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# Discrete Deterministic and Stochastic Dynamical Systems with Delay - Applications

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## 1. Introduction

The dynamical systems with discrete time and delay are obtained by the discretization of the systems of differential equations with delay, or by modeling some processes in which the time variable is  $n \in \mathbf{N}$  and the state variables at the moment  $n - m$ , where  $m \in \mathbf{N}$ ,  $m \geq 1$ , are taken into consideration.

The processes from this chapter have as mathematical model a system of equations given by:

$$x_{n+1} = f(x_n, x_{n-m}, \alpha), \quad (1)$$

where  $x_n = x(n) \in \mathbf{R}^p$ ,  $x_{n-m} = x(n - m) \in \mathbf{R}^p$ ,  $\alpha \in \mathbf{R}$  and  $f : \mathbf{R}^p \times \mathbf{R}^p \times \mathbf{R} \rightarrow \mathbf{R}^p$  is a seamless function,  $n, m \in \mathbf{N}$  with  $m \geq 1$ . The properties of function  $f$  ensure that there is solution for system (1). The system of equations (1) is called *system with discrete-time and delay*. The analysis of the processes described by system (1) follows these steps.

*Step 1.* Modeling the process.

*Step 2.* Determining the fixed points for (1).

*Step 3.* Analyzing a fixed point of (1) by studying the sign of the characteristic equation of the linearized equation in the neighborhood of the fixed point.

*Step 4.* Determining the value  $\alpha = \alpha_0$  for which the characteristic equation has the roots  $\mu_1(\alpha_0) = \mu(\alpha_0)$ ,  $\mu_2(\alpha_0) = \bar{\mu}(\alpha_0)$  with their absolute value equal to 1, and the other roots with their absolute value less than 1 and the following formulas:

$$\left. \frac{d|\mu(\alpha)|}{d\alpha} \right|_{\alpha=\alpha_0} \neq 0, \quad \mu(\alpha_0)^k \neq 1, \quad k = 1, 2, 3, 4$$

hold.

*Step 5.* Determining the local center manifold  $W_{loc}^c(0)$ :

$$y = zq + \bar{z}\bar{q} + \frac{1}{2}w_{20}z^2 + w_{11}z\bar{z} + \frac{1}{2}w_{02}\bar{z}^2 + \dots$$

where  $z = x_1 + ix_2$ , with  $(x_1, x_2) \in V_1 \subset \mathbf{R}^2$ ,  $0 \in V_1$ ,  $q$  an eigenvector corresponding to the eigenvalue  $\mu(0)$  and  $w_{20}, w_{11}, w_{02}$  are vectors that can be determined by the invariance

condition of the manifold  $W_{loc}^c(0)$  with respect to the transformation  $x_{n-m} = x^1, \dots, x_n = x^m, x_{n+1} = x^{m+1}$ . The restriction of system (1) to the manifold  $W_{loc}^c(0)$  is:

$$z_{n+1} = \mu(\alpha_0)z_n + \frac{1}{2}g_{20}z_n^2 + g_{11}z_n\bar{z}_n + \frac{1}{2}g_{02}\bar{z}_n^2 + g_{21}z_n^2\bar{z}_n/2, \quad (2)$$

where  $g_{20}, g_{11}, g_{02}, g_{21}$  are the coefficients obtained using the expansion in Taylor series including third-order terms of function  $f$ .

System (2) is topologically equivalent with the prototype of the 2-dimensional discrete dynamic system that characterizes the systems with a Neimark–Sacker bifurcation.

*Step 6.* Representing the orbits for system (1). The orbits of system (1) in the neighborhood of the fixed point  $x^*$  are given by:

$$x_n = x^* + z_n q + \bar{z}_n \bar{q} + \frac{1}{2}r_{20}z_n^2 + r_{11}z_n\bar{z}_n + \frac{1}{2}r_{02}\bar{z}_n^2 \quad (3)$$

where  $z_n$  is a solution of (2) and  $r_{20}, r_{11}, r_{02}$  are determined with the help of  $w_{20}, w_{11}, w_{02}$ .

The properties of orbit (3) are established using the Lyapunov coefficient  $l_1(0)$ . If  $l_1(0) < 0$  then orbit (3) is a stable invariant closed curve (supercritical) and if  $l_1(0) > 0$  then orbit (3) is an unstable invariant closed curve (subcritical).

The perturbed stochastic system corresponding to (1) is given by:

$$x_{n+1} = f(x_n, x_{n-m}, \alpha) + g(x_n, x_{n-m})\xi_n, \quad (4)$$

where  $x_n = x_{0n}, n \in I = \{-m, -m+1, \dots, -1, 0\}$  is the initial segment to be  $\mathcal{F}_0$ -measurable, and  $\xi_n$  is a random variable with  $E(\xi_n) = 0, E(\xi_n^2) = \sigma > 0$  and  $\alpha$  is a real parameter.

System (4) is called discrete-time stochastic system with delay.

For the stochastic discrete-time system with delay, the stability in mean and the stability in square mean for the stationary state are done.

This chapter is organized as follows. In Section 2 the discrete-time deterministic and stochastic dynamical systems are defined. In Section 3 the Neimark-Sacker bifurcation for the deterministic and stochastic Internet control congestion with discrete-time and delay is studied. Section 4 presents the deterministic and stochastic economic games with discrete-time and delay. In Section 5, the deterministic and stochastic Kaldor model with discrete-time is analyzed. Finally some conclusions and future prospects are provided.

For the models from the above sections we establish the existence of the Neimark-Sacker bifurcation and the normal form. Then, the invariant curve is studied. We also associate the perturbed stochastic system and we analyze the stability in square mean of the solutions of the linearized system in the fixed point of the analyzed system.

## 2. Discrete-time dynamical systems

### 2.1 The definition of the discrete-time, deterministic and stochastic systems

We intuitively describe the dynamical system concept. We suppose that a physical or biologic or economic system etc., can have different states represented by the elements of a set  $S$ . These states depend on the parameter  $t$  called *time*. If the system is in the state  $s_1 \in S$ , at the moment  $t_1$  and passes to the moment  $t_2$  in the state  $s_2 \in S$ , then we denote this transformation by:

$$\Phi_{t_1, t_2}(s_1) = s_2$$

and  $\Phi_{t_1,t_2} : S \rightarrow S$  is called *evolution operator*. In the deterministic evolutive processes the evolution operator  $\Phi_{t_1,t_2}$ , satisfies the Chapman-Kolmogorov law:

$$\Phi_{t_3,t_2} \circ \Phi_{t_2,t_1} = \Phi_{t_3,t_1}, \quad \Phi_{t,t} = id_S.$$

For a fixed state  $s_0 \in S$ , application  $\Phi : \mathbb{R} \rightarrow S$ , defined by  $t \rightarrow s_t = \Phi_t(s_0)$ , determines a curve in set  $S$  that represents the evolution of state  $s_0$  when time varies from  $-\infty$  to  $\infty$ .

An evolutive system in the general form is given by a subset of  $S \times S$  that is the graphic of the system:

$$F_i(t_1, t_2, s_1, s_2) = 0, \quad i = 1..n$$

where  $F_i : \mathbb{R}^2 \times S \rightarrow \mathbb{R}$ .

In what follows, the arithmetic space  $\mathbb{R}^m$  is considered to be the states' space of a system, and the function  $\Phi$  is a  $C^r$ -class differentiable application.

An *explicit differential dynamical system of  $C^r$  class*, is the homomorphism of groups  $\Phi : (\mathbb{R}, +) \rightarrow (Diff^r(\mathbb{R}^m), \circ)$  so that the application  $\mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  defined by  $(t, x) \rightarrow \Phi(t)(x)$  is a differentiable of  $C^r$ -class and for all  $x \in \mathbb{R}^m$  fixed, the corresponding application  $\Phi(x) : \mathbb{R} \rightarrow \mathbb{R}^m$  is  $C^{r+1}$ -class.

A differentiable dynamical system on  $\mathbb{R}^m$  describes the evolution in continuous time of a process. Due to the fact that it is difficult to analyze the continuous evolution of the state  $x_0$ , the analysis is done at the regular periods of time, for example at  $t = -n, \dots, -1, 0, 1, \dots, n$ . If we denote by  $\Phi_1 = f$ , we have:

$$\begin{aligned} \Phi_1(x_0) &= f(x_0), \Phi_2(x_0) = f^{(2)}(x_0), \dots, \Phi_n(x_0) = f^{(n)}(x_0), \\ \Phi_{-1}(x_0) &= f^{(-1)}(x_0), \dots, \Phi_{-n}(x_0) = f^{(-n)}(x_0), \end{aligned}$$

where  $f^{(2)} = f \circ f, \dots, f^{(n)} = f \circ \dots \circ f, f^{(-n)} = f^{(-1)} \circ \dots \circ f^{(-1)}$ .

Thus,  $\Phi$  is determined by the diffeomorphism  $f = \Phi_1$ .

A  $C^r$ -class differential dynamical system with discrete time on  $\mathbb{R}^m$ , is the homomorphism of groups  $\Phi : (\mathbb{Z}, +) \rightarrow (Diff^r(\mathbb{R}^m), \circ)$ .

The orbit through  $x_0 \in \mathbb{R}^m$  of a dynamical system with discrete-time is:

$$O_f(x_0) = \{\dots, f^{(-n)}(x_0), \dots, f^{(-1)}(x_0), x_0, f(x_0), \dots, f^{(n)}(x_0), \dots\} = \{f^{(n)}(x_0)\}_{n \in \mathbb{Z}}.$$

Thus  $O_f(x_0)$  represents a sequences of images of the studied process at regular periods of time.

For the study of a dynamical system with discrete time, the structure of the orbits'set is analyzed. For a dynamical system with discrete time with the initial condition  $x_0 \in \mathbb{R}^m$  ( $m = 1, 2, 3$ ) we can represent graphically the points of the form  $x_n = f^n(x_0)$  for  $n$  iterations of the thousandth or millionth order. Thus, a visual geometrical image of the orbits'set structure is created, which suggests some properties regarding the particularities of the system. Then, these properties have to be approved or disproved by theoretical or practical arguments.

An *explicit dynamical system with discrete time* has the form:

$$x_{n+1} = f(x_{n-p}, x_n), \quad n \in \mathbb{N}, \tag{5}$$

where  $f : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m, x_n \in \mathbb{R}^m, p \in \mathbb{N}$  is fixed, and the initial conditions are  $x_{-p}, x_{1-p}, \dots, x_0 \in \mathbb{R}^m$ .

For system (5), we use the change of variables  $x^1 = x_{n-p}$ ,  $x^2 = x_{n-(p-1)}$ , ...,  $x^p = x_{n-1}$ ,  $x^{p+1} = x_n$ , and we associate the application

$$F : (x^1, \dots, x^{p+1}) \in \mathbf{R}^m \times \dots \times \mathbf{R}^m \rightarrow \mathbf{R}^m \times \dots \times \mathbf{R}^m$$

given by:

$$F : \begin{pmatrix} x^1 \\ \vdots \\ x^p \\ x^{p+1} \end{pmatrix} \rightarrow \begin{pmatrix} x^2 \\ \vdots \\ x^{p+1} \\ f(x^1, x^{p+1}) \end{pmatrix}.$$

Let  $(\Omega, \mathcal{F})$  be a measurable space, where  $\Omega$  is a set whose elements will be noted by  $\omega$  and  $\mathcal{F}$  is a  $\sigma$ -algebra of subsets of  $\Omega$ . We denote by  $\mathcal{B}(\mathbf{R})$   $\sigma$ -algebra of Borelian subsets of  $\mathbf{R}$ . A *random variable* is a measurable function  $X : \Omega \rightarrow \mathbf{R}$  with respect to the measurable spaces  $(\Omega, \mathcal{F})$  and  $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$  (Kloeden et al., 1995).

A probability measure  $P$  on the measurable space  $(\Omega, \mathcal{F})$  is a  $\sigma$ -additive function defined on  $\mathcal{F}$  with values in  $[0, 1]$  so that  $P(\Omega) = 1$ . The triplet  $(\Omega, \mathcal{F}, P)$  is called a *probability space*.

An arbitrary family  $\zeta(n, \omega) = \zeta(n)(\omega)$  of random variables, defined on  $\Omega$  with values in  $\mathbf{R}$ , is called *stochastic process*. We denote  $\zeta(n, \omega) = \zeta(n)$  for any  $n \in \mathbf{N}$  and  $\omega \in \Omega$ . The functions  $X(\cdot, \omega)$  are called the trajectories of  $X(n)$ . We use  $E(\zeta(n))$  for the mean value and  $E(\zeta(n)^2)$  the square mean value of  $\zeta(n)$  denoted by  $\zeta_n$ .

