_i(x_i), i=1,\cdots,n$ and h_i is defined in (6). The orbits of TCNN are then given by the iterations $\mathbf{x}(t+1)=\mathbf{F}(t,\mathbf{x}(t))$ with components $x_i(t+1)=F_i(t,\mathbf{x}(t))$. Note that $\mathbf{y}=H(\mathbf{x})=(h_1(x_1),\cdots,h_n(x_n))$ is a diffeomorphism on $\mathbf{R}^n$. Let W_0 denote the $n\times n$ matrix obtained from the connection matrix W with its diagonal entries being replaced by zeros. Restated, $W_0=W-\mathrm{diag}[W]$. For given $0<\gamma<1$, choose $0<b<1$ such that $|\frac{1-\gamma}{b}|<1$. We consider the following time-dependent energy-like function:

$$\begin{aligned}V(t,\mathbf{x})=&-\frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}w_{ij}h_i(x_i)h_j(x_j)-\sum_{i=1}^{n}a_ih_i(x_i)\\&+(1-\alpha)\sum_{i=1}^{n}\int_0^{h_i(x_i)}h_i^{-1}(\eta)d\eta+b^t.\end{aligned}\quad(11)$$

Theorem (Chen & Shih, 2004b). Assume that W_0 is a cycle-symmetric matrix, and either one of the following condition holds,
(i) $0\le\alpha\le\frac{1}{3}$ and $W_0+4(1-\alpha)\varepsilon I$ is positive definite;
(ii) $\frac{1}{3}\le\alpha\le1$ and $W_0+8\alpha\varepsilon I$ is positive definite;
(iii) $\alpha\ge1$ and $W_0+8\varepsilon I$ is positive definite.
Then there exists an $n_0\in\mathbb{N}$ so that $V(t,\mathbf{x})$ defined by (11) is a Lyapunov function for the TCNN (5)-(7) on $\mathbb{N}_0\times\mathbb{R}^n$.

3.2 Global consensus through a competing-component approach

Grossberg (1978) considered a class of competitive systems of the form

$$\dot{x}_i=a_i(\mathbf{x})[b_i(x_i)-C(x_1,x_2,\cdots,x_n)],\ \ i=1,2,\cdots,n,\quad(12)$$

where $a_i\ge0$, $\partial C/\partial x_i\ge0$, $\mathbf{x}=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n$. In such a system, n is the number of competing populations, $a_i(\mathbf{x})$ refers to competitive balance, $b_i(x_i)$ represents interpopulation signal functions, and $C(x)$ stands for mean competition function, or adaptation level. System (12) was proposed as a mathematical model for the resolution to a dilemma in science for hundred of years: How do arbitrarily many individuals, populations, or states, each obey unique and personal laws, succeed in harmoniously interacting with each other to

form some sort of stable society, or collective mode of behavior. Systems of the form (12) include the generalized Volterra-Lotka systems and an inhibitory network (Hirsch, 1989). A suitable Lyapunov function for system (12) is not known, hence the Lyapunov method and LaSalle invariant principle are invalid. The work in (Grossberg, 1978) employed a skillful competing-component analysis to prove that for system (12), any initial value $\mathbf{x}(0) \geq 0$ (i.e. $x_i(0) \geq 0$, for any i) evolves to a limiting pattern $\mathbf{x}(\infty) = (x_1(\infty), x_2(\infty), \cdots, x_n(\infty))$ with $0 \leq x_i(\infty) := \lim_{t\to\infty} x_i(t) < \infty$, under some conditions on a_i, b_i, C.
System (12) can be approximated, via Euler's difference scheme or delta-operator circuit implementation (Harrer & Nossek,1992), by

$$x_i((k+1)\delta) = x_i(k\delta) + \delta a_i(\mathbf{x}(k\delta))[b_i(x_i(k\delta)) - C(\mathbf{x}(k\delta))],$$

where one takes $x_i(k\delta)$ as the k-th iteration of x_i. In this subsection, let us review the competing-component analysis for convergent dynamics reported in (Shih & Tseng, 2009). Consider the following discrete-time model,

$$x_i(k+1) = x_i(k) + \beta a_i(\mathbf{x}(k))[b_i(x_i(k)) - C(\mathbf{x}(k))], \tag{13}$$

where $i = 1, 2, \cdots, n$, $k \in \mathbb{N}_0 := \{0\} \bigcup \mathbb{N}$. We first consider the theory for (13) with $\beta = 1$, i.e.

$$x_i(k+1) = x_i(k) + a_i(\mathbf{x}(k))[b_i(x_i(k)) - C(\mathbf{x}(k))]. \tag{14}$$

The results can then be extended to (13). First, let us introduce the following definition for the convergent property of discrete-time systems.

Definition. A discrete-time competitive system $\mathbf{x}(k+1) = F(\mathbf{x}(k))$ is said to achieve *global consensus* (or *global pattern formation, global convergence*) if, given any initial value $\mathbf{x}(0) \in \mathbb{R}^n$, the limit $x_i(\infty) := \lim_{k\to\infty} x_i(k)$ exists, for all $i = 1, 2, \cdots, n$.

The following conditions are needed for the main results.
Condition (A1): Each $a_i(\mathbf{x})$ is continuous, and

$$0 < a_i(\mathbf{x}) \leq 1, \text{ for all } \mathbf{x} \in \mathbb{R}^n, i = 1, 2, \cdots, n.$$

Condition (A2): $C(\mathbf{x})$ is bounded and continuously differentiable with bounded derivatives; namely, there exist constants M_1, M_2, r_j such that for all $\mathbf{x} \in \mathbb{R}^n$,

$$M_1 \leq C(\mathbf{x}) \leq M_2,$$

$$0 \leq \frac{\partial C}{\partial x_j}(\mathbf{x}) \leq r_j, \; j = 1, 2, \cdots, n.$$

Condition (A3): $b_i(\xi)$ is continuously differentiable, strictly decreasing and there exist $d_i > 0$, $l_i, u_i \in \mathbb{R}$ such that for all $i = 1, 2, \cdots n$,

$$-d_i \leq b_i'(\xi) < 0, \text{ for all } \xi \in \mathbb{R},$$

$$b_i(\xi) > M_2, \text{ for } \xi \leq l_i, \text{ and } b_i(\xi) < M_1, \text{ for } \xi \geq u_i.$$

Condition (A4): For $i = 1, 2, \cdots, n$,

$$0 < d_i \le 1 - \sum_{j=1}^{n} r_j < 1.$$

Theorem (Shih & Tseng, 2009). System (14) with a_i, b_i, and C satisfying conditions (A1)-(A4) achieves global consensus.

The proof of this theorem consists of three lemmas which depict the properties for the following terms:

$$\begin{aligned}
&g_i(k) = b_i(x_i(k)) - C(\mathbf{x}(k)),\ \Delta g_i(k) = g_i(k+1) - g_i(k),\\
&\hat{g}(k) = \max\{g_i(k) : i = 1, 2, \cdots, n\},\ \check{g}(k) = \min\{g_i(k) : i = 1, 2, \cdots, n\},\\
&I(k) = \min\{i : g_i(k) = \hat{g}(k)\},\ J(k) = \min\{i : g_i(k) = \check{g}(k)\},\\
&\hat{x}(k) = x_{I(k)}(k),\ \check{x}(k) = x_{J(k)}(k),\\
&\hat{b}(k) = b_{I(k)}(\hat{x}(k)),\ \check{b}(k) = b_{J(k)}(\check{x}(k)),\\
&\Delta\hat{b}(k) = \hat{b}(k+1) - \hat{b}(k),\ \Delta\check{b}(k) = \check{b}(k+1) - \check{b}(k),\\
&\Delta b_i(x_i(k)) = b_i(x_i(k+1)) - b_i(x_i(k)).
\end{aligned}$$

Let us recall some of the key lemmas to get a flavor of this approach.

