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Robust Control Approach for Combating the Bullwhip Effect in Periodic-Review Inventory Systems with Variable Lead-Time

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1. Introduction

It is well known that cost-efficient management of production and goods distribution systems in varying market conditions requires implementation of an appropriate inventory control policy (Zipkin, 2000). Since the traditional approaches to inventory control, focused mainly on the statistical analysis of long-term variables and (static) optimization performed on averaged values of various cost components, are no longer sufficient in modern production-inventory systems, new solutions are being proposed. In particular, due to the resemblance of inventory management systems to engineering processes, the methods of control theory are perceived as a viable alternative to the traditional approaches. A summary of the initial control-theoretic proposals can be found in (Axsäter, 1985), whereas more recent results are discussed in (Ortega & Lin, 2004) and (Sarimveis et al., 2008). However, despite a considerable research effort, one of the utmost important, yet still unresolved (Geary et al., 2006) problems observed in supply chain is the bullwhip effect, which manifests itself as an amplification of demand variations in order quantities.

We consider an inventory setting in which the stock at a distribution center is used to fulfill an unknown, time-varying demand imposed by customers and retailers. The stock is replenished from a supplier which delivers goods with delay according to the orders received from the distribution center. The design goal is to generate ordering decisions such that the entire demand can be satisfied from the stock stored at the distribution center, despite the latency in order procurement, referred to as lead-time delay. The latency may be subject to significant fluctuations according to the goods availability at the supplier and transportation time uncertainty. When demand is entirely fulfilled any cost associated with backorders, lost sales, and unsatisfied customers is eliminated. Although a number of researchers have recognized the need to explicitly consider the delay in the controller design and stability analysis of inventory management systems, e.g. Hoberg et al. (2007), robustness issues related to simultaneous delay and demand fluctuations remain to a large extent unexplored (Dolgui & Prodhon, 2007). A few examples constitute the work of Boukas et al. (2000), where an H_∞-norm-based controller has been designed for a production-inventory system with uncertain processing time and input delay, and Blanchini et al. (2003), who concentrated on the stability analysis of a production system with uncertain demand and process setup. Both papers are devoted to the control of manufacturing

systems, rather than optimization of goods flow in supply chain, and do not consider rate smoothening as an explicit design goal. On the contrary, in this work, we focus on the supply chain dynamics and provide formal methods for obtaining a smooth, non-oscillatory ordering signal, what is imperative for reducing the bullwhip effect (Dejonckheere et al., 2003).

From the control system perspective we may identify three decisive factors responsible for poor dynamical performance of supply chains and the bullwhip effect: 1) abrupt order changes in response to demand fluctuations, typical for the traditional order-up-to inventory policies, as discussed in (Dejonckheere et al., 2003); 2) inherent delay between placing of an order and shipment arrival at the distribution center which may span several review periods; and finally, 3) unpredictable variations of lead-time delay. Therefore, to avoid (or combat) the bullwhip effect, the designed policy should smoothly react to the changes in market conditions, and generate order quantities which will not fluctuate excessively in subsequent review intervals even though demand exhibits large and unpredictable variations. This is achieved in this work by solving a *dynamical optimization* problem with quadratic performance index (Anderson & Moore, 1989). Next, in order to eliminate the negative influence of delay variations, a compensation technique is incorporated into the basic algorithm operation together with a saturation block to explicitly account for the supplier capacity limitations. It is shown that in the inventory system governed by the proposed policy the stock level never exceeds the assigned warehouse capacity, which means that the potential necessity for an expensive emergency storage outside the company premises is eliminated. At the same time the stock is never depleted, which implies the 100% service level. The controller demonstrates robustness to model uncertainties and bounded external disturbance. The applied compensation mechanism effectively throttles undesirable quantity fluctuations caused by lead-time changes and information distortion thus counteracting the bullwhip effect.

2. Problem formulation

We consider an inventory system faced by an unknown, bounded, time-varying demand, in which the stock is replenished with delay from a supply source. Such setting, illustrated in Fig. 1, is frequently encountered in production-inventory systems where a common point (distribution center), linked to a factory or external, strategic supplier, is used to provide goods for another production stage or a distribution network. The task is to design a control strategy which, on one hand, will minimize lost service opportunities (occurring when there is insufficient stock at the distribution center to satisfy the current demand), and, on the other hand, will ensure smooth flow of goods despite model uncertainties and external disturbances. The principal obstacle in providing such control is the inherent delay between placing of an order at the supplier and goods arrival at the center that may be subject to significant fluctuations during the control process. Another factor which aggravates the situation is a possible inconsistency of the received shipments with regard to the sequence of orders. Indeed, it is not uncommon in practical situations to obtain the goods from an earlier order after the shipment arrival from a more recent one. In addition, we may encounter other types of disturbances affecting the replenishment process related to organizational issues and quality of information (Zomerdijk & de Vries, 2003) (e.g. when a shipment arrives on time but is registered in another review period, or when an incorrect order is issued from

the distribution center). The time-varying latency of fulfilling of an order will be further referred to as lead-time or lead-time delay.
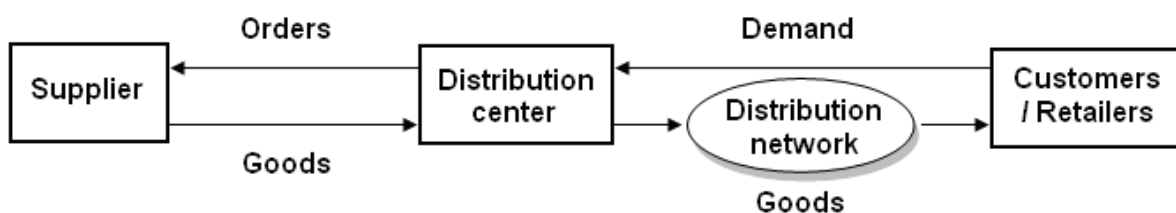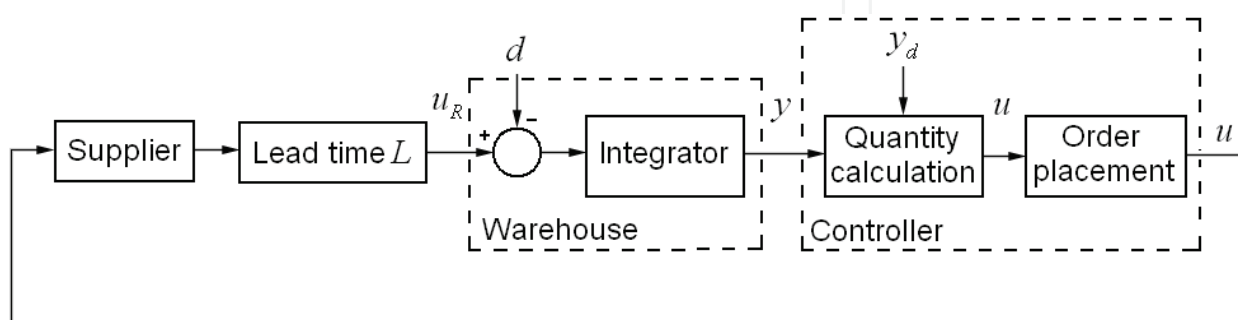

Fig. 1. Inventory system with a strategic supplier

Fig. 2. System model

The schematic diagram of the analyzed periodic-review inventory system is depicted in Fig. 2. The stock replenishment orders u are issued at regular time instants kT, where T is the review period and $k = 0, 1, 2,...$, on the basis of the on-hand stock (the current stock level in the warehouse at the distribution center) $y(kT)$, the target stock level y_d, and the history of previous orders. Each non-zero order placed at the supplier is realized with lead-time delay $L(k)$, assumed to be a multiple of the review period, i.e. $L(k) = n(k)T$, where $n(k)$ and its nominal value $\bar{n}$ are positive integers satisfying

$$(1-\delta)\bar{n} \leq n(k) \leq (1+\delta)\bar{n} \tag{1}$$

and $0 \leq \delta < 1$. Notice that (1) is the only constraint imposed on delay variations, which means that within the indicated interval the actual delay of a shipment may accept any statistical distribution. This implies that consecutive shipments sent by the supplier may arrive out of order at the distribution center and concurrently with other shipments which were sent earlier or afterwards. Since the presented model does not require stating the cause of lead-time variations, neither specification of a particular function $n(k)$ or its distribution, it allows for conducting the robustness study in a broad spectrum of practical situations with uncertain latency in delivering orders.

