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# Stability of Linear Continuous Singular and Discrete Descriptor Time Delayed Systems

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## 1. Introduction

The problem of investigation of time delay systems has been exploited over many years. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability. Consequently, the problem of stability analysis for this class of systems has been one of the main interests for many researchers. In general, the introduction of time delay factors makes the analysis much more complicated.

When the general time delay systems are considered, in the existing stability criteria, mainly two ways of approach have been adopted. Namely, one direction is to contrive the stability condition which does not include the information on the delay, and the other is the method which takes it into account. The former case is often called the delay-independent criteria and generally provides simple algebraic conditions. In that sense the question of their stability deserves great attention. We must emphasize that there are a lot of systems that have the phenomena of time delay and singular characteristics simultaneously. We denote such systems as *the singular (descriptor) differential (difference) systems with time delay*.

These systems have many special properties. If we want to describe them more exactly, to design them more accurately and to control them more effectively, we must pay tremendous endeavor to investigate them, but that is obviously a very difficult work. In recent references authors have discussed such systems and got some consequences. But in the study of such systems, there are still many problems to be considered.

## 2. Time delay systems

### 2.1 Continuous time delay systems

#### 2.1.1 Continuous time delay systems – stability in the sense of Lyapunov

The application of *Lyapunov's* direct method (LDM) is well exposed in a number of very well known references. For the sake of brevity contributions in this field are omitted here. The part of only interesting paper of (*Tissir & Hmamed* 1996), in the context of these investigations, will be presented later.

### 2.1.2 Continuous time delay systems – stability over finite time interval

A linear, multivariable time-delay system can be represented by differential equation:

$$\dot{\mathbf{x}}(t) = A_0 \mathbf{x}(t) + A_1 \mathbf{x}(t - \tau), \quad (1)$$

and with associated function of initial state:

$$\mathbf{x}(t) = \boldsymbol{\Psi}_x(t), \quad -\tau \leq t \leq 0. \quad (2)$$

Equation (1) is referred to as *homogenous*,  $\mathbf{x}(t) \in \mathbb{R}^n$  is a state space vector,  $A_0, A_1$ , are constant system matrices of appropriate dimensions, and  $\tau$  is pure time delay,  $\tau = \text{const.}, (\tau > 0)$ .

Dynamical behavior of the system (1) with initial functions (2) is defined over continuous time interval  $\mathfrak{S} = \{t_0, t_0 + T\}$ , where quantity  $T$  may be either a positive real number or symbol  $+\infty$ , so finite time stability and practical stability can be treated simultaneously. It is obvious that  $\mathfrak{S} \in \mathbb{R}$ . Time invariant sets, used as bounds of system trajectories, satisfy the assumptions stated in the previous chapter (section 2.2).

#### STABILITY DEFINITIONS

In the context of finite or practical stability for particular class of *nonlinear singularly perturbed multiple time delay systems* various results were, *for the first time*, obtained in Feng, Hunsarg (1996). It seems that their definitions are very similar to those in Weiss, Infante (1965, 1967), clearly adopted to time delay systems.

It should be noticed that those definitions are significantly different from definition presented by the authors of this chapter.

In the context of finite time and practical stability for linear continuous time delay systems, various results were first obtained in (Debeljkovic et al. 1997.a, 1997.b, 1997.c, 1997.d), (Nenadic et al. 1997).

In the paper of (Debeljkovic et al. 1997.a) and (Nenadic et al. 1997) some basic results of the area of finite time and practical stability were extended to the particular class of linear continuous time delay systems. Stability sufficient conditions dependent on delay, expressed in terms of time delay fundamental system matrix, have been derived. Also, in the circumstances when it is possible to establish the suitable connection between fundamental matrices of linear time delay and non-delay systems, presented results enable an efficient procedure for testing practical as well the finite time stability of time delay system.

Matrix measure approach has been, for the first time applied, in (Debeljkovic et al. 1997.b, 1997.c, 1997.d, 1997.e, 1998.a, 1998.b, 1998.d, 1998.d) for the analysis of practical and finite time stability of linear time delayed systems. Based on Coppel's inequality and introducing matrix measure approach one provides a very simple delay - dependent sufficient conditions of practical and finite time stability with no need for time delay fundamental matrix calculation.

In (Debeljkovic et al. 1997.c) this problem has been solved for forced time delay system.