Lemma. Consider system (14) with a_i, b_i, and C satisfying conditions (A1)-(A4). Then
(i) for function $\hat{g}$, either case ($\hat{g}$-(i)) or case ($\hat{g}$-(ii)) holds, where
($\hat{g}$-(i)): $\hat{g}(k) < 0$, for all $k \in \mathbb{N}_0$,
($\hat{g}$-(ii)): $\hat{g}(k) \ge 0$, for all $k \ge K_1$, for some $K_1 \in \mathbb{N}_0$;
(ii) for function $\check{g}$, either case ($\check{g}$-(i)) or case ($\check{g}$-(ii)) holds, where
($\check{g}$-(i)): $\check{g}(k) > 0$, for all $k \in \mathbb{N}_0$,
($\check{g}$-(ii)): $\check{g}(k) \le 0$, for all $k \ge K_2$, for some $K_2 \in \mathbb{N}_0$.

Lemma. Consider system (14) with a_i, b_i, and C satisfying conditions (A1)-(A4). Then $\lim_{k\to\infty} \hat{b}(k) = \lim_{k\to\infty} C(\mathbf{x}(k)) = \lim_{k\to\infty} \check{b}(k)$.

4. Complicated dynamics

In this section, we summarize some analytic theories on chaotic dynamics for multi-dimensional maps. There are several definitions for chaos. Let us introduce the representative one by Devaney (1989):

Definition. Let (X, d) be a metric space. A map $F : \Omega \subset X \to \Omega$ is said to be *chaotic* on Ω if
(i) F is topologically transitive in Ω,
(ii) the periodic points of F in Ω are dense in Ω,
(iii) F has sensitive dependence on initial conditions in Ω.

It was shown in (Banks, et al., 1992) that condition (iii) holds under conditions (i) and (ii), if F is continuous in Ω. Let us recall Li-Yorke's theorem.

Theorem (Li & Yorke, 1975). Let J be an interval and let $f : J \to J$ be continuous. Assume there is a point $a \in J$ for which the points $b = f(a)$, $c = f^2(a)$ and $d = f^3(a)$, satisfy

$$d \leq a < b < c \text{ (or } d \geq a > b > c).$$

Then for every $k = 1, 2, \cdots$, there is a periodic point in J having period k. Furthermore, there is an uncountable set $S \subset J$ (containing no periodic points), which satisfies:
(i) $\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0$, and $\liminf_{n\to\infty} |f^n(p) - f^n(q)| = 0$, for every $p, q \in S$ with $p \neq q$; (ii) $\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0$, for every $p \in S$ and periodic point $q \in J$.
Indeed, if there is a periodic point of period 3, then the hypothesis of the theorem will be satisfied. The notion of scrambled set can be generalized to metric space (X, d).

Definition. Let (X, d) be a metric space and $F : X \to X$ be a continuous map. A subset S of X is called a *scrambled set* of F, if for any two different points $\mathbf{x}, \mathbf{y} \in S$,

$$\liminf_{n\to\infty} d(F^n(\mathbf{x}), F^n(\mathbf{y})) = 0, \ \limsup_{n\to\infty} d(F^n(\mathbf{x}), F^n(\mathbf{y})) > 0.$$

A map F is said to be *chaotic in the sense of Li-Yorke* if it has an uncountable scrambled set. It was shown in (Huang & Ye, 2002) that for a compact matric space (X, d), if a map F is chaotic in the sense of Devaney then F is also chaotic in the sense of Li-Yorke.
Let us consider a differentiable map

$$F : \mathbb{R}^n \to \mathbb{R}^n; \tag{15}$$

we denote $\mathbf{x}_k = F^k(\mathbf{x}_0)$ for $k \in \mathbb{N}$ and $\mathbf{x}_0 \in \mathbb{R}^n$, and by $B_r(\mathbf{x})$ and $B_r^*(\mathbf{x})$ the closed balls in $\mathbb{R}^n$ with center at $\mathbf{x}$ and radius $r > 0$ under Euclidean norm $\|\cdot\|$ and certain norm $\|\cdot\|_*$, respectively.

Definition. Suppose $\mathbf{z}$ is a hyperbolic fixed point of a diffeomorphism map $F : \mathbb{R}^n \to \mathbb{R}^n$, and some eigenvalues of $DF(\mathbf{z})$ are greater than one in magnitude and the others smaller than one in magnitude. If the stable manifold and the unstable manifold of F at $\mathbf{z}$ intersect transversally at some point $\mathbf{x}_0$, the orbit $\{\mathbf{x}_k\}_{k=-\infty}^{\infty}$ of F is called a *transversal homoclinic orbit*.
For a diffeomorphism F, Smale discovered an elegant and significant result:

Theorem (Smale, 1967). If the diffeomorphism map F has a transversal homoclinic orbit, then there exists a Cantor set $\Lambda \subset \mathbb{R}^n$ on which F^m is topologically conjugate to a full shift of a symbolic dynamical system with N symbols, for some positive integer m.

Remark. The above theorem can be generalized to maps which are not diffeomorphisms under some extended definition of transversal homoclinic orbits, see Theorem 5.2 and Section 7 in (Hale & Lin, 1986) and Theorem 5.1 in (Steinlein & Walther, 1990).

4.1 On Marotto's theorem

Analytical theory on chaotic dynamics for multi-dimensional systems is quite limited; yet some important progresses have been made. In 1978, Marotto introduced the notion of snapback repeller and extended Li-Yorke's theorem to multi-dimensional maps. This result plays an important role in the study of chaos for higher but finite-dimensional noninvertible maps.

The point $\mathbf{z} \in R^n$ is called an *expanding fixed point* of F in $B_r(\mathbf{z})$, if F is differentiable in $B_r(\mathbf{z})$, $F(\mathbf{z}) = \mathbf{z}$ and

$$|\lambda(\mathbf{x})| > 1, \text{ for all eigenvalues } \lambda(\mathbf{x}) \text{ of } DF(\mathbf{x}), \text{ for all } \mathbf{x} \in B_r(\mathbf{z}). \tag{16}$$

If F is not a one-to-one function in $\mathbb{R}^n$ and $\mathbf{z}$ is an expanding fixed point of F in $B_r(\mathbf{z})$, then there may exist a point $\mathbf{x}_0 \in B_r(\mathbf{z})$ with $\mathbf{x}_0 \neq \mathbf{z}$ such that $F^\ell(\mathbf{x}_0) = \mathbf{z}$ for some positive integer ℓ. The original definition of snapback repeller is as follows.

Definition (Marotto, 1978). Assume that $\mathbf{z}$ is an expanding fixed point of F in $B_r(\mathbf{z})$ for some $r > 0$. Then $\mathbf{z}$ is said to be a *snapback repeller* of F if there exists a point $\mathbf{x}_0 \in B_r(\mathbf{z})$ with $\mathbf{x}_0 \neq \mathbf{z}$, $F^\ell(\mathbf{x}_0) = \mathbf{z}$ and $\det(DF^\ell(\mathbf{x}_0)) \neq 0$ for some positive integer ℓ; see Fig. 1.