The imposed demand (the number of items requested from inventory in period k) is modeled as an *a priori* unknown, bounded function of time $d(kT)$,

$$0 \leq d(kT) \leq d_{\max}. \tag{2}$$

Notice that this definition of demand is quite general and it accounts for any standard distribution typically analyzed in the considered problem. If there is a sufficient number of items in the warehouse to satisfy the imposed demand, then the actually met demand $h(kT)$

(the number of items sold to customers or sent to retailers in the distribution network) will be equal to the requested one. Otherwise, the imposed demand is satisfied only from the arriving shipments, and additional demand is lost (we assume that the sales are not backordered, and the excessive demand is equivalent to a missed business opportunity). Thus, we may write

$$0 \leq h(kT) \leq d(kT) \leq d_{\max}. \tag{3}$$

The dynamics of the on-hand stock y depends on the amount of arriving shipments $u_R(kT)$ and on the satisfied demand h. Assuming that the warehouse is initially empty, i.e. $y(kT) = 0$ for $k < 0$, and the first order is placed at $kT = 0$, then for any $kT \geq 0$ the stock level at the distribution center may be calculated from the following equation

$$y(kT) = \sum_{j=0}^{k-1} u_R(jT) - \sum_{j=0}^{k-1} h(jT) = \sum_{j=0}^{k-1} u[jT - L(j)] - \sum_{j=0}^{k-1} h(jT). \tag{4}$$

Let us introduce a function $\xi(kT) = \xi_+(kT) - \xi_-(kT)$, where

- $\xi_+(kT)$ represents the sum of these surplus items which arrive at the distribution center by the time kT earlier than expected since their delay experienced in the neighborhood of kT is smaller than the nominal one, and

- $\xi_-(kT)$ denotes the sum of items which should have arrived by the time kT, but which cannot reach the center due to the (instantaneous) delay greater than the nominal one.

Assuming that the order quantity is bounded by some positive value $u_{\max}$ (e.g. the maximum number of items the supplier can accumulate and send in one review period), which is commonly encountered in practical systems, then on the basis of (1),

$$\forall_{k \geq 0} \ |\xi(kT)| \leq \xi_{\max} = u_{\max} \delta \overline{L}, \tag{5}$$

where $\overline{L} = \overline{n}T$ is the nominal lead-time. With this notation we can rewrite (4) in the following way

$$y(kT) = \sum_{j=0}^{k-1} u[(j - \overline{n})T] + \xi(kT) - \sum_{j=0}^{k-1} h(jT). \tag{6}$$

It is important to realize that because lead-time is bounded, it suffices to consider the effects caused by its variations (represented by function $\xi(\cdot)$ in the model) only in the neighborhood of kT implied by (1). Since the summing operation is commutative, all the previous shipments, i.e. those arriving before $(k - \delta\overline{n})T$, can be added as if they had actually reached the distribution center on time and this will not change the overall quantity of the received items. In other words, delay variations of shipments acquired in the far past do not inflict perturbation on the current stock.

The discussed model of inventory management system can also be presented in the state space. The state-space realization facilitates adaptation of formal design techniques, and is selected as a basis for the control law derivation described in detail in Section 3.

State-space representation

In order to proceed with a formal controller design we describe the discrete-time model of the considered inventory system in the state space:

$$\mathbf{x}\big[(k+1)T\big] = \mathbf{A}\mathbf{x}(kT) + \mathbf{b}u(kT) + \mathbf{v}_1 h(kT) + \mathbf{v}_2 \xi(kT),$$
$$y(kT) = \mathbf{q}^T \mathbf{x}(kT),$$

(7)

where $\mathbf{x}(kT) = [x_1(kT)\ x_2(kT)\ x_3(kT)\ ...\ x_n(kT)]^T$ is the state vector with $x_1(kT) = y(kT)$ representing the stock level in period k and the remaining state variables $x_j(kT) = u[(k - n + j - 1)T]$ for any $j = 2, ... , n$ equal to the delayed input signal u. $\mathbf{A}$ is $n \times n$ state matrix, $\mathbf{b}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{q}$ are $n \times 1$ vectors

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

(8)

and the system order $n = \bar{n} + 1$. For convenience of the further analysis, we can rewrite the model in the alternative form

$$\begin{cases} x_1\big[(k+1)T\big] = x_1(kT) + x_2(kT) - h(kT) + \xi(kT), \\ x_2\big[(k+1)T\big] = x_3(kT), \\ \quad \vdots \\ x_{n-1}\big[(k+1)T\big] = x_n(kT), \\ x_n\big[(k+1)T\big] = u(kT). \end{cases}$$

(9)

Relation (9) shows how the effects of delay are accounted for in the model by a special choice of the state space in which the state variables contain the information about the most recent order history. The desired system state is defined as

$$\mathbf{x_d} = \begin{bmatrix} x_{d1} \\ x_{d2} \\ \vdots \\ x_{dn-1} \\ x_{dn} \end{bmatrix} = \begin{bmatrix} x_{d1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

(10)

where $x_{d1} = y_d$ denotes the demand value of the first state variable, i.e. the target stock level. By choosing the desired state vector as

$$\mathbf{x_d} = [y_d\ 0\ 0\ ...\ 0]^T$$

we want the first state variable (on-hand stock) to reach the level y_d, and to be kept at this level in the steady-state. For this to take place all the state variables $x_2...x_n$ should be zero once $x_1(kT)$ becomes equal to y_d, precisely as dictated by (10).

In the next section, equations (7)–(10) describing the system behavior and interactions among the principal system variables (ordering signal, on-hand stock level and imposed demand) will be used to develop a discrete control strategy goverining the flow of goods between the supplier and the distribution center.

3. Proposed inventory policy

In this section, we formulate a new inventory management policy and discuss its properties related to handling the flow of goods. First, the nominal system is considered, and the controller parameters are selected by solving a linear-quadratic (LQ) optimization problem. Afterwards, the influence of perturbation is analyzed and an enhanced, nonlinear control law is formulated which demonstrates robustness to delay and demand variations. The key element in the improved controller structure is the compensator which reduces the effects caused by delay fluctuations and information distortion.

3.1 Optimization problem

From the point of view of optimizing the system dynamics, we may state the aim of the control action as bringing the currently available stock to the target level without excessive control effort. Therefore, we seek for a control $u_{opt}(kT)$, which minimizes the following cost functional

$$J(u) = \sum_{k=0}^{\infty} \left\{ u^2(kT) + w \left[y_d - y(kT) \right]^2 \right\}, \tag{11}$$

where w is a positive constant applied to adjust the influence of the controller command and the output variable on the cost functional value. Small w reduces excessive order quantities, but lowers the controller dynamics. High w, in turn, implies fast tracking of the reference stock level at the expense of large input signals. In the extreme case, when $w \to \infty$, the term $y_d - y(kT)$ prevails and the developed controller becomes a dead-beat scheme. From the managerial point of view the application of a quadratic cost structure in the considered problem of inventory control has similar effects as discussed in (Holt et al., 1960) in the context of production planning. It allows for a satisfactory tradeoff between fast reaction to the changes in market conditions (reflected in demand variations) and smoothness of ordering decisions. As a result, the controller will track the target inventory level y_d with good dynamics, yet, at the same time, it will prevent rapid demand fluctuations from propagating in supply chain. A huge advantage of our approach based on dynamical optimization over the results proposed in the past is that the smoothness of ordering decisions is ensured by the controller structure itself. This allows us to avoid signal filtering and demand averaging, typically applied to decrease the degree of ordering variations in supply chain, and thus to avoid errors and inaccuracies inherently implied by these techniques.