The perturbed stochastic of system (5) is:

$$x_{n+1} = f(x_{n-p}, x_n) + g(x_n)\zeta_n, \quad n \in \mathbf{N}$$

where  $g : \mathbf{R}^n \rightarrow \mathbf{R}^n$  and  $\zeta_n$  is a random variable which satisfies the conditions  $E(\zeta_n) = 0$  and  $E(\zeta_n^2) = \sigma > 0$ .

## 2.2 Elements used for the study of the discrete-time dynamical systems

Consider the following discrete-time dynamical system defined on  $\mathbf{R}^m$ :

$$x_{n+1} = f(x_n), \quad n \in \mathbf{N} \tag{6}$$

where  $f : \mathbf{R}^m \rightarrow \mathbf{R}^m$  is a  $C^r$  class function, called *vector field*.

Some information, regarding the behavior of (6) in the neighborhood of the fixed point, is obtained studying the associated linear discrete-time dynamical system.

Let  $x_0 \in \mathbf{R}^m$  be a fixed point of (6). The system

$$u_{n+1} = Df(x_0)u_n, \quad n \in \mathbf{N}$$

where

$$Df(x_0) = \left( \frac{\partial f^i}{\partial x^j} \right) (x_0), \quad i, j = 1..m$$

is called *the linear discrete-time dynamical system* associated to (6) and the fixed point  $x_0 = f(x_0)$ . If the characteristic polynomial of  $Df(x_0)$  does not have roots with their absolute values equal to 1, then  $x_0$  is called a *hyperbolic fixed point*.

We have the following classification of the hyperbolic fixed points:

1.  $x_0$  is a stable point if all characteristic exponents of  $Df(x_0)$  have their absolute values less than 1.
2.  $x_0$  is an unstable point if all characteristic exponents of  $Df(x_0)$  have their absolute values greater than 1.
3.  $x_0$  is a saddle point if a part of the characteristic exponents of  $Df(x_0)$  have their absolute values less than 1 and the others have their absolute values greater than 1.

The orbit through  $x_0 \in \mathbb{R}^m$  of a discrete-time dynamical system generated by  $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is *stable* if for any  $\epsilon > 0$  there exists  $\delta(\epsilon)$  so that for all  $x \in B(x_0, \delta(\epsilon))$ ,  $d(f^n(x), f^n(x_0)) < \epsilon$ , for all  $n \in \mathbb{N}$ .

The orbit through  $x_0 \in \mathbb{R}^m$  is *asymptotically stable* if there exists  $\delta > 0$  so that for all  $x \in B(x_0, \delta)$ ,  $\lim_{n \rightarrow \infty} d(f^n(x), f^n(x_0)) = 0$ .

If  $x_0$  is a fixed point of  $f$ , the orbit is formed by  $x_0$ . In this case  $O(x_0)$  is stable (asymptotically stable) if  $d(f^n(x), x_0) < \epsilon$ , for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} f^n(x) = x_0$ .

Let  $(\Omega, \mathcal{F}, P)$  be a probability space. The perturbed stochastic system of (6) is the following system:

$$x_{n+1} = f(x_n) + g(x_n)\xi_n$$

where  $\xi_n$  is a random variable that satisfies  $E(\xi_n) = 0$ ,  $E(\xi_n^2) = \sigma$  and  $g(x_0) = 0$  with  $x_0$  the fixed point of the system (6).

The linearized of the discrete stochastic dynamical system associated to (6) and the fixed point  $x_0$  is:

$$u_{n+1} = Au_n + \xi_n Bu_n, \quad n \in \mathbb{N} \tag{7}$$

where

$$A = \left( \frac{\partial f^i}{\partial x^j} \right) (x_0), \quad B = \left( \frac{\partial g^i}{\partial x^j} \right) (x_0), \quad i, j = 1..m.$$

We use  $E(u_n) = E_n$ ,  $E(u_n u_n^T) = V_n$ ,  $u_n = (u_n^1, u_n^2, \dots, u_n^m)^T$ .

**Proposition 2.1.** (i) The mean values  $E_n$  satisfy the following system of equations:

$$E_{n+1} = AE_n, \quad n \in \mathbb{N} \tag{8}$$

(ii) The square mean values satisfy:

$$V_{n+1} = AV_n A^T + \sigma B V_n B^T, \quad n \in \mathbb{N} \tag{9}$$

**Proof.** (i) From (7) and  $E(\xi_n) = 0$  we obtain (8).

(ii) Using (7) we have:

$$u_{n+1} u_{n+1}^T = Au_n u_n^T A^T + \xi_n (Au_n u_n^T B^T + Bu_n u_n^T A^T) + \xi_n^2 Bu_n u_n^T B^T. \tag{10}$$

By (10) and  $E(\xi_n) = 0$ ,  $E(\xi_n^2) = \sigma$  we get (9).

Let  $\bar{A}$  be the matrix of system (8), respectively (9). The characteristic polynomial is given by:

$$P_2(\lambda) = \det(\lambda I - \bar{A}).$$

For system (8), respectively (9), the analysis of the solutions can be done by studying the roots of the equation  $P_2(\lambda) = 0$ .

### 2.3 Discrete-time dynamical systems with one parameter

Consider a discrete-time dynamical system depending on a real parameter  $\alpha$ , defined by the application:

$$x \rightarrow f(x, \alpha), \quad x \in \mathbb{R}^m, \alpha \in \mathbb{R} \quad (11)$$

where  $f : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^m$  is a seamless function with respect to  $x$  and  $\alpha$ . Let  $x_0 \in \mathbb{R}^m$  be a fixed point of (11), for all  $\alpha \in \mathbb{R}$ . The characteristic equation associated to the Jacobian matrix of the application (11), evaluated in  $x_0$  is  $P(\lambda, \alpha) = 0$ , where:

$$P(\lambda, \alpha) = \lambda^m + c_1(\alpha)\lambda^{m-1} + \dots + c_{m-1}(\alpha)\lambda + c_m(\alpha).$$

The roots of the characteristic equation depend on the parameter  $\alpha$ .

The fixed point  $x_0$  is called *stable* for (11), if there exists  $\alpha = \alpha_0$  so that equation  $P(\lambda, \alpha_0) = 0$  has all roots with their absolute values less than 1. The existence conditions of the value  $\alpha_0$ , are obtained using Schur Theorem (Lorenz, 1993).

If  $m = 2$ , the necessary and sufficient conditions that all roots of the characteristic equation  $\lambda^2 + c_1(\alpha)\lambda + c_2(\alpha) = 0$  have their absolute values less than 1 are:

$$|c_2(\alpha)| < 1, \quad |c_1(\alpha)| < |c_2(\alpha) + 1|.$$

If  $m = 3$ , the necessary and sufficient conditions that all roots of the characteristic equation

$$\lambda^3 + c_1(\alpha)\lambda^2 + c_2(\alpha)\lambda + c_3(\alpha) = 0$$

have their absolute values less than 1 are:

$$1 + c_1(\alpha) + c_2(\alpha) + c_3(\alpha) > 0, 1 - c_1(\alpha) + c_2(\alpha) - c_3(\alpha) > 0$$

$$1 + c_2(\alpha) - c_3(\alpha)(c_1(\alpha) + c_3(\alpha)) > 0, 1 - c_2(\alpha) + c_3(\alpha)(c_1(\alpha) - c_3(\alpha)) > 0, |c_3(\alpha)| < 1.$$

The *Neimark–Sacker (or Hopf) bifurcation* is the value  $\alpha = \alpha_0$  for which the characteristic equation  $P(\lambda, \alpha_0) = 0$  has the roots  $\mu_1(\alpha_0) = \mu(\alpha_0)$ ,  $\mu_2(\alpha_0) = \bar{\mu}(\alpha_0)$  in their absolute values equal to 1, and the other roots have their absolute values less than 1 and:

$$\text{a) } \left. \frac{d|\mu(\alpha)|}{d\alpha} \right|_{\alpha=\alpha_0} \neq 0. \quad \text{b) } \mu^k(\alpha_0) \neq 1, k = 1, 2, 3, 4$$

hold.

For the discrete-time dynamical system

$$x(n+1) = f(x(n), \alpha)$$

with  $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ , the following statement is true:

**Proposition 2.2.** ((Kuznetsov, 1995), (Mircea et al., 2004)) *Let  $\alpha_0$  be a Neimark-Sacker bifurcation. The restriction of (11) to two dimensional center manifold in the point  $(x_0, \alpha_0)$  has the normal form:*

$$\eta \rightarrow \eta e^{i\theta_0} \left( 1 + \frac{1}{2}d|\eta|^2 \right) + \mathcal{O}(\equiv \Delta)$$

where  $\eta \in \mathbb{C}$ ,  $d \in \mathbb{C}$ . If  $c = \text{Re } d \neq 0$  there is a unique limit cycle in the neighborhood of  $x_0$ . The expression of  $d$  is:

$$d = \frac{1}{2}e^{-i\theta_0} < v^*, C(v, v, \bar{v}) + 2B(v, (I_m - A)^{-1}B(v, \bar{v})) + B(v, (e^{2i\theta_0}I_m - A)^{-1}B(v, v)) > 0$$

where  $Av = e^{i\theta_0}v$ ,  $A^T v^* = e^{-i\theta_0}v^*$  and  $\langle v^*, v \rangle = 1$ ;  $A = \left(\frac{\partial f}{\partial x}\right)_{(x_0, \alpha_0)}$ ,  $B = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(x_0, \alpha_0)}$  and  $C = \left(\frac{\partial^3 f}{\partial x^3}\right)_{(x_0, \alpha_0)}$ .

The center manifold in  $x_0$  is a two dimensional submanifold in  $\mathbb{R}^m$ , tangent in  $x_0$  to the vectorial space of the eigenvectors  $v$  and  $v^*$ .

The following statements are true:

**Proposition 2.3.** (i) If  $m = 2$ , the necessary and sufficient conditions that a Neimark–Sacker bifurcation exists in  $\alpha = \alpha_0$  are:

$$|c_2(\alpha_0)| = 1, |c_1(\alpha_0)| < 2, c_1(\alpha_0) \neq 0, c_1(\alpha_0) \neq 1, \left.\frac{dc_2(\alpha)}{d\alpha}\right|_{\alpha=\alpha_0} > 0.$$

(ii) If  $m = 3$ , the necessary and sufficient conditions that a Neimark–Sacker bifurcation exists in  $\alpha = \alpha_0$ , are:

$$\begin{aligned} &|c_3(\alpha_0)| < 1, c_2(\alpha_0) = 1 + c_1(\alpha_0)c_3(\alpha_0) - c_3(\alpha_0)^2, \\ &\frac{c_3(\alpha_0)(c_1(\alpha_0)c_3'(\alpha_0) + c_1'(\alpha_0)c_3(\alpha_0) - c_2'(\alpha_0) - 2c_3(\alpha_0)c_3'(\alpha_0))}{1 + 2c_3^2(\alpha_0) - c_1(\alpha_0)c_3(\alpha_0)} > 0, \\ &|c_1(\alpha_0) - c_3(\alpha_0)| \neq 0, |c_1(\alpha_0) - c_3(\alpha_0)| \neq 1. \end{aligned}$$

In what follows, we highlight the normal form for the Neimark–Sacker bifurcation.

**Theorem 2.1.** (The Neimark–Sacker bifurcation). Consider the two dimensional discrete-time dynamical system given by:

$$x \rightarrow f(x, \alpha), \quad x \in \mathbb{R}^2, \alpha \in \mathbb{R} \tag{12}$$

with  $x = 0$ , fixed point for all  $|\alpha|$  small enough and

$$\mu_{12}(\alpha) = r(\alpha)e^{\pm i\varphi(\theta)}$$

where  $r(0) = 1$ ,  $\varphi(0) = \theta_0$ . If the following conditions hold:

$$c_1 : r'(0) \neq 0, \quad c_2 : e^{ik\theta_0} \neq 1, k = 1, 2, 3, 4$$

then there is a coordinates' transformation and a parameter change so that the application (12) is topologically equivalent in the neighborhood of the origin with the system:

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &\rightarrow \begin{pmatrix} \cos \theta(\beta) & -\sin \theta(\beta) \\ \sin \theta(\beta) & \cos \theta(\beta) \end{pmatrix} \left[ (1 + \beta) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \right. \\ &\left. + (y_1^2 + y_2^2) \begin{pmatrix} a(\beta) & -b(\beta) \\ b(\beta) & a(\beta) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right] + \mathcal{O}(\|y\|^\Delta), \end{aligned}$$

where  $\theta(0) = \theta_0$ ,  $a(0) = \text{Re}(e^{-i\theta_0}C_1(0))$ , and

$$C_1(0) = \frac{g_{20}(0)g_{11}(0)(1 - 2\mu_0)}{2(\mu_0^2 - \mu_0)} + \frac{|g_{11}(0)|^2}{1 - \bar{\mu}_0} + \frac{|g_{02}(0)|^2}{2(\mu_0^2 - \bar{\mu}_0)} + \frac{g_{21}(0)}{2}$$

$\mu_0 = e^{i\theta_0}$ ,  $g_{20}, g_{11}, g_{02}, g_{21}$  are the coefficients obtained using the expansion in Taylor series including third-order terms of function  $f$ .



