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Orbital Stability of Periodic Traveling Wave Solutions

Jaime Angulo Pava[1] and Fábio Natali[2]
[1]*Department of Mathematics, IME-USP - São Paulo - SP*
[2]*Department of Mathematics, Universidade Estadual de Maringá - Maringá - PR*
Brazil

1. Introduction

The theory of stability of periodic traveling waves associated with evolution partial differential equations of dispersive type has increased significantly in the last five years. A good number of researchers are interested in solving a rich variety of new mathematical problems due to physical importance related to them. This subject is often studied in relation to perturbations of symmetric classes, e.g., the class of periodic functions with the same minimal period as the underlying wave. However, it is possible to consider a stability study with general non-periodic perturbations, e.g., by the class of spatially localized perturbations $L^2(\mathbb{R})$ or by the class of bounded uniformly continuous perturbations $C_b(\mathbb{R})$ (see Mielke (1997), Gardner (1993)-(1997) and Gallay&Hărăguş (2007)).
Here our purpose is to consider the nonlinear stability and linear instability of periodic traveling waveforms. From our experience with nonlinear dispersive equations we know that traveling waves, when they exist, are of fundamental importance in the development of a broad range of disturbance. Then we expect the issue of stability of periodic waves to be of interest and it inspires future developments in this fascinating subject.
It is well known that such theory has started with the pioneering work of Benjamin (1972) regarding the periodic steady solutions called *cnoidal waves*. Its waveform profile was found first by Korteweg&de-Vries (1895) for the currently called Korteweg-de Vries equation (KdV henceforth)

$$u_t + uu_x + u_{xxx} = 0, \tag{1}$$

where $u = u(x,t)$ is a real-valued function of two variables $x, t \in \mathbb{R}$. The cnoidal traveling wave solution, $u(x,t) = \varphi_c(x - ct)$, has a profile determined in the form

$$\varphi_c(\xi) = \beta_2 + (\beta_3 - \beta_2)\mathrm{cn}^2\left(\sqrt{\frac{\beta_3 - \beta_1}{12}}\xi; k\right), \tag{2}$$

where β_i's are real constants and *cn* represents the Jacobi elliptic function *cnoidal*. Among the physical application associated with equation (1) we can mention the propagation of shallow-water waves with weakly non-linear restoring forces, long internal waves in a density-stratified ocean, ion-acoustic waves in a plasma, acoustic waves on a crystal lattice, and so on. Thus, the study of qualitative properties of these nonlinear periodic waves represents a fundamental piece for the understanding of the dynamic associated to this equation. A first stability approach for the cnoidal wave profile as in (2) was determined by Benjamin in (1972). But only years later a complete study was carry out by Angulo et al. (2006). Indeed, by extending the general stability theory due to Grillakis et al. (1987) to the periodic case it was obtained that the orbit generated by the solution φ_c,

$$\Omega_{\varphi_c} = \{\varphi_c(\cdot + y) : y \in \mathbb{R}\}, \tag{3}$$

remains stable by the periodic flow of the KdV equation, more specifically, for initial data close enough to Ω_{φ_c} the solution of the KdV starting in this point will remain close enough to to Ω_{φ_c} for all time. Many ingredients are basic for obtaining this remarkable behavior of the cnoidal waves. One of the cornerstones is the spectral structure associated to the self-adjoint operator on $L^2_{per}([0, L])$ (here L represents the minimal period of φ_c)

$$\mathcal{L}_{kdv} = -\frac{d^2}{dx^2} + c - \varphi_c, \tag{4}$$

which is a Schrödinger operator with a periodic potential.
In this case, the existence of a unique negative eigenvalue and simple and the non-degeneracy of the eigenvalue zero is required. We recall that $\mathcal{L}_{kdv}$ has a compact resolvent and so zero is an isolated eigenvalue. It is well known that the determination of these spectral informations in the periodic case is not an easy task. By taking advantage of the cnoidal profile of φ_c, the eigenvalue problem for $\mathcal{L}_{kdv}$ is reduced to study the classical Lamé problem

$$\frac{d^2}{dx^2}\psi + [\rho - n(n+1)k^2 sn^2(x;k)]\psi = 0, \tag{5}$$

on the space $L^2_{per}([0, 2K(k)])$, for $n \in \mathbb{N}$, $sn(\cdot;k)$ denoting the Jacobi elliptic function *snoidal* and K representing the complete elliptic integral of first kind. Therefore the Floquet theory arises in a crucial form in the stability analysis. The existence of a finite number of instability intervals associated to (5) and an oscillation Sturm analysis will imply the required spectral structure for $\mathcal{L}_{kdv}$. Next, by supposing that φ_c has mean zero property we consider the manifold $M = \{f : \int f^2 dx = \int \varphi_c^2 dx, \int f dx = 0\}$. Then the condition $\frac{d}{dc}\int \varphi_c^2(x)dx > 0$ will imply that

$$\langle \mathcal{L}_{kdv} f, f\rangle \geqq \beta \|f\|^2_{H^1_{per}} \quad \text{for every } f \in T_{\varphi_c}M \cap [\frac{d}{dx}\varphi_c]^{\perp}, \tag{6}$$

where $T_{\varphi_c}M$ represents the tangent space to M in φ_c and $\beta > 0$. Then, from the continuity of the functional $E(f) = \int (f')^2 - \frac{1}{3}f^3 dx$ and from the Taylor theorem we have the following stability property of Ω_{φ_c}: *there is* $\eta > 0$ *and* $D > 0$ *such that*

$$E(u) - E(\varphi_c) \geqq D \inf_{g \in \Omega_{\varphi_c}} \|u - g\|^2_{H^1_{per}} \tag{7}$$

for u satisfying that $\inf_{g\in\Omega_{\varphi_c}} \|u - g\|_{H^1_{per}} < \eta$ *and* $F(u) \equiv \frac{1}{2}\int u^2 dx = \frac{1}{2}\int \varphi_c^2 dx, \int u dx = 0$.
In other words, φ_c is a constraint local minimum of E. Then, since E and F are conserved quantities by the continuous KdV-flow, $t \to u(t)$, we obtain from (7) that the orbit Ω_{φ_c} is stable by initial perturbation in the manifold M. For general perturbations of Ω_{φ_c} we need to have the existence of a smooth curve of traveling waves, $c \to \varphi_c$, and to use the triangular inequality. We call attention that mean zero constraint can be eliminated in the definition of

the manifold M, because the KdV equation is invariant under the Galilean transformation $v(x,t) = u(x+\gamma t, t) - \gamma$, where γ is any real number. That is, if u solves (1), then so does v. In this point some comments on the speed-wave associated to the cnoidal wave solution φ_c deserves to be hold. If we are looking for φ_c having mean zero, we obtain a curve $c \in (0,\infty) \to \varphi_c \in H^1_{per}([0,L])$. But, for instance, if φ_c is positive we obtain a curve $c \in (4\pi^2/L^2, \infty) \to \varphi_c \in H^1_{per}([0,L])$ for any $L > 0$ (see Angulo (2009)).

From analysis above we see that spectral information about the operator in (4) is fundamental for a stability study. Indeed, the second order differential operator appearing in equation (4) is the key to apply the Floquet theory, but from our experience with nonlinear dispersive evolution equations we know that depending of the periodic potential the study can be tricky. Moreover, the Floquet theory is not useful for more general linear operators that arise in the study of nonlinear dispersive equations. For instance, a general kind of dispersive equations can be

$$u_t + u^p u_x - (\mathcal{M}u)_x = 0, \tag{8}$$

where $p \in \mathbb{N}$ and $\mathcal{M}$ is a Fourier multiplier operator defined by

$$\widehat{\mathcal{M}f}(n) = \beta(n)\widehat{f}(n), \quad n \in \mathbb{Z}, \tag{9}$$

with β being a measurable, locally bounded, even function on $\mathbb{R}$, and satisfying the conditions, $A_1|n|^{m_1} \leq \beta(n) \leq A_2(1+|n|)^{m_2}$, for $m_1 \leq m_2$, $|n| \geq k_0$, $\beta(n) > b$ for all $n \in \mathbb{Z}$, and $A_i > 0$. Then, the following unbounded linear self-adjoint operator $\mathcal{L}_{\mathcal{M}} : D(\mathcal{L}_{\mathcal{M}}) \to L^2_{per}([0,L])$

$$\mathcal{L}_{\mathcal{M}} = (\mathcal{M} + c) - \varphi_c^p, \tag{10}$$

arises in the study of traveling wave solutions of the form $u(x,t) = \varphi_c(x - ct)$ for equation (8). Here the profile $\varphi = \varphi_c$ must satisfy the following nonlinear equation

$$(\mathcal{M} + c)\varphi - \frac{1}{p+1}\varphi^{p+1} = A_\varphi, \tag{11}$$

where A_φ is a constant of integration which can be assumed to be zero and the wave-speed c is chosen such that $\mathcal{M} + c$ is a positive operator. Equation (8) with $p = 1$, contains two important models in internal water-wave research: The Benjamin-Ono equation (BO henceforth), $\mathcal{M} = \mathcal{H}\partial_x$, where $\mathcal{H}$ denotes the periodic Hilbert transform defined via the Fourier transform as $\widehat{\mathcal{H}f}(n) = -isgn(n)\widehat{f}(n)$, $n \in \mathbb{Z}$. So, we have that $\mathcal{M}$ has associated the symbol $\beta(n) = |n|$. The other model is the Intermediate Long Wave equation (ILW henceforth), where the pseudo-differential operator $\mathcal{M}$ has associated the symbol $\beta_h(n) = n\coth(nh) - \frac{1}{h}$, $h \in (0, +\infty)$.

Recently, Angulo&Natali (2008) established a new approach for studying the general linear operator $\mathcal{L}_{\mathcal{M}}$ in (10) within the framework of the theory of stability for even and positive periodic traveling waves (see Section 3). Indeed, by using Fourier techniques associated to positive linear operators was obtained that the positivity of the Fourier coefficients associated to φ_c together with a specific positivity property called $PF(2)$ for the Fourier coefficients of the power function φ_c^p, will imply the existence of a unique negative eigenvalue and simple and the non-degeneracy of the eigenvalue zero. Therefore, one of the advantage of Angulo&Natali's approach is the possibility of studying non-local linear operators such as that associated to the BO equation (see Section 5)

$$\mathcal{L}_{bo} = \mathcal{H}\partial_x + c - \varphi_c. \tag{12}$$

We also note that in the case of the critical KdV equation ($p = 4$ and $\mathcal{M} = -\partial_x^2$ in (8)) Angulo&Natali's approach was applied successfully for obtaining the relevant result that there is a family of periodic traveling waves, $c \to \varphi_c$, such that they are stable if the wave-speed $c \in (\pi^2/L^2, r_0/L^2)$ and unstable if $c \in (r_0/L^2, +\infty)$, where $r_0 > 0$ does not depend on L (see Angulo&Natali (2009)). In the case of operators of type Schrödinger,

$$\mathcal{L} = -\frac{d^2}{dx^2} + c - \varphi_c^p, \tag{13}$$

Neves (2009) (see Section 6) and Johnson (2009) have obtained other criterium for obtaining the required spectral information in a stability study. Their approach are different from those ones that we shall establish in this chapter. For instance, Johnson uses tools from ordinary differential equations and Evans function methods. Its stability approach works for perturbations restricted to the manifold of initial data u_0 such that $\int u_0^2(x)dx = \int \varphi_c^2(x)dx$ and $\int u_0(x)dx = \int \varphi_c(x)dx$.
Other important piece of information in a stability study is the existence of solutions for the nonlinear equation (11). For $\mathcal{M} = -\partial_x^2$ is obvious that the quadrature method is the most natural tool to be used (see subsection 5.2). Therefore, the theory of elliptic integrals and Jacobian elliptic functions arise in a very natural way. For $\mathcal{M}$ being a non-local operator the existence problem is not an easy task. In this point the use of Fourier methods can be very useful. Indeed, suppose that φ_c represents a solitary wave solution for equation (11) ($A_\varphi = 0$) with $\widehat{\varphi_c}^{\mathbb{R}}$ representing its Fourier transform on the line, then the Poisson Summation Theorem produces a periodic function ψ given by formula

$$\psi(\xi) = \sum_{n\in\mathbb{Z}} \varphi_c(\xi + Ln) = \frac{1}{L}\sum_{n\in\mathbb{Z}} \widehat{\varphi_c}^{\mathbb{R}}\left(\frac{n}{L}\right) e^{\frac{2\pi i n\xi}{L}}. \tag{14}$$