Applying the standard framework proposed in (Zabczyk, 1974), to system (7)–(8), the control $u_{opt}(kT)$ minimizing criterion (11) can be presented as

$$u_{opt}(kT) = -\mathbf{g}\mathbf{x}(kT) + r, \tag{12}$$

where

$$\mathbf{g} = \mathbf{b}^T \mathbf{K} \left(\mathbf{I_n} + \mathbf{b}\mathbf{b}^T \mathbf{K} \right)^{-1} \mathbf{A},$$

$$r = \mathbf{b}^T \left[\mathbf{K} \left(\mathbf{I_n} + \mathbf{b}\mathbf{b}^T \mathbf{K} \right)^{-1} \mathbf{b}\mathbf{b}^T - \mathbf{I_n} \right] \mathbf{k}, \tag{13}$$

$$\mathbf{k} = -\mathbf{A}^T \left[\mathbf{K} \left(\mathbf{I_n} + \mathbf{b}\mathbf{b}^T \mathbf{K} \right)^{-1} \mathbf{b}\mathbf{b}^T - \mathbf{I_n} \right] \mathbf{k} - w\mathbf{q}y_d,$$

and semipositive, symmetric matrix $\mathbf{K_{n \times n}}$, $\mathbf{K}^T = \mathbf{K} \geq 0$, is determined according to the following Riccati equation

$$\mathbf{K} = \mathbf{A}^T \mathbf{K} \left(\mathbf{I_n} + \mathbf{bb}^T \mathbf{K} \right)^{-1} \mathbf{A} + w \mathbf{qq}^T. \qquad (14)$$

Finding the parameters of the LQ optimal controller for the considered system with delay is a challenging task, as it involves solving an nth order matrix Riccati equation. Nevertheless, by applying the approach presented in (Ignaciuk & Bartoszewicz, 2010) we are able to solve the problem analytically and obtain the control law in a closed form. Below we summarize major steps of the derivation.

3.2 Solution to the optimization problem
We begin with the most general form of matrix $\mathbf{K}$ which can be presented as

$$\mathbf{K}_0 = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n} \\ k_{12} & k_{22} & k_{23} & \dots & k_{2n} \\ k_{13} & k_{23} & k_{33} & \dots & k_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{1n} & k_{2n} & k_{3n} & \dots & k_{nn} \end{bmatrix}. \qquad (15)$$

In the first iteration, we place $\mathbf{K}_0$ directly in (14), and after substituting matrix $\mathbf{A}$ and vector $\mathbf{b}$ as defined by (8), we seek for similarities between the elements k_{ij} on either side of the equality sign in (14). In this way we find the relations among the first four elements in the upper left corner of $\mathbf{K}$: $k_{12} = k_{22} = k_{11} - w$ (note that $k_{21} = k_{12}$ since $\mathbf{K}$ is symmetric). Consequently, after the first analytical iteration, we obtain the following form of $\mathbf{K}$

$$\mathbf{K}_1 = \begin{bmatrix} k_{11} & k_{11} - w & k_{13} & \dots & k_{1n} \\ k_{11} - w & k_{11} - w & k_{23} & \dots & k_{2n} \\ k_{13} & k_{23} & k_{33} & \dots & k_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{1n} & k_{2n} & k_{3n} & \dots & k_{nn} \end{bmatrix}. \qquad (16)$$

Now we substitute $\mathbf{K}_1$ given by (16) into the expression on the right hand side of (14) and compare with its left hand side. This allows us to represent the elements k_{i3} ($i = 1, 2, 3$) in terms of k_{11} as $k_{13} = k_{23} = k_{33} = k_{11} - 2w$. Concisely in matrix form we have

$$\mathbf{K}_2 = \begin{bmatrix} k_{11} & k_{11} - w & k_{11} - 2w & k_{14} & \dots & k_{1n} \\ k_{11} - w & k_{11} - w & k_{11} - 2w & k_{24} & \dots & k_{2n} \\ k_{11} - 2w & k_{11} - 2w & k_{11} - 2w & k_{34} & \dots & k_{3n} \\ k_{14} & k_{24} & k_{34} & k_{44} & \dots & k_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{1n} & k_{2n} & k_{3n} & k_{4n} & \dots & k_{nn} \end{bmatrix}. \qquad (17)$$

We proceed with the substitutions until a general pattern is determined, i.e. until all the elements of $\mathbf{K}$ can be expressed as functions of k_{11} and the system order n. We get $k_{ij} = k_{11} -$

$(j-1)w$ for $j \geq i$ (the upper part of $\mathbf{K}$) and $k_{ij} = k_{11} - (i-1)w$ for $j < i$ (the lower part of $\mathbf{K}$). In matrix form

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{11}-w & k_{11}-2w & \dots & k_{11}-(n-1)w \\ k_{11}-w & k_{11}-w & k_{11}-2w & \dots & k_{11}-(n-1)w \\ k_{11}-2w & k_{11}-2w & k_{11}-2w & \dots & k_{11}-(n-1)w \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{11}-(n-1)w & k_{11}-(n-1)w & k_{11}-(n-1)w & \dots & k_{11}-(n-1)w \end{bmatrix}. \tag{18}$$

If we substitute (18) into the right hand side of equation (14) and compare the first element in the upper left corner of the matrices on either side of the equality sign, we get the expression from which we can determine k_{11}:

$$k_{11} = nw + 1 - \left[k_{11} - (n-1)w + 1 \right]^{-1}. \tag{19}$$

Equation (19) has two roots

$$k_{11}' = \sqrt{w}\left[(2n-1)\sqrt{w} - \sqrt{w+4} \right]/2 \text{ and } k_{11}'' = \sqrt{w}\left[(2n-1)\sqrt{w} + \sqrt{w+4} \right]/2. \tag{20}$$

Since $\quad \det(\mathbf{K}) = w^{n-1}[k_{11} - (n-1)w]$, $\quad$ only $\quad k_{11}'' = \sqrt{w}\left[(2n-1)\sqrt{w} + \sqrt{w+4} \right]/2 \geq (n-1)w$

guarantees that $\mathbf{K}$ is semipositive definite. Consequently, we get matrix $\mathbf{K}$ (18) with $k_{11} = k_{11}''$. This concludes the solution of the Riccati equation.
Having found $\mathbf{K}$, we evaluate $\mathbf{g}$,

$$\mathbf{g} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \end{bmatrix} \left\{ 1 - \left[k_{11} - (n-1)w + 1 \right]^{-1} \right\}. \tag{21}$$

Vector $\mathbf{k}$ is determined by substituting matrix $\mathbf{K}$ given by (18) into the last equation in set (13). We obtain

$$\mathbf{k} = \begin{bmatrix} k_1 & k_1 + wy_d & k_1 + 2wy_d & \dots & k_1 + (n-1)wy_d \end{bmatrix}^T, \tag{22}$$

where

$$k_1 = -wy_d \left\{ n + \left[k_{11} - (n-1)w \right]^{-1} \right\}. \tag{23}$$

Then, using the second equation in set (13), and substituting (23), we calculate r,

$$r = -\frac{k_1 + (n-1)wy_d}{k_{11} - (n-1)w + 1} = \frac{wy_d}{k_{11} - (n-1)w}. \tag{24}$$

Finally, using (21) and (24), the optimal control $u_{opt}(kT)$ can be presented in the following way:

$$u_{opt}(kT) = -\mathbf{g}\mathbf{x}(kT) + r = -\left(1 - \frac{1}{k_{11} - (n-1)w + 1} \right) \sum_{j=1}^{n} x_j(kT) + \frac{wy_d}{k_{11} - (n-1)w}. \tag{25}$$