## 2.4 The Neimark-Sacker bifurcation for a class of discrete-time dynamical systems with delay

A two dimensional discrete-time dynamical system with delay is defined by the equations

$$\begin{aligned}x_{n+1} &= x_n + f_1(x_n, y_n, \alpha) \\ y_{n+1} &= y_n + f_2(x_{n-m}, y_n, \alpha)\end{aligned}\quad (13)$$

where  $\alpha \in \mathbf{R}$ ,  $f_1, f_2 : \mathbf{R}^3 \rightarrow \mathbf{R}$  are seamless functions, so that for any  $|\alpha|$  small enough, the system  $f_1(x, y, \alpha) = 0, f_2(x, y, \alpha) = 0$ , admits a solution  $(\bar{x}, \bar{y})^T \in \mathbf{R}^2$ .

Using the translation  $x_n \rightarrow x_n + \bar{x}, y_n \rightarrow y_n + \bar{y}$ , and denoting the new variables with the same notations  $x_n, y_n$ , system (13) becomes:

$$\begin{aligned}x_{n+1} &= x_n + f(x_n, y_n, \alpha) \\ y_{n+1} &= y_n + g(x_{n-m}, y_n, \alpha)\end{aligned}\quad (14)$$

where:

$$f(x_n, y_n, \alpha) = f_1(x_n + \bar{x}, y_n + \bar{y}, \alpha); g(x_{n-m}, y_n, \alpha) = f_2(x_{n-m} + \bar{x}, y_n + \bar{y}, \alpha).$$

With the change of variables  $x^1 = x_{n-m}, x^2 = x_{n-(m-1)}, \dots, x^m = x_{n-1}, x^{m+1} = x_n, x^{m+2} = y_n$ , application (14) associated to the system is:

$$\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^{m+1} \\ x^{m+2} \end{pmatrix} \rightarrow \begin{pmatrix} x^2 \\ \vdots \\ x^{m+1} + f(x^{m+1}, x^{m+2}, \alpha) \\ x^{m+2} + g(x^1, x^{m+2}, \alpha) \end{pmatrix}.\quad (15)$$

We use the notations:

$$\begin{aligned}a_{10} &= \frac{\partial f}{\partial x^{m+1}}(0, 0, \alpha), a_{01} = \frac{\partial f}{\partial x^{m+2}}(0, 0, \alpha), \\ b_{10} &= \frac{\partial g}{\partial x^1}(0, 0, \alpha), b_{01} = \frac{\partial g}{\partial x^{m+2}}(0, 0, \alpha) \\ a_{20} &= \frac{\partial^2 f}{\partial x^{m+1} \partial x^{m+1}}(0, 0, \alpha), a_{11} = \frac{\partial^2 f}{\partial x^{m+1} \partial x^{m+2}}(0, 0, \alpha), \\ a_{02} &= \frac{\partial^2 f}{\partial x^{m+2} \partial x^{m+2}}(0, 0, \alpha), a_{30} = \frac{\partial^3 f}{\partial x^{m+1} \partial x^{m+1} \partial x^{m+1}}(0, 0, \alpha), \\ a_{21} &= \frac{\partial^3 f}{\partial x^{m+1} \partial x^{m+1} \partial x^{m+2}}(0, 0, \alpha), a_{12} = \frac{\partial^3 f}{\partial x^{m+1} \partial x^{m+2} \partial x^{m+2}}(0, 0, \alpha), \\ a_{03} &= \frac{\partial^3 f}{\partial x^{m+2} \partial x^{m+2} \partial x^{m+2}}(0, 0, \alpha)\end{aligned}\quad (16)$$

$$\begin{aligned}
 b_{20} &= \frac{\partial^2 g}{\partial x^1 \partial x^1}(0, 0, \alpha), & b_{11} &= \frac{\partial^2 g}{\partial x^1 \partial x^{m+2}}(0, 0, \alpha), \\
 b_{02} &= \frac{\partial^2 g}{\partial x^{m+2} \partial x^{m+2}}(0, 0, \alpha), & b_{30} &= \frac{\partial^3 g}{\partial x^1 \partial x^1 \partial x^1}(0, 0, \alpha), \\
 b_{21} &= \frac{\partial^3 g}{\partial x^1 \partial x^1 \partial x^{m+2}}(0, 0, \alpha), & b_{12} &= \frac{\partial^3 g}{\partial x^1 \partial x^{m+2} \partial x^{m+2}}(0, 0, \alpha), \\
 b_{03} &= \frac{\partial^3 g}{\partial x^{m+2} \partial x^{m+2} \partial x^{m+2}}(0, 0, \alpha).
 \end{aligned}
 \tag{17}$$

With (16) and (17) from (15) we have:

**Proposition 2.4.** ((Mircea et al., 2004)) (i) *The Jacobian matrix associated to (15) in  $(0, 0)^T$  is:*

$$A = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 + a_{10} & a_{01} \\ b_{10} & 0 & \dots & 0 & 1 + b_{01} \end{pmatrix}.
 \tag{18}$$

(ii) *The characteristic equation of A is:*

$$\lambda^{m+2} - (2 + a_{10} + b_{01})\lambda^{m+1} + (1 + a_{10})(1 + b_{01})\lambda - a_{01}b_{10} = 0.
 \tag{19}$$

(iii) *If  $\mu = \mu(\alpha)$  is an eigenvalue of (19), then the eigenvector  $q \in \mathbb{C}^{m+2}$ , solution of the system  $Aq = \mu q$ , has the components:*

$$q_1 = 1, q_i = \mu^{i-1}, i = 2, \dots, m + 1, q_{m+2} = \frac{b_{10}}{\mu - 1 - b_{01}}.
 \tag{20}$$

*The eigenvector  $p \in \mathbb{C}^{m+2}$  defined by  $A^T p = \bar{\mu} p$  has the components*

$$\begin{aligned}
 p_1 &= \frac{(\bar{\mu} - 1 - a_{10})(\bar{\mu} - 1 - b_{01})}{m(\bar{\mu} - 1 - a_{10})(\bar{\mu} - 1 - b_{01}) + \bar{\mu}(2\bar{\mu} - 2 - a_{10} - b_{01})}, p_i = \frac{1}{\bar{\mu}^{i-1}} p_1, i = 2, \dots, m, \\
 p_{m+1} &= \frac{1}{\bar{\mu}^{m-1}(\bar{\mu} - 1 - a_{10})} p_1, p_{m+2} = \frac{\bar{\mu}}{b_{10}} p_1.
 \end{aligned}
 \tag{21}$$

*The vectors  $q, p$  satisfy the condition:*

$$\langle q, p \rangle = \sum_{i=1}^{m+2} q_i \bar{p}_i = 1.$$

**The proof** is obtained by straight calculation from (15) and (18).

The following hypotheses are taken into account:

H<sub>1</sub>. The characteristic equation (19) has one pair of conjugate eigenvalues  $\mu(\alpha), \bar{\mu}(\alpha)$  with their absolute values equal to one, and the other eigenvalues have their absolute values less than one.

H<sub>2</sub>. The eigenvalues  $\mu(\alpha), \bar{\mu}(\alpha)$  intersect the unit circle for  $\alpha = 0$ , and satisfy the transversality condition

$$\frac{d}{d\alpha} |\mu(\alpha)|_{\alpha=0} \neq 0.$$

H<sub>3</sub>. If  $\arg(\mu(\alpha)) = \theta(\alpha)$ , and  $\theta_0 = \theta(0)$ , then  $e^{i\theta_0 k} \neq 1, k = 1, 2, 3, 4$ .

From H<sub>2</sub> we notice that for all  $|\alpha|$  small enough,  $\mu(\alpha)$  is given by:

$$\mu(\alpha) = r(\alpha)e^{i\theta(\alpha)}$$

with  $r(0) = 1, \theta(0) = \theta_0, r'(0) \neq 0$ . Thus  $r(\alpha) = 1 + \beta(\alpha)$  where  $\beta(0) = 0$  and  $\beta'(0) \neq 0$ . Taking  $\beta$  as a new parameter, we have:

$$\mu(\beta) = (1 + \beta)e^{i\theta(\beta)} \quad (22)$$

with  $\theta(0) = \theta_0$ . From (22) for  $\beta < 0$  small enough, the eigenvalues of the characteristic equation (19) have their absolute values less than one, and for  $\beta > 0$  small enough, the characteristic equation has an eigenvalue with its absolute value greater than one. Using the center manifold Theorem (Kuznetsov, 1995), application (15) has a family of invariant manifolds of two dimension depending on the parameter  $\beta$ . The restriction of application (15) to this manifold contains the essential properties of the dynamics for (13). The restriction of application (15) is obtained using the expansion in Taylor series until the third order of the right side of application (15).

## 2.5 The center manifold, the normal form

Consider the matrices:

$$A_1 = \begin{pmatrix} a_{20} & a_{11} \\ a_{11} & a_{02} \end{pmatrix}, C_1 = \begin{pmatrix} a_{30} & a_{21} \\ a_{21} & a_{12} \end{pmatrix}, D_1 = \begin{pmatrix} a_{21} & a_{12} \\ a_{12} & a_{03} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} b_{20} & b_{11} \\ b_{11} & b_{02} \end{pmatrix}, C_2 = \begin{pmatrix} b_{30} & b_{21} \\ b_{21} & b_{12} \end{pmatrix}, D_2 = \begin{pmatrix} b_{21} & b_{12} \\ b_{12} & b_{03} \end{pmatrix}$$

with the coefficients given by (16) and (17).

Denoting by  $x = (x^1, \dots, x^{m+2}) \in \mathbf{R}^{m+2}$ , application (15), is written as  $x \rightarrow F(x)$ , where  $F(x) = (x^2, \dots, x^m, x^{m+1} + f(x^{m+1}, x^{m+2}, \alpha), x^{m+2} + g(x^1, x^{m+2}, \alpha))$ .

