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How to Prove Period-Doubling Bifurcations Existence for Systems of any Dimension - Applications in Electronics and Thermal Field

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1. Introduction

When a stable period-k cycle ($k \geq 1$) loses its stability varying one of its parameters μ from a particular value $\mu = \mu_0$ and when a stable period-$2k$ cycle appears at $\mu = \mu_0$, then we generally have a period-doubling bifurcation.
The variation of this system parameter μ in a larger interval can highlight this phenomenon several times: this is called a cascade of period-doubling bifurcations.
Figure 1 represents bifurcation diagrams and illustrates this type of bifurcations.

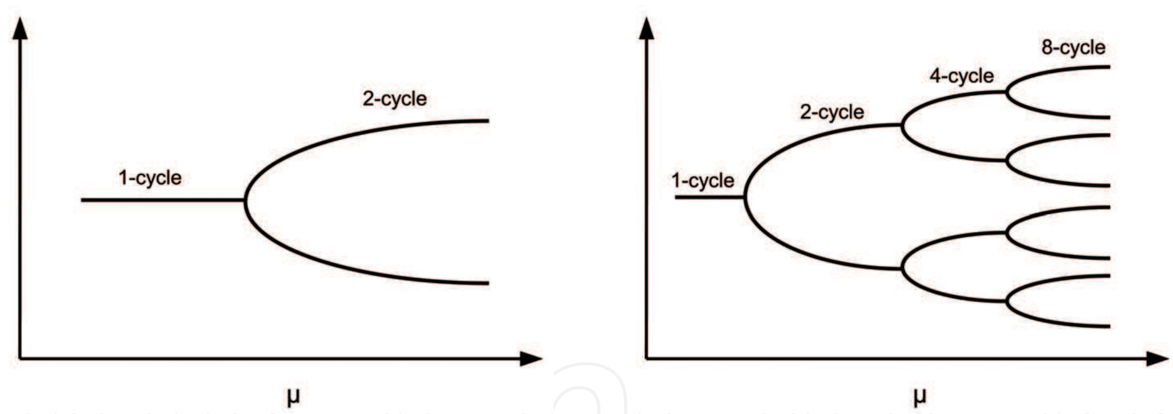


Fig. 1. Illustration of a period-doubling bifurcation with the crossing of a period-1 cycle to a period-2 cycle (on the left) and of a period-doubling bifurcations cascade (on the right).

We can observe this phenomenon in many fields like:

- medicine: onset of a heart attack, epilepsy, neural network... (Aihara et al., 1998), (Smith & Cohen, 1984)
- demography: evolution of animal populations considering for example prey and predator populations (Holmes et al., 1994), (Murray, 1989),
- stock market study (Gleick, 1991),
- sociology: human behaviors study...
- mechanics: system oscillations... (Chung et al., 2003)

In this chapter, we focus on two applications: one in the thermal field with a thermostat with an anticipative resistance (Cébron, 2000) and the second in electronics with a DC/DC converter (Zhusubaliyev & Mosekilde, 2003).
Besides the fact that the bifurcations study allows to know the system behavior, it presents other advantages. Indeed, it can be a useful way to study the system robustness with respect to incertitudes related to the estimation of parameters. It can detect the apparition of chaos. Moreover, bifurcations study can become a practical tool to detect the more influential parameters on the system and so to know what parameter requires an accurate estimates of its value. So, for example, it can save time and money on some costly experiments.
Many authors have already studied period-doubling bifurcations (Baker & Gollub, 1990), (Demazure, 1989), (Guckenheimer & Holmes, 1991), (Kuznetsov, 2004), (Robinson, 1999), (Zhusubaliyev & Mosekilde, 2003)... but most of the time, they focus on one-dimensional systems or they have limited their work to numerical and graphical studies with a bifurcation diagram. So, the theoretical proof of the existence of period-doubling bifurcations for systems of any dimension N, $N \geq 1$, was lacking.
Therefore, here, using some indices given in an exercise in (Robinson, 1999) and following work begun in (Quémard, 2007a), we propose a generalization to any dimension N, $N \geq 1$, of the period-doubling bifurcation theorem. This result is introduced in (Robinson, 1999) for only one-dimensional systems. A proof is also proposed.
Then, we present the studied particular class of hybrid dynamical systems whose two industrial applications (thermostat with an anticipative resistance and DC/DC converter) are chosen to apply this new theorem.
Finally, we conclude this chapter giving some prospects for the future.

2. Period-doubling bifurcation theorem

2.1 Generalization of the period-doubling bifurcation theorem to systems of any dimension

We propose in this paragraph the generalization to any dimension N, $N \geq 1$, of the period-doubling bifurcation theorem. This result was initially found in (Robinson, 1999) for only one-dimensional systems. To do this, we use some indices given in (Robinson, 1999) in an exercise and we complete the work initially realized in (Quémard, 2007a).

Theorem 2.1 (*Generalization of the period-doubling bifurcation theorem*)

Assume that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is a C^r-class ($r \geq 3$) function. We will write $f_\mu(x) = f(x, \mu)$. We assume that f satisfies the following conditions:

1. *The point x_0 is a fixed point of f_μ for the parameter value $\mu = \mu_0$ i.e. $f(x_0, \mu_0) = f_{\mu_0}(x_0) = x_0$.*
2. *The Jacobian matrix of f_{μ_0} at x_0 that is noted $Df_{\mu_0}(x_0)$ has for eigenvalues $\lambda_1(\mu_0) = -1$ and $\lambda_j(\mu_0)$, $j = 2, ..., N$ with $|\lambda_j(\mu_0)| \neq 1$.*

   Let v^1 be a right eigenvector of $Df_{\mu_0}(x_0)$ associated to eigenvalue $\lambda_1(\mu_0)$. We set $V =< v^1 >$. Let v^2,...,v^N be the $N-1$ vectors which form a basis of V', direct sum of the characteristic subspaces (on the right) of $Df_{\mu_0}(x_0)$ different than V. So, we have in particular $V \oplus V' = \mathbb{R}^N$.
3. *Let $x(\mu)$ be the curve of f_μ fixed points near $x(\mu_0)$. We note $\lambda_j(\mu)$, $j = 1, ..., N$, the eigenvalues of the matrix composed of the first partial derivatives of f_μ with respect to*

x, $\partial_x f_\mu(x(\mu))$. We have:

$$\alpha = \frac{d}{d\mu}\lambda_1(\mu)|_{\mu_0} \neq 0.$$

4. *Let:*

$$\beta = \tfrac{1}{3!}w^1 D^3 f_{\mu_0}(x_0)(v^1,v^1,v^1) + \tfrac{1}{4}w^1 D^2 f_{\mu_0}(x_0)(v^1,(Df_{\mu_0}+Id_{\mathbb{R}^N})U)$$
$$+\tfrac{1}{4}w^1 D^2 f_{\mu_0}(x_0)(v^1, D^2 f_{\mu_0}(x_0)(v^1,v^1)) \neq 0,$$

with:

$$U = \begin{pmatrix} 0 \\ -\left[\left(\Pi D f_{\mu_0}(x_0)\,(v^2\ldots v^N)\right)^2 - Id_{V'}\right]^{-1}\Pi(Df_{\mu_0}(x_0)+Id_{\mathbb{R}^N})D^2 f_{\mu_0}(x_0)(v^1,v^1)\end{pmatrix},$$

and Π which corresponds to the projection of $\mathbb{R}^N$ on $V' =< v^2...v^N >$ in parallel to $V =< v^1 >$,

Then, there is a period-doubling bifurcation at (x_0,μ_0). More specificially, there is a differentiable curve of fixed points $x(\mu)$, passing through x_0 at μ_0 such that stability of the fixed point changes at μ_0 (depends on α sign). Moreover, there is also a differentiable curve γ passing through (x_0,μ_0) such that $\gamma\backslash(x_0,\mu_0)$ is the union of period-2 orbits. The curve is tangent to $< v^1 > \times\{\mu_0\}$ at (x_0,μ_0) so γ is the graph of a function of x, $\mu = m(x)$ with $m'(x_0) = 0$ and $m''(x_0) = \frac{-2\beta}{\alpha} \neq 0$. Finally period-2 cycles are on one side of $\mu = \mu_0$ and their stability depends on β sign.

Remark 2.2

When parameter μ is fixed at μ_0, function f_{μ_0} only depends on x. So, we can note $Df_{\mu_0}(x)$ and we call this matrix, jacobian matrix of f_{μ_0}. Nevertheless, if μ is not fixed, we note $\partial_x f_\mu(x)$ and call this matrix, matrix of the first derivatives of f_μ with respect to x.

2.2 Theorem proof

2.2.1 Existence of $x(\mu)$, curve of fixed points of f_μ crossing through x_0 at μ_0

We set $t : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}^N$
$(x,\mu) \longmapsto f(x,\mu) - x.$

Function t is clearly a $\mathcal{C}^r$-class function ($r \geq 3$) on $\mathbb{R}^N \times \mathbb{R}$. Moreover, we have $t(x_0,\mu_0) = 0$ and $\det(\partial_x t(x_0,\mu_0)) \neq 0$ since, by assumption, $Df_{\mu_0}(x_0)$ has not 1 as eigenvalue.
So, we can apply the implicit functions theorem *i.e.* we can solve $t(x,\mu) = 0$ in x near (x_0,μ_0) that gives the existence of $x(\mu)$, fixed points curve of f_μ near μ_0 with, in particular, $x(\mu_0) = x_0$.

2.2.2 Study of the fixed points stability near μ_0

To work on this part, we have to introduce some notations. Let $v^1(\mu)$ be a right eigenvector of $\partial_x f_\mu(x(\mu))$ associated to eigenvalue $\lambda_1(\mu)$ and with $v^1 = v^1(\mu_0)$ associated to $\lambda_1 = \lambda_1(\mu_0) = -1$. Then, we set $V(\mu) =< v^1(\mu) >$ and $V'(\mu)$ the direct sum of the characteristic sub-spaces of $\partial_x f_\mu(x(\mu))$ different than $V(\mu)$.
Let $\mathcal{B}'_\mu = (v^2(\mu),\ldots,v^N(\mu))$ be a basis of $V'(\mu)$. As we have $\mathbb{R}^N = V(\mu)\bigoplus V'(\mu)$, $\mathcal{B}_\mu = (v^1(\mu),\ldots,v^N(\mu))$ is a basis of $\mathbb{R}^N$. Finally, let $\Pi(\mu)$ be the projection on $V'(\mu)$ in parallel to

$V(\mu)$. When all those elements are applied at μ_0, we will just note the name of the element without parenthesis (for example, $V(\mu_0) = V$).
Firstly, we need to compute the matrix of $\partial_x f_\mu(x(\mu))$ in basis $\mathcal{B}_\mu$. To do this, we set $\{w^j\}_{j=1,\ldots,N}$ the dual basis of $\{v^j\}_{j=1,\ldots,N}$ such that $w^j v^i = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise.} \end{cases}$
So, here, $w^1(\mu)$ represents a left eigenvector of $\partial_x f_\mu(x(\mu))$ associated to $\lambda_1(\mu)$ and $w^1(\mu) \in V'(\mu)^\perp$. We have:

$$\partial_x f_{\mu_{\mathcal{B}_\mu}}(x(\mu)) = \mathrm{Mat}_{\mathcal{B}_\mu}(\partial_x f_\mu(x(\mu))) = \begin{pmatrix} w^1(\mu) \\ \vdots \\ w^N(\mu) \end{pmatrix} \partial_x f_\mu(x(\mu)) \left(v^1(\mu) \ldots v^N(\mu) \right)$$

$$= \begin{pmatrix} \lambda_1(\mu) & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & \mathrm{Mat}_{\mathcal{B}'_\mu}(\Pi(\mu) D f_\mu(x(\mu))) & \\ 0 & & & \end{pmatrix}.$$