Note that ψ has a minimal period L. Now, from our experience with dispersive evolution equations we know that the profile ψ does not give for every c a solution for equation (11). Indeed, we have only that for a specific range of the solitary wave-speed, c, it will produce that ψ is in fact a periodic traveling wave solution. In other words, there are an interval I and a smooth wave-speed mapping, $c \in I \to v(c)$, such that ψ satisfies $(\mathcal{M} + v(c))\psi - \frac{1}{p+1}\psi^{p+1} = 0$. An example where equality (14) can be used is in obtaining the well-know Benjamin's periodic traveling wave solution for the BO equation (see subsection 5.1). We note from formula (14) that a good knowledge of the Fourier transform $\widehat{\varphi_c}^{\mathbb{R}}$ is necessary for obtaining an explicit profile of ψ and that the Fourier coefficients of ψ are depending of the discretization of $\widehat{\varphi_c}^{\mathbb{R}}$ to the enumerable set $\{n/L\}_{n\in\mathbb{Z}}$.
We note that in our approach we consider the minimal period associated to the periodic traveling wave solutions completely arbitrary. Our analysis is not restricted to small or large wavelength. We also note that the stability theory to be established here it can be applied to a sufficiently wide range of non-linear dispersive models, such as the nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^p u = 0 \tag{15}$$

with $u = u(x,t) \in \mathbb{C}$ and $p = 2, 3, 4, ...$, and for the generalized Benjamin-Bona-Mahony equations

$$u_t + u_x + u^p u_x + \mathcal{M}u_t = 0, \tag{16}$$

for $p \geqq 1$, $p \in \mathbb{N}$, and $\mathcal{M}$ given by (9) (see Angulo et al. (2010)).

We will also be interested in this chapter in the *linear instability* of periodic traveling wave solutions. By using the theoretical framework of Weinstein (1986) and Grillakis (1988) we show that there is a family of periodic traveling wave for the cubic Schrödinger equation ($p = 2$ in (15)) with a minimal period L which are orbitally stable in $H^1_{per}([0, L])$ but linearly unstable in $H^1_{per}([0, jL])$, for $j \geqq 2$ (see subsection 5.3.1). In the general case of equations in (8) we establish a criterium of linear instability developed recently by Angulo&Natali (2010) (see Section 7).

In the last section of this chapter, we establish some results about the existence and stability of periodic-peakon for the following nonlinear Schrödinger equation (NLS-δ equation),

$$iu_t + u_{xx} + \gamma\delta(x)u + |u|^2u = 0, \tag{17}$$

defined for functions on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Here the symbol δ denotes the Dirac delta distribution, $(\delta, \psi) = \psi(0)$, and $\gamma \in \mathbb{R}$ is denominated the coupling constant or strength attached to the point source located at $x = 0$.

2. Notation

For any complex number $z \in \mathbb{C}$, we denote by $\Re(z)$ and $\Im(z)$ the real part and imaginary part of z, respectively. For $s \in \mathbb{R}$, the Sobolev space $H^s_{\text{per}}([0, L])$ consists of all periodic distributions f such that $\|f\|^2_{H^s} = L \sum\limits_{k=-\infty}^{\infty} (1 + n^2)^s |\widehat{f}(n)|^2 < \infty$. For simplicity, we will use the notation H^s_{per} and $H^0_{\text{per}} = L^2_{\text{per}}$. The Fourier transform of a periodic distribution Ψ is the function $\widehat{\Psi} : \mathbb{Z} \to \mathbb{C}$ defined by the formula $\widehat{\Psi}(n) = \frac{1}{L}\langle \Psi, \Theta_{-n}\rangle$, $n \in \mathbb{Z}$, for $\Theta_n(x) = \exp(2\pi inx/L)$. So, if Ψ is a periodic function with period L, we have $\widehat{\Psi}(n) = \frac{1}{L}\int_0^L \Psi(x)e^{-\frac{2n\pi xi}{L}}\,dx$. The normal elliptic integral of first type (see Byrd&Friedman (1971)) is defined by

$$\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\phi, k)$$

where $y = \sin\phi$ and $k \in (0, 1)$. k is called the modulus and ϕ the argument. When $y = 1$, we denote $F(\pi/2, k)$ by $K = K(k)$. The Jacobian elliptic functions are denoted by $sn(u; k)$, $cn(u; k)$ and $dn(u; k)$ (called, snoidal, cnoidal and dnoidal, respectively), and are defined via the previous elliptic integral. More precisely, let $u(y; k) := u = F(\phi, k)$, then $y = sin\phi := sn(u; k)$, $cn(u; k) = \sqrt{1 - sn^2(u; k)}$ and $dn(u; k) = \sqrt{1 - k^2 sn^2(u; k)}$. We have the following asymptotic formulas: $sn(x; 1) = tanh(x)$, $cn(x; 1) = sech(x)$ and $dn(x; 1) = sech(x)$.

3. Positivity properties of the Fourier transform in the nonlinear stability theory

The approach contained in Angulo& Natali (2008) introduces a new criterium for obtaining that the self-adjoint operator $\mathcal{L}_{\mathcal{M}}$ in (10) possesses exactly one negative eigenvalue which is simple and the eigenvalue zero is simple with eigenfunction $\frac{d}{dx}\varphi$. These specific spectral properties are obtained provided that φ is an even positive periodic function with *a priori* minimal period, and such that $\widehat{\varphi}(n) > 0$ for every $n \in \mathbb{Z}$ and $(\widehat{\varphi^p}(n))_{n\in\mathbb{Z}} \in PF(2)$-discrete class which we shall define below.

We start our approach by defining for all $\theta \geq 0$, the convolution operator $S_\theta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by

$$S_\theta \alpha(n) = \frac{1}{\omega_\theta(n)} \sum_{j=-\infty}^{\infty} \mathcal{K}(n-j)\alpha_j = \frac{1}{\omega_\theta(n)} (\mathcal{K} * \alpha)_n,$$

where $\omega_\theta(n) = \beta(n) + \theta + c$, $\mathcal{K}(n) = \widehat{\varphi_c^p}(n)$, $n \in \mathbb{Z}$. Here we have chosen c such that $c > -b$ where $b \in \mathbb{R}$ satisfies $\beta(n) > b$ for all $n \in \mathbb{Z}$. Then we have $\omega_\theta(n) > 0$ for all $n \in \mathbb{Z}$. It follows that the space X defined by

$$X = \{\alpha \in \ell^2(\mathbb{Z}); ||\alpha||_{X,\theta} := (\sum_{n=-\infty}^{\infty} |\alpha_n|^2 \omega_\theta(n))^{\frac{1}{2}} < \infty\},$$

is a Hilbert space with norm $||\alpha||_{X,\theta}$ and inner product $< \alpha^1, \alpha^2 >_{X,\theta} = \sum_{n=-\infty}^{\infty} \alpha_n^1 \overline{\alpha_n^2} \omega_\theta(n)$.

The next Proposition is a consequence of the theory of self-adjoint operators with a compact resolvent.

Proposition 3.1. *For every $\theta \geq 0$, we have the following*

(a) If $\alpha \in \ell^2$ is an eigensequence of S_θ for a non-zero eigenvalue, then $\alpha \in X$.

(b) The restriction of S_θ to X is a compact, self-adjoint operator with respect to the norm $|| \cdot ||_{X,\theta}$.

(c) 1 is an eigenvalue of S_θ (as an operator of X) if and only if $-\theta$ is an eigenvalue of $\mathcal{L}_\mathcal{M}$ (as an operator of L^2_{per}). Furthermore, both eigenvalues have the same multiplicity.

(d) S_θ has a family of eigensequences $(\psi_{i,\theta})_{i=0}^{\infty}$ forming an orthonormal basis of X with respect to the norm $|| \cdot ||_{X,\theta}$. The eigensequences correspond to real eigenvalues $(\lambda_i(\theta))_{i=0}^{\infty}$ whose only possible accumulation point is zero. Moreover, $|\lambda_0(\theta)| \geq |\lambda_1(\theta)| \geq |\lambda_2(\theta)| \geq \cdots$.

Proof. See Angulo & Natali (2008). □

Definition 3.1. *We say that a sequence $\alpha = (\alpha_n)_{n \in \mathbb{Z}} \subseteq \mathbb{R}$ is in the class $PF(2)$ discrete if*

i) $\alpha_n > 0$, for all $n \in \mathbb{Z}$,

ii) $\alpha_{n_1-m_1}\alpha_{n_2-m_2} - \alpha_{n_1-m_2}\alpha_{n_2-m_1} \geqq 0$, for $n_1 < n_2$ and $m_1 < m_2$,

iii) $\alpha_{n_1-m_1}\alpha_{n_2-m_2} - \alpha_{n_1-m_2}\alpha_{n_2-m_1} > 0$, if $n_1 < n_2$, $m_1 < m_2$, $n_2 > m_1$, and $n_1 < m_2$.

Example: The sequence $a_n = e^{-\eta|n|}$, $n \in \mathbb{Z}$, $\eta > 0$, belongs to $PF(2)$ discrete class. Indeed, the conditions $ii)$ and $iii)$ in Definition 3.1 are equivalents to

$$\begin{aligned} &1)\ |n_1 - m_1| + |n_2 - m_2| \leqq |n_1 - m_2| + |n_2 - m_1|, \text{ if } n_1 < n_2 \text{ and } m_1 < m_2, \text{and} \\ &2)\ |n_1 - m_1| + |n_2 - m_2| < |n_1 - m_2| + |n_2 - m_1|, \text{ if } n_1 < n_2,\ m_1 < m_2, \\ &\quad n_2 > m_1 \text{ and } n_1 < m_2, \end{aligned} \tag{18}$$

which are immediately verified. In section 4 we will use this example in the stability theory of periodic traveling wave solutions for the BO equation.

The next result will also be useful in section 4.

Theorem 3.1. *Let α^1 and α^2 be two even sequences in the class $PF(2)$ discrete, then the convolution $\alpha^1 * \alpha^2 \in PF(2)$ discrete (if the convolution makes sense).*

Proof. See Karlin (1968). □

We present the main result of this section.

Theorem 3.2. *Let φ_c be an even positive solution of (11) with $A_\varphi = 0$. Suppose that $\widehat{\varphi_c}(n) > 0$ for every $n \in \mathbb{Z}$, and $(\widehat{\varphi_c^p}(n))_{n\in\mathbb{Z}} \in PF(2)$ discrete. Then $\mathcal{L}_{\mathcal{M}}$ in (10) possesses exactly a unique negative eigenvalue which is simple, and zero is a simple eigenvalue with eigenfunction $\frac{d}{dx}\varphi_c$.*

Proof. The complete proof of this theorem is very technical and long (see Angulo&Natali (2008) for details), so we only give a sketch of it divided in three basic steps as follows.