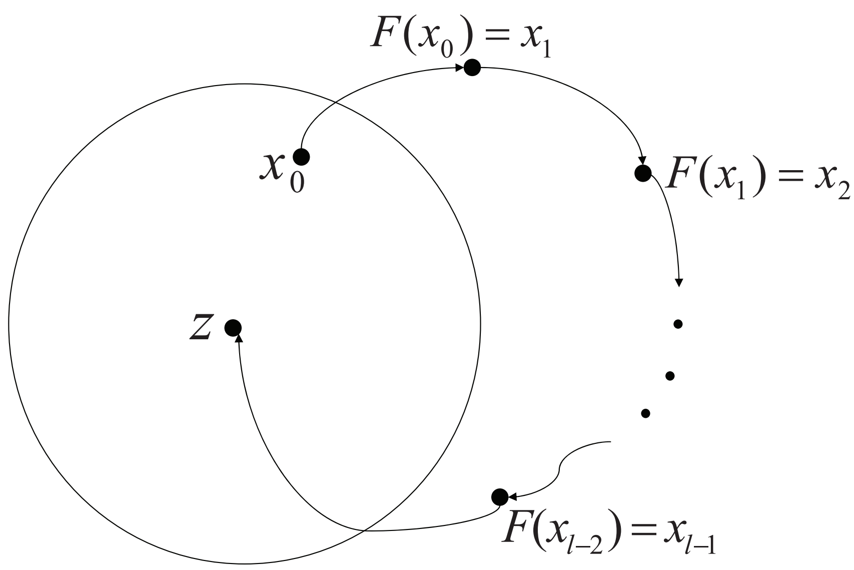

Fig. 1

It is straightforward to see that a snapback repeller gives rise to an orbit $\{\mathbf{x}_k\}_{k=-\infty}^{\infty}$ of F with $\mathbf{x}_k = \mathbf{z}$, for $k \geq \ell$, and $\mathbf{x}_k \to \mathbf{z}$ as $k \to -\infty$. Roughly speaking, the property of this orbit is analogous to the one for homoclinic orbit. In addition, the map F is locally one-to-one at each point $\mathbf{x}_k$, since $\mathbf{x}_0 \in B_r(\mathbf{z})$ and $\det(DF^\ell(\mathbf{x}_0)) \neq 0$. This leads to the trivial transversality, i.e., the unstable manifold $\mathbb{R}^n$ of full dimension intersects transversally the zero- dimensional stable manifold of $\mathbf{z}$. Therefore, snapback repeller may be regarded as a special case of a fixed point with a transversal homoclinic orbit if the latter is generalized to mappings which are not one-to-one.

Theorem (Marotto, 1978). If F possesses a snapback repeller, then F is chaotic in the following sense: There exist (i) a positive integer N, such that F has a point of period p, for each integer $p \geq N$, (ii) a scrambled set of F, i.e., an uncountable set S containing no periodic points of F, such that
(a) $F(S) \subset S$,
(b) $\limsup_{k\to\infty} \|F^k(\mathbf{x}) - F^k(\mathbf{y})\| > 0$, for all $\mathbf{x}, \mathbf{y} \in S$, with $\mathbf{x} \neq \mathbf{y}$,
(c) $\limsup_{k\to\infty} \|F^k(\mathbf{x}) - F^k(\mathbf{y})\| > 0$, for all $\mathbf{x} \in S$ and periodic point $\mathbf{y}$ of F,
(iii) an uncountable subset S_0 of S, such that $\liminf_{k\to\infty} \|F^k(\mathbf{x}) - F^k(\mathbf{y})\| = 0$, for every $\mathbf{x}, \mathbf{y} \in S_0$.

Remark. As the implication of this theorem yields the existence of uncountable scrambled set, we may say that existence of snapback repeller implies chaos in the sense of Li-Yorke.
However, there is a technical flaw in the original derivation. Consider the following two statements:
(A): All eigenvalues of the Jacobian $DF(\mathbf{z})$ are greater than one in norm.
(B): There exist some $s > 1$ and $r > 0$ such that

$$\|F(\mathbf{x}) - F(\mathbf{y})\| > s\|\mathbf{x} - \mathbf{y}\|, \text{ for all } \mathbf{x}, \mathbf{y} \in B_r(\mathbf{z}). \tag{17}$$

That (A) implies (B) may not be true for the Euclidean norm in multi-dimension. In addition, if $\mathbf{z}$ is a fixed point and there exists a norm $\|\cdot\|_*$, such that

$$|\lambda(\mathbf{x})| > 1, \text{ for all eigenvalues } \lambda(\mathbf{x}) \text{ of } DF(\mathbf{x}), \text{ for all } \mathbf{x} \in B_r^*(\mathbf{z}),$$

then

$$\|F(\mathbf{x}) - F(\mathbf{y})\|_* > s \cdot \|\mathbf{x} - \mathbf{y}\|_*, \text{ for all } \mathbf{x}, \mathbf{y} \in B_r^*(\mathbf{z}), \tag{18}$$

still may not be satisfied. This is due to that the norm constructed for such a property depends on the matrix $DF(\mathbf{x})$ which varies at different points $\mathbf{x}$, as the mean-value inequality is applied. Several researchers have made efforts in modifying the definition of snapback repeller to validate the theorem. In 2005, Marotto gave a revised definition of snapback repeller. Note that a fixed point $\mathbf{z}$ of F is *repelling* if all eigenvalues of $DF(\mathbf{z})$ exceed one in norm. For a repelling fixed point $\mathbf{z}$, if there exist a norm $\|\cdot\|_*$ on $\mathbb{R}^n$ and $s > 1$ such that (18) holds, then $B_r^*(\mathbf{z})$ is called a *repelling neighborhood* of $\mathbf{z}$. Note that if $\mathbf{z}$ is a repelling fixed point of F, then one can find a norm $\|\cdot\|_*$ and $r > 0$ so that $B_r^*(\mathbf{z})$ is a repelling neighborhood of $\mathbf{z}$, see (Robinson, 1999).

Definition (Marotto, 2005). Let $\mathbf{z}$ be a repelling fixed point of F. Suppose that there exist a point $\mathbf{x}_0 \neq \mathbf{z}$ in a repelling neighborhood of $\mathbf{z}$ and an integer $\ell > 1$, such that $\mathbf{x}_\ell = \mathbf{z}$ and $\det(DF(\mathbf{x}_k)) \neq 0$ for $1 \leq k \leq \ell$. Then $\mathbf{z}$ is called a *snapback repeller* of F.
The point $\mathbf{x}_0$ in the definition is called a *snapback point* of F. While Marotto's theorem holds under the modified definition, its application becomes more inaccessible; indeed, it is a nontrivial task to confirm that some preimage of a repelling fixed point lies in the repelling neighborhood of this fixed point. From practical view point, condition (16) which was adopted in his original version, is obviously easier to examine than finding the repelling neighborhood for a fixed point. In (Liao & Shih, 2011), two directions have been proposed to confirm that a repelling fixed point is a snapback repeller for multi-dimensional maps. The first one is to find the repelling neighborhood $\mathcal{U}$ of the repeller $\mathbf{z}$ which is based on a computable norm. This is the key part in applying Marotto's theorem for practical application, as one can then attempt to find a snapback point $\mathbf{x}_0$ of $\mathbf{z}$ in $\mathcal{U}$, i.e., $F^\ell(\mathbf{x}_0) = \mathbf{z}$, $\mathbf{x}_0 \in \mathcal{U}$ and $\mathbf{x}_0 \neq \mathbf{z}$, for some $\ell > 1$. The second direction is applying a sequential graphic-iteration scheme to construct the preimages $\{\mathbf{z}_{-k}\}_{k=1}^\infty$ of $\mathbf{z}$, such that $F(\mathbf{z}_{-k}) = \mathbf{z}_{-k+1}$, $k \geq 2$, $F(\mathbf{z}_{-1}) = \mathbf{z}$, $\lim_{k\to\infty} F(\mathbf{z}_{-k}) = \mathbf{z}$. Such an orbit $\{\mathbf{z}_{-k}\}_{k=1}^\infty$ is a homoclinic orbit for the repeller $\mathbf{z}$, in the generalized sense, as mentioned above. The existence of such a homoclinic orbit leads to the existence of a snapback point in the repelling neighborhood of repeller $\mathbf{z}$. Therefore, without finding the repelling region of the fixed point, Marotto's theorem still holds by using

the second method. More precisely, two methodologies were derived to establish the existence of snapback repellers:
(i) estimate the radius of repelling neighborhood for a repelling fixed point, under Euclidean norm,
(ii) construct the homoclinic orbit for a repelling fixed point by using a sequential graphic-iteration scheme.
In some practical applications, one can combine (i) and (ii) to achieve the application of Marotto's theorem. These two methodologies can then be combined with numerical computations and the technique of interval computing which provides rigorous computation precision, to conclude chaotic dynamics for the systems, such as the transiently chaotic neural network (TCNN) and the predator-prey system (Liao & Shih, 2011). Let us recall the results therein.