Substituting $k_{11} = \sqrt{w}\left[(2n-1)\sqrt{w} + \sqrt{w+4}\right]/2$, we arrive at

$$u_{opt}(kT) = \alpha\left[y_d - x_1(kT) - \sum_{j=2}^{n} x_j(kT)\right], \tag{26}$$

where the gain $\alpha = (\sqrt{w(w+4)} - w)/2$. From (9) the state variables x_j ($j = 2, 3,..., n$) may be expressed in terms of the control signal generated at the previous $n - 1$ samples as

$$x_j(kT) = u\left[(k-n+j-1)T\right]. \tag{27}$$

Recall that we introduced the notation $x_1(kT) = y(kT)$. Then, substituting (27) into (26), we obtain

$$u_{opt}(kT) = \alpha\left[y_d - y(kT) - \sum_{j=k-\bar{n}}^{k-1} u(jT)\right], \tag{28}$$

which completes the design of the inventory policy for the nominal system. The policy can be interpreted in the following way: the quantity to be ordered in each period is proportional to the difference between the target and the current stock level ($y_d - y(kT)$), decreased by the amount of open orders (the quantity already ordered at the supplier, but which has not yet arrived at the warehouse due to lead-time delay). It is tuned in a straightforward way by the choice of a single parameter a, i.e. smaller a implies more dampening of demand variations (for a detailed discussion on the selection of a refer to (Ignaciuk & Bartoszewicz, 2010)).

3.3 Stability analysis of the nominal system

The nominal discrete-time system is asymptotically stable if all the roots of the characteristic polynomial of the closed-loop state matrix $\mathbf{A_c} = [\mathbf{I_n} - \mathbf{b}(\mathbf{c}^T\mathbf{b})^{-1}\mathbf{c}^T]\mathbf{A}$ are located within the unit circle on the z-plane. The roots of the polynomial

$$\det(z\mathbf{I_n} - \mathbf{A_c}) = z^n + (\alpha-1)z^{n-1} = z^{n-1}\left[z - (1-\alpha)\right], \tag{29}$$

are located inside the unit circle, if $0 < a < 2$. Since for every n and for every w the gain satisfies the condition $0 < a \leq 1$, the system is asymptotically stable. Moreover, since irrespective of the value of the tuning coefficient w the roots of (29) remain on the nonnegative real axis, no oscillations appear at the output. By changing w from 0 to ∞, the nonzero pole moves towards the origin of the z-plane, which results in faster convergence to the demand state. In the limit case when $w = \infty$, all the closed-loop poles are at the origin ensuring the fastest achievable response in a discrete-time system offered by a dead-beat scheme.

3.4 Robustness issues

The order calculation performed according to (28) is based on the nominal delay which constitutes an estimate of the true (variable) lead-time set according to the contracting agreement with the supplier. The controller designed for the nominal system is robust with

respect to demand fluctuations, yet may generate negative orders in the presence of lead-time variations. In order to eliminate this deficiency and at the same time account for the supplier capacity limitations, we introduce the following modification into the basic algorithm

$$u(kT) = \begin{cases} 0, & \text{if } \varphi(kT) < 0, \\ \varphi(kT), & \text{if } 0 \le \varphi(kT) \le u_{\max}, \\ u_{\max}, & \text{if } \varphi(kT) > u_{\max}, \end{cases} \tag{30}$$

where $u_{\max} > d_{\max}$ is a constant denoting the maximum order quantity that can be provided by the supplier in a single review period. Function $\varphi(\cdot)$ is defined as

$$\varphi(kT) = \alpha \left\{ y_d - y(kT) - \sum_{j=k-\bar{n}}^{k-1} u(jT) + \varepsilon \sum_{j=0}^{k-1} \left[u_R(jT) - u(jT - \bar{L}) \right] \right\}. \tag{31}$$

It consists of two elements:
- LQ optimal controller as given by (28), and
- delay variability compensator tuned by the coefficient $\varepsilon \in [0, 1]$, which accumulates the information about the differences between the number of items which actually arrived at the distribution center and those which were expected to arrive.

3.5 Properties of the robust policy
The properties of the designed nonlinear policy (30)–(31) will be formulated as two theorems and analyzed with respect to the most adverse conditions (the extreme fluctuations of demand and delay). The first proposition shows how to adjust the warehouse storage space to always accommodate the entire stock and in this way eliminate the risk of (expensive) emergency storage outside the company premises. The second theorem states that with an appropriately chosen target stock level there will be always goods in the warehouse to meet the entire demand.

Theorem 1. If policy (30)–(31) is applied to system (7)–(8), then the stock level at the distribution center is always upper-bounded, i.e.

$$\underset{k \ge 0}{\forall} \; y(kT) \le y_{\max} = y_d + u_{\max} + (1 + \varepsilon)\xi_{\max}. \tag{32}$$

Proof. Based on (4), (5), and the definition of function $\xi(\cdot)$, the term compensating the effects of delay variations in (31) satisfies the following relation

$$\sum_{j=0}^{k-1} \left[u_R(jT) - u(jT - \bar{L}) \right] = \sum_{j=0}^{k-1} \left\{ u\left[jT - L(j) \right] - u(jT - \bar{L}) \right\} = \sum_{j=0}^{k-1} \xi(jT) = \xi(kT). \tag{33}$$

Therefore, we may rewrite function $\varphi(\cdot)$ as

$$\varphi(kT) = \alpha \left[y_d - y(kT) - \sum_{j=k-\bar{n}}^{k-1} u(jT) + \varepsilon \xi(kT) \right]. \tag{34}$$

It follows from the algorithm definition and the system initial conditions that the warehouse at the distribution center is empty for any $k \leq (1-\delta)\bar{n}$. Consequently, it is sufficient to show that the proposition holds for all $k > (1-\delta)\bar{n}$. Let us consider some integer $l > (1-\delta)\bar{n}$ and the value of $\varphi(\cdot)$ at instant lT. Two cases ought to be analyzed: the situation when $\varphi(lT) \geq 0$, and the circumstances when $\varphi(lT) < 0$.

Case 1. We investigate the situation when $\varphi(lT) \geq 0$. Directly from (34), we get

$$y(lT) \leq y_d + \varepsilon\xi(lT) - \sum_{j=l-\bar{n}}^{l-1} u(jT). \tag{35}$$

Since u is always nonnegative, we have

$$y(lT) \leq y_d + \varepsilon\xi(lT). \tag{36}$$

Moreover, since $\xi(lT) \leq \xi_{max}$, we obtain

$$y(lT) \leq y_d + \varepsilon\xi_{max} \leq y_{max}, \tag{37}$$

which ends the first part of the proof.