I- Since S_θ is a compact-self-adjoint operator on X, it follows that

$$\lambda_0(\theta) = \pm \sup_{||\alpha||_X=1} |< S_\theta \alpha, \alpha >_X|. \tag{19}$$

Let $\psi(\theta) := \psi$ be an eigensequence of S_θ corresponding to $\lambda_0(\theta) := \lambda_0$. We will show that ψ is one-signed, that is, either $\psi(n) \leq 0$ or $\psi(n) \geq 0$. By contradiction, suppose ψ takes both negative and positive values. By hypotheses the kernel $\mathcal{K} = (\mathcal{K}(n)) = (\widehat{\varphi_c^p}(n))$ *is positive*, then

$$\begin{aligned} S_\theta|\psi|(n) &= \tfrac{1}{\omega_\theta(n)} \textstyle\sum_{j=-\infty}^{\infty} \mathcal{K}(n-j)\psi^+(j) + \tfrac{1}{\omega_\theta(n)} \textstyle\sum_{j=-\infty}^{\infty} \mathcal{K}(n-j)\psi^-(j) \\ &> \left| \tfrac{1}{\omega_\theta(n)} \textstyle\sum_{j=-\infty}^{\infty} \mathcal{K}(n-j)\psi^+(j) - \tfrac{1}{\omega_\theta(n)} \textstyle\sum_{j=-\infty}^{\infty} \mathcal{K}(n-j)\psi^-(j) \right|, \end{aligned}$$

where ψ^+ e ψ^- are the positive and negative parts of ψ respectively. It follows that

$$< S_\theta(|\psi|), |\psi| >_{X,\theta} > \sum_{n=-\infty}^{\infty} |\lambda_0||\psi(n)|^2 \omega_\theta(n) = |\lambda_0| \|\psi\|^2_{X,\theta}.$$

Hence, if we assume that $||\psi||_X = 1$, we obtain $< S_\theta(|\psi|), |\psi| >_X > |\lambda_0|$, which contradicts (19). Then, there is an eigensequence ψ_0 which is nonnegative. Now, since $\mathcal{K}$ *is a positive sequence* and $S_\theta(\psi_0) = \lambda_0\psi_0$, we have $\psi_0(n) > 0$, $\forall n \in \mathbb{Z}$. Therefore, ψ_0 can not be orthogonal to any non-trivial one-signed eigensequence in X, which implies that λ_0 is a simple eigenvalue. Notice that the preceding argument also shows that $-\lambda_0$ can not be an eigenvalue of S_θ, therefore it follows that $|\lambda_1| < \lambda_0$.

II- The next step will be to study the behavior of the eigenvalue $\lambda_1(\theta)$. In fact, it considers the following set of indices,

$$\triangle = \{(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z};\ n_1 < n_2\}.$$

Denoting $\overline{n} = (n_1, n_2)$ and $\overline{m} = (m_1, m_2)$, we define for $\overline{n}, \overline{m} \in \triangle$ the following sequence

$$\mathcal{K}_2(\overline{n}, \overline{m}) := \mathcal{K}(n_1 - m_1)\mathcal{K}(n_2 - m_2) - \mathcal{K}(n_1 - m_2)\mathcal{K}(n_2 - m_1).$$

By hypothesis $\mathcal{K} \in PF(2)$ discrete, hence $\mathcal{K}_2 > 0$. Let $\ell^2(\triangle)$ be defined as

$$\ell^2(\triangle) = \left\{ \alpha = (\alpha_{\overline{n}})_{\overline{n}\in\triangle}; \sum\sum\nolimits_{\triangle} |\alpha_{\overline{n}}|^2 := \sum_{n_1\in\mathbb{Z}} \sum_{\substack{n_1<n_2 \\ n_2\in\mathbb{Z}}} |\alpha(n_1, n_2)|^2 < +\infty \right\},$$

and define the operator $S_{2,\theta} : \ell^2(\triangle) \to \ell^2(\triangle)$ by

$$S_{2,\theta} g(\overline{n}) = \sum\sum_{\triangle} G_{2,\theta}(\overline{n}, \overline{m}) g(\overline{m}),$$

where $G_{2,\theta}(\overline{n}, \overline{m}) = \frac{\mathcal{K}_2(\overline{n}, \overline{m})}{\omega_\theta(n_1)\omega_\theta(n_2)}$. It also consider, the space

$$W = \left\{ \alpha \in \ell^2(\triangle); ||\alpha||_{W,\theta} := \left(\sum\sum_{\triangle} |\alpha(\overline{n})|^2 \omega_\theta(n_1)\omega_\theta(n_2) \right)^{\frac{1}{2}} < \infty \right\}.$$

Then W is a Hilbert space with norm $|| \cdot ||_{W,\theta}$ given above and with inner product

$$< \alpha^1, \alpha^2 >_{W,\theta} = \sum\sum_{\triangle} \alpha^1(\overline{n}) \overline{\alpha^2(\overline{n})} \omega_\theta(n_1)\omega_\theta(n_2).$$

Remark 3.1. 1) *We can show, analogous to Proposition 3.1, that $S_{2,\theta}|_W$ is a self-adjoint, compact operator. Therefore, the eigenvalues associated to it operator can be enumerated in order of decreasing absolute value, that is, $|\mu_0(\theta)| \geq |\mu_1(\theta)| \geq |\mu_2(\theta)| \geq \cdots$.*
2) *We also obtain that $\mu_0(\theta) := \mu_0$ is positive, simple and $|\mu_1| < \mu_0$.*

Definition 3.2. *Let α^1, $\alpha^2 \in \ell^2(\mathbb{Z})$, we define the wedge product $\alpha^1 \wedge \alpha^2$ in $\triangle$ by $(\alpha^1 \wedge \alpha^2)(n_1, n_2) = \alpha^1(n_1)\alpha^2(n_2) - \alpha^1(n_2)\alpha^2(n_1)$.*

We have the following results from Definition 3.2.

Lemma 3.1. *1) Let $A = \{\alpha^1 \wedge \alpha^2;$ for α^1, $\alpha^2 \in X$, $\alpha^1 \wedge \alpha^2 \in \ell^2(\triangle)\}$. Then A is dense in W.*
2) Let $\alpha^1, \alpha^2 \in \ell^2(\mathbb{Z})$. Then $S_{2,\theta}(\alpha^1 \wedge \alpha^2) = S_\theta \alpha^1 \wedge S_\theta \alpha^2$.

Proof. See Karlin (1964), Karlin (1968) and Albert (1992) . □

The following Lemma is the key to characterize the second eigenvalue λ_1.

Lemma 3.2. *For all $\theta \geqq 0$ we have:*
a) $\mu_0(\theta) = \lambda_0(\theta)\lambda_1(\theta)$, and then $\lambda_1(\theta) > 0$.
b) $\lambda_1(\theta)$ is simple.

Proof. See Angulo&Natali (2008). □

III- Final step. For $i = 0, 1$, we have that the differentiable curve $\theta \to \lambda_i(\theta)$ satisfies $\frac{d}{d\theta}\lambda_i(\theta) < 0$ and $\lim_{\theta\to\infty} \lambda_0(\theta) = 0$. From $\widehat{\varphi_c}(n) > 0$ for all $n \in \mathbb{Z}$, it follows $\lambda_1(0) = 1$. Since $\lambda_0(0) > \lambda_1(0) = 1$, there is a unique $\theta_0 \in (0, +\infty)$ such that $\lambda_0(\theta_0) = 1$. From Proposition 3.1, we obtain that $\kappa \equiv -\theta_0$ is a negative eigenvalue of $\mathcal{L}_{\mathcal{M}}$ which is simple. For $i \geq 2$ and $\theta > 0$ we have that $\lambda_i(\theta) \leq \lambda_1(\theta) < \lambda_1(0) = 1$, so 1 can not be eigenvalue of S_θ for all $\theta \in (0, +\infty) \setminus \{\theta_0\}$, since 1 is an eigenvalue only for $\theta = 0$ and $\theta = \theta_0$. Then $\mathcal{L}_{\mathcal{M}}$ has a unique negative eigenvalue which is simple. Finally, since $\lambda_1(0) = 1$ and λ_1 is a simple eigenvalue it follows that $\theta = 0$ is a simple eigenvalue of $\mathcal{L}_{\mathcal{M}}$ by Proposition 3.1. This shows the theorem. □

Remark 3.2. *In Theorem 3.2 the Fourier transform of φ_c and φ_c^p must be calculated in the minimal period L of φ_c.*

3.1 Construction of periodic functions in $PF(2)$ discrete class

In this subsection we show a method for building non-trivial periodic functions such that its Fourier transform belongs to the $PF(2)$ discrete class. We start with the $PF(2)$ continuous class.

Definition 3.3. *We say that a function $\mathcal{K} : \mathbb{R} \to \mathbb{R}$ is in the $PF(2)$ continuous class if*

i) $\mathcal{K}(x) > 0$ *for* $x \in \mathbb{R}$,

ii) $\mathcal{K}(x_1 - y_1)\mathcal{K}(x_2 - y_2) - \mathcal{K}(x_1 - y_2)K(x_2 - y_1) \geqq 0$, *for* $x_1 < x_2$ *and* $y_1 < y_2$*; and*

iii) strict inequality holds in ii) whenever the intervals (x_1, x_2) *and* (y_1, y_2) *intersect.*

The following result is immediate.

Proposition 3.2. *Suposse $\mathcal{K}$ is in the $PF(2)$ continuous class. Then for $\alpha(n) \equiv \mathcal{K}(n)$ we have that the sequence $(\alpha(n))_{n\in\mathbb{Z}}$ is in the $PF(2)$ discrete class.*

Next, we have the following Theorem (see Albert&Bona (1991)).

Theorem 3.3. *Suppose f is a positive, twice-differentiable function on $\mathbb{R}$ satisfying*

$$\frac{d^2}{dx^2}(log\, f(x)) < 0 \qquad for\;\; x \neq 0, \qquad (logarithmically\ concave) \tag{20}$$

then $f \in PF(2)$.

Now we illustrate Theorem 3.3. Indeed, let us consider the solitary wave solution associated to the KdV and modified KdV equation ($p = 2$ and $\mathcal{M} = -\partial_x^2$ in (8)),

$$\phi_{c,p}(\xi) = \Big[\frac{(p+1)(p+2)c}{2}\Big]^{1/p} \operatorname{sech}^{2/p}\Big(\frac{p\sqrt{c}}{2}\xi\Big), \qquad c > 0, \;\; p = 1, 2. \tag{21}$$

Then the Fourier transforms are given by

$$\widehat{\phi_{c,1}}(\xi) = 12\pi\, \frac{\xi}{\sinh(\pi\xi/\sqrt{c})}, \quad \widehat{\phi_{c,2}}(\xi) = \sqrt{\frac{3}{2}}\pi \operatorname{sech}\Big(\frac{\pi\xi}{2\sqrt{c}}\Big). \tag{22}$$

Hence, since $\widehat{\phi_{c,i}}$, $i = 1, 2$, are logarithmically concave functions it follows from Theorem 3.3 that they belong to $PF(2)$. Moreover, from Proposition 3.2 we have that the sequences $(\widehat{\phi_{c,i}}(n))_{n\in\mathbb{Z}}$, $i = 1, 2$, belong to $PF(2)$ discrete class.

Next, for one better convenience of the reader, we establish the Poisson Summation Theorem.

Theorem 3.4. *Let $\widehat{f}^{\mathbb{R}}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx$ and $f(x) = \int_{-\infty}^{\infty} \widehat{f}^{\mathbb{R}}(\xi)e^{2\pi i x\xi}d\xi$ satisfy*

$$|f(x)| \leq \frac{A}{(1+|x|)^{1+\delta}} \quad and \quad |\widehat{f}^{\mathbb{R}}(\xi)| \leq \frac{A}{(1+|\xi|)^{1+\delta}},$$

where $A > 0$ and $\delta > 0$ (then f and $\widehat{f}$ can be assumed continuous functions). Thus, for $L > 0$

$$\sum_{n=-\infty}^{\infty} f\,(x + Ln) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \widehat{f}^{\mathbb{R}}\left(\frac{n}{L}\right) e^{\frac{2\pi i n x}{L}}.$$

The two series above converge absolutely.

Proof. See for example Stein&Weiss (1971). □

From Theorem 3.4 and formulas in (22) we have that the periodization of the solitary wave solutions in (21), $p = 1, 2$, produces the following periodic functions

$$\psi_i(\xi) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \widehat{\phi_{c,i}}\left(\frac{n}{L}\right) e^{\frac{2\pi i n \xi}{L}}, \qquad i = 1, 2, \tag{23}$$

such that its Fourier transform belongs to the $PF(2)$ discrete class.