Repelling neighborhood:
Proposition (Liao & Shih, 2011). Consider a continuously differentiable map $F : \mathbb{R}^n \to \mathbb{R}^n$ with fixed point $\mathbf{z}$. Let

$$\begin{aligned} s_1 &:= \sqrt{\text{minimal eigenvalue of } (DF(\mathbf{z}))^T DF(\mathbf{z})}\,, \\ \eta_r &:= \max_{\mathbf{w}\in B_r(\mathbf{z})} \|B(\mathbf{w},\mathbf{z})\|_2 \\ &= \max_{\mathbf{w}\in B_r(\mathbf{z})} \sqrt{\text{maximal eigenvalue of } (B(\mathbf{w},\mathbf{z}))^T B(\mathbf{w},\mathbf{z})}, \end{aligned}$$

where $B(\mathbf{w},\mathbf{z}) := DF(\mathbf{w}) - DF(\mathbf{z})$. If there exists a $r > 0$ such that

$$s_1 - \eta_r > 1, \tag{19}$$

then $B_r(\mathbf{z})$ is a repelling neighborhood for $\mathbf{z}$, under the Euclidean norm.

There is a second approach which is based on the estimate of the first and second derivatives of F. This estimate is advantageous for quadratic maps since their second derivatives are constants. Let $\sigma_i(\mathbf{x})$ and $\beta_{ij}(\mathbf{x})$ be defined as

$$\begin{aligned} \sigma_i(\mathbf{x}) &:= \sqrt{\text{eigenvalues of } (DF(\mathbf{x}))^T DF(\mathbf{x})}, \\ \beta_{ij}(\mathbf{x}) &:= \text{ eigenvalues of Hessian matrix } H_{F_i}(\mathbf{x}) = [\partial_k \partial_l F_i(\mathbf{x})]_{k\times l}, \end{aligned}$$

where $i, j = 1, 2, \cdots, n$. Let α_r and β_r be defined as

$$\alpha_r := \min_{\mathbf{x}\in B_r(\mathbf{z})} \min_{1\le i\le n} \{\sigma_i(\mathbf{x})\} \tag{20}$$

$$\beta_r := \max_{1\le i\le n} \max_{\mathbf{x}\in B_r(\mathbf{z})} \max_{1\le j\le n} |\beta_{ij}(\mathbf{x})|. \tag{21}$$

Proposition (Liao & Shih, 2011). Consider a C^2 map $F = (F_1, \cdots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ with fixed point $\mathbf{z}$. Let α_r and β_r be defined in (20) and (21). If there exists $r > 0$, such that

$$\alpha_r - r\sqrt{n}\beta_r > 1, \tag{22}$$

then $B_r(\mathbf{z})$ is a repelling neighborhood of $\mathbf{z}$, under the Euclidean norm.
The conditions (19) and (22) are computable numerically and the value r can be found from numerical computation. Furthermore, if there exists a snapback point $\mathbf{x}_0$ in $B_r(\mathbf{z})$, i.e., $\mathbf{x}_0 \in B_r(\mathbf{z})$, and $F^{\ell}(\mathbf{x}_0) = \mathbf{z}$ for some integer $\ell > 1$, then $\mathbf{z}$ is a snapback repeller. Hence, the map F is chaotic in the sense of Marotto.

Sequential graphic-iteration scheme:
We recall an approach which is developed to exploit the existence of snapback repeller, without estimating the repelling neighborhood. In particular, it is a scheme to construct homoclinic orbits for repelling fixed point $\overline{\mathbf{x}}$ of F: $\{\overline{\mathbf{x}}_{-j} : j \in \mathbb{N}\}$ with $F(\overline{\mathbf{x}}_{-1}) = \overline{\mathbf{x}}$, $F(\overline{\mathbf{x}}_{-j}) = \overline{\mathbf{x}}_{-j+1}$, for $j \geq 2$, and $\lim_{j\to\infty} F(\overline{\mathbf{x}}_{-j}) = \overline{\mathbf{x}}$.

Theorem (Liao & Shih, 2011). Assume that there exists a compact, connected, convex region $\Omega = \Pi_{i=1}^{n}\Omega_i \subset \mathbb{R}^n$, so that the C^1 map $F = (F_1, F_2, \cdots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$|\frac{\partial F_i}{\partial x_i}(\mathbf{x})| > 1 + \sum_{j=1, j\neq i}^{n} |\frac{\partial F_i}{\partial x_j}(\mathbf{x})|, \text{ for all } i = 1, \cdots, n, \ \mathbf{x} \in \Omega,$$

and has a repelling fixed point $\overline{\mathbf{x}}$ in $\Omega \subset \mathbb{R}^n$. For $i = 1, \cdots, n$, set

$$\hat{f}_{i,(1)}(\xi) := \sup\{F_i(x'_1, \cdots, x'_{i-1}, \xi, x'_{i+1}, \cdots, x'_n) : x'_j \in \Omega_j, j \in \{1, \cdots, n\}/\{i\}\},$$
$$\check{f}_{i,(1)}(\xi) := \inf\{F_i(x'_1, \cdots, x'_{i-1}, \xi, x'_{i+1}, \cdots, x'_n) : x'_j \in \Omega_j, j \in \{1, \cdots, n\}/\{i\}\},$$

for $\xi \in \mathbb{R}^1$. Also assume that $\hat{f}_{i,(1)}$ and $\check{f}_{i,(1)}$ both have fixed points in Ω_i, for all $i = 1, \cdots, n$, and

$$\overline{\mathbf{x}}_{-\ell+1} \in \mathbb{R}^n \setminus \Omega, \ \overline{\mathbf{x}}_{-\ell} \in \text{int}(\Omega)$$

hold, for some $\ell \geq 2$. Then there exist a sequence of nested regions $\{\Omega_{(k)}\}_{k=1}^{\infty}$ with $\Omega_{(k+1)} \subseteq \Omega_{(k)} \subset \Omega$, and preimages $\overline{\mathbf{x}}_{-k-1} \in \Omega_{(k)}$ of $\overline{\mathbf{x}}$ under F, $k \in \mathbb{N}$. If furthermore, $\|\Omega_{i,(k)}\| \to 0$, as $k \to \infty$, for all $i = 1, \cdots, n$, then $\{\overline{\mathbf{x}}_{-k}\}_{k=1}^{\infty}$ is a homoclinic orbit for $\overline{\mathbf{x}}$. Moreover, if

$$\det\left(DF(\overline{\mathbf{x}}_{-k})\right) \neq 0, \text{ for } 1 \leq k \leq \ell - 1$$

holds, then $\overline{\mathbf{x}}$ is a snapback repeller and F is chaotic in the sense of Marotto's theorem.

Remark. (i) The existence of this homoclinic orbit guarantees the existence of the snapback point without finding the repelling neighborhood. (ii) The conditions in the above theorem are formulated for DF and the one-dimensional maps $\hat{f}_{i,(1)}(\xi)$ and $\check{f}_{i,(1)}(\xi)$ (the upper and lower maps), hence they are easy to examine in applications. For example, for TCNN map, we can find explicit and computable conditions such that all conditions in the theorem are satisfied.