The following statements hold:

**Proposition 2.5.** (i) *The expansion in Taylor series until the third order of function  $F(x)$  is:*

$$F(x) = Ax + \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + \mathcal{O}(|x|^\Delta), \quad (23)$$

where  $A$  is the matrix (18), and

$$B(x, x) = (0, \dots, 0, B^1(x, x), B^2(x, x))^T,$$

$$C(x, x, x) = (0, \dots, 0, C^1(x, x, x), C^2(x, x, x))^T,$$

where:

$$\begin{aligned}
 B^1(x, x) &= (x^{m+1}, x^{m+2})A_1 \begin{pmatrix} x^{m+1} \\ x^{m+2} \end{pmatrix}, B^2(x, x) = (x^1, x^{m+2})A_2 \begin{pmatrix} x^1 \\ x^{m+2} \end{pmatrix}, \\
 C^1(x, x, x) &= (x^{m+1}, x^{m+2})(x^{m+1}C_1 + x^{m+2}D_1) \begin{pmatrix} x^{m+1} \\ x^{m+2} \end{pmatrix}, \\
 C^2(x, x, x) &= (x^1, x^{m+2})(x^1C_2 + x^{m+2}D_2) \begin{pmatrix} x^1 \\ x^{m+2} \end{pmatrix}.
 \end{aligned}
 \tag{24}$$

(ii) Any vector  $x \in \mathbb{R}^{m+2}$  admits the decomposition:

$$x = zq + \bar{z}\bar{q} + y, \quad z \in \mathbb{C} \tag{25}$$

where  $zq + \bar{z}\bar{q} \in T_{\text{center}}$ ,  $y \in T_{\text{stable}}$ ;  $T_{\text{center}}$  is the vectorial space generated by the eigenvectors corresponding to the eigenvalues of the characteristic equation (19) with their absolute values equal to one and  $T_{\text{stable}}$  is the vectorial subspace generated by the eigenvectors corresponding to the eigenvalues of the characteristic equation (19) with their absolute values less than 1. Moreover:

$$z = \langle p, x \rangle, \quad y = x - \langle p, x \rangle q - \langle \bar{p}, x \rangle \bar{q}. \tag{26}$$

(iii)  $F(x)$  given by (23) has the decomposition:

$$F(x) = F_1(z, \bar{z}) + F_2(y)$$

where

$$\begin{aligned}
 F_1(z, \bar{z}) &= G_1(z)q + \overline{G_1(z)}\bar{q} + \langle p, N(zq + \bar{z}\bar{q} + y) \rangle \bar{q} + \langle \bar{p}, N(zq + \bar{z}\bar{q} + y) \rangle \bar{q} \\
 G_1(z) &= \mu z + \langle p, N(zq + \bar{z}\bar{q} + y) \rangle \\
 F_2(y) &= Ay + N(zq + \bar{z}\bar{q} + y) - \langle p, N(zq + \bar{z}\bar{q} + y) \rangle q - \langle \bar{p}, N(zq + \bar{z}\bar{q} + y) \rangle \bar{q}
 \end{aligned}
 \tag{27}$$

and

$$\begin{aligned}
 N(zq + \bar{z}\bar{q} + y) &= \frac{1}{2}B(zq + \bar{z}\bar{q} + y, zq + \bar{z}\bar{q} + y) + \\
 &+ \frac{1}{6}C(zq + \bar{z}\bar{q} + y, zq + \bar{z}\bar{q} + y, zq + \bar{z}\bar{q} + y) + O(zq + \bar{z}\bar{q} + y)
 \end{aligned}
 \tag{28}$$

(iv) The two-dimensional differential submanifold from  $\mathbb{R}^{m+2}$ , given by  $x = zq + \bar{z}\bar{q} + V(z, \bar{z})$ ,  $z \in \mathcal{V}_0 \subset \mathbb{C}$ , where  $V(z, \bar{z}) = \bar{V}(z, \bar{z})$ ,  $\langle p, V(z, \bar{z}) \rangle = 0$ ,  $\frac{\partial V(z, \bar{z})}{\partial z}(0, 0) = 0$ , is tangent to the vectorial space  $T_{\text{center}}$  in  $0 \in \mathbb{C}$ .

**Proof.** (i) Taking into account the expression of  $F(x)$  we obtain the expansion in Taylor series until the third order (23).

(ii) Because  $\mathbb{R}^{m+2} = T_{\text{center}} \oplus T_{\text{stable}}$  and  $\langle p, y \rangle = 0$ , for any  $y \in T_{\text{stable}}$ , we obtain (25) and (26).

(iii) Because  $F(x) \in \mathbb{R}^{m+2}$ , with decomposition (25) and  $\langle p, q \rangle = 1$ ,  $\langle \bar{p}, q \rangle = 0$ , we have (27).

(iv) Using the definition of the submanifold, this submanifold is tangent to  $T_{\text{center}}$ .

The center manifold in  $(0,0)^T \in \mathbb{R}^2$  is a two dimensional submanifold from  $\mathbb{R}^{m+2}$  tangent to  $T_{\text{center}}$  at  $0 \in \mathbb{C}$  and invariant with respect to the applications  $G_1$  and  $F_2$ , given by (27). If  $x = zq + \bar{z}\bar{q} + V(z, \bar{z})$ ,  $z \in \mathcal{V}_0 \subset \mathbb{C}$  is the analytical expression of the tangent submanifold to  $T_{\text{center}}$ , the invariant condition is written as:

$$V(G_1(z), \overline{G_1(z)}) = F_2(V(z, \bar{z})). \quad (29)$$

From (27), (28) and (29) we find that  $x = zq + \bar{z}\bar{q} + V(z, \bar{z})$ ,  $z \in \mathcal{V}_0$  is the center manifold if and only if the relation:

$$V(\mu z + \langle p, N(zq + \bar{z}\bar{q} + V(z, \bar{z})) \rangle, \bar{\mu}\bar{z} + \langle \bar{p}, N(zq + \bar{z}\bar{q} + V(z, \bar{z})) \rangle) = AV(z, z) + N(zq + \bar{z}\bar{q} + V(z, \bar{z})) - \langle p, N(zq + \bar{z}\bar{q} + V(z, \bar{z})) \rangle q - \langle \bar{p}, N(zq + \bar{z}\bar{q} + V(z, \bar{z})) \rangle \bar{q} \quad (30)$$

holds.

In what follows we consider the function  $V(z, \bar{z})$  of the form:

$$V(z, \bar{z}) = \frac{1}{2}w_{20}z^2 + w_{11}z\bar{z} + w_{02}\bar{z}^2 + \mathcal{O}(|z|^3), \quad z \in \mathcal{V}, \bar{z} \in \mathcal{A} \quad (31)$$

**Proposition 2.6.** (i) If  $V(z, \bar{z})$  is given by (31), and  $N(zq + \bar{z}\bar{q} + y)$ , with  $y = V(z, \bar{z})$  is given by (28), then:

$$G_1(z) = \mu z + \frac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + g_{02}\bar{z}^2 + \frac{1}{2}g_{21}z^2\bar{z} + \dots \quad (32)$$

where:

$$\begin{aligned} g_{20} &= \langle p, B(q, q) \rangle, & g_{11} &= \langle p, B(q, \bar{q}) \rangle, & g_{02} &= \langle p, B(\bar{q}, \bar{q}) \rangle \\ g_{21} &= \langle p, B(\bar{q}, w_{20}) \rangle + 2 \langle p, B(q, w_{11}) \rangle + \langle p, C(q, q, \bar{q}) \rangle. \end{aligned} \quad (33)$$

(ii) If  $V(z, \bar{z})$  is given by (31), relation (30) holds, if and only if  $w_{20}, w_{11}, w_{02}$  satisfy the relations:

$$(\mu^2 I - A)w_{20} = h_{20}, \quad (I - A)w_{11} = h_{11}, \quad (\bar{\mu}^2 I - A)w_{02} = h_{02} \quad (34)$$

where:

$$\begin{aligned} h_{20} &= B(q, q) - \langle p, B(q, q) \rangle q - \langle \bar{p}, B(q, q) \rangle \bar{q} \\ h_{11} &= B(q, \bar{q}) - \langle p, B(q, \bar{q}) \rangle q - \langle \bar{p}, B(q, \bar{q}) \rangle \bar{q} \\ h_{02} &= B(\bar{q}, \bar{q}) - \langle p, B(q, \bar{q}) \rangle q - \langle \bar{p}, B(\bar{q}, \bar{q}) \rangle \bar{q}. \end{aligned}$$

**Proof.** (i) Because  $B(x, x)$  is a bilinear form,  $C(x, x, x)$  is a trilinear form, and  $y = V(z, \bar{z})$ , from (28) and the expression of  $G_1(z)$  given by (27), we obtain (32) and (33).

(ii) In (30), replacing  $V(z, \bar{z})$  with (32) and  $N(zq + \bar{z}\bar{q} + V(z, \bar{z}))$  given by (28), we find that  $w_{20}, w_{11}, w_{02}$  satisfy the relations (31).

Let  $q \in \mathbb{R}^{m+2}$ ,  $p \in \mathbb{R}^{m+2}$  be the eigenvectors of the matrices  $A$  and  $A^T$  corresponding to the eigenvalues  $\mu$  and  $\bar{\mu}$  given by (20) and (21) and:

$$\begin{aligned} a &= B^1(q, q), \quad b = B^2(q, q), \quad a_1 = B^1(q, \bar{q}), \quad b_1 = B^2(q, \bar{q}), \quad C_1 = C^1(q, q, \bar{q}), \quad C_2 = C^2(q, q, \bar{q}), \\ r_{20}^1 &= B^1(\bar{q}, w_{20}), \quad r_{20}^2 = B^2(\bar{q}, w_{20}), \quad r_{11}^1 = B^1(q, w_{11}), \quad r_{11}^2 = B^2(q, w_{11}), \end{aligned} \quad (35)$$

where  $B^1, B^2, C^1, C^2$ , are applications given by (24).

**Proposition 2.7.** (i) The coefficients  $g_{20}, g_{11}, g_{02}$  given by (33) have the expressions:

$$g_{20} = p_{m+1}a + p_{m+2}b, \quad g_{11} = p_{m+1}a_1 + p_{m+2}b_1, \quad g_{02} = p_{m+1}\bar{a} + p_{m+2}\bar{b}. \tag{36}$$

(ii) The vectors  $h_{20}, h_{11}, h_{02}$  given by (34) have the expressions:

$$\begin{aligned} h_{20} &= (0, \dots, 0, a, b)^T - (p_{m+1}a + p_{m+2}b)q - (\bar{p}_{m+1}a + \bar{p}_{m+2}b)\bar{q} \\ h_{11} &= (0, \dots, 0, a, b)^T - (p_{m+1}a_1 + p_{m+2}b_1)q - (\bar{p}_{m+1}a_1 + \bar{p}_{m+2}b_1)\bar{q} \\ h_{02} &= \bar{h}_{20}. \end{aligned} \tag{37}$$

(iii) The systems of linear equations (34) have the solutions:

$$\begin{aligned} w_{20} &= \left( v_{20}^1, \mu^2 v_{20}^1, \dots, \mu^{2m} v_{20}^1, \frac{a + (\mu^2 - a_{10})\mu^{2m} v_{20}^1}{a_{01}} \right)^T - \frac{p_{m+1}a + p_{m+2}b}{\mu^2 - \mu} q - \frac{\bar{p}_{m+1}a + \bar{p}_{m+2}b}{\mu^2 - \bar{\mu}} \bar{q} \\ w_{11} &= \left( v_{11}^1, v_{11}^1, \dots, v_{11}^1, \frac{a_1 + (1 - a_{10})v_{11}^1}{a_{01}} \right)^T - \frac{p_{m+1}a_1 + p_{m+2}b_1}{1 - \mu} q - \frac{\bar{p}_{m+1}a_1 + \bar{p}_{m+2}b_1}{1 - \bar{\mu}} \bar{q} \\ w_{02} = \bar{w}_{20}, v_{20}^1 &= \frac{aa_{01} - b(\mu^2 - b_{01})}{(\mu^2 - a_{10})(\mu^2 - b_{01})\mu^{2m} - b_{10}a_{01}}, v_{11}^1 = \frac{b_1 a_{01} - a_1(1 - b_{01})}{(1 - a_{10})(1 - b_{01}) - b_{10}a_{01}}. \end{aligned}$$

(iv) The coefficient  $g_{21}$  given by (33) has the expression:

$$g_{21} = p_{m+1}r_{20}^1 + p_{m+2}r_{20}^2 + 2(p_{m+1}r_{11}^1 + p_{m+2}r_{11}^2) + p_{m+1}C_1 + p_{m+2}C_2. \tag{38}$$

**Proof.** (i) The expressions from (36) are obtained from (33) using (35).

(ii) The expressions from (37) are obtained from (34) with the notations from (35).

(iii) Because  $\mu^2, \bar{\mu}^2, 1$  are not roots of the characteristic equation (19) then the linear systems (34) are determined compatible systems. The relations (37) are obtained by simple calculation.

(iv) From (33) with (35) we obtain (38).

Consider the discrete-time dynamical system with delay given by (13), for which the roots of the characteristic equation satisfy the hypotheses  $H_1, H_2, H_3$ . The following statements hold:

**Proposition 2.8.** (i) The solution of the system (13) in the neighborhood of the fixed point  $(\bar{x}, \bar{y}) \in \mathbb{R}^2$ , is:

$$\begin{aligned} x_n &= \bar{x} + q_{m+1}z_n + \bar{q}_{m+1}\bar{z}_n + \frac{1}{2}w_{20}^{m+1}z_n^2 + w_{11}^{m+1}z_n\bar{z}_n + \frac{1}{2}w_{02}^{m+1}\bar{z}_n^2 \\ y_n &= \bar{y} + q_{m+2}z_n + \bar{q}_{m+2}\bar{z}_n + \frac{1}{2}w_{20}^{m+2}z_n^2 + w_{11}^{m+2}z_n\bar{z}_n + \frac{1}{2}w_{02}^{m+2}\bar{z}_n^2 \\ x_{n-m} = u_n &= \bar{x} + q_1z_n + \bar{q}_1\bar{z}_n + \frac{1}{2}w_{20}^1z_n^2 + w_{11}^1z_n\bar{z}_n + \frac{1}{2}w_{02}^1\bar{z}_n^2 \end{aligned} \tag{39}$$

where  $z_n$  is a solution of the equation:

$$z_{n+1} = \mu z_n + \frac{1}{2}g_{20}z_n^2 + g_{11}z_n\bar{z}_n + \frac{1}{2}g_{02}\bar{z}_n^2 + \frac{1}{2}g_{21}z_n^2\bar{z}_n \tag{40}$$

and the coefficients from (40) are given by (36) and (38).