Then, we call $x_{0_\mathcal{B}}$ the column matrix of vector $x_0 = x(\mu_0)$ written in basis $\mathcal{B} = \mathcal{B}_{\mu_0}$ with $x_{0_\mathcal{B}} = (x^1_{0_\mathcal{B}}, \ldots, x^N_{0_\mathcal{B}})^T = (w^1 \ldots w^N)^T x_0$ and $x_{\mathcal{B}_\mu}(\mu)$ the column matrix of vector $x(\mu)$ written in basis $\mathcal{B}_\mu$ with $x_{\mathcal{B}_\mu}(\mu) = (x^1_{\mathcal{B}_\mu}(\mu), \ldots, x^N_{\mathcal{B}_\mu}(\mu))^T = (w^1(\mu) \ldots w^N(\mu))^T x(\mu)$ where x_0 and $x(\mu)$ are considered relatively to the canonical basis.
Similarly, we note $f_{\mu_{\mathcal{B}_\mu}} = \left(f^1_{\mu_{\mathcal{B}_\mu}} \ldots f^N_{\mu_{\mathcal{B}_\mu}} \right)^T = (w^1(\mu) \ldots w^N(\mu))^T f_\mu$ the column matrix of vector $f_\mu = (f^1_\mu \ldots f^N_\mu)^T$ written in basis $\mathcal{B}_\mu$.
Now, we define the following function Ψ :

$$\begin{aligned} \Psi : \mathbb{R}^N \times \mathbb{R} &\longrightarrow \mathbb{R}^{N-1} \\ (x, \mu) &\longmapsto \Pi(\mu)(f^2_\mu(x) - x), \end{aligned} \tag{1}$$

where $f^2_\mu = f_\mu \circ f_\mu$. We have near $\mu = \mu_0$, $\Psi(x(\mu), \mu) = 0$ since, by assumption, $x(\mu)$ is a fixed point of f_μ near μ_0. Moreover:

$$\frac{\partial \Psi}{\partial x^2_{\mathcal{B}_\mu} \ldots x^N_{\mathcal{B}_\mu}}(x(\mu), \mu) = \Pi(\mu) \left[\left(\frac{\partial f_{\mu_{\mathcal{B}_\mu}}}{\partial x^1_{\mathcal{B}_\mu} \ldots x^N_{\mathcal{B}_\mu}}(x(\mu)) \right)_{\mathcal{B}_\mu} \left(\frac{\partial f_{\mu_{\mathcal{B}_\mu}}}{\partial x^2_{\mathcal{B}_\mu} \ldots x^N_{\mathcal{B}_\mu}}(x(\mu)) \right)_{\mathcal{B}_\mu} - \begin{pmatrix} 0 \ldots 0 \\ I_{N-1} \end{pmatrix}_{\mathcal{B}_\mu} \right]$$

$$= \left(\frac{\partial f^2_{\mu_{\mathcal{B}_\mu}} \ldots f^N_{\mu_{\mathcal{B}_\mu}}}{\partial x^2_{\mathcal{B}_\mu} \ldots x^N_{\mathcal{B}_\mu}}(x(\mu)) \right)^2_{\mathcal{B}'_\mu} - Id_{V'(\mu)}. \tag{2}$$

Since $|\lambda_j(\mu_0)| \neq 1 \ \forall j \geq 2$, we can conclude that, near μ_0, $\frac{\partial \Psi}{\partial x^2_{\mathcal{B}_\mu} \ldots x^N_{\mathcal{B}_\mu}}(x(\mu), \mu)$ is an invertible matrix. Thus, for a fixed μ near μ_0, we can apply the implicit functions theorem near $x(\mu)$ and we can solve $\Psi(x, \mu) = 0$ in terms of $x^1_{\mathcal{B}_\mu}$ *i.e.* there exists a function φ_μ,

defined near $x^1_{\mathcal{B}_\mu}$ such that $\varphi_{\mu_{\mathcal{B}'_\mu}}(x^1_{\mathcal{B}_\mu}) = (x^2_{\mathcal{B}_\mu}\dots x^N_{\mathcal{B}_\mu})^T = (\varphi^2_{\mu_{\mathcal{B}'_\mu}}(x^1_{\mathcal{B}_\mu})\dots\varphi^N_{\mu_{\mathcal{B}'_\mu}}(x^1_{\mathcal{B}_\mu}))^T$ with $\Psi((x^1_{\mathcal{B}_\mu}, \varphi_{\mu_{\mathcal{B}'_\mu}}(x^1_{\mathcal{B}_\mu}))_{\mathcal{B}_\mu}, \mu) = 0$.
Derivating Ψ with respect to $x^1_{\mathcal{B}_\mu}$, we have:

$$\frac{\partial\Psi}{\partial x^1_{\mathcal{B}_\mu}}(x,\mu) + \frac{\partial\Psi}{\partial x^2_{\mathcal{B}_\mu}\dots x^N_{\mathcal{B}_\mu}}(x,\mu)\frac{\partial\varphi_\mu}{\partial x^1_{\mathcal{B}_\mu}}(x^1_{\mathcal{B}_\mu}) = 0 \text{ at } x = (x^1_{\mathcal{B}_\mu}, \varphi_{\mu_{\mathcal{B}'_\mu}}(x^1_{\mathcal{B}_\mu}))_{\mathcal{B}_\mu}.$$

Thus, using the form of the $\partial_x f_\mu(x(\mu))$ matrix in basis $\mathcal{B}_\mu$, we obtain:

$$\frac{\partial\Psi}{\partial x^1_{\mathcal{B}_\mu}}(x(\mu),\mu) = \Pi(\mu)\left[\left(\partial_x f_{\mu_{\mathcal{B}_\mu}}(x(\mu))\frac{\partial f_{\mu_{\mathcal{B}_\mu}}}{\partial x^1_{\mathcal{B}_\mu}}(x(\mu)) - \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\right)_{\mathcal{B}_\mu}\right] = 0_{V'}$$

and since $\frac{\partial\Psi}{\partial x^2_{\mathcal{B}_\mu}\dots x^N_{\mathcal{B}_\mu}}(x(\mu),\mu)$ is invertible near μ_0, we conclude that $\frac{\partial\varphi_\mu}{\partial x^1_{\mathcal{B}_\mu}}(x^1_{\mathcal{B}_\mu}(\mu)) = 0_{V'}$.
Thereafter, in order to obtain 0 as a fixed point near all μ, we introduce, in the basis $\mathcal{B}_\mu$, the following function ϕ_μ near $(0,\mu_0)$:

$$\phi_\mu(y) = \begin{pmatrix} x^1_{\mathcal{B}_\mu}(\mu) + y \\ \varphi_\mu(x^1_{\mathcal{B}_\mu}(\mu)+y)\end{pmatrix}_{\mathcal{B}_\mu} \quad \text{with} \quad \begin{cases}\phi_\mu(0) = x(\mu)\\ \frac{\partial\phi_\mu}{\partial y}(0) = v^1(\mu).\end{cases}$$

Then, we introduce $g(y,\mu) = w^1(\mu)\left[f_\mu(\phi_\mu(y)) - \phi_\mu(y)\right]$. We have $g(0,\mu) = w^1(\mu)\left[f_\mu(x(\mu)) - x(\mu)\right] = 0$ since $x(\mu)$ is a fixed point of f_μ and $\phi_\mu(0) = x(\mu)$.
Since $\frac{\partial g}{\partial y}(0,\mu) = w^1(\mu)[\partial_x f_\mu(x(\mu)) - I_{\mathbb{R}^N}]\frac{\partial\phi_\mu}{\partial y}(0) = \lambda_1(\mu) - 1$, the calculation of $\frac{\partial g^2}{\partial\mu\partial y}(0,\mu)$ permits us to study the stability change of the fixed points along the fixed points curve. We have near μ_0: $\frac{\partial^2 g}{\partial\mu\partial y}(0,\mu) = \frac{d\lambda_1}{d\mu}(\mu) \neq 0$ by the third condition of the theorem.
Conclusion: the sign of $\alpha = \frac{d\lambda_1}{d\mu}(\mu_0)$ determines which side of the plan $\mu = \mu_0$ the fixed point will be attractive or repulsive.

2.2.3 Existence of a differentiable curve γ passing through (x_0,μ_0) such that $\gamma\backslash(x_0,\mu_0)$ is the union of period-2 cycles

We set $h(y,\mu) = w^1(\mu)\left[f^2_\mu(\phi_\mu(y)) - \phi_\mu(y)\right]$.
We have $h(0,\mu) = w^1(\mu)\left[f_\mu\circ f_\mu(x(\mu)) - x(\mu)\right] = 0$. Here, we search y not null, solution of $h(y,\mu) = 0$ which will give us a fixed point $\phi_\mu(y)$ different than $\phi_\mu(0) = x(\mu)$ for f^2_μ. To do this, we introduce:

$$M(y,\mu) = \begin{cases}\frac{h(y,\mu)}{y} \text{ if } y \neq 0\\ \lim\limits_{y\to 0}\frac{h(y,\mu)}{y} = \frac{\partial h}{\partial y}(0,\mu) \text{ if } y = 0.\end{cases}$$

We compute $M(0,\mu) = \frac{\partial h}{\partial y}(0,\mu) = w^1(\mu)\left[(\partial_x f_\mu(x(\mu)))^2 - Id_{\mathbb{R}^N}\right]\frac{\partial\phi_\mu}{\partial y}(0) = \lambda_1^2(\mu) - 1$. So, $M(0,\mu_0) = 0$.

Then, we compute $M_\mu(0,\mu_0) = \frac{\partial M}{\partial \mu}(0,\mu_0)$:

$$M_\mu(0,\mu_0) = 2\frac{d\lambda_1}{d\mu}(\mu_0)\lambda_1(\mu_0) = -2\alpha \neq 0 \text{ (third theorem condition).}$$

Thus, we can apply the implicit functions theorem and we obtain the existence of a differentiable function $\mu = m(y)$ (whose curve is noted γ, $\mu_0 = m(0)$) such that $M(y, m(y)) = 0$ near $(0,\mu_0)$. Then, $\phi_{m(y)}(y)$ is a period-2 fixed point of f_μ.

2.2.4 Calculation of $m'(0)$

Differentiating function M, we have:

$$M_y(0,\mu_0) + M_\mu(0,\mu_0)m'(0) = 0 \Rightarrow m'(0) = -\frac{M_y(0,\mu_0)}{M_\mu(0,\mu_0)}.$$

We know $M_\mu(0,\mu_0) = -2\alpha$ so it remains to compute $M_y(0,\mu_0) = \frac{\partial M}{\partial y}(0,\mu_0) = \lim\limits_{y\to 0} \frac{h(y,\mu_0)}{y^2}$. A limited development of h near $y = 0$ gives:

$$h(y,\mu_0) = h(0,\mu_0) + \frac{\partial h}{\partial y}(0,\mu_0)y + \frac{1}{2!}\frac{\partial^2 h}{\partial y^2}(0,\mu_0)y^2 + \mathcal{O}(y^3),$$

and permits us to conclude $M_y(0,\mu_0) = \frac{1}{2}\frac{\partial^2 h}{\partial y^2}(0,\mu_0)$. Moreover, derivating twice function h with respect to y, we obtain at $y = 0$, $\mu = \mu_0$:

$$\begin{aligned}\frac{\partial^2 h}{\partial y^2}(0,\mu_0) &= w^1 \frac{\partial}{\partial y}\left[Df_{\mu_0}(f_{\mu_0}(\phi_{\mu_0}(y)))\right]|_{y=0} Df_{\mu_0}(x_0)v^1 \\ &+ w^1 Df_{\mu_0}(x_0)\frac{\partial}{\partial y}\left[Df_{\mu_0}(\phi_{\mu_0}(y))\right]|_{y=0}v^1 + w^1(Df_{\mu_0}(x_0))^2\frac{\partial^2\phi_{\mu_0}}{\partial y^2}(0) - w^1\frac{\partial^2\phi_{\mu_0}}{\partial y^2}(0).\end{aligned}$$

Since $w^1(Df_{\mu_0}(x_0))^2\frac{\partial^2\phi_{\mu_0}}{\partial y^2}(0) - w^1\frac{\partial^2\phi_{\mu_0}}{\partial y^2}(0) = 0$ (because $w^1 Df_{\mu_0}(x_0) = -w^1$) and:

$$\begin{aligned}&w^1 \frac{\partial}{\partial y}\left[Df_{\mu_0}(f_{\mu_0}(\phi_{\mu_0}(y)))\right]|_{y=0} Df_{\mu_0}(x_0)v^1 + w^1 Df_{\mu_0}(x_0)\frac{\partial}{\partial y}\left[Df_{\mu_0}(\phi_{\mu_0}(y))\right]|_{y=0}v^1 \\ &= -w^1 D^2 f_{\mu_0}(x_0)(Df_{\mu_0}(x_0)v^1, v^1) - w^1 D^2 f_{\mu_0}(x_0)(v^1, v^1) = 0,\end{aligned}$$

we obtain $\frac{\partial^2 h}{\partial y^2}(0,\mu_0) = 0$ and we finally conclude that $M_y(0,\mu_0) = 0$, so $m'(0) = 0$. Therefore, γ is tangent to $< v^1 > \times\{\mu_0\}$ at (x_0,μ_0).