4.2 Applications and extensions

We review some applications of snapback repeller and chaotic dynamics in (Marotto, 1979a, 1979b). Consider a two-dimensional mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $F(x, y) = (f(x), x)$, with $f : \mathbb{R} \to \mathbb{R}$ being differentiable.

Lemma (Marotto, 1979a). (i) If f has a stable periodic point z of period p, then $F(x,y) = (f(x),x)$ has a stable periodic point (z,y_0) of period p where $y_0 = f^{p-1}(z)$. (ii) If f has a snapback repeller, then $F(x,y) = (f(x),x)$ has a transversal homoclinic orbit.
Using these results, one can investigate the dynamics of the following difference equation:

$$x_{k+1} = f(x_k, bx_{k-1}), \tag{23}$$

where $b, x_k \in \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. We rewrite (23) into the following two-dimensional system:

$$\begin{aligned} x_{k+1} &= f(x_k, by_k) \\ y_{k+1} &= x_k. \end{aligned} \tag{24}$$

Moreover, when $b = 0$, (23) is the following scalar problem

$$x_{k+1} = f(x_k, 0). \tag{25}$$

It was shown that the dynamics of (23) or (24) are determined by those of (25), if b is close to 0:

Theorem (Marotto, 1979a). (i) If (25) has a stable periodic point x_0 of period p, then there exists $\epsilon > 0$ such that (24) has a stable periodic point $(x(b), y(b))$ of period p for all $|b| < \epsilon$. In this case $(x(b), y(b))$ is a uniquely defined, continuous function of b with $x(0) = x_0$. (ii) If (25) has a snapback repeller, then (24) has a transversal homoclinic orbit for all $|b| < \epsilon$, for some $\epsilon > 0$.

Next, let us consider another class of two-dimensional map $G : \mathbb{R}^2 \to \mathbb{R}^2$ which is determined by two scalar equations $f(x)$ and $g(y)$ where $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable and $G(x,y)$ is defined by $G(x,y) = (f(x), g(y))$.

Lemma (Marotto, 1979a). (i) If one of the mappings f and g has a snapback repeller and the other has an unstable fixed point, then $G(x,y) = (f(x), g(y))$ has a snapback repeller. (ii) If one of the mappings f and g has a snapback repeller and the other has a stable fixed point, then $G(x,y) = (f(x), g(y))$ has a transversal homoclinic orbit.

Now, we consider the dynamics for systems of the form:

$$\begin{aligned} x_{k+1} &= f(x_k, by_k) \\ y_{k+1} &= g(cx_k, y_k), \end{aligned} \tag{26}$$

where $f, g : \mathbb{R}^2 \to \mathbb{R}$ are differentiable, and $b, c \in \mathbb{R}$ are close to 0. If $b = c = 0$, then (26) can be simplified to the uncoupled system:

$$x_{k+1} = f(x_k, 0) \tag{27}$$

$$y_{k+1} = g(0, y_k). \tag{28}$$

Theorem (Marotto,1979a). (i) If one of the (27) and (28) has a snapback repeller and the other has an unstable fixed point, then (26) has a snapback repeller for all $|b|, |c| < \epsilon$, for some $\epsilon > 0$.

(ii) If one of the (27) and (28) has a snapback repeller and the other has a stable fixed point, then (26) has a transversal homoclinic orbit for all $|b|, |c| < \epsilon$, for some $\epsilon > 0$.

Remark. By examining the simplified systems, the above results exhibit the dynamics of system (24) or (26) under some small perturbations of certain parameters. However, these theorems do not provide any indication about the estimate of ϵ.

Next, let us recall the Hénon map

$$x_{k+1} = y_k + 1 - ax_k^2$$
$$y_{k+1} = bx_k,$$

which can be equivalently written as

$$\begin{aligned} u_{k+1} &= bv_k + 1 - au_k^2 =: f(u_k, bv_k) \\ v_{k+1} &= u_k, \end{aligned} \qquad (29)$$

where $f(u,v) = v + 1 - au^2$. It was shown in (Marotto, 1979b) that $u_{k+1} = f(u_k, 0)$ has a snapback repeller, when $a > 1.55$. Hence (29) has a transversal homolinic orbit for all $a > 1.55$ and $|b| < \epsilon$, for some $\epsilon > 0$.

In (Li, et al., 2008), they considered a one-parameter family of maps H_λ on $\mathbb{R}^n \times \mathbb{R}^m$ with $H_0(\mathbf{x}, \mathbf{y}) = (F(\mathbf{x}), G(\mathbf{x}))$ and continuous $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^m$ or $H_0(\mathbf{x}, \mathbf{y}) = (F(\mathbf{x}), G(\mathbf{x}, \mathbf{y}))$ with continuous maps $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. They used the covering relations method proposed by Zgliczyński in (Zgliczyński, 1996, 1997) to prove that if $n = 1$ and F has a positive topological entropy, or if $n > 1$ and F has a snapback repeller, then any small perturbation H_λ of H_0 has a positive topological entropy. Without using hyperbolicity, the covering relations method still provides a way to verify the existence of periodic points, the symbolic dynamics and the positive topological entropy. Moreover, they also applied this method to obtain a new proof for García's result (García, 1986) that if a map has a snapback repeller then it has a positive topological entropy. One can obtain similar results by using this method with other structure, such as a hyperbolic horseshoe.
Since the definition of snapback repeller proposed by Marotto relies on the norm, the following definition independent of norm was proposed.

Definition (Li, et al., 2008). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function. A fixed point $\mathbf{z}$ for F is called a *snapback repeller* if (i) all eigenvalues of the derivative $DF(\mathbf{z})$ are greater than one in absolute value and (ii) there exists a sequence $\{\mathbf{x}_{-i}\}_{i\in\mathbb{N}}$ such that $\mathbf{x}_{-1} \neq \mathbf{z}$, $\lim_{i\to\infty} \mathbf{x}_{-i} = \mathbf{z}$, and for all $i \in \mathbb{N}$, $F(\mathbf{x}_{-i}) = \mathbf{x}_{-i+1}$, $F(\mathbf{x}_{-1}) = \mathbf{z}$ and $\det(DF(\mathbf{x}_{-i})) \neq 0$.

Remark. Although the above definition is independent of norm on the phase space, it requires the existence of the pre-images for the repeller. The sequential graphic-iteration scheme outlined above provides a methodology for such a construction.

Note that item (i) implies that there exist a norm $\|\cdot\|_*$ on $\mathbb{R}^n$, $r > 0$ and $s > 1$, such that $\|F(\mathbf{x}) - F(\mathbf{y})\|_* > s\|\mathbf{x} - \mathbf{y}\|_*$ for all $\mathbf{x}, \mathbf{y} \in B_r^*(\mathbf{z})$. Hence F is one-to-one on $B_r^*(\mathbf{z})$ and $F(B_r^*(\mathbf{z})) \supset B_r^*(\mathbf{z})$. Therefore, if there exists a point $\mathbf{x}_0 \in B_r^*(\mathbf{z})$ such that $F^\ell(\mathbf{x}_0) = \mathbf{z}$ and $\det(DF^\ell(\mathbf{x}_0)) \neq 0$ for some positive integer ℓ, then item (ii) of the above definition is satisfied. In addition, in (Li & Chen, 2003), they showed that this norm can be chosen to be the Euclidean norm on $\mathbb{R}^n$, under the condition that all eigenvalues of $(DF(\mathbf{z}))^T DF(\mathbf{z})$ are greater than one. However, this condition is more restrictive, due to that a repelling fixed point has the potential to be a snapback repeller, without satisfying this condition.