4. Orbital stability definition and main theorem

In this section we establish the definition of stability which we are interested in this chapter and a general stability theorem for periodic traveling waves.

Definition 4.1. *Let φ be a periodic traveling wave solution of (11) with minimal period L and consider $\tau_r\varphi(x) = \varphi(x+r)$, $x \in \mathbb{R}$ and $r \in \mathbb{R}$. We define the set $\Omega_\varphi \subset H^{\frac{m_2}{2}}_{per}$, the orbit generated by φ, as $\Omega_\varphi = \{g;\ g = \tau_r\varphi$, for some $r \in \mathbb{R}\}$. For any $\eta > 0$, let us define the set $U_\eta \subset H^{\frac{m_2}{2}}_{per}$ by $U_\eta = \{f;\ \inf_{g\in\Omega_\varphi} \|f-g\|_{H^{\frac{m_2}{2}}_{per}} < \eta\}$. With this terminology, we say that φ is (orbitally) stable in $H^{\frac{m_2}{2}}_{per}$ by the flow generated by equation (8) if,*

(i) there is s_0 such that $H^{s_0}_{per} \subseteq H^{\frac{m_2}{2}}_{per}$ and the initial value problem associated to (8) is globally well-posed in $H^{s_0}_{per}$.

(ii) For every $\varepsilon > 0$, there is $\delta > 0$ such that for all $u_0 \in U_\delta \cap H^{s_0}_{per}$, the solution u of (8) with $u(0,x) = u_0(x)$ satisfies $u(t) \in U_\varepsilon$ for all $t \in \mathbb{R}$.

Remark 4.1. *We have some comments about Definition 4.1:*

1. *Definition 4.1 is based on the translation symmetry associated to model (8).*
2. *In Definition 4.1 we are introducing other space, $H^{s_0}_{per}$, because to obtain a global well-posed theory in the energy space $H^{\frac{m_2}{2}}_{per}$ can not be an easy task. For instance, in the case of the regularized Benjamin-Ono equation (equation (16) with $p = 1$ and $\mathcal{M} = \mathcal{H}\partial_x$) it is possible to have a global well-posed theory in the space $H^{s_0}_{per}$ with $s_0 > \frac{1}{2}$, but global well-posed in $H^{\frac{1}{2}}_{per}$ remains an open problem (see Angulo et al. (2010)).*
3. *Definition 4.1 was given for equations in (8), but naturally it is also valid for those ones in (16). Stability definition for the Schrödinger models (15) and (17) is different to that given in definition 4.1, since we have two symmetries (translations and rotations) and one symmetry (rotations) for that models, respectively (see Theorem 5.5 and Theorem 8.1 below).*

The proof of the following general stability theorem can be shown by using techniques due to Benjamin (1972), Bona (1975), Weinstein (1986) or Grillakis *et al.* (1987) (see also Angulo (2009))

Theorem 4.1. *Let φ_c be a periodic traveling wave solution of (11) and suppose that part* (i) *of the Definition 4.1 holds. Suppose also that the operator $\mathcal{L}_\mathcal{M}$ in (10) possesses exactly a unique negative eigenvalue which is simple, and zero is a simple eigenvalue with eigenfunction $\frac{d}{dx}\varphi_c$. Choose $\chi \in L^2_{per}$ such that $\mathcal{L}_\mathcal{M}\chi = \varphi_c$, and define $I = (\chi, \varphi_c)_{L^2_{per}}$. If $I < 0$, then φ_c is stable in $H^{\frac{m_2}{2}}_{per}$.*

Remark 4.2. *In our cases the function χ in Theorem 4.1 will be chosen as $\chi = -\frac{d}{dc}\varphi_c$. Then, $I < 0$ if and only if $\frac{d}{dc}\int \varphi_c^2(\xi)d\xi > 0$.*

5. Stability of periodic traveling wave solutions for some dispersive models

In this section we are interested in applying the theory obtained in Section 3 to obtain the stability of specific periodic traveling waves associated to the following models: the BO equation, the modified KdV and the cubic Schrödinger equation.

5.1 Stability for the BO equation

We start by finding a smooth curve $c \to \varphi_c$ of solutions associated with the following non-local differential equation

$$\mathcal{H}\varphi_c' + c\varphi_c - \frac{1}{2}\varphi_c^2 = 0. \tag{24}$$

Here we present an approach based on the Poisson Summation Theorem for obtaining an explicit solution to equation (24). Indeed, it we consider, the solitary wave solution associated to BO equation, namely, $\phi_\omega(x) = \dfrac{4\omega}{1+\omega^2x^2}$, with $\omega > 0$. Since its Fourier transform is given by $\widehat{\phi_\omega}^{\mathbb{R}}(x) = 4\pi e^{\frac{-2\pi}{\omega}|x|}$, we obtain from Theorem 3.4 the following periodic wave of minimal period L

$$\psi_\omega(x) = \frac{4\pi}{L}\sum_{n=0}^{+\infty}\varepsilon_n e^{\frac{-2\pi n}{\omega L}}\cos\left(\frac{2n\pi x}{L}\right) = \frac{4\pi}{L}\frac{\sinh\left(\frac{2\pi}{\omega L}\right)}{\cosh\left(\frac{2\pi}{\omega L}\right) - \cos\left(\frac{2\pi x}{L}\right)}, \tag{25}$$

where $\varepsilon_n = 1$ for $n = 0$, and $\varepsilon_n = 2$ for $n \geqq 1$. Next we see that the profile ψ_ω represents a periodic solution for (24) with $\omega = \omega(c)$ and $c > 2\pi/L$. Let φ_c, $c > 0$, be a smooth periodic solution of (24) with minimal period L, then φ_c can be expressed as a Fourier series

$$\varphi_c(x) = \sum_{n=-\infty}^{+\infty} a_n e^{\frac{2n\pi i x}{L}}. \tag{26}$$

Now, from (24), we get

$$\left[\frac{2\pi|n|}{L} + c\right]a_n = \frac{1}{2}\sum_{m=-\infty}^{+\infty} a_{n-m}a_m.$$

We consider $a_n \equiv 4\pi e^{-\gamma|n|/L}$, $n \in \mathbb{Z}$, $\gamma \in \mathbb{R}$. Substituting a_n in the last identity we obtain

$$\sum_{m=-\infty}^{+\infty} a_{n-m}a_m = \frac{16\pi^2}{L^2}e^{-\gamma|n|}\left[|n| + 1 + 2\sum_{k=1}^{+\infty} e^{-2\gamma k}\right] = \frac{16\pi^2}{L^2}e^{-\gamma|n|}(|n| + \coth\gamma).$$

Then,

$$c + \frac{2\pi|n|}{L} = \frac{4\pi}{L}\cdot\frac{1}{2}(|n| + \coth\gamma). \tag{27}$$

Consider $\gamma = 2\pi/(\omega L)$. Then for $c > 2\pi/L$ we choose the solitary wave-speed $\omega = \omega(c) > 0$ such that

$$\tanh(\gamma) = \frac{2\pi}{cL}. \tag{28}$$

Therefore, from uniqueness of the Fourier series we obtain $\varphi_c = \psi_{\omega(c)}$. Hence, since the mapping $c \to \gamma(c) = \tanh^{-1}(2\pi/(cL))$ is a differentiable function for $c > 2\pi/L$, it follows that $c \in \left(\frac{2\pi}{L}, +\infty\right) \mapsto \varphi_c \in H^n_{per}([0,L])$, $n \in \mathbb{N}$, is a smooth curve of periodic traveling wave solutions for the BO equation. From our analysis we have then the following Fourier expansion for φ_c

$$\varphi_c(x) = \frac{4\pi}{L} \sum_{n=-\infty}^{+\infty} e^{-\gamma|n|} e^{\frac{2\pi i n x}{L}}, \tag{29}$$

with γ satisfying (28). Then, we obtain immediately that $(\widehat{\varphi_c}(n))_{n\in\mathbb{Z}} \in PF(2)$ discrete class (see (18)) and from Theorem 3.2, that the operator in (12) possesses exactly a unique negative eigenvalue which is simple and whose kernel is generated by $\frac{d}{dx}\varphi_c$. Next we calculate the sign of the quantity $I = -\frac{1}{2}\frac{d}{dc}||\varphi_c||^2_{L^2_{per}}$. Indeed, from (29) and Parseval Theorem it follows

$$I = -\frac{L}{2}\frac{d}{dc}||\widehat{\varphi_c}||^2_{\ell^2} = -\frac{8\pi^2}{L}\frac{d}{dc}\sum_{n=-\infty}^{\infty} e^{-2\gamma|n|} = -\frac{32\pi^3}{c^2L^2}\frac{1}{1-\left(\frac{2\pi}{cL}\right)^2}\sum_{n=-\infty}^{\infty} |n| e^{-2\gamma|n|} < 0. \tag{30}$$

Hence, from Theorem 4.1 we obtain the orbital stability of the periodic solutions (29) in $H^{\frac{1}{2}}_{per}$ by the periodic flow of the BO equation.

Remark 5.1. *The periodic global well-posed theory for the BO in $H^{\frac{1}{2}}_{per}$ has been shown by Molinet (2008) and Molinet&Ribaut (2009).*

5.2 Stability for the mKdV equation

Next we study the modified KdV equation written as

$$u_t + 3u^2 u_x + u_{xxx} = 0. \tag{31}$$

In this case, the periodic traveling wave solution $u(x,t) = \varphi_c(x - ct)$ satisfies the equation

$$\varphi_c'' - c\varphi_c + \varphi_c^3 = 0. \tag{32}$$

On this time we are going to use the quadrature method to determine a profile for φ_c (see Angulo et al. (2010) for the use of the Poisson Summation Theorem). Thus, multiplying equation (32) by φ_c' and integrating once we deduce the following differential equation in quadrature form

$$[\varphi_c']^2 = \frac{1}{2}\left[-\varphi_c^4 + 2c\varphi_c^2 + 4B_{\varphi_c}\right], \tag{33}$$

where B_{φ_c} is a nonzero constant of integration. The periodic solutions arise of the specific form of the roots associated with the polynomial $F(t) = -t^4 + 2ct^2 + 4B_{\varphi_c}$. We start by considering F with four real roots such that $-\eta_1 < -\eta_2 < 0 < \eta_2 < \eta_1$, then we obtain

$$[\varphi_c']^2 = \frac{1}{2}(\eta_1^2 - \varphi_c^2)(\varphi_c^2 - \eta_2^2). \tag{34}$$

By looking for positive solutions we have $\eta_2 \leqq \varphi_c \leqq \eta_1$ and from (34), $2c = \eta_1^2 + \eta_2^2$ and $4B_{\varphi_c} = -\eta_1^2\eta_2^2$. Next, for $\phi_c \equiv \varphi_c/\eta_1$ and $\phi_c^2 = 1 - k^2\sin^2\psi$ we obtain from (34) the following elliptic integral equation $F(\psi(\xi), k) = \eta_1\xi/\sqrt{2}$, with $k^2 = (\eta_1^2 - \eta_2^2)/\eta_1^2$. Therefore, from the definition of the Jacobi elliptic function snoidal, sn, it follows that for $l = \eta_1/\sqrt{2}$, $\sin(\psi(\xi)) = \text{sn}(l\xi;k)$, and hence $\phi_c(\xi) = \sqrt{1-k^2\text{sn}^2(l\xi;k)} = \text{dn}(l\xi;k)$. Then if we return back to the initial variable φ_c, we obtain the so-called **dnoidal wave** solutions:

$$\varphi_c(\xi) \equiv \varphi_c(\xi;\eta_1,\eta_2) = \eta_1 \,\text{dn}\Big(\frac{\eta_1}{\sqrt{2}}\,\xi;k\Big) \tag{35}$$

with

$$k^2 = \frac{\eta_1^2-\eta_2^2}{\eta_1^2}, \qquad \eta_1^2+\eta_2^2 = 2c, \qquad 0<\eta_2<\eta_1. \tag{36}$$