2.2.5 Calculation of $m''(0)$

As $\mu = m(y) = m(0) + m'(0)y + \frac{1}{2}m''(0)y^2 + \mathcal{O}(y^3) = \mu_0 + \frac{1}{2}m''(0)y^2 + \mathcal{O}(y^3)$, we know that a sufficient condition to have γ on only one side of $\mu = \mu_0$ is that $m''(0) \neq 0$.
To compute $m''(0)$, by differentiating twice function M, we obtain:

$$M_{yy}(y,m(y)) + 2M_{\mu y}(y,m(y))m'(y) + M_{\mu\mu}(y,m(y))(m'(y))^2 + M_\mu(y,m(y))m''(y) = 0.$$

Evaluating this equation at $y = 0$ and using $m'(0) = 0$, we have:

$$m''(0) = -\frac{M_{yy}(0, \mu_0)}{M_\mu(0, \mu_0)}.$$

As we know $M_\mu(0, \mu_0) = -2\alpha$, it remains to compute $M_{yy}(0, \mu_0) = \frac{1}{3}\frac{\partial^3 h}{\partial y^3}(0, \mu_0)$ (we have used a limited development of h to order three near $y = 0$). We already know:

$$\begin{cases} \frac{\partial h}{\partial y}(y, \mu_0) = w^1 R(y, \mu_0)\frac{\partial \phi_{\mu_0}}{\partial y}(y) \\ \frac{\partial^2 h}{\partial y^2}(y, \mu_0) = w^1 \frac{\partial R}{\partial y}(y, \mu_0)\frac{\partial \phi_{\mu_0}}{\partial y}(y) + w^1 R(y, \mu_0)\frac{\partial^2 \phi_{\mu_0}}{\partial y^2}(y), \end{cases}$$

with $R(y, \mu_0) = Df_{\mu_0}(f_{\mu_0}(\phi_{\mu_0}(y)))Df_{\mu_0}(\phi_{\mu_0}(y)) - Id_{\mathbb{R}^N}$.
Then, derivating $\frac{\partial^2 h}{\partial y^2}(y, \mu_0)$ with respect to y and applying it at $y = 0$, we obtain:

$$\frac{\partial^3 h}{\partial y^3}(0, \mu_0) = w^1 \frac{\partial^2 R}{\partial y^2}(0, \mu_0)\frac{\partial \phi_{\mu_0}}{\partial y}(0) + 2w^1 \frac{\partial R}{\partial y}(0, \mu_0)\frac{\partial^2 \phi_{\mu_0}}{\partial y^2}(0) + w^1 R(0, \mu_0)\frac{\partial^3 \phi_{\mu_0}}{\partial y^3}(0). \quad (3)$$

In order to alleviate notations, we study each term of $\frac{\partial^3 h}{\partial y^3}(0, \mu_0)$ separately.
• For the first element, we have:

$$w^1 \frac{\partial^2 R}{\partial y^2}(0, \mu_0)v^1 = w^1 \frac{\partial}{\partial y}\left[D^2 f_{\mu_0}(f_{\mu_0}(\phi_{\mu_0}(y)))\left(Df_{\mu_0}(\phi_{\mu_0}(y))\frac{\partial \phi_{\mu_0}}{\partial y}(y), Df_{\mu_0}(\phi_{\mu_0}(y))v^1\right)\right]$$
$$+w^1 \frac{\partial}{\partial y}\left[Df_{\mu_0}(f_{\mu_0}(\phi_{\mu_0}(y)))D^2 f_{\mu_0}(\phi_{\mu_0}(y))\left(\frac{\partial \phi_{\mu_0}}{\partial y}(y), v^1\right)\right],$$

with the convention for all function f and all vectors u_1 and $u_2 \in \mathbb{R}^N$:
$D^2 f(x)(u_1, u_2) = D^2 f(x)(u_2, u_1) = \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}(x)u_1^i u_2^j$.
So, after some calculations and simplifications, we obtain:

$$w^1 \frac{\partial^2 R}{\partial y^2}(0, \mu_0)v^1 = -2w^1 D^3 f_{\mu_0}(x_0)\left(v^1, v^1, v^1\right) - 3w^1 D^2 f_{\mu_0}(x_0)\left(v^1, D^2 f_{\mu_0}(x_0)\left(v^1, v^1\right)\right)$$
$$-w^1 D^2 f_{\mu_0}(x_0)\left(v^1, (Df_{\mu_0}(x_0) + Id_{\mathbb{R}^N})\frac{\partial^2 \phi_{\mu_0}}{\partial y^2}(0)\right).$$

• Then, we study the second term of (3) and we finally have:

$$w^1 \frac{\partial R}{\partial y}(0, \mu_0)\frac{\partial^2 \phi_{\mu_0}}{\partial y^2}(0) = -w^1 D^2 f_{\mu_0}(x_0)\left(v^1, Df_{\mu_0}(x_0)\frac{\partial^2 \phi_{\mu_0}}{\partial y^2}(0)\right) - w^1 D^2 f_{\mu_0}(x_0)\left(v^1, \frac{\partial^2 \phi_{\mu_0}}{\partial y^2}(0)\right).$$

• Finally, the last term of (3) gives:

$$w^1 R(0, \mu_0)\frac{\partial^3 \phi_{\mu_0}}{\partial y^3}(0) = w^1((Df_{\mu_0}(x_0))^2 - 1)\frac{\partial^3 \phi_{\mu_0}}{\partial y^3}(0) = w^1 \frac{\partial^3 \phi_{\mu_0}}{\partial y^3}(0) - w^1 \frac{\partial^3 \phi_{\mu_0}}{\partial y^3}(0) = 0,$$

since $w^1 \left(Df_{\mu_0}(x_0)\right)^2 = w^1$.

From this, we can conclude:

$$\frac{\partial^3 h}{\partial y^3}(0,\mu_0) = -2w^1 D^3 f_{\mu_0}(x_0)\left(v^1,v^1,v^1\right) - 3w^1 D^2 f_{\mu_0}(x_0)\left(v^1, D^2 f_{\mu_0}(x_0)\left(v^1,v^1\right)\right)$$
$$-3w^1 D^2 f_{\mu_0}(x_0)\left(v^1, \left(Df_{\mu_0}(x_0) + Id_{\mathbb{R}^N}\right)\frac{\partial^2 \phi_{\mu_0}}{\partial y^2}(0)\right).$$

Now, it remains to find a relation between $\frac{\partial^2 \phi_{\mu_0}}{\partial y^2}(0)$ and $f_{\mu_0}(x_0)$ in order that $M_{yy}(0,\mu_0)$ be only a function of $f_{\mu_0}(x_0)$. To do this, we differentiate again function Ψ defined by (1) with respect to $x^1_{\mathcal{B}_\mu}$. We obtain:

$$\frac{\partial^2 \Psi}{\partial x^{1^2}_{\mathcal{B}_\mu}}(x,\mu) + 2\frac{\partial^2 \Psi}{\partial x^1_{\mathcal{B}_\mu}\partial x^2_{\mathcal{B}_\mu}\ldots x^N_{\mathcal{B}_\mu}}(x,\mu)\frac{\partial \varphi_\mu}{\partial x^1_{\mathcal{B}_\mu}}(x^1_{\mathcal{B}_\mu}) + \frac{\partial^2 \Psi}{(\partial x^2_{\mathcal{B}_\mu}\ldots x^N_{\mathcal{B}_\mu})^2}(x,\mu)\left(\frac{\partial \varphi_\mu}{\partial x^1_{\mathcal{B}_\mu}}(x^1_{\mathcal{B}_\mu})\right)^2$$
$$+\frac{\partial \Psi}{\partial x^2_{\mathcal{B}_\mu}\ldots x^N_{\mathcal{B}_\mu}}(x,\mu)\frac{\partial^2 \varphi_\mu}{\partial x^{1^2}_{\mathcal{B}_\mu}}(x^1_{\mathcal{B}_\mu}) = 0, \tag{4}$$

at x which satisfies $\Psi(x,\mu) = 0$ *i.e.* $x = (x^1_{\mathcal{B}_\mu}, \varphi_{\mu_{\mathcal{B}'_\mu}}(x^1_{\mathcal{B}_\mu}))_{\mathcal{B}_\mu}$.
Applied at (x_0,μ_0), relation (4) becomes:

$$\frac{\partial^2 \Psi}{\partial x^{1^2}_{\mathcal{B}}}(x_0,\mu_0) + \frac{\partial \Psi}{\partial x^2_{\mathcal{B}}\ldots x^N_{\mathcal{B}}}(x_0,\mu_0)\frac{\partial^2 \varphi_{\mu_0}}{\partial x^{1^2}_{\mathcal{B}}}(x^1_{0_{\mathcal{B}}}) = 0.$$

As $\frac{\partial \Psi}{\partial x^2_{\mathcal{B}}\ldots x^N_{\mathcal{B}}}(x_0,\mu_0)$ given by (2) is an invertible matrix, it remains to compute $\frac{\partial^2 \Psi}{\partial x^{1^2}_{\mathcal{B}}}(x_0,\mu_0)$. We have:

$$\frac{\partial^2 \Psi}{\partial x^{1^2}_{\mathcal{B}}}(x_0,\mu_0) = \Pi\left[D^2 f_{\mu_0}(x_0)\left(\frac{\partial f_{\mu_0}}{\partial x^1_{\mathcal{B}}}(x_0), \frac{\partial f_{\mu_0}}{\partial x^1_{\mathcal{B}}}(x_0)\right) + Df_{\mu_0}(x_0)\frac{\partial^2 f_{\mu_0}}{\partial x^{1^2}_{\mathcal{B}}}(x_0)\right]$$
$$= \Pi\left[\left(Df_{\mu_0}(x_0) + Id_{\mathbb{R}^N}\right) D^2 f_{\mu_0}(x_0)(v^1,v^1)\right]$$

since by definition, $\frac{\partial^2 f_{\mu_0}}{\partial x^{1^2}_{\mathcal{B}}}(x_0) = D^2 f_{\mu_0}(x_0)(v^1,v^1)$.
Finally, all these calculations lead to write:

$$\frac{\partial^2 \varphi_{\mu_0}}{\partial x^{1^2}_{\mathcal{B}}}(x^1_{0_{\mathcal{B}}}) = -\left[\left(\Pi Df_{\mu_0}(x_0)(v^2\ldots v^N)\right)^2 - Id_{V'}\right]^{-1}\Pi\left(Df_{\mu_0}(x_0) + Id_{\mathbb{R}^N}\right) D^2 f_{\mu_0}(x_0)(v^1,v^1).$$

Thus, we obtain $\frac{\partial^2 \phi_{\mu_0}}{\partial y^2}(0)$ as a function of $f_{\mu_0}(x_0)$ since $\frac{\partial^2 \phi_{\mu_0}}{\partial y^2}(0) = \begin{pmatrix} 0 \\ \frac{\partial^2 \varphi_{\mu_0}}{\partial x^{1^2}_{\mathcal{B}}}(x^1_{0_{\mathcal{B}}}) \end{pmatrix}_{\mathcal{B}}$.
We can conclude that $\frac{\partial^3 h}{\partial y^3}(0,\mu_0) = 3M_{yy}(0,\mu_0) = -12\beta \neq 0$ (fourth assumption of the theorem) so $m''(0) = -\frac{-4\beta}{-2\alpha} = -2\frac{\beta}{\alpha} \neq 0$. This confirms that γ, curve of period-2 fixed points, is on one side of $\mu = \mu_0$.

2.2.6 Stability study of period-2 cycles

To study the stability of the period-2 cycles, we study function $\frac{\partial h}{\partial y}(y, m(y))$ near $y = 0$ ($\mu = \mu_0$). To do this, we give the limited development of $\frac{\partial h}{\partial y}(y, m(y))$ near 0:

$$\frac{\partial h}{\partial y}(y, m(y)) = \frac{\partial h}{\partial y}(0, \mu_0) + \frac{\partial^2 h}{\partial y^2}(0, \mu_0)y + \frac{\partial^2 h}{\partial \mu \partial y}(0, \mu_0)(\mu - \mu_0)$$
$$+ \frac{1}{2!}\frac{\partial^3 h}{\partial y^3}(0, \mu_0)y^2 + \mathcal{O}(y^3) + \mathcal{O}((\mu - \mu_0)^2) + \mathcal{O}(y^2(\mu - \mu_0)).$$

We know $\frac{\partial h}{\partial y}(0, \mu_0) = 0$, $\frac{\partial^2 h}{\partial y^2}(0, \mu_0) = 0$, $\frac{\partial^2 h}{\partial \mu \partial y}(0, \mu_0) = -2\alpha$ and $\frac{\partial^3 h}{\partial y^3}(0, \mu_0) = 3M_{yy}(0, \mu_0) = -12\beta$.
Moreover, we have:

$$\frac{\partial^2 h}{\partial \mu \partial y}(0, \mu_0)(m(y) - m(0)) = M_\mu(0, \mu_0)(\frac{1}{2}m''(0)y^2 + \mathcal{O}(y^3)) = 2\beta y^2 + \mathcal{O}(y^3)).$$

Finally, we find $\frac{\partial h}{\partial y}(y, m(y)) = -4\beta y^2 + \mathcal{O}(y^3))$. This confirms that the stability of period-2 cycles depends ont the β sign. This completes the proof of the theorem.