Theorem (Li, et al., 2008). Let H_λ be a one-parameter family of continuous maps on $\mathbb{R}^n \times \mathbb{R}^m$ such that $H_\lambda(\mathbf{x}, \mathbf{y})$ is continuous as a function of $\lambda \in \mathbb{R}^l$ and $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m$. Assume that $H_0(\mathbf{x}, \mathbf{y}) = (F(\mathbf{x}), G(\mathbf{x}))$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and has a snapback repeller and $G : \mathbb{R}^n \to \mathbb{R}^m$. Then H_λ has a positive topological entropy for all λ sufficiently close to 0.

Theorem (Li, et al.,2008). Let H_λ be a one-parameter family of continuous maps on $\mathbb{R}^n \times \mathbb{R}^m$ such that $H_\lambda(\mathbf{z})$ is continuous as a function of $\lambda \in \mathbb{R}^l$ and $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m$. Assume that $H_0(\mathbf{x}, \mathbf{y}) = (F(\mathbf{x}), G(\mathbf{x}, \mathbf{y}))$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and has a snapback repeller, $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, and $G(\mathbb{R}^n \times S) \subset \text{int}(S)$ for some compact set $S \subset \mathbb{R}^m$ homeomorphic to the closed unit ball in $\mathbb{R}^m$. Then H_λ has a positive topological entropy for all λ sufficiently close to 0.

Moreover, it was shown in (Li & Lyu,2009) that if F has a snapback repeller and G is a small C^1 perturbation of F, then G has a snapback repeller, positive topological entropy, as the implicit function theorem is applied. Moreover, G is chaotic in the sense of Li-Yorke. More precisely,

Theorem (Li & Lyu, 2009). Let F be a C^1 map on $\mathbb{R}^n$ with a snapback repeller. If G is a C^1 map on $\mathbb{R}^n$ such that $\|F - G\| + \|DF - DG\|_\star$ is small enough, where $\|\cdot\|_\star$ is the operator norm on the space of linear maps on $\mathbb{R}^n$ induced by the Euclidean norm $\|\cdot\|$, then G has a snapback repeller, exhibits Li-Yorke chaos, and has positive topological entropy.

Corollary (Li & Lyu, 2009). Let $F_\mu(\mathbf{x})$ be a one-parameter family of C^1 maps with $\mathbf{x} \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^l$. Assume that $F_\mu(\mathbf{x})$ is C^1 as a function jointly of $\mathbf{x}$ and μ and that F_{μ_0} has a snapback repeller. Then map F_μ has a snapback repeller, exhibits Li-Yorke chaos, and has positive topological entropy, for all μ sufficiently close to μ_0.
In (Shi & Chen, 2004, 2008), they generalized the definitions of expanding fixed point, snapback repeller, homoclinic orbit, and heteroclinic orbit for a continuously differentiable map from $\mathbb{R}^n$ to general metric spaces as follows. Herein, $B_r^d(\mathbf{x})$ denotes the closed balls of radius r centered at $\mathbf{x} \in X$ under metric d, i.e.

$$B_r^d(\mathbf{z}) := \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{z}) \leq r\}.$$

In the following, we introduce the coupled-expanding map.
Definition (Shi & Chen, 2008). Let $F : D \subset X \to X$ be a map where (X, d) is a metric space. If there exists $\ell \geq 2$ subsets V_i, $1 \leq i \leq \ell$, of D with $V_i \cap V_j = \partial_D V_i \cap \partial_D V_j$ for each pair of (i, j), $1 \leq i \neq j \leq \ell$, such that

$$F(V_i) \supset \cup_{j=1}^{\ell} V_j,\ 1 \leq i \leq \ell,$$

where $\partial_D V_i$ is the relative boundary of V_i with respect to D, then F is said to be *coupled-expanding* in V_i, $1 \leq i \leq \ell$. Moreover, the map F is said to be *strictly coupled-expanding* in V_i, $1 \leq i \leq \ell$, if $d(V_i, V_j) > 0$, for all $1 \leq i \neq j \leq \ell$.

Definition (Shi & Chen, 2004). Let $F : X \to X$ be a map on metric space (X, d). (i) A point $\mathbf{z} \in X$ is called an *expanding fixed point* (or a *repeller*) of F in $B_{r_0}^d(\mathbf{z})$ for some constant $r_0 > 0$, if

$F(\mathbf{z}) = \mathbf{z}$ and

$$d(F(\mathbf{x}), F(\mathbf{y})) \geq \lambda d(\mathbf{x}, \mathbf{y}), \text{ for all } \mathbf{x}, \mathbf{y} \in B^d_{r_0}(\mathbf{z})$$

for some constant $\lambda > 1$. Moreover, $\mathbf{z}$ is called a *regular expanding fixed point* of F in $B^d_{r_0}(\mathbf{z})$ if $\mathbf{z}$ is an interior point of $F(\text{int}(B^d_{r_0}(\mathbf{z})))$.
(ii) Let $\mathbf{z}$ be an expanding fixed point of F in $B^d_{r_0}(\mathbf{z})$ for some $r_0 > 0$. Then $\mathbf{z}$ is said to be a *snapback repeller* of F if there exists a point $\mathbf{x}_0 \in \text{int}(B^d_{r_0}(\mathbf{z}))$ with $\mathbf{x}_0 \neq \mathbf{z}$ and $F^{\ell}(\mathbf{x}_0) = \mathbf{z}$ for some positive integer $\ell \geq 2$. Moreover, $\mathbf{z}$ is said to be a *nondegenerate snapback repeller* of F if there exist positive constants μ and δ_0 such that $\text{int}(B^d_{\delta_0}(\mathbf{x}_0)) \subset \text{int}(B^d_{r_0}(\mathbf{z}))$ and

$$d(F^{\ell}(\mathbf{x}), F^{\ell}(\mathbf{y})) \geq \mu d(\mathbf{x}, \mathbf{y}), \text{ for all } \mathbf{x}, \mathbf{y} \in B^d_{\delta_0}(\mathbf{x}_0);$$

$\mathbf{z}$ is called a *regular snapback repeller* of F if $F(\text{int}(B^d_{r_0}(\mathbf{z})))$ is open and there exists a positive constant δ_0^* such that $\text{int}(B^d_{\delta_0^*}(\mathbf{x}_0)) \subset \text{int}(B^d_{r_0}(\mathbf{z}))$ and $\mathbf{z}$ is an interior point of $F^{\ell}(\text{int}(B^d_{\delta}(\mathbf{x}_0)))$ for any positive constant $\delta \leq \delta_0^*$.
(iii) Assume that $\mathbf{z} \in X$ is a regular expanding fixed point of F. Let U be the maximal open neighborhood of $\mathbf{z}$ in the sense that for any $\mathbf{x} \in U$ with $\mathbf{x} \neq \mathbf{z}$, there exists $k_0 \geq 1$ with $F^{k_0}(\mathbf{x}) \notin U$, $F^{-k}(\mathbf{x})$ is uniquely defined in U for all $k \geq 1$, and $F^{-k}(\mathbf{x}) \to \mathbf{z}$ as $k \to \infty$. U is called *the local unstable set* of F at $\mathbf{z}$ and is denoted by $W^u_{loc}(\mathbf{z})$.
(iv) Let $\mathbf{z} \in X$ be a regular expanding fixed point of F. A point $\mathbf{x} \in X$ is called *homoclinic* to $\mathbf{z}$ if $\mathbf{x} \in W^u_{loc}(\mathbf{z})$, $\mathbf{x} \neq \mathbf{z}$, and there exists an integer $m \geq 1$ such that $F^m(\mathbf{x}) = \mathbf{z}$. A homoclinic orbit to $\mathbf{z}$, consisting of a homoclinic point $\mathbf{x}$ with $F^m(\mathbf{x}) = \mathbf{z}$, its backward orbit $\{F^{-j}(\mathbf{x})\}_{j=1}^{\infty}$, and its finite forward orbit $\{F^j(\mathbf{x})\}_{j=1}^{m-1}$, is called *nondegenerate* if for each point $\mathbf{x}_j$ on the homoclinic orbit there exist positive constants r_j and μ_j such that

$$d(F(\mathbf{x}), F(\mathbf{y})) \geq \mu_j d(\mathbf{x}, \mathbf{y}), \text{ for all } \mathbf{x}, \mathbf{y} \in B^d_{r_j}(\mathbf{x}_j).$$