Next we study the fundamental period associated to φ_c. Indeed, since $\text{dn}(u+2K) = \text{dn}\, u$, it follows that φ_c has the fundamental period (wavelength) T_{φ_c}, given by

$$T_{\varphi_c} \equiv \frac{2\sqrt{2}}{\eta_1}\,K(k). \tag{37}$$

Now, by using (36) we have for $c>0$ that $0<\eta_2<\sqrt{c}<\eta_1<\sqrt{2c}$. Hence one can consider (37) as a function of η_2, namely

$$T_{\varphi_c}(\eta_2) = \frac{2\sqrt{2}}{\sqrt{2c-\eta_2^2}}\,K(k(\eta_2)) \qquad \text{with} \qquad k^2(\eta_2) = \frac{2c-2\eta_2^2}{2c-\eta_2^2}. \tag{38}$$

Then, since for $\eta_2 \to 0$ we have $K(k(\eta_2)) \to +\infty$, it follows that $T_{\varphi_c}(\eta_2) \to +\infty$ as $\eta_2 \to 0$. Now, for $\eta_2 \to \sqrt{c}$ we obtain $K(k(\eta_2)) \to \pi/2$. Therefore, $T_{\varphi_c}(\eta_2) \to \pi\sqrt{2}/\sqrt{c}$ as $\eta_2 \to \sqrt{c}$. Finally, since $\eta_2 \to T_{\varphi_c}(\eta_2)$ is a strictly decreasing function we obtain $T_{\varphi_c} > \frac{\pi\sqrt{2}}{\sqrt{c}}$. Then the implicit function theorem implies the following result (see Angulo (2007)).

Theorem 5.1. *Let $L>0$ be arbitrary but fixed. Then there exists a smooth mapping curve $c \in J_0 = \left(\frac{2\pi^2}{L^2}, +\infty\right) \to \varphi_c \in H^n_{per}([0,L])$, such that φ_c satisfies equation (32) and it has the dnoidal profile*

$$\varphi_c(\xi) = \eta_1 dn\Big(\frac{\eta_1}{\sqrt{2}}\xi;k\Big), \;\; \xi \in [0,L]. \tag{39}$$

Here, $c \in J_0 \to \eta_1(c) \in (\sqrt{c},\sqrt{2c})$, $c \in J_0 \to k(c) \in (0,1)$ are smooth.

Our next step is the study of the following periodic eigenvalue problem,

$$\begin{cases} \mathcal{L}_{mkdv}\psi \equiv \left(-\dfrac{d^2}{dx^2}+c-3\varphi_c^2\right)\psi = \lambda\psi \\ \psi(0)=\psi(L),\; \psi'(0)=\psi'(L). \end{cases} \tag{40}$$

Then, we have the following theorem.

Theorem 5.2. *Let φ_c be the dnoidal wave solution given by Theorem 5.1. Then problem (40) defined on $H^2_{per}([0,L])$ has exactly its three first eigenvalues simple, being the eigenvalue zero, the second one with eigenfunction φ_c'. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues which are double.*

Remark 5.2. *The periodic global well-posed theory for the mKdV in H^1_{per} can be found in Colliander et al. (2003).*

The proof of Theorem 5.2 is based on the Floquet theory. So, we have the following classical theorem (see Magnus&Winkler (1976))

Theorem 5.3. *Consider the Lamé's equation*

$$-\chi'' + m(m+1)k^2 sn^2(x;k)\chi = \rho\chi, \tag{41}$$

where m is a real parameter. Then we guarantee the existence of two linearly independent periodic solutions to (41) *with period $2K$ or $4K$ if and only if m is an integer. By letting $l = m$ if m is a non-negative integer and $l = -m-1$ if m is a negative integer then Lamé's equation* (41) *has, at most, $l+1$ intervals of instability (including the interval $(-\infty, \rho_0)$ with ρ_0 being the first eigenvalue). In addition, if m is a non-negative integer then* (41) *has exactly $m+1$ intervals of instability.*

Proof. (Theorem 5.2) Since operator $\mathcal{L}_{mkdv}$ has a compact resolvent its spectrum is a countable infinity set of eigenvalues $\{\lambda_n; n = 0, 1, 2, ...\}$, with

$$\lambda_0 < \lambda_1 \leqq \lambda_2 < \lambda_3 \leqq \lambda_4 < \cdots. \tag{42}$$

We denote by ψ_n the eigenfunction associated to the eigenvalue λ_n. The eigenvalue distribution in (42) is a consequence of the following Oscillation Sturm-Liouville result:

i) ψ_0 has no zeros in $[0, L]$,

ii) ψ_{2n+1} and ψ_{2n+2} have exactly $2n+2$ zeros in $[0, L)$.

Next, since $\mathcal{L}_{mkdv}\varphi_c' = 0$ and φ_c' has two zeros in $[0, L)$ we have that the eigenvalue zero is either λ_1 or λ_2. For determining that $0 = \lambda_1 < \lambda_2$ we will use Theorem 5.3. Indeed, the transformation $Q(x) = \psi(x\sqrt{2}/\eta_1)$ implies from (40) the following Lamé problem for Q,

$$\begin{cases} Q'' + [\rho - 6k^2 sn^2(x,k)]Q = 0 \\ Q(0) = Q(2K(k)), \;\; Q'(0) = Q'(2K(k)), \end{cases} \tag{43}$$

with

$$\rho = 2(\lambda + 3\eta_1^2 - c)/\eta_1^2. \tag{44}$$

Therefore, since problem (43) has exactly 3 intervals of instability we have that the eigenvalues $\{\rho_n; n = 0, 1, 2, ...\}$ will satisfy that ρ_0, ρ_1 and ρ_2 are simples and $\rho_3 = \rho_4$, $\rho_5 = \rho_6, \cdots$. Next, we establish the values of the eigenvalues ρ_i, $i = 1, 2, 3$. Indeed, $\rho_0 = 2[1 + k^2 - \sqrt{1 - k^2 + k^4}]$, $\rho_1 = 4 + k^2$, $\rho_2 = 2[1 + k^2 + \sqrt{1 - k^2 + k^4}]$. Therefore relation (44) implies that for $i = 1, 2, 3$, ρ_i determine the eigenvalues λ_i, respectively. Hence, zero is the second eigenvalue for (40) and it is simple. This shows the theorem. □

Remark 5.3. *We note that a part of the conclusion of Theorem 5.2 can be also obtained via Theorem 3.2. Indeed, the Fourier transform of the dnoidal profile φ_c is given by $\widehat{\varphi_c}(n) = \frac{\sqrt{2}\pi}{L} sech\left(\frac{\pi n}{\sqrt{\omega(c)L}}\right)$, where for $c > \frac{2\pi^2}{L^2}$ we have $\omega(c) = c/(16(2-k^2)K^2(\sqrt{1-k^2}))$. Then since the function $f(x) = \mu sech(\nu x)$ belongs to $PF(2)$ continuous for μ, ν positive (see (22)), it follows $(\widehat{\varphi_c}(n))_{n\in\mathbb{Z}} \in PF(2)$ discrete. Finally, since the convolution of even sequences in $PF(2)$ discrete is in $PF(2)$ discrete (see Theorem 3.1) we obtain that $\widehat{\varphi_c^2} \in PF(2)$ discrete.*

Finally, we calculate the sign of the quantity $D = \frac{1}{2}\frac{d}{dc}||\varphi_c||^2_{L^2_{per}}$. From integral elliptic theory we have

$$||\varphi_c||^2_{L^2_{per}} = \sqrt{2}\eta_1 \int_0^{\frac{\eta_1}{\sqrt{2}}L} dn^2(x;k)dx = \frac{8K(k)}{L}\int_0^K dn^2(x;k)dx = \frac{8}{L}K(k)E(k). \tag{45}$$

Then, since the maps $k \in (0,1) \to K(k)E(k)$ and $c \to k(c)$ are strictly increasing functions, it follows immediately from (45) that $D > 0$. Hence, Theorem 4.1 implies stability in $H^1_{per}([0,L])$ of the dnoidal solutions (39) by the periodic flow of the mkdV equation.

Remark 5.4. *Recently Johnson (2009) has proposed an approach to determine the nonlinear stability of periodic traveling wave for models of KdV type. Next, we would like to show that this theory can not be applied to the smooth curve of dnoidal wave, $c \to \varphi_c$, established by Theorem 5.1. Indeed, from analysis in subsection 5.2 we have for $L > 0$ fixed that $B := B_{\varphi_c}(c) = -\frac{-16(1-k(c)^2)K^2(k(c))}{L^2}$. Now, we note that $\frac{dB}{dc} > 0$ and so φ_c can be seen as a function of the parameter B. In the proof of Lemma 4.2 in Johnson (2009) is deduced that $\frac{d}{dB}\varphi_c$ is a periodic function if and only if the kernel of the operator $\mathcal{L}_{mkdv}$ is double. So, from the equality $\frac{d}{dB}\varphi_c = \frac{dc}{dB}\frac{d\varphi_c}{dc}$, we deduce that $\frac{d}{dB}\varphi_c$ is periodic since $\frac{dc}{dB} > 0$ and $\frac{d\varphi_c}{dc}$ is a periodic function by construction. Therefore from Johnson's Lemma we obtain that $ker(\mathcal{L}_{mkdv})$ is double which is a contradiction.*

5.3 Stability and instability for the cubic Schrödinger equation

In this subsection we are interested in studying stability properties of two specific families de traveling wave solutions for the cubic Schrödinger equation (NLS henceforth)

$$iu_t + u_{xx} + |u|^2 u = 0, \tag{46}$$

namely, the dnoidal and cnoidal solutions.

5.3.1 The dnoidal case

The stability analysis associated to the mKdV equation (31) gives us the basic information to study the stability of periodic standing-wave solutions for the NLS equation (46) in the form $u(x,t) = e^{ict}\varphi_c(x)$. Indeed, Theorem 5.1 implies the existence of a smooth curve $c \to e^{ict}\varphi_c$ of periodic traveling wave solutions for the NLS with a profile given by the dnoidal function. Now, since the NLS has two basic symmetries, rotations and translations, we have that the orbit to be studied here will be $\mathcal{O}_{\varphi_c} = \{e^{i\theta}\varphi_c(\cdot + y) : y \in \mathbb{R}, \theta \in [0,2\pi)\}$. Therefore, from Weinstein (1986) and Grillakis et. al (1987) we need to study the spectrum of the following linear operators: $\mathcal{L}_{mkdv}$ in (40), which henceforth we denote by $\mathcal{L}^-$, and the operator $\mathcal{L}^+$ defined by

$$\mathcal{L}^+ = -\frac{d^2}{dx^2} + c - \varphi_c^2. \tag{47}$$

The following theorem is related to the specific structure of $\mathcal{L}^+$.

Theorem 5.4. *The self-adjoint operator $\mathcal{L}^+$ defined on $H^2_{per}([0,L])$ is a nonnegative operator. The eigenvalue zero is simple with eigenfunction associated φ_c and the remainder of the eigenvalues are double.*

Proof. Since $\varphi_c > 0$ and $\mathcal{L}^+\varphi_c = 0$, it follows from the theory of self-adjoint operators that zero is simple and it is the first eigenvalue. Now, by Theorem 5.3 we obtain that $\mathcal{L}^+$ has exactly two instability intervals and so the remainder of the spectrum of $\mathcal{L}^+$ is constituted by eigenvalues which are double. This finishes the Theorem. ☐

Now, from Angulo (2007) we have the following stability theorem for the NLS equation.