3. Presentation of the studied particular class of hybrid dynamical systems

3.1 General presentation

We consider the following hybrid dynamical system (h.d.s.) of order N, $N \geq 1$:

$$\begin{cases} \dot{X}(t) = A\big(q(\xi(t))\big)X(t) + V\big(q(\xi(t))\big), \\ \xi(t) = cst - WX(t), \end{cases} \tag{5}$$

where A is a stable square matrix of order N, V and X are column matrices of order N and W is a row matrix of order N, all these matrices having real entries. Moreover, cst is a real constant. We suppose that X and so ξ are continuous.
In this model, the discrete variable is q which can take two values u_1, u_2 according to ξ which follows a hysteresis phenomenon described by figure 2.

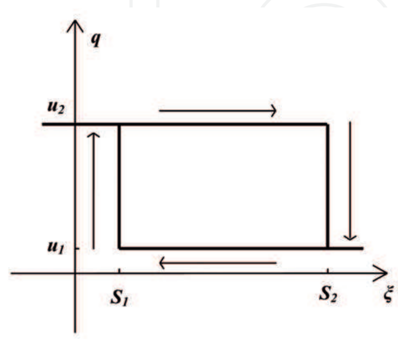


Fig. 2. Hysteresis phenomenon followed by discrete variable q.

If ξ reaches its lower threshold S_1 by decreasing value then q changes its value from u_1 to u_2. Similarly, if ξ reaches its upper threshold S_2 by increasing value then q changes its value from

u_2 to u_1. In those conditions, multifunction $q(\xi)$ is explicitly given by:

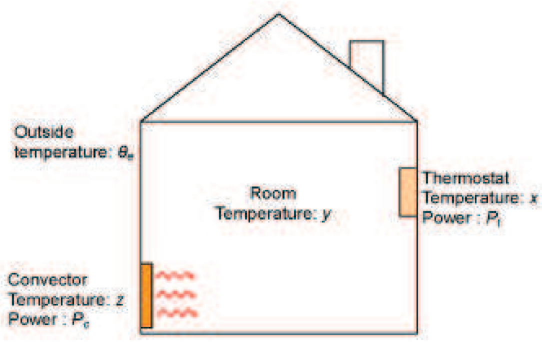
$$\begin{cases} q(\xi(t)) = u_1 \text{ if } \xi(t^-) = S_2 \text{ and } q(\xi(t^-)) = u_2 \\ q(\xi(t)) = u_2 \text{ if } \xi(t^-) = S_1 \text{ and } q(\xi(t^-)) = u_1 \\ q(\xi(t)) = q(\xi(t^-)) \text{ otherwise.} \end{cases} \tag{6}$$

In the first two cases, t is called switching time and so, S_1 and S_2 are respectively called lower and upper switching thresholds.
A lot of applications of many fields and of all dimensions belong to this h.d.s. class. In this paper, we will study two of these applications:

- the first of dimension three: a thermostat with an anticipative resistance,
- the second of dimension four: a DC/DC converter.

3.2 Application 1: thermostat with an anticipative resistance

The first considered application is the one of a thermostat wtih an anticipative resistance which controls a convector located in the same room (Cébron, 2000). The thermal processus is given by figure 3 (on the left). We note x, y and z (in K) the temperatures respectively of the



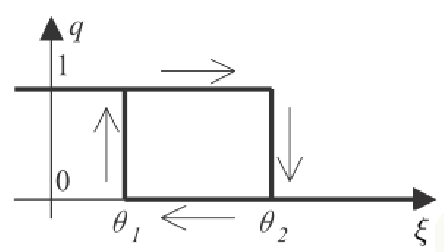


Fig. 3. thermal processus (on the left) and hysteresis phenomenon (on the right).

thermostat, of the room and of the convector. The functioning principle of such a thermostat is the following: powers of the thermostat P_t and of the convector P_c (in W) are active when $q = 1$ and inactive when $q = 0$. If initially $q = 1$, as P_t is active, the desired temperature is reached firstly by the thermostat temperature before the room temperature that makes q changes its value from 1 to 0. Thus, the introduction of the anticipative resistance reduces the amplitude of y. This presents an interest of energy saving.
The Fourier law and a power assessment (Saccadura, 1998) give the following differential system of dimension three with the same form than (5):

$$\begin{cases} \dot{X}(t) = AX(t) + q(\xi(t))B + C, \\ \xi(t) = LX(t), \end{cases}$$

where:

$$A = \begin{pmatrix} -a & a & 0 \\ e & -(b+d+e) & b \\ 0 & c & -c \end{pmatrix}, B = \begin{pmatrix} p_t \\ 0 \\ p_c \end{pmatrix}, C = \begin{pmatrix} 0 \\ d.\theta_e \\ 0 \end{pmatrix}, L = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T,$$

and

$$a = \frac{1}{m_t C_t R_t}, \; b = \frac{1}{m_p C_p R_c}, \; c = \frac{1}{m_c C_c R_c}, \; d = \frac{1}{m_p C_p R_m}, \; e = \frac{1}{m_p C_p R_t}, \; p_t = \frac{P_t}{m_t C_t}, \; p_c = \frac{P_c}{m_c C_c}.$$

Coefficients R_t, R_c, R_m (in K.W^{-1}) are thermal resistances, C_t, C_p, C_c (in J.kg^{-1}.K^{-1}) are heat capacities and m_t, m_p, m_c (in kg) are masses according to indices t, p, c and m which respectively represent the thermostat, the convector, the room and the house wall. Moreover, θ_e (in K) corresponds to the outside temperature.
Here, the discrete variable q follows the hysteresis phenomenon described in figure 3 where $u_1 = 0$, $u_2 = 1$, $S_1 = \theta_1$ and $S_2 = \theta_2$.

3.3 Application 2: DC/DC converter

The second studied application is the one of a DC/DC converter (Zhusubaliyev & Mosekilde, 2003), (Lim & Hamill, 1999). The electrical equivalent circuit is given by figure 4. This circuit

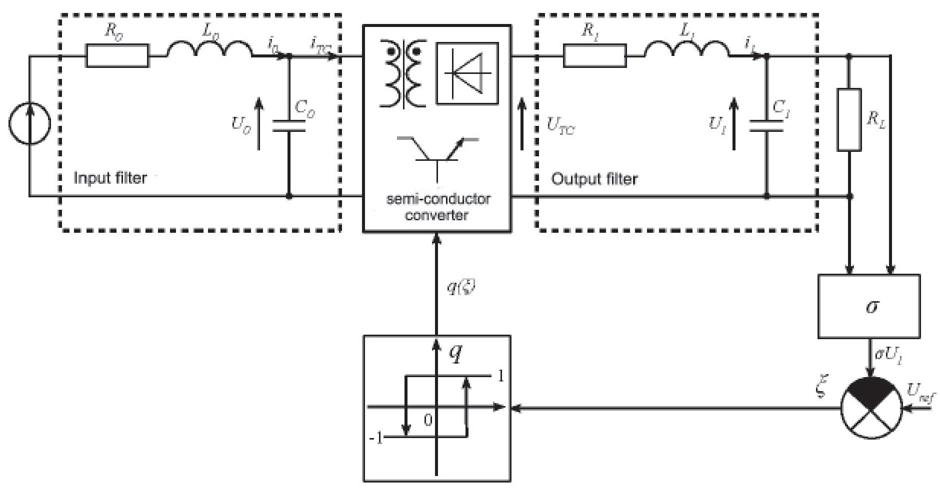


Fig. 4. Electrical equivalent circuit of a DC/DC converter.

includes a converter DC voltage generator and two filters LC (input and output). The output voltage of the circuit, given by σU_1, $0 < \sigma < 1$, with σ the sensor gain, will be compared to the reference signal U_{ref} (in V). The difference of these two quantities, noted $\xi = U_{\text{ref}} - \sigma U_1$, called deviation signal, is applied to the relay element with hysteresis in order to form square pulses to control the converter switching elements. Here, $u_1 = -1$, $u_2 = 1$, $S_1 = -\chi_0$, $S_2 = \chi_0$. Thus, electronical laws give the following differential system of order four which takes the form of (5):

$$\begin{cases} \dot{X} = A\big(q(\xi(t))\big)X(t) + V, \\ \xi(t) = U_{\text{ref}} - UX(t), \end{cases}$$

where:

$$A(q) = \begin{pmatrix} -\eta & -\eta & 0 & 0 \\ \gamma & 0 & -\frac{\gamma}{2}(1+q) & 0 \\ 0 & \frac{\mu}{2}(1+q) & -\nu & -\mu \\ 0 & 0 & \frac{\lambda}{\alpha} & -\frac{\lambda}{\beta} \end{pmatrix}, \; V = \begin{pmatrix} \eta\Omega \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; U = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sigma E^* \end{pmatrix}^T,$$

with $\eta = \frac{R_0}{L_0}$, $\Omega = \frac{E_0}{E^*}$, $\gamma = \frac{1}{C_0 R_0}$, $\mu = \frac{R_0}{L_1}$, $\nu = \frac{R_1}{L_1}$, $\lambda = \frac{1}{C_1 R^*}$, $\beta = \frac{R_L}{R^*}$, $\alpha = \frac{R_0}{R^*}$ where R^* is a normalization resistance taken equal to 1. Moreover, $x_1 = \frac{R_0 i_0}{E^*}$, $x_2 = \frac{U_0}{E^*}$, $x_3 = \frac{R_0 i_1}{E^*}$, $x_4 = \frac{U_1}{E^*}$, where E^* is a voltage which permits us to work with dimensionless variables and is equal to 1 here.

Coefficients L_0 and L_1 (in H) are the inductances, C_0 and C_1 (in F) are capacities, R_0, R_1 and R_L (in Ω) are losses in the inductances, R_c is th load resistor. Moreover, i_0 and i_1 (in A) are currents in the inductance coils. Values U_0 and U_1 (in V) are voltages on the condensers of capacities C_0 and C_1 respectively according to indices 0 and 1. These indices respectively represent the elements of the input filter and of the output filter. Finally, E_0 is the input voltage.

3.4 Determination of period-k cycles equations ($k \geq 1$)

In this paragraph, we remain the results established in (Quémard et al., 2005), (Quémard et al., 2006), (Quémard, 2007b).

From general system (5), as we have $\xi(t_n) = S_1$ or S_2 (t_n is the n-th switching time if it exists), the trajectory follows a cycle by construction. So, it is rather natural to study the limit cycles existence for the general system. Moreover, the existence of such cycles for those non linear systems has already been proved in (Zhusubaliyev & Mosekilde, 2003), (Girard, 2003) for example.

Let t_0 be an initial given time and $t_1 < t_2 < ... < t_n < t_{n+1} < ...$ the increasing suite of successive switching times on $[t_0, +\infty[$, necessarily distincts because the definition of $q(\xi(t))$ implies $\xi(t_n) \neq \xi(t_{n-1})$.

To simplify notations, we set $q_n = q(\xi(t_n))$ and we have $q(\xi(t)) = q_n$ on $[t_n, t_{n+1}[$. Similarly, we set $\xi_n = \xi(t_n)$, $A_n = A(q_n)$, $V_n = V(q_n)$. A classical integration of (5) gives on interval $[t_n, t_{n+1}[$:

$$X(t) = \mathrm{e}^{(t-t_n)A_n}\Gamma_n - A_n^{-1}V_n, \tag{7}$$

where $\Gamma_n \in \mathbb{R}^N$ correspond to the integration constants, functions of n.

Thus, introducing notation $\sigma_n = t_n - t_{n-1} > 0$, $n \geq 1$ and considering the continuity assumption at t_n, we obtain:

$$\begin{cases} \Gamma_n = \mathrm{e}^{\sigma_n A_{n-1}}\Gamma_{n-1} + A_n^{-1}V_n - A_{n-1}^{-1}V_{n-1}, \forall n \geq 1 \\ \Gamma_0 = X(t_0) + A_0^{-1}V_0. \end{cases} \tag{8}$$

Then, we set $\forall n \geq 1$, $\xi_n = f(S_1, S_2, q_{n-1}, q_n)$ (f function from $\mathbb{R}^4$ to $\mathbb{R}$ in order to have $\xi_n = S_1$ or S_2 according to the hysteresis variable q). By definition, we also can write $\forall n \geq 1$, $\xi_n = cst - WX(t_n) = cst - W(\Gamma_n - A_n^{-1}V_n)$. So, combining those two expressions for ξ_n, we finally obtain:

$$\forall n \geq 1, cst - W(\Gamma_n - A_n^{-1}V_n) - f(S_1, S_2, q_{n-1}, q_n) = 0. \tag{9}$$

Resolution of system (5), (6) with unknowns $X(t)$, $(t_n)_{n\in\mathbb{N}}$ is equivalent to the one of system (8), (6) with unknowns $(\Gamma_n)_{n\geq 1}$, $(\sigma_n)_{n\geq 1}$. Nevertheless, it is very difficult to explicitly solve this system with theoretical way (Jaulin et al., 2001), (Zhusubaliyev & Mosekilde, 2003) so we content ourselves with a numerical resolution.