A homoclinic orbit is called *regular* if for each point $\mathbf{x}_j$ on the orbit, there exists a positive constant $\tilde{r}_j$ such that for any positive constant $r \leq \tilde{r}_j$, $F(\mathbf{x}_j)$ is an interior point of $F(\text{int}(B^d_r(\mathbf{x}_j)))$. Otherwise, it is called *singular*. A point $\mathbf{x}$ is called *heteroclinic* to $\mathbf{z}$, if $\mathbf{x} \in W^u_{loc}(\mathbf{z})$ and there exists a $m \geq 1$ such that $F^m(\mathbf{x})$ lies on a different periodic orbit from $\mathbf{z}$.
Notice that if a map F on $\mathbb{R}^n$ has a snapback repeller, and is continuously differentiable in some neighborhood of $\mathbf{x}_j = F^j(\mathbf{x}_0)$, for $0 \leq j \leq \ell - 1$, then the snapback repeller is regular and nondegenerate. For continuously differentiable finite-dimensional maps, the definition of snapback repeller has been extended in (Shi & Chen, 2004, 2008) to the maps in the general metric space, through introducing the two classifications: regular and singular, nondegenerate and degenerate. It was proved that a map F is a strict coupled-expansion and chaotic in the sense of both Devaney and Li-Yorke if F has a nondegenerate and regular snapback repeller or a nondegenerate and regular homoclinic orbit to an expanding fixed point. Moreover, if F is C^1 in $\mathbb{R}^n$ and has a snapback repeller under Marotto's definition, then the snapback repeller is nondegenerate and regular. Therefore, F is chaotic in the sense of Marotto, Devaney, and Li-Yorke. In addition, more general scenario for degenerate and regular snapback repeller, was studied in (Shi & Yu, 2008).

4.3 Some remarks

We summarize some results concerning the above-mentioned notions.
(i) For a compact metric space (X, d), chaos in the sense of Devaney implies chaos in the sense of Li-Yorke.
(ii) If a map F has a snapback repeller $\mathbf{z}$, then F is chaotic in the sense of Marotto and Li-Yorke.
(iii) If a map $F : \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\sigma : \Sigma_2^+ \to \Sigma_2^+$, for some Cantor set Λ, then F is chaotic on Λ in the sense of Devaney and Li-Yorke.
(iv) For a complete metric space (X, d) and a map $F : X \to X$, if F has a regular nondegenerate snapback repeller $\mathbf{z} \in X$, then there exists a Cantor set Λ so that $F^m : \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\sigma : \Sigma_2^+ \to \Sigma_2^+$, for some integer m. Consequently, F^m is chaotic on Λ in the sense of Devaney and Li-Yorke.
(v) For a complete metric space (X, d) and a C^1 map $F : X \to X$, if F has a Marotto's snapback repeller $\mathbf{z}$, then $\mathbf{z}$ is also a regular nondegenerate snapback repeller. Hence, F^m is chaotic in the sense of Devaney and Li-Yorke, for some integer m.
(vi) If a map F has a transversal homoclinic orbit, then there exists a Cantor set Λ so that $F^m : \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\sigma : \Sigma_2^+ \to \Sigma_2^+$, for some integer m. Consequently, F^m is chaotic on Λ in the sense of Devaney and Li-Yorke.

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6. References

Arik, S. (2000) Global asymptotic stability of a class of dynamical neural networks, *IEEE Trans. Circuits Syst. I*, Vol. 47: 568–571.

Banks, J., Brooks, J., Cairns, G., Davis, G. & Stacey, P. (1992) On Devaney's definition of chaos, *Amer. Math. Monthly*, Vol. 99: 332–334.

Bauer, F., Atay, F. M. & Jost, J. (2009) Synchronization in discrete-time networks with general pairwise coupling, *Nonlinearity*, Vol. 22: 2333–2351.

Bauer, F., Atay, M.F. &, Jost J. (2010) Synchronized chaos in networks of simple units, *Europhys. Lett.*, Vol 89: 20002.

Buric, N. & Todorovic, D. (2003) Dynamics of Fitzhugh-Nagumo excitable systems with delayed coupling, *Phys. Rev. E*, Vol. 67: 066222.

Campbell, S. A. (2006) Time delays in neural systems, in McIntosh, A. R. and Jirsa, V. K.(ed.), *Handbook of Brain Connectivity*, Springer-Verlag, New York, pp. 65–90.

Chen, H. C., Hung, Y. C., Chen, C. K. & Liao, T. L. (2006) Image-processing algorithms realized by discrete-time cellular neural networks and their circuit implementations, *Chaos, Solitons and Fractals*, Vol. 29 (No. 5): 1100–1108.

Chen, L. & Aihara, K. (1995) Chaotic simulated annealing for combinatorial optimization, *Neural Networks*, Vol. 8: 915–930.

Chen, L. & Aihara, K. (1997) Chaos and asymptotical stability in discrete-time neural networks, *Physica D*, Vol. 104: 286–325.

Chen, L. & Aihara, K. (1999) Global searching ability of chaotic neural networks, *IEEE Trans. Circuits Syst. I: Fundamental Theory and Applications*, Vol. 46: 97–993.

Chen, S. S. & Shih, C. W. (2002) Transversal homoclinic orbits in a transiently chaotic neural network, *Chaos*, Vol. 12 (No.3): 654–671.

Chen, S. S. & Shih, C. W. (2004a) Dynamics for discrete-time cellular neural networks, *Intern. J. Biburc. Chaos*, Vol. 14 (No. 8): 2667–2687.

Chen, S. S. & Shih, C. W. (2004b) Asymptotic behaviors in a transiently chaotic neural networks, *Discrete and Continuous Dynamical systems*, Vol. 10: 805–826.

Chen, S. S. & Shih, C. W. (2008) Solving TSP via Neural Networks, in Federico Greco(ed), *Travelling Salesman Problem*, I-Tech Education and Publishing, pp. 117–134.

Chen, S. S. & Shih, C. W. (2009) Transiently chaotic neural networks with piecewise linear output functions, *Chaos, Solitons and Fractals*, Vol. 39: 717–730.

Chua, L. O. & Yang, L. (1988) Cellular neural networks, theory, *IEEE Trans. Circuits Syst.*, Vol. 35: 1257–72.

Chua, L. O. (1998) *CNN: A Paradigm for Complexity*, World Scientific.

Devaney, R.L. (1989) *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, New York.

Forti, M. & Tesi, A. (1995) New conditions for global stability of neural networks with application to linear and quadratic programming problems, *IEEE Transations on Circuit Systems-I: Fundament Theory and Application*, Vol. 42 (No. 7): 354–366.

García, B.C. (1986) Chaos and topological entropy in dimension $n > 1$, *Erg. Theory Dyn. Syst.*, Vol. 6: 163–165.

Grossberg, S. (1978), Competition, decision, consensus, *J. Math. Anal. Appl.*, Vol. 66: 470 –493.

Hale, J. K. & Lin, X. (1986) Symbolic dynamics and nonlinear semiflow, *Annali di Matematica Pura ed Applicata*, Vol. 144: 229–259.