Theorem 5.5. *Let φ_c be the dnoidal wave solution given by Theorem 5.1. Then the orbit $\mathcal{O}_{\varphi_c}$ is stable in $H^1_{per}([0,L])$ by the periodic flow of the NLS equation. Indeed, for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if the initial data u_0 satisfies*

$$\inf_{(y,\theta)\in\mathbb{R}\times[0,2\pi)} \|u_0 - e^{i\theta}\varphi_c(\cdot + y)\|_1 < \delta,$$

then the solution $u(t)$ of the NLS equation with $u(0) = u_0$ satisfies

$$\inf_{(y,\theta)\in\mathbb{R}\times[0,2\pi)} \|u(t) - e^{i\theta}\varphi_c(\cdot + y)\|_1 < \epsilon,$$

for all $t \in \mathbb{R}$, $\theta = \theta(t)$ and $y = y(t)$.

Remark 5.5. *The periodic global well-posed theory for the NLS equation in H^1_{per} has been shown by Bourgain (1999).*

Theorem 5.5 establishes that the dnoidal solutions are stable by periodic perturbations of the same minimal period L in $H^1_{per}([0,L])$. Now, since φ_c is also a periodic traveling wave solution for the NLS equation in every interval $[0, jL]$, for $j \in \mathbb{N}$ and $j \geqq 2$, it is natural to ask about its stability in $H^1_{per}([0, jL])$. As we see below they will be nonlinearly unstable (in fact, they are linearly unstable). We start our analysis with the following elementary result.

Lemma 5.1. *Define P_j and Q_j as the number of negatives eigenvalues of $\mathcal{L}^-$ and $\mathcal{L}^+$, respectively, with periodic boundary condition in $[0, jL]$ and $j \geqq 2$. Then $Q_j = 0$ and $P_j = 2j$ or $P_j = 2j - 1$.*

Proof. Since $\varphi_c > 0$ and $\mathcal{L}^+\varphi_c(x) = 0$, $x \in [0, jL]$, by the Oscillation Sturm-Liouville result for Hill equations, we obtain that zero must be the first eigenvalue and therefore for all $j \geqq 2$, $Q_j = 0$. Next, since $\mathcal{L}^-\varphi_c'(x) = 0$, $x \in [0, jL]$, and the number of zeros (nodes) of φ_c' in the semi-open interval $[0, jL)$ is $2j$, the Oscillation Theorem implies that the eigenvalues corresponding to the zero eigenvalue are λ_{2j} or λ_{2j-1}. Therefore, we have $P_j = 2j$ or $P_j = 2j - 1$. This finishes the Lemma. ☐

A theoretical framework for proving nonlinear instability from a linear instability result for nonlinear Schrödinger type's equation was developed in Grillakis (1988), Jones (1988) and Grillakis et al. (1990). In those approach one deduces instability when the number of negative eigenvalues of $\mathcal{L}^-$ exceeds the number of negative eigenvalues of $\mathcal{L}^+$ by more than one (see Theorem 5.8 below). The parts of the instability theorems in Grillakis (1988) that are needed for obtaining a linear instability of φ_c, connect P_j, Q_j and the existence of real eigenvalues of the operator (the linearized Hamiltonian)

$$N = \begin{pmatrix} 0 & \mathcal{L}^+ \\ -\mathcal{L}^- & 0 \end{pmatrix}. \tag{48}$$

First, define: 1)K_j - the orthogonal projection on $(\ker \mathcal{L}^-)^\perp$; 2) R_j - the operator $R_j = K_j\mathcal{L}^-K_j$; 3) S_j - the number of negative eigenvalues of R_j; 4) $I_{real}(N_j)$ - the number of pairs of nonzero real eigenvalues of N considered on $[0, jL]$.

Theorem 5.6. *[Grillakis (1988), Jones (1988)] For $j \geqq 1$ we have*

1) If $|S_j - Q_j| = m_j > 0$, then $I_{real}(N_j) \geqq m_j$.

2) If $S_j = Q_j$ and $\{f \in L^2_{per}([0, jL]) : (R_i f, f) < 0$ and $((\mathcal{L}^+)^{-1} f, f) < 0\} = \emptyset$, then $I_{real}(N_j) \geqq 1$.

The following result gives a condition for obtaining the number S_j.

Theorem 5.7. *[Grillakis (1988)] If $\frac{d}{dc}\int_0^{jL} \varphi_c^2(x)dx > 0$ then $S_j = P_j - 1$.*

The following theorem is the main result of this section.

Theorem 5.8. *[Instability] Consider the dnoidal solution φ_c given by Theorem 5.1. Then the orbit $\mathcal{R}_{\varphi_c} = \{e^{iy}\varphi_c : y \in \mathbb{R}\}$ is $H^1_{per}([0, jL])$-unstable for all $j \geqq 2$, by the flow of the periodic NLS equation.*

Proof. The strategy of the proof is initially to show that the orbit $\mathcal{R}_{\varphi_c}$ is *linearly unstable*. For this, we rewrite equation (15) in the Hamiltonian form

$$\frac{d}{dt}\mathbf{u}(t) = JG'(\mathbf{u}(t)), \tag{49}$$

where $\mathbf{u} = (\Re(u), \Im(u))^t$, J is the skew-symmetric, one-one and onto matrix given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{50}$$

and

$$G(\mathbf{u}) = \int \left[\frac{1}{2}|\mathbf{u}'|^2 - \frac{1}{4}|\mathbf{u}|^4\right] dx, \tag{51}$$

which is a conservation law to (46). Next, for the linearization of (49) around the orbit $\mathcal{R}_{\varphi_c}$ we proceed as follows: we write $\Psi_c = (0, \varphi_c)^t$ and define

$$\mathbf{v}(t) = T_p(-ct)\mathbf{u}(t) - \Psi_c. \tag{52}$$

Here $T_p(s)$ acts as a rotation matrix. Hence, if $T_p'(0)$ denotes the infinitesimal generator of $T_p(s)$, from the properties: $T_p(s)T_p'(0) = T_p'(0)T_p(s)$, $T_p(-s)JT_p(s) = J$, $G'(T_p(s)\mathbf{u}) = T_p(s)H'(\mathbf{u})$, $J^{-1}T_p'(0)\mathbf{u} = -F'(\mathbf{u})$ ($F(\mathbf{u}) = \frac{1}{2}\int |\mathbf{u}|^2 dx$) we obtain, via Taylor Theorem,

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= J[G'(\mathbf{v} + \Psi_c) + cF'(\mathbf{v} + \Psi_c)] \\ &= J[G''(\Psi_c)\mathbf{v} + cF''(\Psi_c)\mathbf{v} + G'(\Psi_c) + cF'(\Psi_c) + O(\|\mathbf{v}\|^2)] = N\mathbf{v} + O(\|\mathbf{v}\|^2), \end{aligned} \tag{53}$$

where in the last equality we are taking into account that Ψ_c is a critical point of $G + cF$ and J is a bounded linear operator. Here, N is the linear operator defined in (48). Therefore, we are interested in the problem of determining a growing mode solution $\mathbf{v}(t) = e^{\lambda t}\Phi(x)$ with $\Re(\lambda) > 0$ of the *linearized problem*

$$\frac{d\mathbf{v}}{dt} = N\mathbf{v}. \tag{54}$$

We note that the eigenvalues of N appear in conjugate pairs. Now, since $\frac{d}{dc}\int_0^L \varphi_c^2(x)dx > 0$ (see (45)) it follows from Lemma 5.1, Theorem 5.6 and Theorem 5.7 that $m_j = 2j - 1$, or, $2j - 2$. Therefore for $j \geqq 2$, the number $I_{real}(N_j) \geqq m_j > 0$. Then the zero solution of (54) is unstable, which implies that the orbit $\mathcal{R}_{\varphi_c}$ is nonlinearly unstable (see Grillakis (1988) and Grillakis et al. (1990)). $\square$

Remark 5.6. *1. The Instability criterium in Grillakis et al. (1990) can not be used for studying the instability of the orbit $\mathcal{R}_{\varphi_c}$. In fact, we denote by $n(H_c)$ the number of negative eigenvalues of the diagonal linear operator $H_c \equiv J^{-1}N$. Since the kernel of H_c is generated by φ_c and $\frac{d}{dx}\varphi_c$, and $\frac{d}{dc}\int_0^{jL}\varphi_c^2(x)dx > 0$, we have that if $n(H_c) - 1$ is odd then the orbit $\mathcal{R}_{\varphi_c}$ is nonlinearly unstable in $H^1_{per}([0, jL])$. Now, from Lemma 5.1 we have for $j \geqq 2$ that $n(H_c) - 1 = P_j - 1 = 2j - 1$, or, $2j - 2$. But as we will see latter $P_j = 2j - 1$ (see Section 6), therefore we have that $n(H_c)$ is always even.*

2. *The analysis in this subsection can be applied to study the following forced periodic nonlinear Schrödinger equation*

$$iu_t - u_{xx} - \gamma|u|^2u = \varepsilon exp(-i\Omega^2 t + i\alpha) - i\delta u,$$

where ε is the small forcing amplitude, δ is the small damping coefficient, Ω^2 is the forcing frequency and α is an arbitrary phase (see Shlizerman&Rom-Kedar (2010)) .

5.3.2 The cnoidal case

Equation (32) has other family of periodic solutions determined by the Jacobi elliptic function *cnoidal*. Indeed, now suppose that equation (33) now is written in the quadrature form

$$[\varphi_c']^2 = \frac{1}{2}(a^2 + \varphi_c^2)(b^2 - \varphi_c^2).$$

For $b > 0$ we have that $-b \leqq \varphi_c \leqq b$ and so $2c = b^2 - a^2$ and $4B_{\varphi_c} = a^2b^2 > 0$. Therefore, it is possible to obtain (see Angulo (2007)) that the profile

$$\varphi_c(\xi) = bcn(\beta\xi; k) \tag{55}$$

is a solution of (32) with $k^2 = b^2/(a^2 + b^2)$ and $\beta = \sqrt{(a^2 + b^2)/2}$. Then by using the implicit function theorem we have the following result (see Angulo (2007)).

Theorem 5.9. *Let $L > 0$ arbitrary but fixed. Then we have two branches of cnoidal wave solutions for the NLS equation. Indeed,*

1) *there are a strictly increasing smooth function $c \in (0, +\infty) \to b(c) \in (\sqrt{2c}, +\infty)$ and a smooth curve $c \in (0, +\infty) \to \varphi_{c,1} \in H^1_{per}([0, L])$ of solutions for equation (32) with*

$$\varphi_{c,1}(\xi) = b\ cn\Big(\sqrt{b^2 - c}\ \xi; k\Big). \tag{56}$$

Here the modulus $k = k(c)$ satisfies $k^2 = b^2/(2b^2 - 2c)$ and $k'(c) > 0$,

2) *there are a strictly decreasing smooth function $c \in (-\frac{4\pi^2}{L^2}, 0) \to a(c) \in (\sqrt{-2c}, +\infty)$ and a smooth curve $c \in \Big(-\frac{4\pi^2}{L^2}, 0\Big) \to \varphi_{c,2} \in H^1_{per}([0, L])$ of solutions for equation (32) with*

$$\varphi_{c,2}(\xi) = \sqrt{a^2 + 2c}\ cn\Big(\sqrt{a^2 + c}\ \xi; k\Big). \tag{57}$$

Here the modulus $k = k(c)$ satisfies $k^2 = (a^2 + 2c)/(2a^2 + 2c)$ and $k'(c) > 0.$

Now, for the cnoidal case it makes necessary to study the behavior of the first eigenvalues related to the following self-adjoint operators

$$\mathcal{L}_i^- = -\frac{d^2}{dx^2} + c - 3\varphi_{c,i}^2, \quad \mathcal{L}_i^+ = -\frac{d^2}{dx^2} + c - \varphi_{c,i}^2, \quad i = 1, 2. \tag{58}$$

The next theorem gives the specific structure for $\mathcal{L}_i^{\pm}$.