(ii) There is a complex change variable, so that equation (40) becomes:

$$w_{n+1} = \mu(\beta)w_n + C_1(\beta)w_n^2\bar{w}_n + \mathcal{O}(|\mu|^\Delta) \quad (41)$$

where:

$$C_1(\beta) = \frac{g_{20}(\beta)g_{11}(\beta)(\bar{\mu}(\beta) - 3 - 2\mu(\beta))}{2(\mu(\beta)^2 - \mu(\beta))(\bar{\mu}(\beta) - 1)} + \frac{|g_{11}(\beta)|^2}{1 - \bar{\mu}(\beta)} + \frac{|g_{02}(\beta)|^2}{2(\mu^2(\beta) - \bar{\mu}(\beta))} + \frac{g_{21}(\beta)}{2}.$$

(iii) Let  $l_0 = \operatorname{Re}(e^{-i\theta_0}C_1(0))$ , where  $\theta_0 = \arg(\mu(0))$ . If  $l_0 < 0$ , in the neighborhood of the fixed point  $(\bar{x}, \bar{y})$  there is an invariant stable limit cycle.

**Proof.** (i) From Proposition 2.6, application (15) associated to (13) has the canonical form (40). A solution of system (40) leads to (39).

(ii) In equation (40), making the following complex variable change

$$z = w + \frac{g_{20}}{2(\mu^2 - \mu)}w^2 + \frac{g_{11}}{|\mu|^2 - \mu}w\bar{w} + \frac{g_{02}}{2(\bar{\mu}^2 - \mu)}\bar{w}^2 + \\ + \frac{g_{30}}{6(\mu^3 - \mu)}w^3 + \frac{g_{12}}{2(\bar{\mu}|\mu|^2 - \mu)}w\bar{w}^2 + \frac{g_{03}}{6(\bar{\mu}^3 - \mu)}\bar{w}^3,$$

for  $\beta$  small enough, equation (41) is obtained. The coefficients  $g_{20}, g_{11}, g_{02}$  are given by (36) and

$$g_{30} = p_{m+1}C^1(q, q, q) + p_{m+2}C^2(q, q, q), \\ g_{12} = p_{m+1}C^1(q, \bar{q}, \bar{q}) + p_{m+2}C^2(q, \bar{q}, \bar{q}) \\ g_{03} = p_{m+1}C^1(\bar{q}, \bar{q}, \bar{q}) + p_{m+2}C^2(\bar{q}, \bar{q}, \bar{q}).$$

(iii) The coefficient  $C_1(\beta)$  is called *resonant cubic coefficient*, and the sign of the coefficient  $l_0$ , establishes the existence of a stable or unstable invariant limit cycle (attractive or repulsive) (Kuznetsov, 1995).

### 3. Neimark-Sacker bifurcation in a discrete time dynamic system for Internet congestion.

The model of an Internet network with one link and single source, which can be formulated as:

$$\dot{x}(t) = k(w - af(x(t - \tau))) \quad (42)$$

where:  $k > 0$ ,  $x(t)$  is the sending rate of the source at the time  $t$ ,  $\tau$  is the sum of forward and return delays,  $w$  is a target (set-point), and the congestion indication function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is increasing, nonnegative, which characterizes the congestion. Also, we admit that  $f$  is nonlinear and its third derivative exists and it is continuous.

The model obtained by discretizing system (42) is given by:

$$x_{n+1} = x_n - akf(x_{n-m}) + kw \quad (43)$$

for  $n, m \in \mathbb{N}$ ,  $m > 0$  and it represents the dynamical system with discrete-time for Internet congestion with one link and a single source.

Using the change of variables  $x^1 = x_{n-m}, \dots, x^m = x_{n-1}, x^{m+1} = x_n$ , the application associated to (43) is:

$$\begin{pmatrix} x^1 \\ \vdots \\ x^m \\ x^{m+1} \end{pmatrix} \rightarrow \begin{pmatrix} x^2 \\ \vdots \\ x^{m+1} \\ kw - akf(x^1) + x^{m+1} \end{pmatrix}. \tag{44}$$

The fixed point of (44) is  $(x^*, \dots, x^*)^T \in \mathbb{R}^{m+1}$ , where  $x^*$  satisfies relation  $w = af(x^*)$ . With the translation  $x \rightarrow x + x^*$ , application (44) can be written as:

$$\begin{pmatrix} x^1 \\ \vdots \\ x^m \\ x^{m+1} \end{pmatrix} \rightarrow \begin{pmatrix} x^2 \\ \vdots \\ x^{m+1} \\ kw - akg(x^1) + x^{m+1} \end{pmatrix} \tag{45}$$

where  $g(x^1) = f(x^1 + x^*)$ .

The following statements hold:

**Proposition 3.1.** ((Mircea et al., 2004)) (i) The Jacobian matrix of (45) in  $0 \in \mathbb{R}^{m+1}$  is

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -ak\rho_1 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{46}$$

where  $\rho_1 = g'(0)$ .

(ii) The characteristic equation of  $A$  is:

$$\lambda^{m+1} - \lambda^m + ak\rho_1 = 0. \tag{47}$$

(iii) If  $\mu \in \mathbb{C}$  is a root of (47), the eigenvector  $q \in \mathbb{R}^{m+1}$  that corresponds to the eigenvalue  $\mu$ , of the matrix  $A$ , has the components:

$$q_i = \mu^{i-1}, \quad i = 1, \dots, m + 1$$

and the components of the eigenvector  $p \in \mathbb{R}^{m+1}$  corresponding to  $\bar{\mu}$  of the matrix  $A^T$  are:

$$p_1 = -\frac{ak\rho_1}{\bar{\mu}^{m+1} - mak\rho_1}, \quad p_i = \frac{1}{\bar{\mu}^{i-1}} p_1, \quad i = 2, \dots, m - 1, \quad p_m = \frac{\bar{\mu}^2 - \bar{\mu}}{p\rho_1} p_1, \quad p_{m+1} = -\frac{\bar{\mu}}{ak\rho_1} p_1.$$

The vectors  $p \in \mathbb{R}^{m+1}, q \in \mathbb{R}^{m+1}$  satisfy the relation  $\sum_{i=1}^{m+1} \bar{p}_i q_i = 1$ .

The following statements hold:



**Proposition 3.2.** (i) If  $m = 2$ , equation (47) becomes:

$$\lambda^3 - \lambda^2 + ak\rho_1 = 0. \quad (48)$$

Equation (48) has two complex roots with their absolute values equal to 1 and one root with the absolute value less than 1, if and only if  $k = \frac{\sqrt{5}-1}{2a\rho_1}$ . For  $k = k_0 = \frac{\sqrt{5}-1}{2a\rho_1}$ , equation (48) has the roots:

$$\begin{aligned} \lambda_{1,2} &= \exp(\pm i\theta(k_0)i), \\ \theta(k_0) &= \arccos \frac{1 + \sqrt{5}}{4}. \end{aligned} \quad (49)$$

(ii) With respect to the change of parameter

$$k = k(\beta) = k_0 + g(\beta)$$

where:

$$g(\beta) = \frac{\sqrt{1+4(1+\beta)^6} - (1+\beta)^2 - \sqrt{5} + 1}{2k_0\rho_1}$$

equation (49) becomes:

$$\lambda^3 - \lambda^2 + ak(\beta)\rho_1 = 0. \quad (50)$$

The roots of equation (50) are:

$$\mu_{1,2}(\beta) = (1+\beta)\exp(\pm i\omega(\beta)), \quad \lambda(\beta) = -\frac{ak(\beta)\rho_1}{(1+\beta)^2}$$

where:

$$\omega(\beta) = \arccos \frac{(1+\beta)^2 + \sqrt{1+4(1+\beta)^6}}{4(1+\beta)^2}.$$

(iii) The eigenvector  $q \in \mathbf{R}^3$ , associated to the  $\mu = \mu(\beta)$ , for the matrix  $A$  has the components:

$$q_1 = 1, \quad q_2 = \mu, \quad q_3 = \mu^2$$

and the eigenvector  $p \in \mathbf{R}^3$  associated to the eigenvalue  $\bar{\mu} = \bar{\mu}(\beta)$  for the matrix  $A^T$  has the components:

$$p_1 = \frac{ak\rho_1}{2ak\rho_1 - \bar{\mu}^3}, \quad p_2 = \frac{\bar{\mu}^2}{\bar{\mu}^3 - 2ak\rho_1}, \quad p_3 = \frac{\bar{\mu}}{\bar{\mu}^3 - 2ak\rho_1}.$$

(iv)  $a_0$  is a Neimark-Sacker bifurcation point.

Using Proposition 3.2, we obtain:

**Proposition 3.3.** The solution of equation (43) in the neighborhood of the fixed point  $x^* \in \mathbf{R}$  is:

$$\begin{aligned} u_n &= x^* + z_n + \bar{z}_n + \frac{1}{2}w_{20}^1 z_n^2 + w_{11}^1 z_n \bar{z}_n + \frac{1}{2}w_{02}^1 \bar{z}_n^2 \\ x_n &= x^* + q_3 z_n + \bar{q}_3 \bar{z}_n + \frac{1}{2}w_{20}^3 z_n^2 + w_{11}^3 z_n \bar{z}_n + \frac{1}{2}w_{02}^3 \bar{z}_n^2 \end{aligned}$$

where:

$$w_{20}^1 = \frac{\mu^2(\mu^2 - 1)h_{20}^1 - (\mu^2 - 1)h_{20}^2 + h_{20}^3}{\mu^6 - \mu^4 + ak\rho_1}, w_{20}^2 = \mu^2 w_{20}^1 - h_{20}^1, w_{20}^3 = \mu^4 w_{20}^1 - \mu^2 h_{20}^1 - h_{20}^2$$

$$w_{11}^1 = \frac{h_{11}^3}{ak\rho_1}, w_{11}^2 = w_{11}^1 - h_{11}^1, w_{11}^3 = w_{11}^1 - h_{11}^1 - h_{11}^2$$

$$h_{20}^1 = 4ak\rho_2(p_3 + \bar{p}_3), h_{20}^2 = ak\rho_1(p_3q_2 + \bar{p}_3\bar{q}_2), h_{20}^3 = 4ak\rho_2(1 + p_3q_3 + \bar{p}_3\bar{q}_3)$$

$$h_{11}^1 = h_{20}^1, h_{11}^2 = h_{20}^2, h_{11}^3 = h_{20}^3$$

and  $z_n \in \mathbb{C}$  is a solution of equation:

$$z_{n+1} = \mu z_n - \frac{1}{2}p_3ak\rho_2(z_n^2 + 2z_n\bar{z}_n + \bar{z}_n^2) + \frac{1}{2}p_3(-ka\rho_1w_{20}^1 - ka\rho_1w_{11}^1 + \rho_3),$$

$$\rho_1 = f'(0), \rho_2 = f''(0), \rho_3 = f'''(0).$$

Let

$$C_1(\beta) = -\frac{p_3a^2k^2\rho_2^2(\bar{\mu} - 3 - 2\mu)}{2(\mu^2 - \mu)(\bar{\mu} - 1)} + \frac{a^2k^2\rho_2^2|p_3|^2}{1 - \bar{\mu}} + \frac{ak|\rho_2p_3|}{2(\mu^2 - \bar{\mu})} + p_3(-ak\rho_1w_{20}^1 - ak\rho_1w_{11}^1 + \rho_3)$$

and

$$l(0) = \text{Re}(\exp(-i\theta(a_0))C_1(0)).$$

If  $l(0) < 0$ , the Neimark–Sacker bifurcation is supercritical (stable).

The model of an Internet network with  $r$  links and a single source, can be analyzed in a similar way.

The perturbed stochastic equation of (43) is:

$$x_{n+1} = x_n - \alpha kf(x_{n-m}) + kw + \xi_n b(x_n - x^*) \tag{51}$$

and  $x^*$  satisfies the relation  $w = af(x^*)$ , where  $E(\xi_n) = 0, E(\xi_n^2) = \sigma > 0$ .