Case 2. In the second part of the proof we analyze the situation when $\varphi(lT) < 0$. First, we find the last instant $l_1T < lT$ when $\varphi(\cdot)$ was nonnegative. According to (34), $\varphi(0) = ay_d > 0$, so the moment l_1T indeed exists, and the value of $y(l_1T)$ satisfies the inequality similar to (35), i.e.

$$y(l_1T) \leq y_d + \varepsilon\xi(l_1T) - \sum_{j=l_1-\bar{n}}^{l_1-1} u(jT). \tag{38}$$

The stock level at instant lT can be expressed as

$$y(lT) = y(l_1T) + \sum_{j=l_1-\bar{n}}^{l-\bar{n}-1} u(jT) + \xi(lT) - \sum_{j=l_1}^{l-1} h(jT), \tag{39}$$

which after applying (38) leads to

$$\begin{aligned}
y(lT) &\leq y_d + \varepsilon\xi(l_1T) - \sum_{j=l_1-\bar{n}}^{l_1-1} u(jT) + \sum_{j=l_1-\bar{n}}^{l-\bar{n}-1} u(jT) + \xi(lT) - \sum_{j=l_1}^{l-1} h(jT) \\
&\leq y_d + \varepsilon\xi(l_1T) + \xi(lT) + \sum_{j=l_1}^{l-\bar{n}-1} u(jT) - \sum_{j=l_1}^{l-1} h(jT).
\end{aligned} \tag{40}$$

The algorithm generated a nonzero quantity for the last time before lT at l_1T, and this value can be as large as u_{max}. Consequently, the sum $\sum_{j=l_1}^{l-\bar{n}-1} u(jT) = u(l_1T) \leq u_{max}$. From inequalities (3) and the condition $\xi(lT) \leq \xi_{max}$ we obtain the following stock estimate

$$\begin{aligned}
y(lT) &\leq y_d + \varepsilon\xi(l_1T) + \xi(lT) + u(l_1T) \\
&\leq y_d + \varepsilon\xi_{max} + \xi_{max} + u_{max} = y_{max},
\end{aligned} \tag{41}$$

which concludes the second part of the reasoning and completes the proof of Theorem 1. □
Theorem 1 states that the warehouse storage space is finite and never exceeds the level of y_{max}. This means that irrespective of the demand and delay variations the system output $y(\cdot)$ is bounded, and the risk of costly emergency storage is eliminated. The second theorem, formulated below, shows that with the appropriately selected target stock y_d we can make the on-hand stock positive, which guarantees the maximum service level in the considered system with uncertain, variable delay.

Theorem 2. If policy (30)–(31) is applied to system (7)–(8), and the target stock level satisfies

$$y_d > u_{max}(\bar{n} + 1/\alpha + 1) + (1+\varepsilon)\xi_{max}, \tag{42}$$

then for any $k \geq (1+\delta)\bar{n} + T_{max}/T$, where $T_{max} = Ty_{max}/(u_{max} - d_{max})$, the stock level is strictly positive.

Proof. The theorem assumption implies that we deal with time instants $kT \geq (1+\delta)\bar{n}T + T_{max}$. Considering some $l \geq (1+\delta)\bar{n} + T_{max}/T$ and the value of signal $\varphi(lT)$, we may distinguish two cases: the situation when $\varphi(lT) < u_{max}$, and the circumstances when $\varphi(lT) \geq u_{max}$.

Case 1. First, we consider the situation when $\varphi(lT) < u_{max}$. We obtain from (34)

$$y(lT) > y_d - \frac{u_{max}}{\alpha} - \sum_{j=l-\bar{n}}^{l-1} u(jT) + \varepsilon\xi(lT). \tag{43}$$

The order quantity is always bounded by u_{max}, which implies

$$y(lT) > y_d - u_{max}/\alpha - u_{max}\bar{n} + \varepsilon\xi(lT). \tag{44}$$

Since $\xi(\cdot) \geq -\xi_{max}$, we get

$$y(lT) > y_d - u_{max}/\alpha - u_{max}\bar{n} - \varepsilon\xi_{max}. \tag{45}$$

Using assumption (42), we get $y(lT) > 0$, which concludes the first part of the proof.

Case 2. In the second part of the proof we investigate the situation when $\varphi(lT) \geq u_{max}$. First, we find the last period $l_1 < l$ when function $\varphi(\cdot)$ was smaller than u_{max}. It comes from Theorem 1 that the stock level never exceeds the value of y_{max}. Furthermore, the demand is limited by d_{max}. Thus, the maximum interval T_{max} during which the controller may continuously generate the maximum order quantity u_{max} is determined as $T_{max} = Ty_{max}/(u_{max} - d_{max})$, and instant l_1T does exist. Moreover, from the theorem assumption we get $l_1T \geq (1+\delta)\bar{n}T$, which means that by the time l_1T the first shipment from the supplier has already reached the distribution center, no matter the delay variation.

The value of $\varphi(l_1T) < u_{max}$. Thus, following similar reasoning as presented in (43)–(45), we arrive at $y(l_1T) > 0$ and

$$\begin{aligned} y(lT) &> y_d - \frac{u_{max}}{\alpha} - \sum_{j=l_1-\bar{n}}^{l_1-1} u(jT) + \varepsilon\xi(l_1T) + \sum_{j=l_1-\bar{n}}^{l-\bar{n}-1} u(jT) + \xi(lT) - \sum_{j=l_1}^{l-1} h(jT) \\ &= y_d - \frac{u_{max}}{\alpha} + \varepsilon\xi(l_1T) + u(l_1T) + \sum_{j=l_1+1}^{l-1} u(jT) - \sum_{j=l-\bar{n}}^{l-1} u(jT) + \xi(lT) - \sum_{j=l_1}^{l-1} h(jT). \end{aligned} \tag{46}$$

Recall that l_1T was the last instant before lT when the controller calculated a quantity smaller than u_{max}. This quantity, $u(l_1T)$, could be as low as zero. Afterwards, the algorithm generates the maximum order and the first sum in (46) reduces to $u_{max}(l-1-l_1)$. Moreover, since for any k, $u(kT) \leq u_{max}$, the second sum is upper-bounded by $u_{max}\overline{n}$, which implies

$$y(lT) > y_d - u_{max} / \alpha + \varepsilon\xi(l_1T) + 0 + u_{max}(l-1-l_1) - u_{max}\overline{n} + \xi(lT) - \sum_{j=l_1}^{l-1} h(jT). \qquad (47)$$

According to (3), the realized demand satisfies $0 \leq h(\cdot) \leq d_{max}$, hence

$$y(lT) > y_d - u_{max} / \alpha + u_{max}(l-1-l_1) - u_{max}\overline{n} + \varepsilon\xi(l_1T) + \xi(lT) - d_{max}(l-l_1). \qquad (48)$$

Since $\xi(lT) \geq -\xi_{max}$, we get

$$y(lT) > y_d - u_{max} / \alpha + u_{max}(l-1-l_1) - u_{max}\overline{n} - \varepsilon\xi_{max} - \xi_{max} - d_{max}(l-l_1). \qquad (49)$$

Finally, using the theorem assumption (42), we may estimate the stock level at instant lT in the following way

$$y(lT) > (u_{max} - d_{max})(l-l_1). \qquad (50)$$

Since $l > l_1$, and by assumption $u_{max} > d_{max}$, we get $y(lT) > 0$. This completes the proof of Theorem 2. ◻

Remark. Theorem 2 defines the warehouse storage space which needs to be provided to ensure the maximum service level. The required warehouse capacity is specified following the worst-case uncertainty analysis (for an instructive insight how this methodology relates to production-distribution systems see e.g. (Blanchini et. al., 2003) and (Sarimveis et al., 2008)). Notice, however, that the value given in (42) scales linearly with the maximum order quantity related to demand by the inequality $u_{max} > d_{max}$. Therefore, in the situation when the mean demand differs significantly from the maximum one, it may be convenient to substitute u_{max} with some positive $d_L < d_{max} < u_{max}$. In such a case the 100% service level is no longer ensured, yet the average stock level, and as a consequence the holding costs, will be reduced.

4. Numerical example

We verify the properties of the nonlinear inventory policy (30)–(31) proposed in this work in a series of simulation tests. The system parameters are chosen in the following way: review period $T = 1$ day, nominal lead-time $\overline{L} = \overline{n}T = 8$ days, tolerance of delay variation $\delta = 0.25$, the maximum daily demand at the distribution center $d_{max} = 50$ items, and the maximum order quantity $u_{max} = 55$ items. In order to provide fast response yet with a smooth ordering signal, the controller gain should not exceed 0.618, which corresponds to the balanced optimziation case with $w = 1$. Since, additionally, we should account for ordering oscillations caused by delay changes, in the tests the gain is adjusted to $a(w) = a(0.5) = 0.5$. We consider two scenarios reflecting the most common market situations.