Moreover, such globally non linear systems can admit zero, one or more solutions (Quémard et al., 2006), (Quémard, 2007b), (Quémard, 2009) that implies the existence of period-k cycles ($k \geq 1$). To determine equations of those cycles, we introduce for all suite $(U_n)_{n\in\mathbb{N}}$ the

following notation $U_n^i = U_{2kn+i}$, $\quad n \geq 0$, for $i = 1, ..., 2k$ with $k \in \mathbb{N}^*$ which corresponds to the cycle period.
Thus, the suite of successive switching times is noted $(\sigma_n^1, \sigma_n^2, \ldots, \sigma_n^{2k-1}, \sigma_n^{2k})_{n\in\mathbb{N}}$ and the one of the successive integration constants is noted $(\Gamma_n^1, \Gamma_n^2, \ldots, \Gamma_n^{2k-1}, \Gamma_n^{2k})_{n\in\mathbb{N}}$. We set $R_n = (\sigma_n^1, \Gamma_n^1, \ldots, \sigma_n^{2k}, \Gamma_n^{2k})$. We suppose that R_n has a limit $R = (\sigma^1, \Gamma^1, \ldots, \sigma^{2k}, \Gamma^{2k})$. In those conditions, at R, system of equations (8), (9) is equivalent to system $H(R, R) = 0$, $\forall n \geq 0$ where $H = (H_1, \ldots, H_{4k})^T$ is a function defined for $i = 1, ..., 2k$ by:

$$\begin{cases} H_i(R,R) = \Gamma^i - \mathrm{e}^{\sigma^i A_{i-1}}\Gamma^{i-1} - A_i^{-1}V_i + A_{i-1}^{-1}V_{i-1} = 0, \\ H_{2k+i}(R,R) = cst - W(\Gamma^i - A_i^{-1}V_i) - f(S_1, S_2, q_{i-1}, q_i) = 0, \end{cases} \tag{10}$$

with index $i = 0$ if i is even and $i = 1$ if i is odd. Moreover, $\Gamma_n^0 = \Gamma_n^{2k}$.
From each $2k$ first equations H_i, $i = 1, ..., 2k$ of (10) and using the first remaining $2k - 1$ equations, we can determine by recurrence an expression of Γ^i, $i = 1, ..., 2k$ which becomes a function of σ^i, $i = 1, ..., 2k$. Then, replacing Γ^i, $i = 1, ..., 2k$ with this expression in the last $2k$ equations H_{2k+i}, $i = 1, ..., 2k$, of system (10), we can obtain, for $i = 1, ..., 2k$, the following system of $2k$ equations F_i for $2k$ unknowns σ^i, $i = 1, ..., 2k$:

$$\begin{aligned} &F_i = -W((I_N - \textstyle\prod_{m=1}^{2k} D_{(i-m+1)\mathrm{mod}(2k)})^{-1}(I_N + \sum_{j=1}^{2k-1}(-1)^j(\prod_{l=1}^{2k-j} D_{(i-l+1)\mathrm{mod}(2k)})) \\ &(A_i^{-1}V_i - A_{i-1}^{-1}V_{i-1}) - A_i^{-1}V_i) - f(S_1, S_2, q_{i-1}, q_i) + cst = 0. \end{aligned} \tag{11}$$

with $D_m = \mathrm{e}^{\sigma^m A_{m-1}}$, $m = 1, \ldots, 2k$ and setting $D_0 = D_{2k}$.
This system represents the period-k cycle equations ($k \geq 1$) and it will be solved numerically for the two considered applications either with the formal calculus (*Maple*) and the interval analysis (*Proj2D*) or with a classical Newton algorithm (*Matlab*). If we apply system (11) to the application of the thermostat, we have, for example, for a period-2 cycle and after setting $K_1 = F_1 - F_4$, $K_2 = F_3 - F_2$, $K_3 = F_2 - F_4$, $K_4 = F_1$, the following equivalent system:

$$\begin{cases} K_1(\sigma^1,\sigma^2,\sigma^3,\sigma^4) = L(I_N - \mathrm{e}^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}(I_N - \mathrm{e}^{\sigma^1 A}) \\ (I_N - \mathrm{e}^{\sigma^4 A} + \mathrm{e}^{(\sigma^3+\sigma^4)A} - \mathrm{e}^{(\sigma^2+\sigma^3+\sigma^4)A})A^{-1}B + \Delta\theta = 0 \\ K_2(\sigma^1,\sigma^2,\sigma^3,\sigma^4) = L(I_N - \mathrm{e}^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}(I_N - \mathrm{e}^{\sigma^3 A}) \\ (I_N - \mathrm{e}^{\sigma^2 A} + \mathrm{e}^{(\sigma^1+\sigma^2)A} - \mathrm{e}^{(\sigma^1+\sigma^2+\sigma^4)A})A^{-1}B + \Delta\theta = 0 \\ K_3(\sigma^1,\sigma^2,\sigma^3,\sigma^4) = L(I_N - \mathrm{e}^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}(\mathrm{e}^{\sigma^2 A}(I_N - \mathrm{e}^{\sigma^1 A} + \mathrm{e}^{(\sigma^1+\sigma^4)A}) \\ -\mathrm{e}^{\sigma^4 A}(I_N - \mathrm{e}^{\sigma^3 A} + \mathrm{e}^{(\sigma^2+\sigma^3)A}))A^{-1}B = 0 \\ K_4(\sigma^1,\sigma^2,\sigma^3,\sigma^4) = \Delta q_1 L(I_N - \mathrm{e}^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}(I_N - \mathrm{e}^{\sigma^1 A} + \mathrm{e}^{(\sigma^1+\sigma^4)A} \\ -\mathrm{e}^{(\sigma^1+\sigma^3+\sigma^4)A})A^{-1}B - (1-q_0)LA^{-1}B - LA^{-1}C - (1-q_0)\theta_1 - q_0\theta_2 = 0. \end{cases} \tag{12}$$

Similarly, wo have for the electronical application the following system for period-2 cycle equations:

$$\begin{cases} K_1(\sigma^1,\sigma^2,\sigma^3,\sigma^4) = U_{ref} - U((I_N - e^{\sigma^1 A_0}e^{\sigma^4 A_1}e^{\sigma^3 A_0}e^{\sigma^2 A_1})^{-1} \\ (I_N - e^{\sigma^1 A_0} + e^{\sigma^1 A_0}e^{\sigma^4 A_1} - e^{\sigma^1 A_0}e^{\sigma^4 A_1}e^{\sigma^3 A_2})(A_1^{-1} - A_0^{-1})V - A_1^{-1}V) - \frac{1}{2}(q_1 - q_0)\chi_0 = 0, \\ K_2(\sigma^1,\sigma^2,\sigma^3,\sigma^4) = U_{ref} - U((I_N - e^{\sigma^2 A_1}e^{\sigma^1 A_0}e^{\sigma^4 A_1}e^{\sigma^3 A_0})^{-1} \\ (I_N - e^{\sigma^2 A_1} + e^{\sigma^2 A_1}e^{\sigma^1 A_0} - e^{\sigma^2 A_1}e^{\sigma^1 A_0}e^{\sigma^4 A_1})(A_0^{-1} - A_1^{-1})V - A_0^{-1}V) - \frac{1}{2}(q_0 - q_1)\chi_0 = 0, \\ K_3(\sigma^1,\sigma^2,\sigma^3,\sigma^4) = U_{ref} - U((I_N - e^{\sigma^3 A_0}e^{\sigma^2 A_1}e^{\sigma^1 A_0}e^{\sigma^4 A_1})^{-1} \\ (I_N - e^{\sigma^3 A_0} + e^{\sigma^3 A_0}e^{\sigma^2 A_1} - e^{\sigma^3 A_0}e^{\sigma^2 A_1}e^{\sigma^1 A_0})(A_1^{-1} - A_0^{-1})V - A_1^{-1}V) - \frac{1}{2}(q_1 - q_0)\chi_0 = 0, \\ K_4(\sigma^1,\sigma^2,\sigma^3,\sigma^4) = U_{ref} - U((I_N - e^{\sigma^4 A_1}e^{\sigma^3 A_0}e^{\sigma^2 A_1}e^{\sigma^1 A_0})^{-1} \\ (I_N - e^{\sigma^4 A_1} + e^{\sigma^4 A_1}e^{\sigma^3 A_0} - e^{\sigma^4 A_1}e^{\sigma^3 A_0}e^{\sigma^2 A_1})(A_0^{-1} - A_1^{-1})V - A_0^{-1}V) - \frac{1}{2}(q_0 - q_1)\chi_0 = 0. \end{cases} \tag{13}$$

3.5 Hybrid Poincaré application

Function f of theorem 2.1 will be the hybrid Poincaré application for our two applications. So, we have to introduce this function for the general system (5) (see (Quémard et al., 2005)).
To do this, we firstly consider the following different ways to write $\xi(t_n) = \xi_n$ given by this system:

$$\begin{cases} \xi_n = f(S_1, S_2, q_{n-1}, q_n), \\ \xi_n = cst - WX(t_n^-) = cst - W(e^{\sigma_n A_{n-1}}\Gamma_{n-1} - A_{n-1}^{-1}V_{n-1}) \end{cases}$$

and that gives:

$$cst - W(e^{\sigma_n A_{n-1}}\Gamma_{n-1} - A_{n-1}^{-1}V_{n-1}) - f(S_1, S_2, q_{n-1}, q_n) = 0. \tag{14}$$

Duration σ_n, $n \geq 1$, implicitly given by equation (14), defines for $n \geq 1$ a function Ψ_{q_n} of Γ_{n-1} such that:

$$\sigma_n = \psi_{q_n}(\Gamma_{n-1}), \quad \forall n \geq 1. \tag{15}$$

Moreover, equation (8), introduced in the last paragraph, defines for $n \geq 1$ a function g_{q_n} of σ_n and of Γ_{n-1} *i.e.*:

$$\Gamma_n = g_{q_n}(\sigma_n, \Gamma_{n-1}), \quad \forall n \geq 1. \tag{16}$$

Then, we set:

$$\begin{cases} \forall n \geq 1, \quad G_{q_n}(.) = g_{q_n}(\psi_{q_n}(.),.), \\ \forall n \geq 2, \quad h_{q_n} = G_{q_n} \circ G_{q_{n-1}}(.). \end{cases} \tag{17}$$

Since $q_n = q_0$ if n is even and $q_n = q_1$ if n is odd, we obtain $h_{q_n} = h_{q_0}$ if n is even and $h_{q_n} = h_{q_1}$ if n is odd ($n \geq 1$). We note:

$$\begin{aligned} h : \mathbb{R}^N &\longrightarrow \mathbb{R}^N \\ \Gamma &\longmapsto \begin{cases} h_{q_0}(\Gamma) \text{ if } n \text{ is even} \\ h_{q_1}(\Gamma) \text{ if } n \text{ is odd.} \end{cases} \end{aligned} \tag{18}$$

Function h defined by (18) corresponds to the hybrid Poincaré application associated to the studied h.d.s. Period-1 cycles are built from fixed points Γ^2 for h_{q_0} and Γ^1 for h_{q_1}. Period-2 cycles correspond to a 2-periodic point Γ^4 (or Γ^2) for h_{q_0} and Γ^1 (or Γ^3) for h_{q_1} characterized by $h_{q_0}(\Gamma^4) = \Gamma^2$, $h_{q_1}(\Gamma^1) = \Gamma^3$ and $h_{q_0} \circ h_{q_0}(\Gamma^4) = h^2_{q_0}(\Gamma^4) = \Gamma^4$, $h_{q_1} \circ h_{q_1}(\Gamma^1) = h^2_{q_1} = \Gamma^1$.

4. Theorem application to the thermostat

For this application to the thermostat presented in section 3, we decide to vary parameter $\mu = R_c$ which corresponds to the convector resistance and we choose for other parameters, fixed values given in table 1.