We study the case  $m = 2$ . Using (46) the linearized equation of (51) has the matrices:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -ak\rho_1 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{pmatrix}$$

Using Proposition 2.1, the characteristic polynomial of the linearized system of (51) is given by:

$$P_2(\lambda) = (\lambda^3 - (1 + \sigma b^2)\lambda^2 - a^2k^2\rho_1^2)(\lambda^3 + ak\rho_1\lambda + a^2k^2\rho_1^2).$$

If the roots of  $P_2(\lambda)$  have their absolute values less than 1, then the square mean values of the solutions for the linearized system of (51) are asymptotically stable. The analysis of the roots for the equation  $P_2(\lambda) = 0$  can be done for fixed values of the parameters.

The numerical simulation can be done for:  $w = 0.1, a = 8$  and  $f(x) = x^2/(20 - 3x)$ .

#### 4. A discrete economic game with delay

The economic game is described by a number of firms that enter the market with a homogeneous consumption product at different moments  $n$ , where  $n \in \mathbf{N}$ . In what follows we consider two firms  $F_1, F_2$  and  $x, y$  the state variables of the model that represent the firms' outputs. The price function of the product (the inverse demand function) is  $p : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ , derivable function with  $\lim_{x \rightarrow \infty} p(x) = 0$ ,  $\lim_{x \rightarrow 0^+} p(x) = \infty$  and  $p'(x) < 0$ . The cost functions are  $C_i : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ ,  $i = 1, 2$ , derivable functions with  $C'_i(x) \neq 0$ ,  $C''_i(x) \geq 0$ . The profit functions of the firms,  $\pi_i : \mathbf{R}_+^2 \rightarrow \mathbf{R}_+$ ,  $i = 1, 2$ , are given by:

$$\pi_1(x, y) = p(x + y)x - C_1(x), \quad \pi_2(x, y) = p(x + y)y - C_2(y).$$

The non-cooperative game  $F_1, F_2$ , denoted by  $\Gamma = (\mathbf{R}_+^2, \pi_1, \pi_2)$  is called *deterministic economic game*. The Nash solution of  $\Gamma$  is called *the solution of the deterministic economic game*.

From the definition of the Nash solution, we find that the solution of the deterministic economic game is given by the quantities  $(\bar{x}, \bar{y})^T \in \mathbf{R}_+^2$  for which the profit of each firm is maximum. Thus, the Nash solution is the solution of the following system:

$$\begin{cases} \pi_{1x} = p'(x + y)x + p(x + y) - C'_1(x) = 0 \\ \pi_{2y} = p'(x + y)y + p(x + y) - C'_2(y) = 0. \end{cases} \quad (52)$$

A solution  $(\bar{x}, \bar{y})^T \in \mathbf{R}_+^2$  of (52) is a (local) maximum for  $\pi_i$ ,  $i = 1, 2$  if and only if:

$$p''(\bar{x} + \bar{y})\bar{x} + 2p'(\bar{x} + \bar{y}) < C''_1(\bar{x}), \quad p''(\bar{x} + \bar{y})\bar{y} + 2p'(\bar{x} + \bar{y}) < C''_2(\bar{y}).$$

At each moment  $n$ ,  $n \in \mathbf{N}$  the firms adjust their quantities  $x_n, y_n$ , proportionally to the marginal profits  $\frac{\partial \pi_1}{\partial x}, \frac{\partial \pi_2}{\partial y}$ . The quantities from the  $n + 1$  moment satisfy the relations:

$$\begin{aligned} x_{n+1} &= x_n + k(p'(x_n + y_n)x_n + p(x_n + y_n) - C'_1(x_n)) \\ y_{n+1} &= y_n + \alpha(p'(x_{n-m} + y_n)y_n + p(x_{n-m} + y_n) - C'_2(y_n)) \end{aligned} \quad (53)$$

where  $m \in \mathbf{N}$ ,  $m \geq 1$ .

System (53) is a discrete dynamic economic game with delay.

With respect to the change of variables  $x^1 = x_{n-m}, \dots, x^m = x_{n-1}, x^{m+1} = x_n, x^{m+2} = y_n$  the application associated to (53) is:

$$\begin{pmatrix} x^1 \\ \vdots \\ x^m \\ x^{m+1} \\ x^{m+2} \end{pmatrix} \rightarrow \begin{pmatrix} x^2 \\ \vdots \\ x^{m+1} \\ x^{m+1} + k(p'(x^{m+1} + x^{m+2})x^{m+1} + p(x^{m+1} + x^{m+2}) - C'_1(x^{m+1})) \\ x^{m+2} + \alpha(p'(x^1 + x^{m+2})x^{m+2} + p(x^1 + x^{m+2}) - C'_2(x^{m+2})) \end{pmatrix}. \quad (54)$$

The fixed point of (54) is the point with the coordinates  $(x_0, \dots, x_0, y_0) \in \mathbf{R}^{m+2}$  where  $(x_0, y_0)$  is the solution of the following system:

$$p'(x + y)x + p(x + y) - C'_1(x) = 0, \quad p'(x + y)y + p(x + y) - C'_2(y) = 0 \quad (55)$$

In what follows we use the notations:

$$\rho_i = p^{(i)}(\bar{x} + \bar{y}), \mu_{i1} = C_1^{(i)}(\bar{x}), \mu_{i2} = C_2^{(i)}(\bar{y}), \quad i = 1, 2, 3, 4 \tag{56}$$

the derivatives of  $i = 1, 2, 3, 4$  order of the functions  $p, C_1, C_2$  in the point  $(\bar{x}, \bar{y})$ ,

$$\begin{aligned} a_{10} &= \rho_2\bar{x} + 2\rho_1 - \mu_{21}, \quad a_{01} = \rho_2\bar{x} + \rho_1, \quad a_{20} = \rho_3\bar{x} + 3\rho_2 - \mu_{31}, \quad a_{11} = \rho_3\bar{x} + 2\rho_2, \\ a_{02} &= \rho_3\bar{x} + \rho_2, \quad a_{30} = \rho_4\bar{x} + 4\rho_3 - \mu_{41}, \quad a_{21} = \rho_4\bar{x} + 3\rho_3, \quad a_{12} = \rho_4\bar{x} + 2\rho_3, \quad a_{03} = \rho_4\bar{x} + 2\rho_3, \end{aligned} \tag{57}$$

$$\begin{aligned} b_{10} &= \rho_2\bar{y} + \rho_1, \quad b_{01} = \rho_2\bar{y} + 2\rho_1 - \mu_{22}, \quad b_{20} = \rho_3\bar{y} + \rho_2, \quad b_{11} = \rho_3\bar{y} + 2\rho_2, \quad b_{02} = \rho_3\bar{y} + 3\rho_2 - \mu_{32}, \\ b_{30} &= \rho_4\bar{y} + 2\rho_3, \quad b_{21} = \rho_4\bar{y} + 2\rho_3, \quad b_{12} = \rho_4\bar{y} + 3\rho_3, \quad b_{03} = \rho_4\bar{y} + 4\rho_3 - \mu_{42}, \quad d_1 = b_{01} + k(a_{10}b_{01} - a_{01}b_{10}). \end{aligned} \tag{58}$$

**Proposition 4.1.** ((Neamtu, 2010)) (i) *The Jacobian matrix of the application (54) in  $(x_0, \dots, x_0, y_0)$  is:*

$$A = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 & 0 \\ 0 & \dots & \dots & 1 + ka_{10} & ka_{01} \\ \alpha b_{10} & 0 & \dots & 0 & 1 + \alpha b_{01} \end{pmatrix}. \tag{59}$$

(ii) *The characteristic equation of A given by (59) is:*

$$\lambda^{m+2} - a\lambda^{m+1} + b\lambda^m - c = 0$$

where

$$a = 2 + ka_{10} + \alpha b_{01}, \quad b = (1 + ka_{10})(1 + \alpha b_{01}), \quad c = k\alpha a_{01}b_{10}.$$

(iii) *The eigenvector  $q \in \mathbf{R}^{m+2}$ , which corresponds to the eigenvalue  $\mu$  of the matrix A and satisfies the system  $Aq = \mu q$ , has the components:*

$$q_i = \mu^{i-1}, \quad i = 1, \dots, m + 1, \quad q_{m+2} = \frac{\alpha b_{10}}{\mu - 1 - \alpha b_{01}}. \tag{60}$$

*The eigenvector  $p \in \mathbf{R}^{m+2}$ , which corresponds to the eigenvalue  $\bar{\mu}$  of the matrix A and satisfies the system  $A^T p = \bar{\mu} p$ , has the components:*

$$\begin{aligned} p_1 &= \frac{(\bar{\mu} - 1 - ka_{10})(\bar{\mu} - 1 - \alpha b_{10})}{m(\bar{\mu} - 1 - ka_{10})(\bar{\mu} - 1 - kb_{01}) + \bar{\mu}(2\bar{\mu} - 2 - ka_{10} - \alpha b_{01})} \\ p_i &= \frac{1}{\bar{\mu}^{i-1}}, \quad i = 2, \dots, m - 2, \quad p_{m+1} = \frac{1}{\bar{\mu}^{m-1}(\bar{\mu} - 1 - ka_{10})} p_1, \quad p_{m+2} = \frac{\bar{\mu}}{\alpha b_{10}} p_1. \end{aligned} \tag{61}$$

*The vectors  $q, p$  given by (60) and (61) satisfy the condition*

$$\langle q, p \rangle = \sum_{i=1}^{m+2} q_i \bar{p}_i = 1.$$

The proof follows by direct calculation.

If  $m = 1$ , the following statements hold:

**Proposition 4.2.** (i) If  $k \neq \frac{b_{01}}{a_{01}b_{10}}$  and

$$(k(a_{01}b_{10} - b_{01}a_{10}) - b_{01})^2 + 4ka_{10}a_{01}b_{10}(b_{01} - ka_{01}k_{10}) \geq 0$$

and  $\alpha_0$  is a solution of the equation:

$$ka_{01}b_{10}(b_{01} - ka_{01}b_{10})\alpha^2 + (k(a_{01}b_{10} - b_{01}a_{10}) - b_{01})\alpha - ka_{10} = 0$$

so that:

$$\alpha_0 k |a_{01}b_{10}| < 1, \quad |(b_{01} - ka_{01}b_{10})\alpha_0 + \lambda + ka_{10}| < 2,$$

then the equation:

$$\lambda^3 - a_0\lambda^2 + b_0\lambda - c_0 = 0 \quad (62)$$

has two roots with their absolute value equal to 1 and one root with its absolute value less than 1, where:

$$a_0 = 2 + ka_{10} + \alpha_0 b_{01}, \quad b_0 = (1 + ka_{10})(1 + \alpha_0 b_{01}), \quad c_0 = ka_{01}b_{10}\alpha_0. \quad (63)$$

(ii) If for  $|\beta|$  small enough,  $\Delta_1(\beta) \geq 0$ , where

$$\begin{aligned} \Delta_1(\beta) = & ((1 + \beta)^2(b_{01}c_0 + ka_{01}b_{10}a_0) - 2c_0ka_{01}b_{10} - (1 + \beta)^4b_{01}(1 + ka_{10}))^2 - \\ & - 4ka_{01}b_{10}(b_{01}(1 + \beta)^2 - ka_{01}b_{10})((1 + \beta)^2a_0c_0 - c_0^2 + (1 + \beta)^4(1 + a_0c_0 - c_0^2) + (1 + \beta)^6), \end{aligned}$$

and  $a_0, b_0, c_0$  given by (63), then there is  $g : \mathbf{R} \rightarrow \mathbf{R}$ , with  $g(0) = 0$ ,  $g'(0) \neq 0$  so that the variable change:

$$\alpha = \alpha(\beta) = \alpha_0 + g(\beta)$$

transforms equation (62) in equation:

$$\lambda^3 - a(\beta)\lambda^2 + b(\beta)\lambda - c(\beta) = 0 \quad (64)$$

Equation (64) admits the solutions:

$$\mu_{1,2}(\beta) = (1 + \beta)e^{\pm i\theta(\beta)}, \quad \theta(\beta) = \arccos \frac{a(\beta)(1 + \beta)^2 - c(\beta)}{2(1 + \beta)^3}, \quad \lambda(\beta) = \frac{c(\beta)}{(1 + \beta)^2} \quad (65)$$

where:

$$\begin{aligned} g(\beta) = & \frac{2c_0ka_{01}b_{10} + (1 + \beta)^4b_{01}(1 + ka_{10}) - (1 + \beta)^2(b_{01}c_0 + ka_{01}b_{10}a_0) + \sqrt{\Delta_1(\beta)}}{2ka_{01}b_{10}(b_{01}|1 + \beta|^2 - ka_{01}b_{10})}, \\ a(\beta) = & a_0 + b_{01}g(\beta), \quad b(\beta) = b_0 + b_{01}(1 + ka_{10})g(\beta), \quad c(\beta) = c_0 + ka_{01}b_{10}g(\beta). \end{aligned}$$