Theorem 5.10. *Let $L > 0$ and $\varphi_{c,i}$, $i = 1, 2$, the cnoidal wave solution given by Theorem 5.9. Then,*

1) *The operators $\mathcal{L}_i^+$ defined on $H^2_{per}([0, L])$ have exactly one negative eigenvalue which is simple, the eigenvalue zero is also simple with eigenfunction $\varphi_{c,i}$. Moreover, the remainder of the spectrum is a discrete set of eigenvalues converging to infinity.*
2) *The operators $\mathcal{L}_i^-$ defined on $H^2_{per}([0, L])$ have exactly two negative eigenvalue which are simple. The eigenvalue zero is the third one, which is simple with eigenfunction $\varphi'_{c,i}$. Moreover, the remainder of the eigenvalues are double and converging to infinity.*

Proof. The proof is a consequence of Theorem 5.3 and the Oscillation Sturm-Liouville Theorem (see Angulo (2007)). □

We note that the stability or instability of the cnoidal solutions $\varphi_{c,i}$ can not be determined by using the same techniques mentioned above for the case of dnoidal solution (see Angulo (2007) for discussion). Indeed, since $\frac{d}{dc}\|\varphi_{c,i}\|^2 > 0$ and for

$$H_{c,i} = \begin{pmatrix} \mathcal{L}_i^- & 0 \\ 0 & \mathcal{L}_i^+ \end{pmatrix} \tag{59}$$

we have $n(H_{c,i}) - 1 = 2$, it follows that the Grillakis et. al (1990) stability approach is not applicable in this case. A similar situation occurs with Grillakis (1988) and Jones (1988) instability theories.

Remark 5.7. *Recently, Natali&Pastor (2008) have determined that the cnoidal wave solution in (55) is orbitally unstable by the periodic flow of the Klein-Gordon equation*

$$u_{tt} - u_{xx} + u - |u|^2 u = 0, \tag{60}$$

by using the abstract theory due to Grillakis et al. (1990). In fact, if one considers a standing wave solution for (60) of the form $u(x, t) = e^{ict}\varphi_c(x)$, $|c| < 1$, we conclude from Theorem 5.10 that the operator

$$\mathcal{L}_{kg} = \begin{pmatrix} \mathcal{L}_{kg}^- & 0 \\ 0 & \mathcal{L}_{kg}^+ \end{pmatrix} \tag{61}$$

has three negative eigenvalues which are simple and the eigenvalue zero is double. Here, $\mathcal{L}_{kg}^- = \begin{pmatrix} -\frac{d^2}{dx^2} + 1 - 3\varphi_c^2 & -c \\ -c & 1 \end{pmatrix}$ and $\mathcal{L}_{kg}^+ = \begin{pmatrix} -\frac{d^2}{dx^2} + 1 - \varphi_c^2 & c \\ c & 1 \end{pmatrix}$. So, since $D = -\frac{d}{dc}\left(c\int_0^L \varphi_c(x)^2 dx\right)$ is negative it follows that $n(\mathcal{L}_{kg}) - 0 = 3$ is an odd number. Then, the approach in Grillakis et al. (1990) can be applied in order to conclude the instability result.

In Section 7 we establish a new criterium for the instability of periodic traveling wave solutions for general nonlinear dispersive equations. An application of this technique shows that the cnoidal wave profile associated to the mKdV is actually unstable.

6. Hill's operators and the stability of periodic waves.

As we have seen in previous sections the study of the spectrum associated to the Hill operator

$$\mathcal{L}_Q = -\frac{d^2}{dx^2} + Q(x), \tag{62}$$

with Q being a periodic potential, is of relevance in the stability's study of periodic traveling wave solutions for nonlinear evolution equations. Recently, Neves (2009), have presented a new technique to establish a characterization of the nonpositive eigenvalues of $\mathcal{L}_Q$ by knowing one of its eigenfunctions. Next, we will give the main points of his theory and we apply it to a specific situation. Indeed, let us consider the Hill equation related to the operator in (62),

$$y''(x) + Q(x)y(x) = 0, \tag{63}$$

where we assume that the potential Q is a $\pi-$periodic function. Denote by y_1 and y_2 two normalized solutions of (63), that is, solutions uniquely determined by the initial conditions $y_1(0) = 1$, $y_1'(0) = 0$, $y_2(0) = 0$, $y_2'(0) = 1$. The characteristic equation associated with (63) is given by

$$\rho^2 - [y_1(\pi) + y_2'(\pi)]\rho + 1 = 0, \tag{64}$$

and the characteristic exponent is a number α which satisfies the equation $e^{i\alpha\pi} = \rho_1$, $e^{-i\alpha\pi} = \rho_2$, where ρ_1 and ρ_2 are the roots of the characteristic equation (64). It is well known from Floquet's Theorem (see Magnus&Winkler (1976)) that if $\rho_1 = \rho_2 = 1$ equation (63) has a nontrivial $\pi-$periodic smooth solution. So, if one considers p such periodic solution and y be another solution which is linearly independent of p, then $y(x+\pi) = \rho_1 y(x) + \theta p(x)$, for θ constant. The case $\theta = 0$ is equivalent to say that y is also a $\pi-$periodic solution. Next, suppose that $z_1 < z_2 < \cdots < z_{2n}$ are the simple zeros of p in the interval $[0, \pi)$. Then from Taylor's formula, p can be written as

$$p(x) = (x - z_i)p'(z_i) + O((x - z_i)^3), \tag{65}$$

and therefore, for x near z_i we deduce, $\frac{x-z_i}{p(x)} = \frac{1}{p'(z_i)+O((x-z_i)^2)}$. Next, we choose in each interval (z_{i-1}, z_i) one point x_i such that $p'(x_i) = 0$. Thus, the zeros z_i of p and x_i of p' intercalated as follows $z_1 < x_1 < z_2 < x_2 < \cdots < z_{2n} < x_{2n}$ and, of course, they shall repeat to the right and to the left by the periodicity of the functions.
Define, for $[x_1, x_1 + \pi)$

$$q(x) = \frac{x - z_i}{p(x)} = \frac{1}{p'(z_i) + O((x - z_i)^3)}, \quad x \in [x_{i-1}, x_1), \tag{66}$$

where $i = 2, \cdots, 2n+1$, $z_{2n+1} = z_1 + \pi$ and $x_{2n+1} = x_1 + \pi$. Next, it is possible to extend q to whole line by periodicity. Moreover we guarantee that function q is a piecewise smooth with jump discontinuities in the points x_i, q is continuous to the right and $\pi-$periodic with $q'(z_i) = 0$, $q'(x_i) = \frac{1}{p(x_i)}$, that is, q' is continuous on whole real line.

Then, we can state the following result which is a new version of the Floquet Theorem for the case $\rho_1 = \rho_2 = 1$.

Theorem 6.1. *If p is a $\pi-$periodic solution of (63), q is the function defined in (66) and*

$$j(x_i) = \frac{q(x_i^+) - q(x_i^-)}{p(x_i)} = -\frac{z_{i+1} - z_i}{p^2(x_i)}.$$

Then, the solution y linearly independent with p such that the Wronskian $W(p,y) = 1$ satisfies

$$y(x+\pi) = y(x) + \theta p(x), \tag{67}$$

where θ is given by

$$\theta = \sum_{x_i \in (0,\pi]} j(x_i) + 2\int_0^{\pi} \frac{q'(x)}{p(x)} dx. \tag{68}$$

In particular, y is $\pi-$periodic if and only if $\theta = 0$.

Proof. See Neves (2009). □

Now, we turn back to the linear operator in (62). We have from Oscillation Theorem (see Magnus&Winkler (1976)) that the spectrum of $\mathcal{L}_Q$ under periodic conditions is formed by an unbounded sequence of real numbers, $\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 \cdots < \lambda_{2n-1} \leq \lambda_{2n} \cdots$, where $\lambda_n' s$ are the roots of the characteristic equation

$$\Delta(\lambda) = y_1(\pi,\lambda) + y_2'(\pi,\lambda) = 2, \tag{69}$$

and $y_1(\cdot,\lambda)$ and $y_2(\cdot,\lambda)$ are the solutions of the differential equation $-y'' + (Q(x) - \lambda)y = 0$ determined by the initial conditions $y_1(0,\lambda) = 1$, $y_1'(0,\lambda) = 0$, $y_2(0,\lambda) = 0$ and $y_2'(0,\lambda) = 1$. We recall that the mapping $\lambda \to \Delta(\lambda)$ is an analytic function.

Now, we know that the spectrum of $\mathcal{L}_Q$ is also characterized by the number of zeros of the eigenfunctions. So, if p is an eigenfunction associated to the eigenvalues λ_{2n-1} or λ_{2n}, then p has exactly $2n$ zeros in the interval $[0,\pi)$. We can enunciate the converse of the previous result with the following statement.

Theorem 6.2. *If p is the eigenfunction of $\mathcal{L}_Q$ associated with the eigenvalue λ_k, $k \geq 1$, and θ is the constant given by Theorem 6.1, then λ_k is simple if and only if $\theta \neq 0$. In addition, if p has $2n$ zeros in the interval $[0,\pi)$, then $\lambda_k = \lambda_{2n-1}$ if $\theta < 0$, and $\lambda_k = \lambda_{2n}$ if $\theta > 0$.*

Proof. See Neves (2009). □

Remark 6.1. *We note that the main idea in the proof of Theorem 6.2 is to determine the sign of $\Delta'(\lambda_k)$, this fact can be obtained from the equality*

$$\Delta'(\lambda_k) = -\theta\Big[\|y_1\|^2 p^2(0) + 2 < y_1, y_2 > p(0)p'(0) + \|y_2\|^2 (p'(0))^2\Big].$$

Indeed, since $\Delta'(\lambda_k)\theta < 0$ we have for $\theta < 0$ that $\lambda_k = \lambda_{2n-1}$ and for $\theta > 0$ that $\lambda_k = \lambda_{2n}$.

As an application of Theorem 6.2 we obtain the spectral information for the linear operator $\mathcal{L}_{mkdv}$ in (40) with φ_c being the dnoidal profile determined by Theorem 5.1. Initially, we write $\mathcal{L}_{mkdv} = -\frac{d^2}{dx^2} + c - 3\varphi_c^2 = -\frac{d^2}{dx^2} + Q(x,c)$, then for this kind of operators we already know that the nonpositive spectrum is invariant with respect to parameter c (see Neves (2008)), so, it is sufficient to establish the spectral condition contained in Theorem 4.1 for a fixed value of

$c \in I = (2\pi^2/L^2, +\infty)$. Then, if one considers $L = \pi$ and the unique value of $c \in I$ such that $k(c) = \frac{1}{2}$, the value of θ in Theorem 6.1 will be $\theta \approx -0.5905625$. Now, we known that φ_c' is an eigenfunction of $\mathcal{L}_{mkdv}$ with eigenvalue $\lambda = 0$ and such that it has two zeros in the interval $[0, \pi)$. We conclude from Theorem 6.2 that $\mathcal{L}_{mkdv}$ possesses one negative eigenvalue which is simple. Moreover, since $\theta \neq 0$ it follows that $\lambda = 0$ is a simple eigenvalue.

Remark 6.2. *Theorem 6.1 and Theorem 6.2 can also be used to show that the operator $\mathcal{L}_{mkdv}$ with periodic boundary conditions on $[0, jL]$, $j \geqq 2$, has the zero eigenvalue as being simple and it is the $2j$-nth eigenvalue. So, the number of negative eigenvalue of $\mathcal{L}_{mkdv}$ is $P_j = 2j - 1$.*

7. Instability of periodic waves

This section is devoted to establish sufficient conditions for the linear instability of periodic traveling wave solutions, $u(x,t) = \varphi_c(x - ct)$, for the general class of dispersive equation in (8). We shall extend the asymptotic perturbation theories in Vock&Hunziker (1982) and Lin (2008) (see also Hislop&Sigal (1996)) to the periodic case.