R_t	R_m	Q_t	Q_c	Q_p	P_t	P_c	θ_e	θ_1	θ_2
1.5	1	50	800	5000	0.8	50	281	293	294

Table 1. Numerical values to illustrate a period-doubling bifurcation for the thermostat.

To plot the bifurcation diagram given by figure 5, we use *Matlab* and for each value of parameter R_c, we solve system (12) with a classical Newton algorithm. Then, we plot the corresponding values of σ^2 and σ^4 (we could have chosen σ^1 and σ^3). If σ^2 and σ^4 have the same values then, the Newton algorithm tends to a period-1 cycle. Otherwise, it tends to a period-2 cycle. It is from value $R_c = R_{c_0} \simeq 1.5453923$ that σ^2 and σ^4 begin to take different

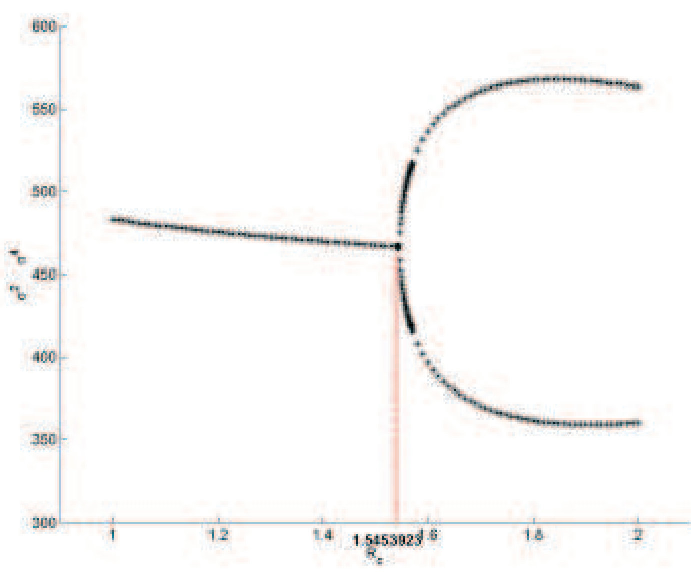


Fig. 5. Bifurcation diagram for the thermostat.

numerical values. For $R_c \leq R_{c_0}$, the Newton algorithm tends to a period-1 cycle and for $R_c > R_{c_0}$, it tends to a period-2 cycle. Figure 6 illustrates this phenomenon.
Before verifying the four conditions of theorem 2.1, we need to compute numerical values for σ^i and Γ^i, $i = 1, 2$ at $R_c = R_{c_0}$. From system (12), we obtain values of σ^i and from these values, we can compute the ones of Γ^i which can be written as a function of σ^i. We obtain $\sigma^1 \simeq 144.473853$, $\sigma^2 \simeq 466.851260$, $\Gamma^1 = (15.172083 \quad 1.049091 \quad -3.221174)^T$, $\Gamma^2 = (-47.749927 \quad -1.221632 \quad 8.971559)^T$.

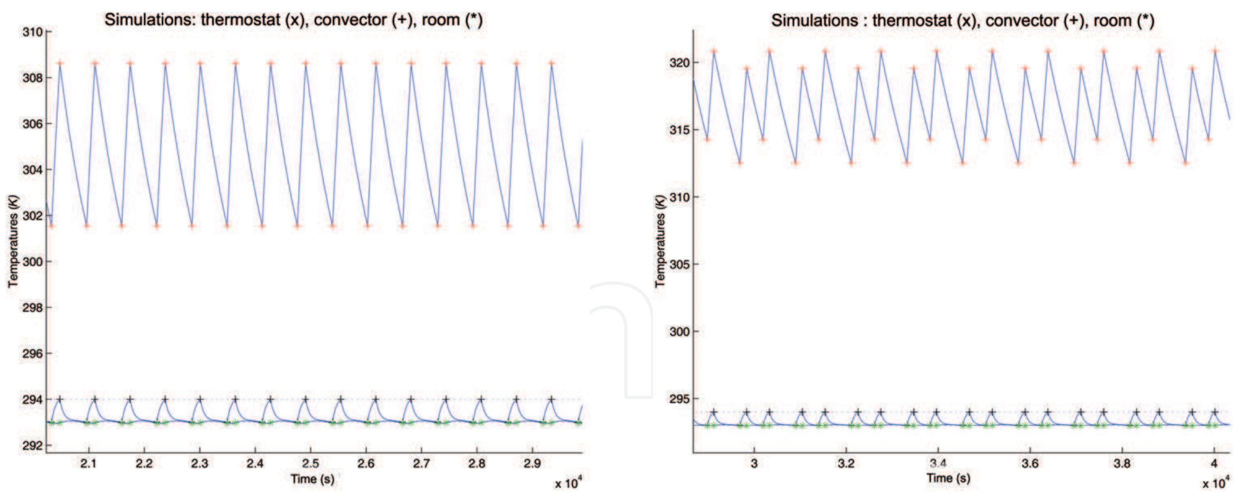


Fig. 6. Period-1 cycle for $R_c \leq R_{c_0}$ (on the left) and period-2 cycle for $R_c > R_{c_0}$ (on the right).

Now, we can begin to apply theorem 2.1 to the adapted Poincaré application h_{q_n} associated to the thermostat which is explicitly given by:

$$
\begin{aligned}
h_{q_n} : \mathbb{R}^N \times \mathbb{R} &\longrightarrow \mathbb{R}^N \\
(\Gamma_{n-2}, R_c) &\longmapsto \begin{cases} h_{q_0}(\Gamma^2_{n-1}, R_c) = h_{q_{0_{R_c}}}(\Gamma^2_{n-1}) \text{ if } n \text{ is even} \\ h_{q_1}(\Gamma^1_{n-1}, R_c) = h_{q_{1_{R_c}}}(\Gamma^1_{n-1}) \text{ if } n \text{ is odd.} \end{cases}
\end{aligned}
$$

Here, we restrict our study to the case n even (the case n odd giving the same results). Let us verify the four assumptions of theorem 2.1.

• **First assumption:**

We have built the Poincaré application in order to have Γ^2 as a fixed point of h_{q_0}. So, $h_{q_{0_{R_{c_0}}}}(\Gamma^2) = G_{q_{0_{R_{c_0}}}} \circ G_{q_{1_{R_{c_0}}}}(\Gamma^2) = G_{q_{0_{R_{c_0}}}}(\Gamma^1) = \Gamma^2$ and the first assumption is satisfied.

• **Second assumption:**

Here, we can omit parameter R_c since it is fixed and so, does not affect the result. Then, the Poincaré application becomes a function only of Γ^2_{n-1}. Thus, we can write $Dh_{q_0}(\Gamma^2) = \partial_{\Gamma^2_{n-1}} h_{q_{0_{R_{c_0}}}}(\Gamma^2)$ the Jacobian matrix of h_{q_0}.

We need to give the expression of the Jacobian matrix Dh_{q_0} so, since it will be also used for the electronical application, we will compute it in the general way. We note $h_{q_n}(\Gamma_{n-2}) = G_{q_n} \circ G_{q_{n-1}}(\Gamma_{n-2})$. Therefore, the Jacobian matrix Dh_{q_n} is given by:

$$
\begin{aligned}
Dh_{q_n}(\Gamma_{n-2}) &= DG_{q_n}(G_{q_{n-1}}(\Gamma_{n-2}))DG_{q_{n-1}}(\Gamma_{n-2}) \\
&= DG_{q_n}(\Gamma_{n-1})DG_{q_{n-1}}(\Gamma_{n-2}).
\end{aligned}
\tag{19}
$$

>From definition (17) of G_{q_n}, we have:

$$
DG_{q_n}(\Gamma_{n-1}) = \frac{\partial g_{q_n}}{\partial \sigma_n}(\sigma_n, \Gamma_{n-1})\frac{\partial \psi_{q_n}}{\partial \Gamma_{n-1}}(\Gamma_{n-1}) + \frac{\partial g_{q_n}}{\partial \Gamma_{n-1}}(\sigma_n, \Gamma_{n-1}). \tag{20}
$$

We easily obtain $\frac{\partial g_{q_n}}{\partial \sigma_n}(\sigma_n, \Gamma_{n-1})$ and $\frac{\partial g_{q_n}}{\partial \Gamma_{n-1}}(\sigma_n, \Gamma_{n-1})$ derivating the second member of equation (8) respectively with respect to σ_n and Γ_{n-1}:

$$\begin{cases} \frac{\partial g_{q_n}}{\partial \sigma_n}(\sigma_n, \Gamma_{n-1}) = A_{n-1} e^{\sigma_n A_{n-1}} \Gamma_{n-1} \\ \frac{\partial g_{q_n}}{\partial \Gamma_{n-1}}(\sigma_n, \Gamma_{n-1}) = e^{\sigma_n A_{n-1}}. \end{cases}$$

Moreover, the calculus of $\frac{\partial \psi_{q_n}}{\partial \Gamma_{n-1}}(\Gamma_{n-1})$ is obtained differentiating implicit equation given by (14) with respect to Γ_{n-1} with $\sigma_n = \Psi_{q_n}(\Gamma_{n-1})$ and gives:

$$-WA_{n-1} e^{\sigma_n A_{n-1}} \Gamma_{n-1} \frac{\partial \psi_{q_n}}{\partial \Gamma_{n-1}}(\Gamma_{n-1}) - W e^{\sigma_n A_{n-1}} = 0.$$

If $-WA_{n-1} e^{\sigma_n A_{n-1}} \Gamma_{n-1} \neq 0$ that we assume, we can deduce:

$$\frac{\partial \psi_{q_n}}{\partial \Gamma_{n-1}}(\Gamma_{n-1}) = \frac{-W e^{\sigma_n A_{n-1}}}{WA_{n-1} e^{\sigma_n A_{n-1}} \Gamma_{n-1}}. \tag{21}$$

Finally, we can write:

$$DG_{q_n}(\Gamma_{n-1}) = \left(I_N - \frac{A_{n-1} e^{\sigma_n A_{n-1}} \Gamma_{n-1} W}{WA_{n-1} e^{\sigma_n A_{n-1}} \Gamma_{n-1}} \right) e^{\sigma_n A_{n-1}}, \tag{22}$$

and we deduce the expression of $Dh_{q_n}(\Gamma_{n-2})$ with (19).
For the first application of the thermostat, we choose an eigenvectors basis and relatively to this basis, we obtain:

$$Dh_{q_0}(\Gamma^2) = \left(I_3 - \frac{A e^{\sigma^2 A} \Gamma^1 L}{L A e^{\sigma^2 A} \Gamma^1} \right) e^{\sigma^2 A} \left(I_3 - \frac{A e^{\sigma^1 A} \Gamma^2 L}{L A e^{\sigma^1 A} \Gamma^2} \right) e^{\sigma^1 A}.$$

Numerically, we have:

$$Dh_{q_0}(\Gamma^2) \simeq \begin{pmatrix} -1.8419626299 & 0.4499899352 & -0.8182826401 \\ -0.0011184622 & 0.0097398043 & 0.0174088456 \\ 1.8430810922 & -0.4597297396 & 0.8008737945 \end{pmatrix},$$

which has three eigenvalues $\lambda_1 = -1, \quad \lambda_2 = 0, \quad \lambda_3 \simeq -0.031348$. One is equal to -1 and the others respect $|\lambda_i| \neq 1$, $i = 2, 3$ so the second assumption of the theorem is verified.
• **Third assumption:** If we reason like in the theorem proof, we have, since $Dh_{q_0}(\Gamma^2)$ has not 1 as eigenvalue, $x_f(R_c)$, curve of fixed points exists with, in particular, $x_f(R_{c_0}) = \Gamma^2$.
Matrix of the first derivatives of $h_{q_{0_{R_c}}}$ with respect to Γ^2_{n-1} at $x_f(R_c)$ takes the following form $\forall n \geq 1$:

$$\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_c}}}(x_f(R_c)) = \left(\frac{\partial h^i_{q_{0_{R_c}}}}{\partial \Gamma^{2j}_{n-1}}(x_f(R_c)) \right)_{i,j=1,\dots,3} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, \tag{23}$$

where Γ^{2j}_{n-1}, $j = 1, \ldots, 3$ represents the j-th component of vector Γ^2_{n-1} and $h^i_{q_{0_{R_c}}}$, $i = 1, \ldots, 3$ is the i-th component of $h_{q_{0_{R_c}}}$.

So, to find eigenvalues of $\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_c}}}(x_f(R_c))$, we have to solve equation $\det(\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_c}}}(x_f(R_c)) - zI_3) = 0$.