For  $m = 1$ , the model of a discrete economic game with delay (53) is written as:

$$\begin{aligned} x_{n+1} = & x_n + k(p'(x_n + y_n)x_n + p(x_n + y_n) - C'_1(x_n)) \\ y_{n+1} = & y_n + \alpha(p'(x_{n-1} + y_n)y_n + p(x_{n-1} + y_n) - C'_2(y_n)). \end{aligned} \quad (66)$$

We have:

**Proposition 4.3.** (i) The solution of (66) in the neighborhood of  $(\bar{x}, \bar{x}, \bar{y}) \in \mathbb{R}^3$  is:

$$\begin{aligned} x_n &= \bar{x} + q_2 z_n + \bar{q}_2 \bar{z}_n + \frac{1}{2} w_{20}^2 z_n^2 + w_{11}^2 z_n \bar{z}_n + \frac{1}{2} w_{02}^2 \bar{z}_n^2, \\ y_n &= \bar{y} + q_3 z_n + \bar{q}_3 \bar{z}_n + \frac{1}{2} w_{20}^3 z_n^2 + w_{11}^3 z_n \bar{z}_n + w_{02}^3 \bar{z}_n^2, \\ u_n &= x_{n-1} = \bar{x} + q_1 z_n + \bar{q}_1 \bar{z}_n + \frac{1}{2} w_{20}^1 z_n^2 + w_{11}^1 z_n \bar{z}_n + w_{02}^1 \bar{z}_n^2, \end{aligned} \tag{67}$$

where  $z_n$  is a solution of the equation:

$$z_n = \mu(\beta) z_n + \frac{1}{2} g_{20} z_n^2 + g_{11} z_n \bar{z}_n + \frac{1}{2} g_{02} \bar{z}_n^2 + \frac{1}{2} g_{21} z_n^2 \bar{z}_n \tag{68}$$

and the coefficients from (67) and (68) are given by (36) and (38) for  $m = 1$ , where  $a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{30}, a_{21}, a_{12}, a_{03}$  are given by (57) and  $q \in \mathbb{R}^3, p \in \mathbb{R}^3$  are given by (60), (61) for  $m = 1$ .

(ii) If  $l_0 = \text{Re}(e^{-i\theta(0)} C_1(0))$ , where  $\theta(\beta)$  is given by (65) and then if  $l_0 < 0$  in the neighborhood of the fixed point  $(\bar{x}, \bar{y})$  there is a stable limit cycle.

For  $m = 2$ , the results are obtained in a similar way to Proposition 2.3.

We will investigate an economic game where the price function is  $p(x) = \frac{1}{x}, x \neq 0$ , and the cost functions are  $C_i(x) = c_i x + b_i$ , with  $c_i > 0, i = 1, 2$ .

The following statements hold:

**Proposition 4.4.** (i) The fixed point of the application (54) is  $(\bar{x}, \dots, \bar{x}, \bar{y}) \in \mathbb{R}^{m+2}$  where:

$$\begin{aligned} \bar{x} &= \frac{c_2}{(c_1 + c_2)^2}, \\ \bar{y} &= \frac{c_1}{(c_1 + c_2)^2}. \end{aligned} \tag{69}$$

(ii) The coefficients (56), (57) and (58) are:

$$\begin{aligned} \rho_1 &= -(c_1 + c_2)^2, \rho_2 = 2(c_1 + c_2)^3, \rho_3 = -6(c_1 + c_2)^4, \rho_4 = 24(c_1 + c_2)^5, \\ a_{10} &= -2c_1(c_1 + c_2), a_{01} = c_2^2 - c_1^2, a_{11} = -2(c_1 + c_2)^3, a_{02} = 2(c_1 + c_2)^2(c_1 - 2c_2), \\ a_{20} &= 6c_1(c_1 + c_2)^2, a_{12} = 12(c_1 + c_2)^3(c_2 - c_1), a_{21} = 6(c_1 + c_2)^3(c_2 - 3c_1), a_{21} = a_{12}, \\ a_{30} &= -24c_1(c_1 + c_2)^3, b_{10} = c_1^2 - c_2^2, b_{01} = -2c_2(c_1 + c_2), b_{20} = 2(c_1 + c_2)^2(c_2 - 2c_1), \\ b_{11} &= 2(c_1 + c_2)^2(2c_2 - c_1), b_{02} = 6c_2(c_1 + c_2)^2, b_{30} = 12(c_1 + c_2)^3(c_1 - c_2), b_{21} = b_{30}, \\ b_{12} &= 6(c_1 + c_2)^3(c_1 - 3c_2), b_{03} = -24c_2(c_1 + c_2)^3. \end{aligned}$$

(iii) The solutions of system (66) in the neighborhood of the fixed point  $(\bar{x}, \bar{x}, \bar{y}) \in \mathbb{R}^3$  is given by (67). The coefficients from (67), (68) are:

$$\begin{aligned} g_{20} &= p_2 a + p_3 b, g_{11} = p_2 a_1 + p_3 b_1, g_{02} = p_2 \bar{a} + p_3 \bar{b}, \\ g_{21} &= p_2 r_{20}^1 + p_3 r_{20}^2 + 2(p_2 r_{11}^1 + p_3 r_{11}^2) + p_2 C^1 + p_2 C^2, \end{aligned}$$

$$\begin{aligned} a &= (q_2, q_3)A_1(q_2, q_3)^T, \\ b &= (q_1, q_3)A_2(q_1, q_3)^T, \\ a_1 &= (q_2, q_3)A_1(\bar{q}_2, \bar{q}_3)^T, \\ b_1 &= (q_1, q_3)A_2(\bar{q}_1, \bar{q}_3)^T, \end{aligned}$$

$$\begin{aligned} C^1 &= (q_2, q_3)(q_2A_{11} + q_3A_{12})(\bar{q}_2, \bar{q}_3)^T, C^2 = (q_1, q_3)(q_1A_{21} + q_3A_{22})(\bar{q}_1, \bar{q}_3)^T, \\ r_{20}^1 &= (\bar{q}_2, \bar{q}_3)A_1(w_{20}^2, w_{20}^3)^T, r_{20}^2 = (\bar{q}_1, q_3)A_2(w_{20}^1, w_{20}^3)^T, \\ r_{11}^1 &= (\bar{q}_2, q_3)A_1(w_{11}^2, w_{11}^3)^T, r_{11}^2 = (q_1, q_3)A_2(w_{11}^1, w_{11}^3)^T, \end{aligned}$$

$$\begin{aligned} w_{20}^1 &= v_{20} - \frac{p_2a + p_3b}{\mu^2 - \mu} - \frac{\bar{p}_2a + \bar{p}_3b}{\mu^2 - \bar{\mu}}, \\ w_{20}^2 &= \mu^2v_{20} - \frac{p_2a + p_3b}{\mu^2 - \mu}q_2 - \frac{\bar{p}_2a + \bar{p}_3b}{\mu^2 - \bar{\mu}}\bar{q}_2, \\ w_{20}^3 &= \frac{a + (\mu^2 - a_{10})\mu^2v_{20}}{a_{01}} - \frac{p_2a + p_3b}{\mu^2 - \mu}q_3 - \frac{\bar{p}_2a + \bar{p}_3b}{\mu^2 - \bar{\mu}}\bar{q}_3, \\ w_{02}^1 &= \bar{w}_{20}^1, w_{02}^2 = \bar{w}_{20}^2, w_{02}^3 = \bar{w}_{20}^3 \end{aligned}$$

$$\begin{aligned} w_{11}^1 &= v_{11} - \frac{p_2a_1 + p_3b_1}{1 - \mu} - \frac{\bar{p}_2a_1 + \bar{p}_3b_1}{1 - \bar{\mu}}, \\ w_{11}^2 &= v_{11} - \frac{p_2a_1 + p_3b_1}{1 - \mu}q_2 - \frac{\bar{p}_2a_1 + \bar{p}_3b_1}{1 - \bar{\mu}}\bar{q}_2, \\ w_{11}^3 &= \frac{a_1 + (1 - a_{10})v_{11}}{a_{01}} - \frac{p_2a_1 + p_3b_1}{1 - \mu}q_3 - \frac{\bar{p}_2a_1 + \bar{p}_3b_1}{1 - \bar{\mu}}\bar{q}_3, \end{aligned}$$

$$\begin{aligned} p_1 &= \frac{(\bar{\mu} - 1 - ka_{10})(\bar{\mu} - 1 - \alpha b_{10})}{(\bar{\mu} - 1 - ka_{10})(\bar{\mu} - 1 - kb_{01}) + \bar{\mu}(2\bar{\mu} - 2 - ka_{10} - \alpha b_{01})} \\ p_2 &= \frac{p_1}{\bar{\mu} - 1 - ka_{10}}, p_3 = \frac{\bar{\mu}}{\alpha b_{10}}p_1, q_1 = 1, q_2 = \mu, q_3 = \frac{\alpha b_{10}}{\mu - 1 - \alpha b_{01}}. \end{aligned}$$

(iv) The variations of the profits in the neighborhood of the fixed point  $(\bar{x}, \bar{y})^T \in \mathbb{R}^2$ , are given by:

$$\pi_{1n} = p(x_n + y_n)x_n - c_1x_n - b_1, \pi_{2n} = p(x_n + y_n)y_n - c_2y_n - b_2.$$

The above model has a similar behavior as the economic models that describe the business cycles (Kuznetsov, 1995), (Mircea et al., 2004).

The model can be analyzed in a similar way for the case  $m > 2$ .

For  $m = 1$ , the stochastic system associated to (53) is given by:

$$\begin{aligned} x_{n+1} &= x_n + k(p'(x_n + y_n)x_n + p(x_n + y_n) - c'_1(x_n)) + \xi_n b_{22}(x_n - \bar{x}) \\ y_{n+1} &= y_n + k(p'(x_{n-1} + y_n)y_n + p(x_{n-1} + y_n) - c'_2(y_n)) + \xi_n b_{33}(y_n - \bar{y}) \end{aligned} \quad (70)$$

where  $(\bar{x}, \bar{y})$  is the solution of (55).

The linearized of (70) has the matrices:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 + ka_{10} & ka_{01} \\ \alpha b_{10} & 0 & 1 + \alpha b_{01} \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix} \tag{71}$$

Using Proposition 2.1, the characteristic polynomial of (70) is given by:

$$P_2(\lambda) = \lambda(\lambda^2 - \lambda(a_{22}a_{33} + \sigma b_{22}b_{33}) - a_{23}a_{33}a_{31})(\lambda(\lambda - a_{22}^2)(\lambda - a_{33}^2 - \sigma b_{33}^2) - a_{23}^2 a_{31}^2), \tag{72}$$

where  $a_{22} = 1 + ka_{10}$ ,  $a_{23} = ka_{01}$ ,  $a_{31} = \alpha b_{10}$ ,  $a_{33} = 1 + \alpha b_{01}$ .

The analysis of the roots for the equation  $P_2(\lambda) = 0$  is done for fixed values of the parameters.

The numerical simulation can be done for  $c_1 = 0.1$ ,  $c_2 = 0.4$ ,  $k = 0.04$ ,  $\sigma = 0.4$ .

### 5. The discrete deterministic and stochastic Kaldor model

The discrete Kaldor model describes the business cycle for the state variables characterized by the income (national income)  $Y_n$  and the capital stock  $K_n$ , where  $n \in \mathbb{N}$ . For the description of the model's equations we use the investment function  $I : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$  denoted by  $I = I(Y, K)$  and the savings function  $S : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ , denoted by  $S = S(Y, K)$  both considered as being differentiable functions (Dobrescu & Opreş, 2009), (Dobrescu & Opreş, 2009).

The discrete Kaldor model describes the income and capital stock variations using the functions  $I$  and  $S$  and it is described by:

$$\begin{aligned} Y_{n+1} &= Y_n + s(I(Y_n, K_n) - S(Y_n, K_n)) \\ K_{n+1} &= K_n + I(Y_n, K_n) - qK_n. \end{aligned} \tag{73}$$

In (73),  $s > 0$  is an adjustment parameter, which measures the reaction of the system to the difference between investment and saving.