We start by denoting $f(u) = u^{p+1}/(p+1)$, then the linearized equation associated to (8) in the traveling frame $(x + ct, t)$ is given by

$$(\partial_t - c\partial_x)u + \partial_x(f'(\varphi_c)u - \mathcal{M}u) = 0. \tag{70}$$

As mentioned in Subsection 5.3, the central point in this type of problems is the existence of a growing mode solution $e^{\lambda t}u(x)$, with $\Re(\lambda) > 0$, for (70). Hence, the function u must satisfy

$$(\lambda - c\partial_x)u + \partial_x(f'(\varphi_c)u - \mathcal{M}u) = 0. \tag{71}$$

Equation (71) gives us the family of operators $\mathcal{A}^\lambda : H^{m_2}_{per}([0,L]) \to L^2_{per}([0,L])$ given by,

$$\mathcal{A}^\lambda u = cu + \frac{c\partial_x}{\lambda - c\partial_x}(f'(\varphi_c)u - \mathcal{M}u). \tag{72}$$

Hence the existence of a growing mode solution is reduced to find $\lambda > 0$ such that $\mathcal{A}^\lambda$ has a nontrivial kernel. For $\mathcal{A}^0 = \mathcal{M} + c - f'(\varphi_c)$ (see (10)), we have the following results:

1) For $\lambda > 0$, $\mathcal{A}^\lambda \to \mathcal{A}^0$ strongly in $L^2_{per}([0,L])$ when $\lambda \to 0^+$.
2) The compact embedding $H^{m_2}_{per}([0,L]) \hookrightarrow L^2_{per}([0,L])$ give us $\sigma_{ess}(\mathcal{A}^\lambda) = \emptyset$ for all $\lambda > 0$.
3) There exists $\Lambda > 0$ such that for all $\lambda > \Lambda$, $\mathcal{A}^\lambda$ has no eigenvalues $z \in \mathbb{C}$ satisfying $\Re(z) \leq 0$.

Definition 7.1. *An eigenvalue $\mu_0 \in \sigma_p(\mathcal{A}^0)$ is stable with respect to the family of perturbations $\mathcal{A}^\lambda$ defined in (72) if the following two conditions hold:*
(i) there is $\delta > 0$ such that the annular region $\mathcal{Q}_\delta := \{z \in \mathbb{C};\ 0 < |z - \mu_0| < \delta\}$ is contained in the $\rho(\mathcal{A}^0)$ and in the region of boundedness for the family $\{\mathcal{A}^\lambda\}$, Δ_b, defined by

$$\Delta_b := \{z \in \mathbb{C};\ ||R_\lambda(z)||_{B(L^2_{per})} \leq M,\ \forall\, 0 < \lambda \ll 1\}.$$

Here $M > 0$ does not depend on λ and $R_\lambda(z) = (\mathcal{A}^\lambda - z)^{-1}$.
(ii) Let Γ be a simple closed curve about $\mu_0 \in \sigma_p(\mathcal{A}^\lambda)$ contained in the resolvent set of $\mathcal{A}^\lambda$ and define the Riesz projector $P_\lambda = \frac{1}{2\pi i}\int_\Gamma R_\lambda(z)dz$. Then

$$\lim_{\lambda \to 0^+} ||P_\lambda - P_{\mu_0}||_{B(L^2_{per})} = 0. \tag{73}$$

Remark 7.1. *It follows from Definition 7.1 that for all* $0 < \lambda \ll 1$, $\mathcal{A}^\lambda$ *has total algebraic multiplicity equal to the* μ_0 *inside* $\mathcal{Q}_\delta$.

The next lemma is the cornerstone of our analysis.

Lemma 7.1. *The following three conditions are equivalent:*

(i) the number $z \in \Delta_b$;

(ii) for all $u \in C^\infty_{per}([0,L])$ *we have* $||(\mathcal{A}^\lambda - z)u||_{L^2_{per}} \geq \varepsilon ||u||_{L^2_{per}} > 0$ *for all* $0 < \lambda \ll 1$;

(iii) the number $z \in \rho(\mathcal{A}^0)$.

Proof. See Angulo&Natali (2010). □

Lemma 7.1 enable us to prove the following result.

Theorem 7.1. *Let* $\mathcal{A}^\lambda$ *be the linear operator defined in (72). Suppose that* μ_0 *is a discrete eigenvalue of* $\mathcal{L}_\mathcal{M}$. *Then* μ_0 *is stable in the sense of the Definition 7.1.*

Proof. See Angulo&Natali (2010). □

Then, we can enunciate the following instability criteria (see Lin (2008) for the solitary wave case).

Theorem 7.2. *Let* φ_c *be a periodic traveling wave solution related to equation (11). We assume that* $\ker(\mathcal{A}^0) = [\varphi_c']$. *Denote by* $n^-(\mathcal{A}^0)$ *the number (counting multiplicity) of negative eigenvalues of the operator* $\mathcal{A}^0$. *Then there is a purely growing mode* $e^{\lambda t}u(x)$ *with* $\lambda > 0$, $u \in H^{m_2}_{per}([0,L])$ *to the linearized equation (70), if one of the following two conditions is true:*

(i) $n^-(\mathcal{A}^0)$ *is even and* $\frac{d}{dc}\int_0^L \varphi_c^2(x)dx > 0$.

(ii) $n^-(\mathcal{A}^0)$ *is odd and* $\frac{d}{dc}\int_0^L \varphi_c^2(x)dx < 0$.

Proof. See Angulo&Natali (2010). □

7.1 Nonlinear instability of cnoidal waves for the mKdV equation.

The arguments presented in Subsection 5.3.2 and from Theorem 7.2 enable us to determine that the cnoidal wave solutions $\varphi_{c,i}$ defined by Theorem 5.9 are linearly unstable for the mKdV equation. Now, we sketch the proof that linear instability implies nonlinear instability of cnoidal waves for the mKdV equation. In fact, we have that the linearized equation (70) takes the form $u_t = J\mathcal{L}_i^- u$, $i = 1,2$, where $J = \partial_x$ and $\mathcal{L}_i^-$ are defined in (58). So, $J\mathcal{L}_i^-$ has a positive real eigenvalue. Next, we define $S : H^1_{per}([0,L]) \to H^1_{per}([0,L])$ as $S(u) = u_\phi(1)$ where $u_\phi(t)$ is the solution of the Cauchy problem,

$$\begin{cases} u_t + 3u^2 u_x - cu_x + u_{xxx} = 0, \\ u(x,0) = \phi(x). \end{cases} \tag{74}$$

Then, it follows that the cnoidal waves $\varphi_{c,i}$ are stationary solutions for (74). Now, from Colliander et al. (2003) follows that the mapping data-solution related to the mKdV equation (74), $\Upsilon_c : H^1_{per}([0,L]) \to C([0,T]; H^1_{per}([0,L]))$ is smooth. Furthermore $S(\varphi_{c,i}) = \varphi_{c,i}$ for $i = 1,2$. Thus, since S is at least a $C^{1,\alpha}$ map defined on a neighborhood of the fixed point $\varphi_{c,i}$, we have from Henry et al. (1982) that there is an element $\mu \in \sigma(S'(\varphi_{c,i}))$ with $|\mu| > 1$ which implies the nonlinear instability in $H^1_{per}([0,L])$ of the cnoidal wave solutions $\varphi_{c,i}$.

8. Stability of periodic-peakon waves for the NLS-δ

Recently Angulo&Ponce (2010) have established a theory of existence and stability of periodic-peakon solutions for the cubic NLS-δ equation in (17) ($p = 2$). More precisely, it was shown the existence of a smooth branch of periodic solutions, $(\omega, Z) \to \varphi_{\omega,Z} \in H^1_{per}([0, 2L])$, for the semi-linear elliptic equation

$$-\varphi''_{\omega,Z} + \omega\varphi_{\omega,Z} - Z\delta(x)\varphi_{\omega,Z} = \varphi^3_{\omega,Z}, \tag{75}$$

such that

$$\begin{array}{l} (1)\ -\varphi''_{\omega,Z}(x) + \omega\varphi_{\omega,Z}(x) = \varphi^3_{\omega,Z}(x) \quad \text{for } x \neq \pm 2nL,\ n \in \mathbb{N}. \\ (2)\ \varphi'_{\omega,Z}(0+) - \varphi'_{\omega,Z}(0-) = -Z\varphi_{\omega,Z}(0), \\ (3)\ \lim_{Z\to 0} \varphi_{\omega,Z} = \varphi_\omega, \end{array} \tag{76}$$

where φ_ω is the dnoidal profile in (39). We note that if $\varphi_{\omega,Z}$ is a solution of (75) then $\varphi_{\omega,Z}(\cdot + y)$ **is not necessarily a solution of** (75). Therefore the stability study for the "*periodic-peakon*" $\varphi_{\omega,Z}$ is for the orbit,

$$\Omega_{\varphi_{\omega,Z}} = \{e^{i\theta}\varphi_{\omega,Z} : \theta \in [0, 2\pi]\}. \tag{77}$$

The profile of $\varphi_{\omega,Z}$ is based in the Jacobi elliptic function *dnoidal* and determined for $\omega > Z^2/4$ by the patterns:

$$\begin{array}{l} (1)\ \text{for } Z > 0,\ \varphi_{\omega,Z}(\xi) = \eta_{1,Z}dn\left(\frac{\eta_{1,Z}}{\sqrt{2}}|\xi| + a; k\right), \\ (2)\ \text{for } Z < 0,\ \varphi_{\omega,Z}(\xi) = \eta_{1,Z}dn\left(\frac{\eta_{1,Z}}{\sqrt{2}}|\xi| - a; k\right), \end{array} \tag{78}$$

where $\eta_{1,Z}$ and k depend of ω and Z. The shift-function a satisfies that $\lim_{Z\to 0} a(\omega, Z) = 0$. So, since the basic symmetry for the NLS-δ equation is the phase-invariance we have the following stability definition for $\Omega_{\varphi_{\omega,Z}}$.

Definition 8.1. *For $\eta > 0$ we put $U_\eta = \{v \in H^1_{per}([0, 2L]);\ \inf_{\theta\in\mathbb{R}} ||v - e^{i\theta}\varphi_{\omega,Z}||_{H^1_{per}} < \eta\}$. The periodic standing wave $e^{i\omega t}\varphi_{\omega,Z}$ is stable if for $\epsilon > 0$ there exists $\eta > 0$ such that for $u_0 \in U_\eta$, the solution $u(t)$ of the NLS-δ equation with $u(0) = u_0$ satisfies $u(t) \in U_\epsilon$ for all $t \in \mathbb{R}$. Otherwise, $e^{i\omega t}\varphi_{\omega,Z}$ is said to be unstable in $H^1_{per}([0, 2L])$.*

The stability result established in Angulo&Ponce (2010) for the family of periodic-peakon in (78) is the following;

Theorem 8.1. *Let $\omega > \frac{\pi^2}{2L^2}$, $\omega > \frac{Z^2}{4}$ and ω large. Then we have:*

1. *For $Z > 0$ the dnoidal-peakon standing wave $e^{i\omega t}\varphi_{\omega,Z}$ is stable in $H^1_{per}([-L, L])$.*
2. *For $Z < 0$ the dnoidal-peakon standing wave $e^{i\omega t}\varphi_{\omega,Z}$ is unstable in $H^1_{per}([-L, L])$.*
3. *For $Z < 0$ the dnoidal-peakon standing wave $e^{i\omega t}\varphi_{\omega,Z}$ is stable in $H^1_{per,even}([-L, L])$.*

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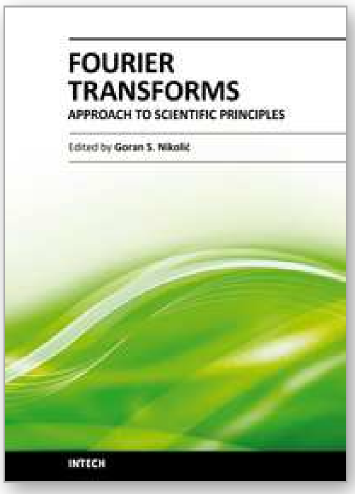

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