Moreover, firstly, we can remark, from equation (9), that we have $-WD\Gamma_n = 0$ *i.e.* $-WDG_{q_n}(\Gamma_{n-1}) = 0$ by definition of G_{q_n}. This means that $-W$ is a left eigenvector of $DG_{q_n}(\Gamma_{n-1})$ associated to eigenvalue 0 and so, 0 is always an eigenvalue for $Dh_{q_n}(\Gamma_{n-2})$ *i.e.* $\det(\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_c}}}(x_f(R_c))) = 0$.

Taking into account this remark, we know that to compute the two other eigenvalues, it remains to solve the following equation:

$$-\lambda^2 + \lambda.\mathrm{Tr}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_c}}}(x_f(R_c))) - \mathrm{Tr}(\mathrm{com}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_c}}}(x_f(R_c)))) = 0, \tag{24}$$

where $\mathrm{com}(M)$ corresponds to the comatrix of M for any square matrix M and with:

$$\begin{cases} \mathrm{Tr}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_c}}}(x_f(R_c))) = a_1 + b_2 + c_3, \\ \mathrm{Tr}(\mathrm{com}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_c}}}(x_f(R_c)))) = b_2c_3 - b_3c_2 + a_1c_3 - a_3c_1 + a_1b_2 - a_2b_1. \end{cases}$$

If $\lambda_1(R_c)$ (with $\lambda_1(R_{c_0}) = -1$) is an eigenvalue of $\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_c}}}(x_f(R_c))$, then $\lambda_1(R_c)$ verifies equation (24). Thus, derivating this obtained equation with respect to parameter R_c and then, applying it at $R_c = R_{c_0}$, we obtain:

$$-2\lambda_1'(R_{c_0})\lambda_1(R_{c_0}) + \lambda_1'(R_{c_0})\mathrm{Tr}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_{c_0}}}}(\Gamma^2)) + \lambda_1(R_{c_0})\frac{\partial}{\partial R_c}(\mathrm{Tr}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_c}}}(x_f(R_c))))_{|R_c=R_{c_0}}$$

$$-\frac{\partial}{\partial R_c}(\mathrm{Tr}(\mathrm{com}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_c}}}(x_f(R_c)))))_{|R_c=R_{c_0}} = 0.$$

Thus, we finally obtain an expression for $\lambda_1'(R_{c_0}) = \frac{d\lambda_1}{dR_c}(R_{c_0})$:

$$\lambda_1'(R_{c_0}) = \frac{\frac{\partial}{\partial R_c}\Big(\mathrm{Tr}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_c}}}(x_f(R_c))) + \mathrm{Tr}(\mathrm{com}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_c}}}(x_f(R_c))))\Big)_{|R_c=R_{c_0}}}{\Big(2 + \mathrm{Tr}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{R_{c_0}}}}(\Gamma^2))\Big)}.$$

Therefore, we developp all computations and applying them to our numerical values, we finally obtain:

$$\lambda_1'(R_{c_0}) = \frac{d\lambda_1}{dR_c}(R_{c_0}) \simeq 16.9072 \neq 0,$$

that satisfies the third assumption of theorem 2.1.

• **Fourth assumption:**

To verify the last assumption of theorem 2.1, we have to compute β. As for the second assumption, we can omit R_c in all expressions because it does not affect the result.

To compute β, we need the calculation of the second and third derivatives of $h^i_{q_0}$, $i = 1, ..., 3$. They have been obtained applying the formula of the n-th derivatives of a function composition with several variables (we do not explicit them here since it is very long and without a big interest).

Moreover, for the second assumption, we have proved that $Dh_{q_0}(\Gamma^2)$ has three different eigenvalues noted λ_i, $i = 1, ..., 3$. So, we choose a basis a right eigenvectors of $Dh_{q_0}(\Gamma^2)$ noted $(v^1 \quad v^2 \quad v^3)$ such that v^i is associated to λ_i, $i = 1, ..., 3$ with, in particular, v^1 associated to -1. We numerically have:

$$v^1 \simeq \begin{pmatrix} 1 \\ 0.018670 \\ -1.018670 \end{pmatrix}, v^2 \simeq \begin{pmatrix} 1.037479 - 0.297064i \\ 0.811752 - 0.232431i \\ -1.849231 + 0.529494i \end{pmatrix}, v^3 \simeq \begin{pmatrix} 0.870614 - 0.066942i \\ 1.816933 - 0.139705i \\ -0.960593 + 0.073860i \end{pmatrix}.$$

Similarly, we take as a dual basis $\{w^j\}_{j=1,...,3}$ of $\{v^j\}_{j=1,...,3}$ the left eigenvectors of matrix $Dh_{q_0}(\Gamma^2, R_{c_0})$ associated to λ_i, $i = 1, ..., 3$ such that $w^i v^j = 1$ if $i = j$ and $w^i v^j = 0$ otherwise. We numerically obtain:

$$w^1 \simeq \begin{pmatrix} 1.885527 \\ -0.448237 \\ 0.861079 \end{pmatrix}, w^2 \simeq \begin{pmatrix} -1.237960 - 0.354468i \\ -0.049794 - 0.014257i \\ -1.216183 - 0.348232i \end{pmatrix}, w^3 \simeq \begin{pmatrix} 0.575651 + 0.044262i \\ 0.575651 + 0.044262i \\ 0.575651 + 0.044262i \end{pmatrix}.$$

Thus, the computation of β becomes possible and numerically gives:

$$\beta \simeq 0.7049 \neq 0,$$

that verifies the fourth and last assumption of theorem 2.1.
• **Conclusion:** These four assumptions theoretically prove, with the period-doubling bifurcation thereom, that there exists at $R_c \simeq R_{c_0} \simeq 1.5453923$ a period-doubling bifurcation which highlights the loss of stability of the stable period-1 cycle and the emergence of a stable period-2 cycle. It confirms that we had graphically seen.

5. Theorem application to the DC/DC converter

Other authors (Zhusubaliyev & Mosekilde, 2003), (Lim & Hamill, 1999) were interested in the problem of period-doubling bifurcations for this electronical application. However, they often study the phenomenon only using bifurcation diagrams.
Here, we propose a theoretical proof but firstly, we can also propose a bifurcation diagram to highlight the crossing of a period-1 cycle to a period-2 cycle with the variation of one parameter.
We choose the fixed following numerical values in table 2 and L_0 is the variable parameter. To plot the bifurcation diagram given by figure 7, we use the same method than the

R_1	R_0	R_L	L_1	C_0	C_1	U_{ref}	σ	χ_0	E_0
2	5	80	0.09	3.10^{-6}	2.10^{-5}	2.4	0.1	0.25	200

Table 2. Numerical values to illustrate a period-doubling bifurcation for the DC/DC converter.

one use for the thermostat solving system (13) with a Newton algorithm for each value of L_0. It is from value $L_0 \simeq L_{0_0} \simeq 0.1888073$ that σ^2 and σ^4 begin to take different values. Figure 8 illustrates that, for $L_0 \leq L_{0_0}$, the Newton algorithm tends to a period-1 cycle and for $L_0 > L_{0_0}$, it tends to a period-2 cycle. As for the thermal application, we need to compute values of σ^i and Γ^i, $i = 1, 2$ at $L_0 = L_{0_0}$. It is done with a Newton algorithm and system (13) to obtain σ^i and the integration constants Γ^i,

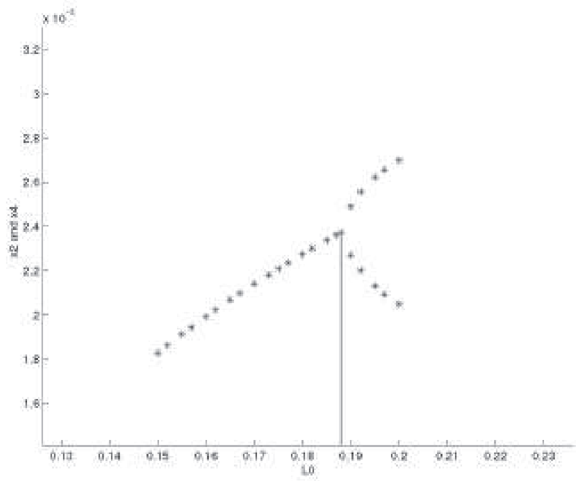


Fig. 7. Bifurcation diagram for the DC/DC converter.

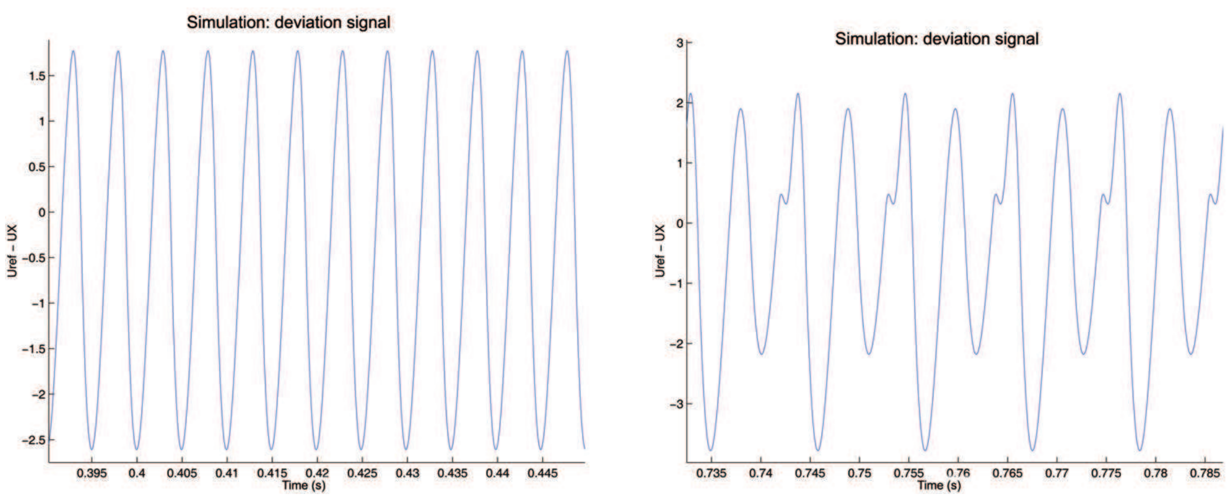


Fig. 8. Period-1 cycle for $L_0 \leq L_{0_0}$ (on the left) and period-2 cycle for $L_0 > L_{0_0}$ (one the right).

$i = 1, 2$ are computed from the obtained values of σ^i since we have seen that they are only functions of σ^i. We numerically obtain: $\sigma^1 \simeq 0.0030227$, $\sigma^2 \simeq 0.0023804$, $\Gamma^1 \simeq (7.690 + 1.156i \quad -54.176 - 386.422i \quad 6.199 + 3.877i \quad -81.201 - 55.029i)^T$, $\Gamma^2 \simeq (-2.266 + 0.742i \quad -38.350 + 113.344i \quad -3.090 + 0.286i \quad 13.250 + 39.466i)^T$.
Thus, we verify the four assumptions of theorem 2.1 using the Poncaré application $h_{q_{0_{L_0}}}$ (we obtain the same results for $h_{q_{1_{L_0}}}$) associated to our DC/DC converter.

• **First assumption:**
The first assumption is clearly staisfied by construction of the Poincaré application associated to our system.

• **Second assumption:**
As for the thermal application, for this assumption, parameter L_0 is fixed to L_{0_0} so does not affect the result. Thus, the Poincaré application can be considered as a function of Γ^2_{n-1}. With numerical values of table 2, of σ^i and of Γ^i, $i = 1, 2$, we numerically compute the Jacobian

matrix of h_{q_0} at Γ^2 and L_{0_0} and we obtain:

$$Dh_{q_0}(\Gamma^2) \simeq \begin{pmatrix} -0.3194-0.2921i & 0.0048-0.0027i & 0.1451-0.0027i & 0.0165-0.0074i \\ 12.0276+6.8908i & -0.3194+0.2921i & 8.0084+9.1076i & 0.0673+1.1643i \\ 0.49640+0.1299i & 0.0091+0.0046i & -0.3067-0.0494i & -0.0126-0.0194i \\ 4.2869-5.3865i & 0.1065-0.0865i & -2.2699+3.4955i & -0.3067+0.4939i \end{pmatrix}.$$

This matrix has four different eigenvalues $\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 \simeq 0.079202$ and $\lambda_4 \simeq -0.331416$. One of this value is equal to -1 and the others verify $|\lambda_i| \neq 1$, $i = 2,3,4$ so the second assumption is satisfied.