We admit Keynes's hypothesis which states that the saving function is proportional to income, meaning that

$$S(Y, K) = pY, \tag{74}$$

where  $p \in (0, 1)$  is the propensity to save with the respect to the income.

The investment function  $I$  is defined by taking into account a certain normal level of income  $u$  and a normal level of capital stock  $\frac{pu}{q}$ , where  $u \in \mathbb{R}$ ,  $u > 0$ . The coefficient  $q \in (0, 1)$  represents the capital depreciation.

In what follows we admit Rodano's hypothesis and consider the form of the investment function as follows:

$$I(Y, K) = pu + r \left( \frac{pu}{q} - K \right) + f(Y - u) \tag{75}$$

where  $r > 0$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable function with  $f(0) = 0$ ,  $f'(0) \neq 0$  and  $f'''(0) \neq 0$ .

System (73) with conditions (74) and (75) is written as:



$$\begin{aligned} Y_{n+1} &= (1 - sp)Y_n - rsK_n + sf(Y_n - u) + spu \left(1 + \frac{r}{q}\right) \\ K_{n+1} &= (1 - r - q)K_n + f(Y_n - u) + pu \left(1 + \frac{r}{q}\right) \end{aligned} \quad (76)$$

with  $s > 0, q \in (0, 1), p \in (0, 1), r > 0, u > 0$ .

The application associated to system (76) is:

$$\begin{pmatrix} y \\ k \end{pmatrix} \rightarrow \begin{pmatrix} (1 - sp)y - rsk + sf(y - u) + spu \left(1 + \frac{r}{q}\right) \\ (1 - r - q)k + f(y - u) + pu \left(1 + \frac{r}{q}\right) \end{pmatrix}. \quad (77)$$

The fixed points of the application (77) with respect to the model's parameters  $s, q, p, r$  are the solutions of the following system:

$$\begin{aligned} py + rk - f(y - u) - pu \left(1 + \frac{r}{q}\right) &= 0 \\ (r + q)k - f(y - u) - pu \left(1 + \frac{r}{q}\right) &= 0 \end{aligned}$$

that is equivalent to:

$$qk - py = 0, \quad p \left(1 + \frac{r}{q}\right) (y - u) = f(y - u). \quad (78)$$

Taking into account that  $f$  satisfies  $f(0) = 0$ , by analyzing (78) we have:

**Proposition 5.1.** (i) The point of the coordinates  $P \left(u, \frac{pu}{q}\right)$  is the fixed point of the application (77).

(ii) If  $f(x) = \arctan x$ , and  $p \left(1 + \frac{r}{q}\right) \geq 1$  then application (77) has an unique fixed point given by  $P \left(u, \frac{pu}{q}\right)$ .

(iii) If  $f(x) = \arctan x$  and  $p \left(1 + \frac{r}{q}\right) < 1$  then the application (77) has the fixed points  $P \left(u, \frac{pu}{q}\right)$ ,  $R \left(y_r, \frac{py_r}{q}\right)$ ,  $Q \left(y_q, \frac{py_q}{q}\right)$ , where  $y_q = 2u - y_r$  and  $y_r$  is the solution of the following equation:

$$\arctan(y - u) = p \left(1 + \frac{r}{q}\right) (y - u)$$

Let  $(y_0, k_0)$  be a fixed point of the application (77). We use the following notations:  $\rho_1 = f'(y_0 - u)$ ,  $\rho_2 = f''(y_0 - u)$ ,  $\rho_3 = f'''(y_0 - u)$  and

$$a_{10} = s(\rho_1 - p), \quad a_{01} = -rs, \quad b_{10} = \rho_1, \quad b_{01} = -q - r.$$

**Proposition 5.2.** (i) The Jacobian matrix of (77) in the fixed point  $(y_0, k_0)$  is:

$$A = \begin{pmatrix} 1 + a_{10} & a_{01} \\ b_{10} & 1 + b_{01} \end{pmatrix}. \tag{79}$$

(ii) The characteristic equation of  $A$  given by (79) is:

$$\lambda^2 - a\lambda + b = 0 \tag{80}$$

where  $a = 2 + a_{10} + b_{01}$ ,  $b = 1 + a_{10} + b_{01} - a_{01}b_{10}$ .

(iii) If  $q + r < 1$ ,  $\rho_1 < 1 + \frac{r(q+r-4)}{(q+r-2)^2}$  and  $s = s_0$ , where:

$$s_0 = \frac{q+r}{(1-q-r)(\rho_1-p)+r}$$

then equation (80) has the roots with their absolute values equal to 1.

(iv) With respect to the change of variable:

$$s(\beta) = \frac{(1+\beta)^2 - 1 + q + r}{(1-q-r)(\rho_1-p)+r}$$

equation (80) becomes:

$$\lambda^2 - a_1(\beta)\lambda + b_1(\beta) = 0 \tag{81}$$

where

$$a_1(\beta) = 2 + \frac{(\rho_1-p)((1+\beta)^2 - 1 + \rho + r)}{(1-q-r)(\rho_1-p)+r} - q - r, \quad b_1(\beta) = (1+\beta)^2.$$

Equation (81) has the roots:

$$\mu_{1,2}(\beta) = (1+\beta)e^{\pm i\theta(\beta)}$$

where

$$\theta(\beta) = \arccos \frac{a_1(\beta)}{2(1+\beta)}.$$

(v) The point  $s(0) = s_0$  is a Neimark-Sacker bifurcation point.

(vi) The eigenvector  $q \in \mathbb{R}^2$ , which corresponds to the eigenvalue  $\mu(\beta) = \mu$  and is a solution of  $Aq = \mu q$ , has the components

$$q_1 = 1, \quad q_2 = \frac{\mu - 1 - a_{10}}{a_{01}}. \tag{82}$$

The eigenvector  $p \in \mathbb{R}^2$ , which corresponds to the eigenvalue  $\bar{\mu}$  and is a solution of  $A^T p = \bar{\mu} p$ , has the components:

$$p_1 = \frac{a_{01}b_{10}}{a_{01}b_{10} + (\bar{\mu} - 1 - a_{10})^2},$$

$$p_2 = \frac{a_{01}(\bar{\mu} - 1 - a_{01})}{a_{01}b_{10} + (\bar{\mu} - 1 - a_{10})^2}. \tag{83}$$

The vectors  $q, p$  given by (82) and (83) satisfy the condition  $\langle q, p \rangle = q_1\bar{p}_1 + q_2\bar{p}_2 = 1$ .

The proof follows by direct calculation using (77).

With respect to the translation  $y \rightarrow y + y_0$ ,  $k \rightarrow k + k_0$ , the application (77) becomes:

$$\begin{pmatrix} y \\ k \end{pmatrix} \rightarrow \begin{pmatrix} (1 - sp)y - rsk + sf(y + y_0 - u) - f(y_0 - u) \\ -(r + q)k + f(y + y_0 - u) - f(y_0 - u) \end{pmatrix}. \quad (84)$$

Expanding  $F$  from (84) in Taylor series around  $0 = (0, 0)^T$  and neglecting the terms higher than the third order, we obtain:

$$F(y, k) = \begin{pmatrix} (1 + a_{10})y + a_{01}k + \frac{1}{2}s\rho_2y^2 + \frac{1}{6}s\rho_3y^3 \\ b_{10}y + b_{01}k + \frac{1}{2}\rho_2y^2 + \frac{1}{6}\rho_3y^3 \end{pmatrix}.$$

**Proposition 5.3.** (i) The canonical form of (84) is:

$$\begin{aligned} z_{n+1} = & \mu(\beta)z_n + \frac{1}{2}(s(\beta)p_1 + p_2)\rho_2(z_n^2 + 2z_n\bar{z}_n + \bar{z}_n^2) + \\ & + \frac{1}{6}(s(\beta)p_1 + p_2)\rho_3(z_n^3 + 3z_n^2\bar{z}_n + 3z_n\bar{z}_n^2 + \bar{z}_n^3). \end{aligned} \quad (85)$$

(ii) The coefficient  $C_1(\beta)$  associated to the canonical form (85) is:

$$\begin{aligned} C_1(\beta) = & \left( \frac{(p(\beta)p_1 + p_2)^2(\bar{\mu} - 3 + 2\mu)}{2(\mu^2 - \mu)(\bar{\mu} - 1)} + \frac{|s(\beta)p_1 + p_2|^2}{1 - \bar{\mu}} + \right. \\ & \left. + \frac{|s(\beta)p_1 + p_2|^2}{2(\mu^2 - \bar{\mu})} \right) \rho_2^2 + \frac{s(\beta)p_1 + p_2}{2} \rho_3 \end{aligned}$$

and  $l_1(0) = \text{Re}(C_1(0)e^{i\theta(0)})$ . If  $l_1(0) < 0$  in the neighborhood of the fixed point  $(y_0, k_0)$  then there is a stable limit cycle. If  $l_1(0) > 0$  there is an unstable limit cycle.

(iii) The solution of (76) in the neighborhood of the fixed point  $(y_0, k_0)$  is:

$$Y_n = y_0 + z_n + \bar{z}_n, \quad K_n = k_0 + q_2z_n + \bar{q}_2\bar{z}_n$$

where  $z_n$  is a solution of (85).

The stochastic system of (76) is given by (Mircea et al., 2010):

$$\begin{aligned} Y_{n+1} = & (1 - sp)Y_n - rsK_n + sf(Y_n - u) + spu \left( 1 + \frac{r}{q} \right) + \xi_n b_{11}(Y_n - u) \\ K_{n+1} = & (1 - r - q)K_n + f(Y_n - u) + pu \left( 1 + \frac{r}{q} \right) + \xi_n b_{22} \left( K_n - \frac{pu}{q} \right) \end{aligned}$$

with  $E(\xi_n) = 0$  and  $E(\xi_n^2) = \sigma$ .

Using (79) and Proposition 5.2, the characteristic polynomial of the linearized system of (5) is given by:

$$P_2(\lambda) = \det \begin{pmatrix} \lambda - (1 + a_{10})^2 - \sigma b_{11}^2 & -a_{01}^2 & -2a_{01}(1 + a_{10}) \\ -b_{10}^2 & \lambda - (1 + b_{01})^2 - \sigma b_{22} & -2b_{10}(1 + b_{01}) \\ -b_{10}(1 + a_{10}) & -a_{01}(1 + b_{01}) & \lambda - (a_{01}b_{10} + (1 + a_{10})(1 + b_{01}) + \sigma b_{11}b_{22}) \end{pmatrix} \quad (86)$$

The analysis of the roots for  $P_2(\lambda) = 0$  can be done for fixed values of the parameters.

## 6. Conclusions

The aim of this chapter is to briefly present some methods used for analyzing the models described by deterministic and stochastic discrete-time equations with delay. These methods are applied to models that describe: the Internet congestion control, economic games and the Kaldor economic model, as well. The obtained results are presented in a form which admits the numerical simulation.

The present chapter contains a part of the authors' papers that have been published in journals or proceedings, to which we have added the stochastic aspects.

The methods used in this chapter allow us to study other models described by systems of equations with discrete time and delay and their associated stochastic models.

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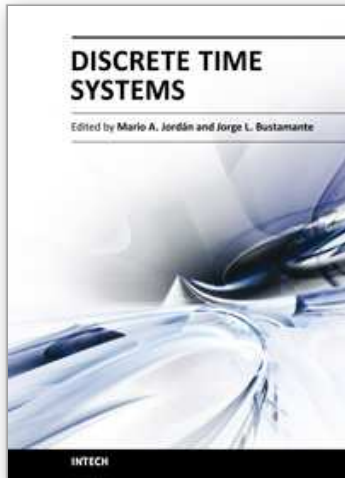
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Discrete-Time Systems comprehend an important and broad research field. The consolidation of digital-based computational means in the present, pushes a technological tool into the field with a tremendous impact in areas like Control, Signal Processing, Communications, System Modelling and related Applications. This book attempts to give a scope in the wide area of Discrete-Time Systems. Their contents are grouped conveniently in sections according to significant areas, namely Filtering, Fixed and Adaptive Control Systems, Stability Problems and Miscellaneous Applications. We think that the contribution of the book enlarges the field of the Discrete-Time Systems with signification in the present state-of-the-art. Despite the vertiginous advance in the field, we also believe that the topics described here allow us also to look through some main tendencies in the next years in the research area.

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