Scenario 1. In the first series of simulations we test the controller performance in response to the demand pattern illustrated in Fig. 3, which shows a trend in the demand with abrupt seasonal changes. It is assumed that lead-time fluctuates according to

$$L(k) = \left\lfloor \left[1 + \delta \sin\left(2\pi kT / \bar{n} \right) \right] \bar{n}T \right\rfloor = \left\lfloor \left[1 + 0.25 \sin\left(\pi k / 4 \right) \right] 8 \right\rfloor, \tag{51}$$

where $\lfloor f \rfloor$ denotes the integer part of f. The actual delay in procurring orders is illustrated in Fig. 4.
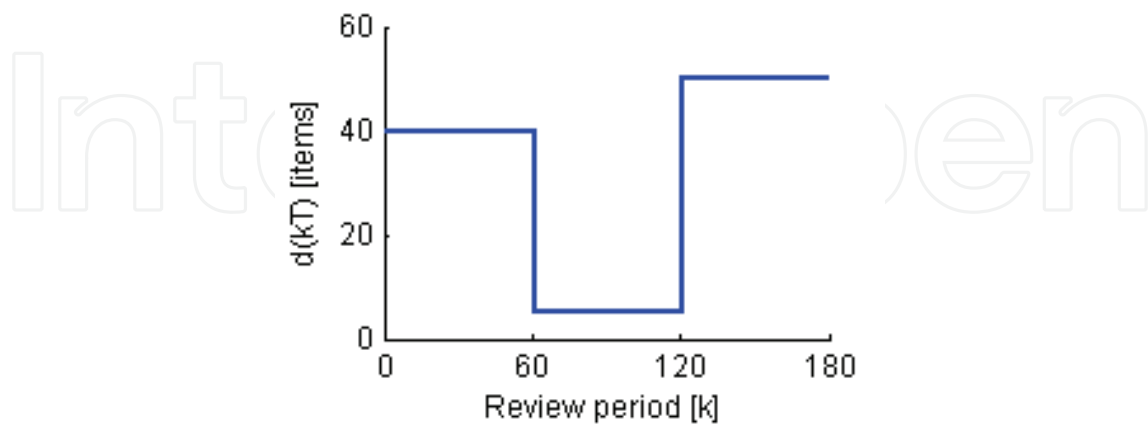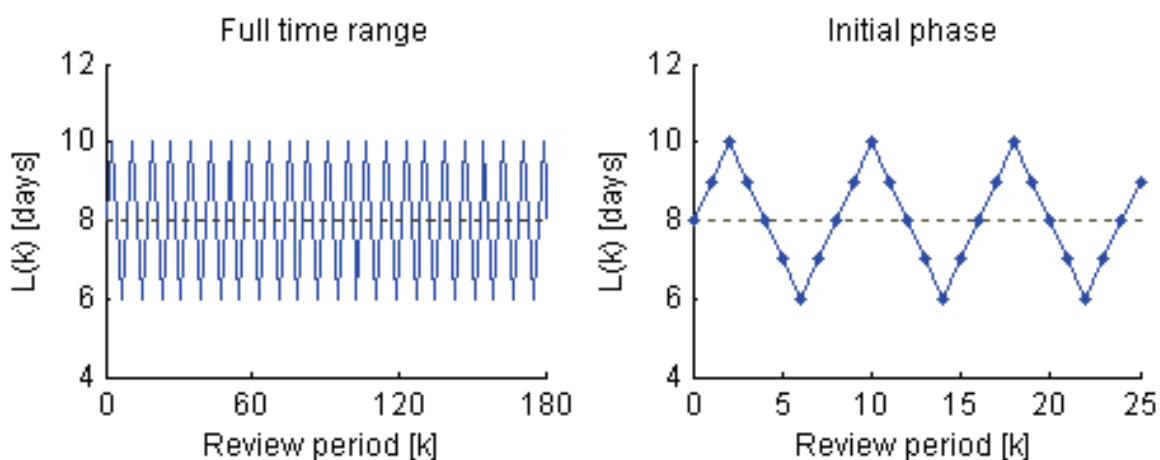

Fig. 3. Market demand – seasonal trend

Fig. 4. Lead-time delay

In order to elaborate on the adverse effects of delay variations, and assess the quality of the proposed compensation mechanism, we run two tests. In the first one (curve (a) in the graphs), we show the controller performance with compensation turned off, i.e. with $\varepsilon = 0$, and in the seond test, we consider the case of a full compensation in action with ε set equal to 1 (curve (b) in the graphs). The target stock level y_d is adjusted according to the guideliness provided by Theorem 2 so that the maximum service level is obtained, and the storage space $y_{\max}$ is reserved according to the condition stipulated in Theorem 1. The actual values used in the simulations are summarized in Table 1.

The test results are shown in Figs. 5–7: the ordering signal generated by the controller in Fig. 5, the received orders in Fig. 6, and the resultant on-hand stock in Fig. 7. It is clear from the graphs that the proposed controller quickly responds to the sudden changes in the demand trend. Moreover, the stock does not increase beyond the warehouse capacity, and it never drops to zero after the initial phase which implies the 100% service level. If we compare the curves representing the case of a full compensation (b) and the case of the

compensation turned-off (a) in Figs. 5 and 7, we can notice that the proposed compensation mechanism eliminates the oscillations of the control signal originating from delay variations. This allows for smooth reaction to the changes in market trend, and an ordering signal which is easy to follow by the supplier. We can learn from Fig. 7 that the obtained smooth ordering signal also permits reducing the on-hand stock while keeping it positive. This means that the maximum service level is achieved, but with decreased holding costs.

Compensation {on/off}	Target stock y_d [items]	Storage space y_{max} [items]
off: $\varepsilon = 0$	720 > 715	885
on: $\varepsilon = 1$	830 > 825	1105

Table 1. Controller parameters in Scenario 1
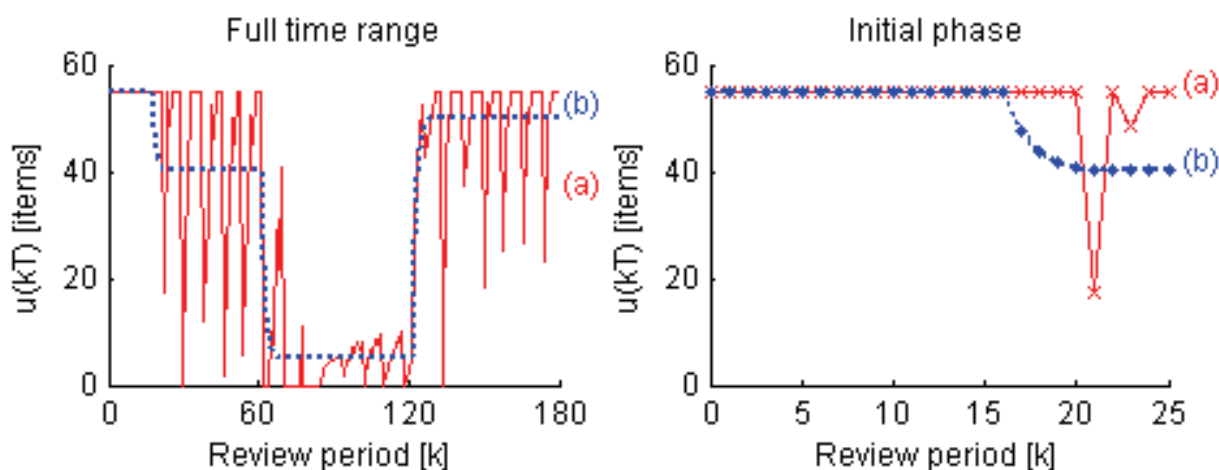
Fig. 5. Generated orders
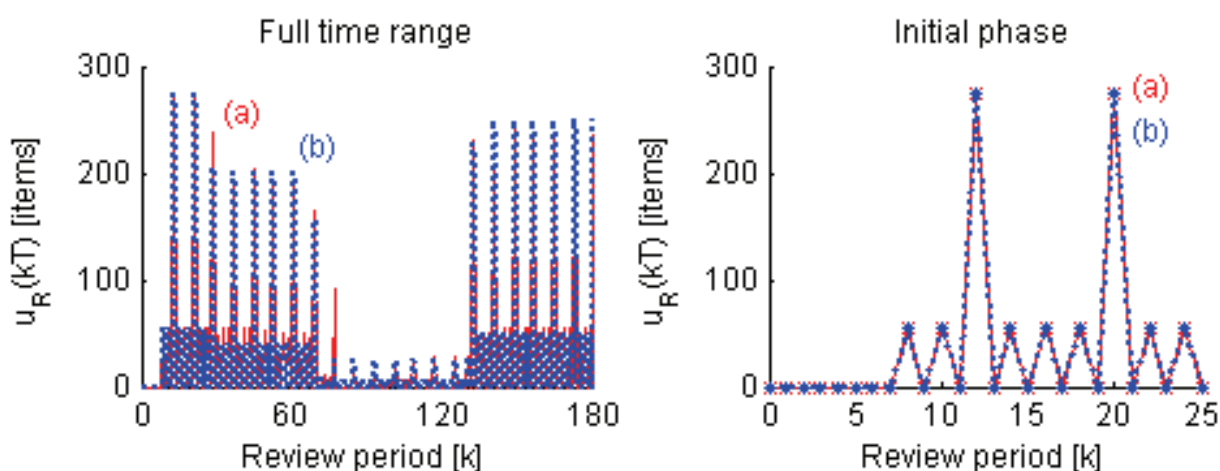
Fig. 6. Received shipments