• **Third assumption:**

Let $x_f(L_0)$ be the curve of the fixed-point of $h_{q_{0_{L_0}}}$. The matrix of the first derivatives of $h_{q_{0_{L_0}}}$ with respect to Γ^2_{n-1} at $x_f(L_0)$ can be written for the DC/DC converter:

$$\partial_{\Gamma^2_{n-1}} h_{q_{0_{L_0}}}(x_f(L_0)) = \left(\frac{\partial h^i_{q_{0_{L_0}}}}{\partial \Gamma^{2j}_{n-1}}(x_f(L_0)) \right)_{i,j=1,\ldots,4} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}.$$

Then, to compute $\lambda_1'(L_{0_0}) = \frac{d\lambda_1}{dL_0}(L_{0_0})$, we use the same method than the one explained for the thermostat. We develop equation $\det(\partial_{\Gamma^2_{n-1}} h_{q_{0_{L_0}}}(x_f(L_0)) - \lambda I_4) = 0$ and we simplify it using the fact that 0 is always an eigenvalue of $\partial_{\Gamma^2_{n-1}} h_{q_{0_{L_0}}}(x_f(L_0))$. Then, assuming that $\lambda_1(L_0)$ is an eigenvalue of $\partial_{\Gamma^2_{n-1}} h_{q_{0_{L_0}}}(x_f(L_0))$ with $\lambda_1(L_{0_0}) = -1$, we obtain at L_{0_0}:

$$\lambda_1'(L_{0_0}) = \frac{\frac{\partial}{\partial L_0}\left(\mathrm{Tr}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{L_0}}}(x_f(L_0))) + M_1(L_0) + \mathrm{Tr}(\mathrm{com}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{L_0}}}(x_f(L_0)))) \right)_{|L_0=L_{0_0}}}{(3+2\mathrm{Tr}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{L_{0_0}}}}(\Gamma^2)) + M_1(L_{0_0}))},$$

where

$$\begin{cases} \mathrm{Tr}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{L_0}}}(x_f(L_0))) = a_1 + b_2 + c_3 + d_4, \\ M_1(L_0) = a_1b_2 - a_2b_1 + a_1c_3 - a_3c_1 + a_1d_4 - a_4d_1 + b_2c_3 - b_3c_2 + b_2d_4 - b_4d_2 + c_3d_4 - c_4d_3, \\ \mathrm{Tr}(\mathrm{com}(\partial_{\Gamma^2_{n-1}} h_{q_{0_{L_0}}}(x_f(L_0)))) = a_1b_2c_3 + a_1b_2d_4 + a_1c_3d_4 + b_2c_3d_4 + b_3c_4d_2 + b_4c_2d_3 + a_2b_4d_1 \\ +a_3b_1c_2 + a_3c_4d_1 + a_4b_1d_2 + a_4c_1d_3 + a_2b_3c_1 - a_1b_4d_2 - a_1c_4d_3 - a_1b_3c_2 - b_2c_4d_3 - b_3c_2d_4 \\ -b_4c_3d_2 - a_3c_1d_4 - a_4b_2d_1 - a_4c_3d_1 - a_3b_2c_1 - a_2b_1c_3 - a_2b_1d_4. \end{cases}$$

We do not detail calculations here since they are the same than the one for the thermostat with index i which varies from 1 to 4. We finally numerically obtain:

$$\lambda_1'(L_{0_0}) \simeq 376.77 + 48.85i \neq 0,$$

that satisfies the third assumption of theorem 2.1.

• **Fourth assumption:**

To verify the last assumption of theorem 2.1, we have to compute β. To do this, we need to know the first, the second and the third derivatives of h_{q_0} with respect to Γ^2_{n-1}. They

are obtained using our knowledge of the derivatives of a function composition with various variables.
Thus, it remains to give a right and a left eigenvectors basis of $Dh_{q_{0_{L_{0_0}}}}(\Gamma^2)$. For the right eigenvectors basis, we choose $(v^1 \quad v^2 \quad v^3 \quad v^4)$ with v^i associated to λ_i, $i = 1, ..., 4$ with, in particular, $\lambda_1 = -1$. We numerically choose:

$$v^1 \simeq \begin{pmatrix} 1 \\ -11.442836 + 48.851737i \\ -0.426519 - 1.349074i \\ -18.867917 - 2.085051i \end{pmatrix}, v^2 \simeq \begin{pmatrix} 0.5166742 - 0.78824872i \\ 17.01036378 + 44.12314420i \\ 0.65810329 - 0.22792681i \\ 9.27738581 + 1.11320087i \end{pmatrix}$$

$$v^3 \simeq \begin{pmatrix} 0.01119889 - 0.01608901i \\ 0.52509800 + 0.08316542i \\ 0.01236110 + 0.00503187i \\ 0.12880400 - 0.12438117i \end{pmatrix}, v^4 \simeq \begin{pmatrix} 0.01627771 - 0.18549512i \\ 2.81524460 + 8.90855146i \\ 0.05802415 + 0.01511695i \\ 0.50027614 - 0.62998584i \end{pmatrix}.$$

Identically, we take as a dual basis $\{w^j\}_{j=1,...,4}$ of $\{v^j\}_{j=1,...,4}$ the left eigenvectors of $Dh_{q_{0_{L_{0_0}}}}(\Gamma^2)$ associated to eigenvalues λ_i, $i = 1, ..., 4$ such that $w^i v^j = 1$ if $i = j$ and $w^i v^j = 0$ otherwise. We numerically have:

$$w^1 \simeq \begin{pmatrix} -6.00780496 + 3.31230746i \\ -0.10257727 - 0.09040793i \\ -0.65155347 - 1.40122597i \\ -0.10158624 + 0.05428312i \end{pmatrix}, w^2 \simeq \begin{pmatrix} 0 \\ 0 \\ 0.45187014 + 0.88898172i \\ 0.04698337 - 0.05759709i \end{pmatrix},$$

$$w^3 \simeq \begin{pmatrix} -0.15269597 - 0.97827008i \\ -0.007293202 - 0.01833645i \\ 0.44704526 + 0.83019484i \\ -0.026511494 + 0.06508790i \end{pmatrix}, w^4 \simeq \begin{pmatrix} 50.68797524 - 15.97723598i \\ 1.02351719 + 0.27277622i \\ 0.82854165 - 35.60644916i \\ -0.93039774 + 2.48628517i \end{pmatrix}.$$

We conclude $\beta \simeq 0.076 - 0.13i \neq 0$, that satisfies the last assumption.
• **Conclusion:**
The fourth assumptions of theorem 2.1 being satisfied, so, the theorem of period-doubling bifurcation can be applied and theoretically prove the existence of a period-doubling bifurcation at $L_0 = L_{0_0}$. It confirms the observations made on the bifurcation diagram of figure 7.

6. Conclusion

We have presented a new tool to theoretically prove the existence of a period-doubling bifurcation for a particular value of the parameters. This is a generalization of the period-doubling bifurcation theorem of systems of any dimension N, $N \geq 1$.
This result has been applied to two applications of industrial interest and of two different dimensions: the one of dimension three with the thermostat with an anticipative resistance and the second in dimension four with the DC/DC converter. This work has confirmed the observations graphically made on the bifurcation diagrams.
Such a bifurcation can appear from three-dimensional systems for the studied h.d.s. class. Indeed, in dimension one, as zero is the single eigenvalue of the Jacobian matrix Dh_{q_n}, we

can directly conclude that there is not exist a bifurcation. Moreover, in dimension two, in (Quémard, 2007a), we have proved that the two eigenvalues of the Jacobian matrix Dh_{q_n} are 0 and $\exp((A_1 + A_2)(\sigma^1 + \sigma^2))$ where A_1 and A_2 are the diagonal elements of matrix A written in an eigenvectors basis and with $A_1 < 0$, $A_2 < 0$. So, since $\sigma^1 > 0$ and $\sigma^2 > 0$, in dimension two, Dh_{q_n} has two eigenvalues which belong to the open unit disk. Thus, there is no such bifurcation in dimension two.
Finally, in this paper, we have directly chosen numerical values for which there exists this type of bifurcation but sometimes, it is very difficult to find them. So, it would be very interesting to find a method which permits to quickly obtain those values. To do this, for exemple in dimension three, we can firstly use a graphical method varying two parameters and solving the period-2 cycle equations system with a Newton algorithm for each value of those parameters. Then, we plot, with two different colors, the corresponding points depending whether the algorithm converges to a period-1 cycle or to a period-2 cycle. It is not very precise but it can give an interval containing searched values. Then, to refine these values, we should build a system with three equations for three unknowns σ^1, σ^2 and the parameter which varies. This system could be solved with a Broyden algorithm for example taking initial values belonging to the obtained interval to ensure the algorithm convergence.
From $\det(\partial_{\Gamma^2_{n-1}} h_{q_0}(\Gamma^2) - zI_3) = 0$ and knowing that 0 and -1 are two solutions at the bifurcation point, we could obtain the first equation. Then, from system (11) applied for period-1 cycles, we can obtain the two others considering the varying parameter as the third variable. This could be a future research work.

7. References

Aihara, K., Takabe, T. & Toyoda, M. (1998). Chaotic Neural Networks, *Physics Letters A* 144: 333–340.

Baker, G.-L. & Gollub, J.-P. (1990). *Chaotic dynamics an introduction*, 2nd edn, Cambridge University Press.

Cébron, B. (2000). *Commande de systèmes dynamiques hybrides*, PhD thesis, LISA, Université d'Angers,.

Chung, K., Chan, C. & Xu, J. (2003). An efficient method for switching branches of period-doubling bifurcations of strongly non-linear autonomous oscillators with many degrees of freedom, *Journal of Sound and Vibration* 267: 787–808.

Demazure, M. (1989). *Catastrophes et bifurcations*, Ellipses.

Girard, A. (2003). Computation and Stability Analysis of Limit Cycles in Piecewise Linear Hybrid Systems, *ADHS'2003 [Proc. of IFAC Conf.]*.

Gleick, J. (1991). *La théorie du chaos*, Champs Flammarion.

Guckenheimer, J. & Holmes, P.-J. (1991). *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, 3rd edn, Springer.

Holmes, E., Lewis, M., Banks, J. & Veit, R. (1994). Partial differential equations in ecology: spatial interactions and population dynamics, *Physics Letters A* 75: 17–29.

Jaulin, L., Kieffer, M., Didrit, . & Walter, E. (2001). *Applied Interval Analysis with Examples in Parameter and State Estimation*, Springer-Verlag.

Kuznetsov, Y. (2004). *Elements of Applied Bifurcation Theory*, Springer.

Lim, Y. & Hamill, D. (1999). Problems of computing Lyapunov exponents in power electronics, *International Symposium on Circuits and Systems, Orlando FL* 5: 297–301.

Murray, J. (1989). *Mathematical biology*, Springer.

Quémard, C. (2007a). *Analyse et optimisation d'une classe particulière de systèmes dynamiques hybrides à commutations autonomes*, PhD thesis, LISA, Université d'Angers,.

Quémard, C. (2007b). Analysis of a Class of Hybrid Dynamical Systems with Hysteresis Phenomenon, *NumAn (Numerical Analysis, Kalamata)*.

Quémard, C. (2009). Switchings Conditions Study for a Particular Class of Hybrid Dynamical Systems, *WCE (London)*.

Quémard, C., Jolly, J.-C. & Ferrier, J.-L. (2005). Search for Cycles in Piecewise Linear Hybrid Dynamical Systems with Autonomous Switchings. Application to a Thermal Device, *IMACS'2005 World Congress (Paris)*.

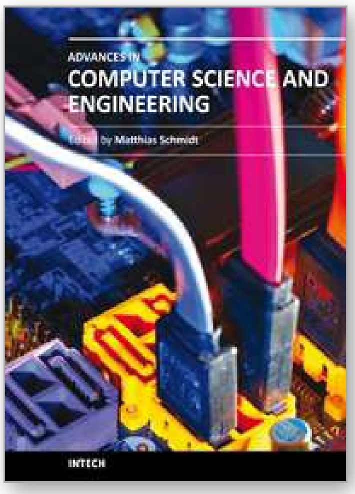
Quémard, C., Jolly, J.-C. & Ferrier, J.-L. (2006). Search for Period-2 Cycles in a Class of Hybrid Dynamical Systems with Autonomous Switchings. Application to a Thermal Device, *ADHS'2006 (Alghero)*.

Robinson, C. (1999). *Dynamical Systems*, 2nd edn, CRC Press.

Saccadura, J.-F. (1998). *Initiation aux transferts thermiques*, Lavoisier.

Smith, J. & Cohen, R. (1984). Simple Finite-Element Model Accounts for Wide Range of Cardiac Dysrhythmias, *National Academy of Sciences of the United States of America* 81: 233–237.

Zhusubaliyev, Z. & Mosekilde, E. (2003). *Bifurcations and Chaos in Piecewise-Smooth Dynamical Systems*, Vol. 44 of *Nonlinear Science*, World Scientific.



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