Hale, J. & Raugel, G. (1992) Convergence in gradient-like systems with applications to PDE, *ZAMP*, Vol. 43: 63–124.

Hänggi, M., Reddy, H. C. & Moschytz, G. S. (1999) Unifying results in CNN theory using delta operator, *IEEE Int. Symp. Circuits Syst.*, Vol. 3: 547–550.

Harrer, H. & Nossek, J. A. (1992) An analog implementation of discrete-time cnns, *IEEE Transactions on Neural Networks*, Vol. 3: 466–476.

Hirsh, M. (1989) Convergent activation dynamics in continuous time networks, *Neural Networks*, Vol. 2: 331–349.

Huang, W. & Ye, X. (2002) Devaney's chaos or 2-scattering implies Li-Yorke's chaos, *Topology Appl.*, Vol. 117: 259–272.

Jost, J. & Joy, M. (2001), Spectral properties and synchronization in coupled map lattices, *Phys. Rev. E*, Vol. 65: 016201.

Kaneko, K. (1984) Period-doubling of kink-antikink patterns, quasi-periodicity in antiferro-like structures and spatial intermittency in coupled map lattice-toward a prelude to a field theory of chaos, *Prog. Theor. Phys.*, Vol. 72: 480–486.

LaSalle, J. P. (1976) The stability of dynamical systems, *Regional Conference Series in Applied Mathematics*, Vol. 25, SIAM, Philadelphia.

Li, C. & Chen, G. (2003). An improved version of the Marotto theorem, *Chaos, Solitons and Fractals*, Vol. 18: 69-77.

Li, M. C., Lyu, M. J. & Zgliczyński, P. (2008) Topological entropy for multidimensional perturbations of snap-back repellers and one-dimensional maps, *Nonlinearity*, Vol. 21: 2555– 2567.

Li, M.C. & Lyu, M.J. (2009) A simple proof for persistence of snap-back repellers, *J. Math. Anal. Appl.*, Vol. 52: 669-671.

Li, T.Y. & Yorke, J.A. (1975) Period three implies chaos, *Amer. Math. Monthly*, Vol. 82: 985–992.

Liang, J.L. & Cao, J. (2004) Exponential stability of continuous-time and discrete-time bidirectional associative memory networks with delays, *Chaos, Solitons and Fractals*, Vol. 22 (No. 4): 773–785.

Liao, K. L. & Shih, C. W. (2011) Snapback repellers and homoclinic orbits for multi-dimensional maps, Submitted.

Liu, Y.R., Wang, Z.D. & Liu, X.H. (2008) Robust stability of discrete-time stochastic neural networks with time-varying delays, *Neurocomputing*, Vol. 71: 823–833.

Lu, W. & Chen, T (2004) Synchronization analysis of linearly coupled networks of discrete time systems, *Physica D*, Vol. 198: 148–168.

Mak, K. L., Peng, J. G., Xu, Z. B. & Yiu, K. F. (2007) A new stability criterion for discrete-time neural networks: Nonlinear spectral radius, *Chaos, Solitons and Fractals*, Vol. 31 (No. 2): 424–436.

Marotto, F. R. (1978) Snap-back repellers imply chaos in $\mathbb{R}^n$, *J. Math. Anal. Appl.*, Vol. 63: 199–223.

Marotto, F. R. (1979a) Perturbations of stable and chaotic difference equations, *J. Math. Anal. Appl.*, Vol. 72: 716–729.

Marotto, F. R. (1979b) Chaotic behavior in the Hénon mapping, *Commun. Math. Phys.*, Vol. 68: 187–194.

Marotto, F. R. (2005) On redefining a snap-back repeller, *Chaos, Solitions and Fractals*, Vol. 25: 25–28.

Peterson, C. & Söderberg, B. (1993) Artificial neural networks, in C. Reeves(ed.), *Modern Heuristic Techniques for Combinatorial Optimization*, Blackwell Scientific Publishing, Oxford, England, pp. 197–242.

Robinson, C. (1999) *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos 2nd edn*, Boca Raton, FL: CRC Press.

Roska, T. & Chua L. O. (1992) Cellular neural networks with nonlinear and delay-type template, *Int. J. Circuit Theory and Appl.*, Vol. 20: 469–481.

Shi, Y. & Chen, G. (2004) Chaos of discrete dynamical systems in complete metric spaces, *Chaos, Solitons and Fractals*, Vol. 22: 555–571.

Shi, T. & Yu, P. (2008) Chaos induced by regular snap-back repellers, *J. Math. Anal. Appl.*, Vol.337: 1480–1494.

Shih, C. W. (2000) Influence of boundary conditions on pattern formation and spatial chaos in lattice systems, *SIAM J. Applied Math.*, Vol. 61 (No. 1): 335–368.

Shih, C. W. & Weng, C. W. (2002) On the templates corresponding to cycle-symmetric connectivity in cellular neural networks, *Intern. Bifurc. Chaos*, Vol. 12: 2957–2966.

Shih, C. W. & Tseng, J. P. (2009) Global consensus for discrete-time competitive systems, *Chaos, Solitons and Fractals*, Vol 41: 302–310.

Smale, S. (1967) Differentiable dynamics systems, *Bull. Amer. Math. Soc.*, Vol. 73: 747–817.

Steinlein, H. & Walther, H. (1990) Hyperbolic sets, transversal homoclinic trajectories, and symbolic dynamics for C^1-maps in Banach spaces, *J. Dyna. Diff. Equa.*, Vol. 3: 325–365.

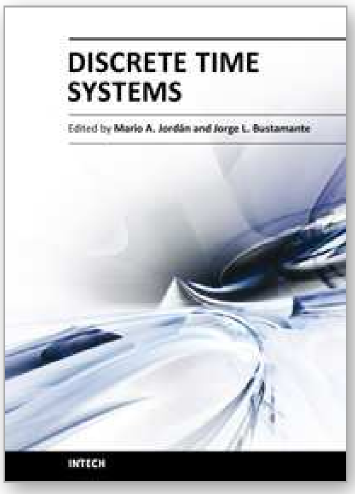
Wang, L. & Zou, X. (2008) Convergence of DiscreteąVTime Neural Networks with Delays, *International Journal of Qualitative Theory of Differential Equations and Applications*, Vol. 2 (No. 1): 24–37.

Wu, J. (2001) *Introduction to Neural Dynamics and Signal Transmission Delay*, Walter de Gruyter, Berlin.

Wu, Z., Su, H., Chu, J. & Zhou, W. (2009) New results on robust exponential stability for discrete recurrent neural networks with time-varying delays, *Neurocomputing*, Vol. 72: 3337–3342.

Yua, J., Zhang, K. & Fei, S. (2010) Exponential stability criteria for discrete-time recurrent neural networks with time-varying delay, *Nonlinear Analysis: Real World Applications*, Vol. 11: 207–216.

Yuan, L., Yuan, Z. & He, Y. (2009) Convergence of non-autonomous discrete-time Hopfield model with delays, *Neurocomputing* ,Vol. 72: 3802–3808.



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Discrete-Time Systems comprehend an important and broad research field. The consolidation of digital-based computational means in the present, pushes a technological tool into the field with a tremendous impact in areas like Control, Signal Processing, Communications, System Modelling and related Applications. This book attempts to give a scope in the wide area of Discrete-Time Systems. Their contents are grouped conveniently in sections according to significant areas, namely Filtering, Fixed and Adaptive Control Systems, Stability Problems and Miscellaneous Applications. We think that the contribution of the book enlarges the field of the Discrete-Time Systems with signification in the present state-of-the-art. Despite the vertiginous advance in the field, we also believe that the topics described here allow us also to look through some main tendencies in the next years in the research area.