Scenario 2. In the second scenario, we investigate the controller behavior in the presence of highly variable stochastic demand. Function $d(\cdot)$ following the normal distribution with mean $d_\mu = 25$ items and standard deviation $d_\delta = 25$ items, $D_{norm}(25, 25)$, is illustrated in Fig. 8.
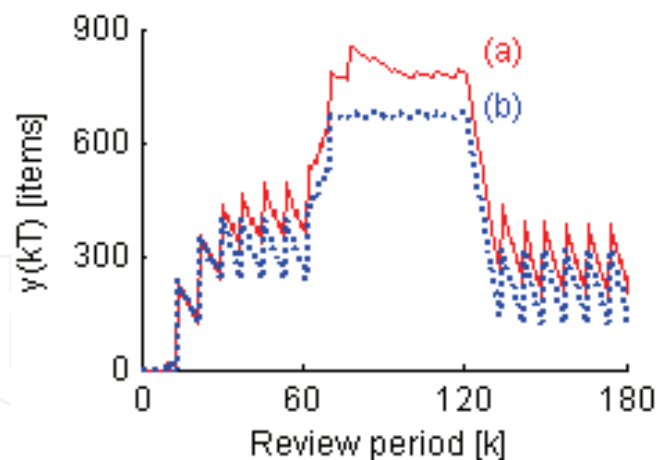
Fig. 7. On-hand stock

Since the mean demand in the stochastic pattern significantly differs from the maximum value, we adjust the target stock according to (42) with $u_{max} > d_{max}$ replaced by d_μ = 25 items. This results in y_d = 375 items (with ε = 1). Although it is no longer guaranteed to satisfy all of the customer demand (the service level decreases to 98%), the holding costs are nearly halved. For the purpose of comparison we also run the tests for a classical order-up-to (OUT) policy (order up to a target value y_{OUT} if the total stock – equal to the on-hand stock plus open orders – drops below y_{OUT}). In order to compare the controllers in a fair way, we apply the same compensation mechanism for the OUT policy as is used for our, LQ-based scheme. We also reduce the value of the target stock level for the OUT policy y_{OUT} setting α = 1 in (42). The controller parameters actually used in the test are grouped in Table 2. Lead-time is assumed to follow the normal distribution D_{norm}(8 days, 2 days). The actual delay in procurring orders is illustrated in Fig. 9.

Policy	Target stock y_d | y_{OUT} [items]	Storage space y_{max} [items]
LQ-based	375	500
OUT	350	475

Table 2. Controller parameters in Scenario 2

The orders generated by both policies are shown in Fig. 10, the received shipments in Fig. 11, and the on-hand stock in Fig. 12. It is evident from the plots that in contrast to the OUT policy (a), our scheme (b) successfully dampens demand fluctuations at the very first stage of supply chain, and it results in a smaller on-hand stock. Performing statistical analysis we obtain 261 items[2] order variance for the OUT policy and 99 items[2] for our controller. Consequently, according to the most popular (Miragliota, 2006) measure of the bullwhip effect proposed by Chen et al. (2000), which is the ratio of variances of orders and demand, we obtain for our scheme 0.44, which corresponds to 2.27 attenuation of demand variations. The ratio of variances for the OUT policy equals 1.16 > 1 which implies amplified variations and the bullwhip effect. This clearly shows the benefits of application of formal control concepts, in particular dynamical optimization and disturbance compensation, in alleviating the adverse effects of uncertainties in supply chain.
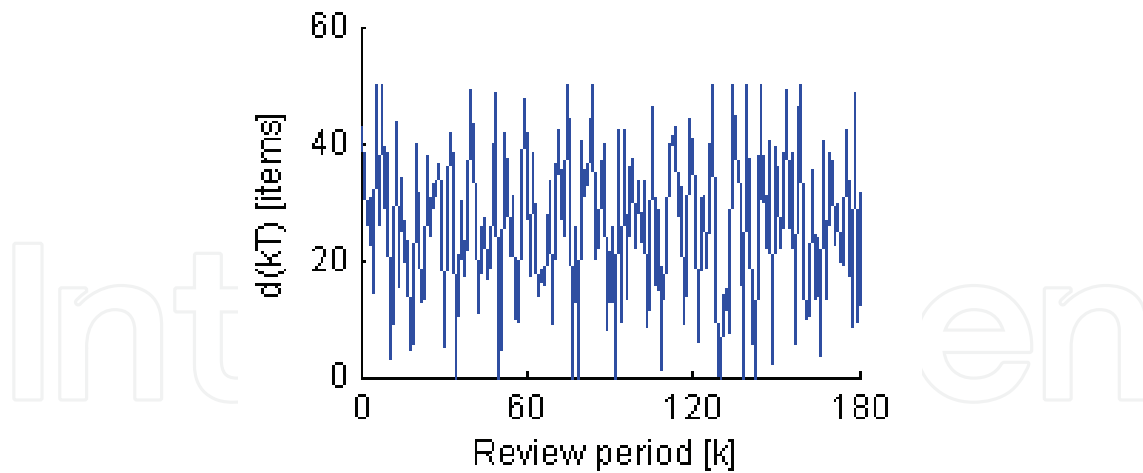
Fig. 8. Market demand following the normal distribution with mean and standard deviation equal to 25 items
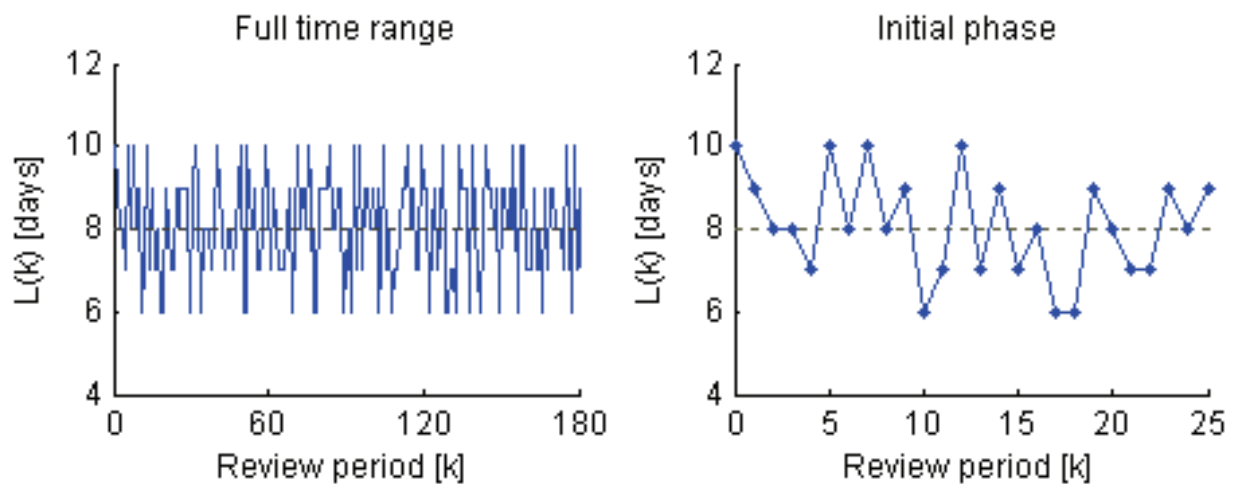

Fig. 9. Lead-time delay following the normal distribution with mean 8 days and standard deviation 2 days
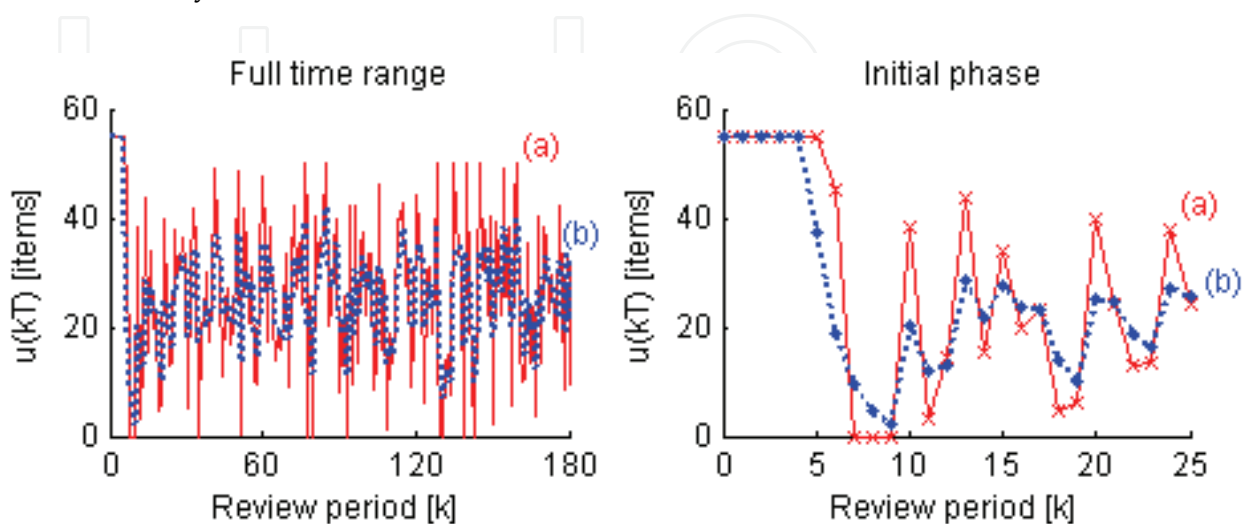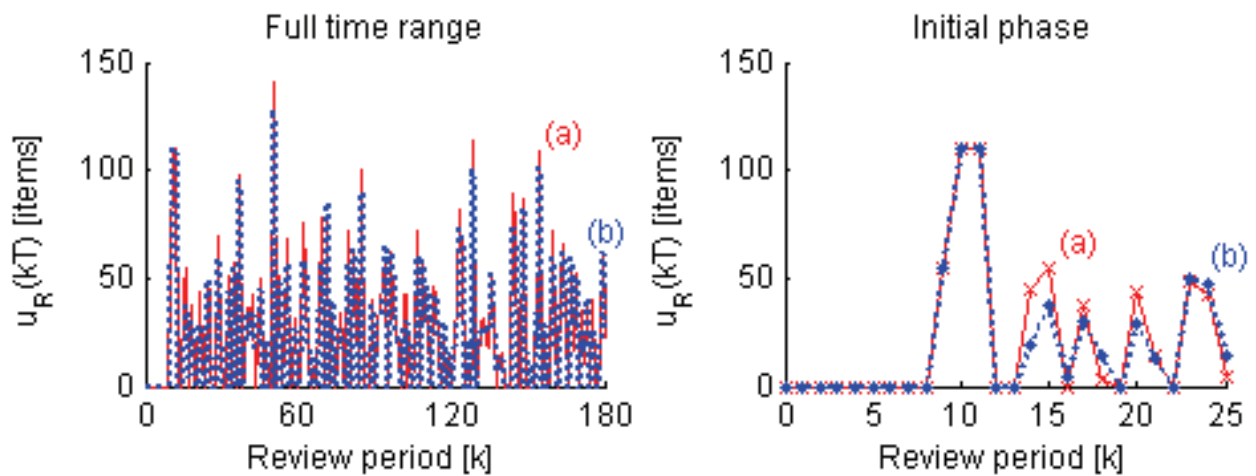

Fig. 10. Generated orders
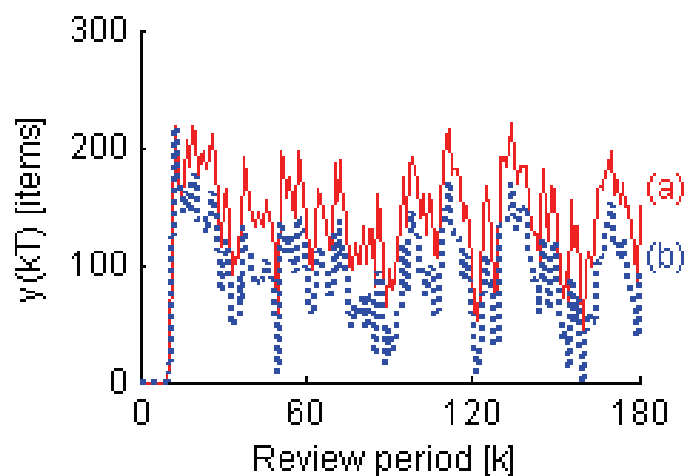
Fig. 11. Received shipments

Fig. 12. On-hand stock

5. Conclusion

In this chapter, we presented a robust supply policy for periodic-review inventory systems. The policy is designed based on sound control-theoretic foundations with the aim of reducing the bullwhip effect. The proposed policy successfully counteracts the increase of order oscillations in the presence of highly variable demand, lead-time fluctuations, and supplier capacity constraints. It guarantees that the incoming shipments will not cause warehouse overflow, implying that emergency storage is never required. Moreover, the presented policy ensures that all of the demand is satisfied from the on-hand stock, thus eliminating the risk of missed service opportunities and necessity for backorders.

6. Acknowledgement

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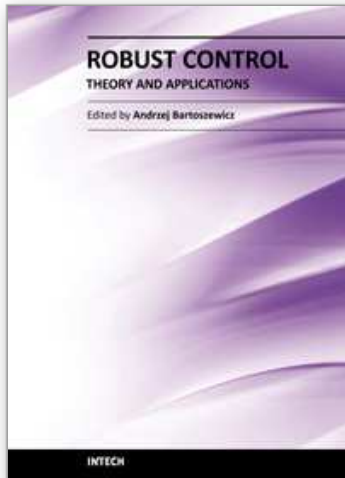
Robust Control, Theory and Applications

Edited by Prof. Andrzej Bartoszewicz

The main objective of this monograph is to present a broad range of well worked out, recent theoretical and application studies in the field of robust control system analysis and design. The contributions presented here include but are not limited to robust PID, H-infinity, sliding mode, fault tolerant, fuzzy and QFT based control systems. They advance the current progress in the field, and motivate and encourage new ideas and solutions in the robust control area.

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