Another approach, based on very well known Bellman-Gronwall Lemma, was applied in (Debeljkovic et al. 1998.c), to provide new, more efficient sufficient delay-dependent conditions for checking finite and practical stability of continuous systems with state delay.

Collection of all previous results and contributions was presented in paper (Debeljkovic et al. 1999) with overall comments and slightly modified Bellman-Gronwall approach.

Finally, modified Bellman-Gronwall principle, has been extended to the particular class of continuous *non-autonomous* time delayed systems operating over the finite time interval, (Debeljkovic et al. 2000.a, 2000.b, 2000.c).

**Definition 2.1.2.1** Time delay system (1-2) is *stable* with respect to  $\{\alpha, \beta, -\tau, T, \|\mathbf{x}\|\}$ ,  $\alpha \leq \beta$ , if for any trajectory  $\mathbf{x}(t)$  condition  $\|\mathbf{x}_0\| < \alpha$  implies  $\|\mathbf{x}(t)\| < \beta \quad \forall t \in [-\Delta, T]$ ,  $\Delta = \tau_{\max}$ , (Feng, Hunsarg 1996).

**Definition 2.1.2.2** Time delay system (1-2) is *stable* with respect to  $\{\alpha, \beta, -\tau, T, \|\mathbf{x}\|\}$ ,  $\gamma < \alpha < \beta$ , if for any trajectory  $\mathbf{x}(t)$  condition  $\|\mathbf{x}_0\| < \alpha$ , implies (Feng, Hunsarg 1996):

- i. Stability w.r.t.  $\{\alpha, \beta, -\tau, T, \|\mathbf{x}\|\}$ ,
- ii. There exist  $t^* \in ]0, T[$  such that  $\|\mathbf{x}(t)\| < \gamma$  for all  $\forall t \in ]t^*, T[$ .

**Definition 2.1.2.3** System (1) satisfying initial condition (2) is finite time stable with respect to  $\{\zeta(t), \beta, \mathfrak{T}\}$  if and only if  $\|\Psi_x(t)\| < \zeta(t)$ , implies  $\|\mathbf{x}(t)\| < \beta$ ,  $t \in \mathfrak{T}$ ,  $\zeta(t)$  being scalar function with the property  $0 < \zeta(t) \leq \alpha$ ,  $-\tau \leq t \leq 0$ ,  $-\tau \leq t \leq 0$ , where  $\alpha$  is a real positive number and  $\beta \in \mathbb{R}$  and  $\beta > \alpha$ , (Debeljkovic et al. 1997.a, 1997.b, 1997.c, 1997.d), (Nenadic et al. 1997).

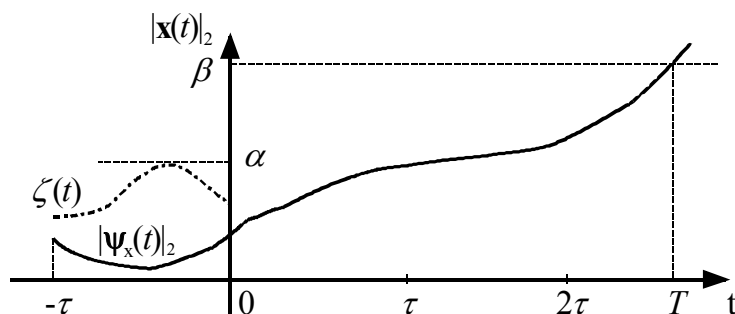


Fig. 2.1 Illustration of preceding definition

**Definition 2.1.2.4** System (1) satisfying initial condition (2) is finite time stable with respect to  $\{\zeta(t), \beta, \tau, \mathfrak{T}, \mu(A_0 \neq 0)\}$  iff  $\Psi_x(t) \in \mathcal{S}_\alpha$ ,  $\forall t \in [-\tau, 0]$ , implies  $\mathbf{x}(t_0, t, \mathbf{x}_0) \in \mathcal{S}_\beta$ ,  $\forall t \in [0, T]$  (Debeljkovic et al. 1997.b, 1997.c).

**Definition 2.1.2.5** System (1) satisfying initial condition (2) is finite time stable with respect to  $\{\alpha, \beta, \tau, \mathfrak{T}, \mu_2(A_0) \neq 0\}$  iff  $\Psi_x(t) \in \mathcal{S}_\alpha$ ,  $\forall t \in [-\tau, 0]$ , implies  $\mathbf{x}(t, t_0, \mathbf{x}_0, \mathbf{u}(t)) \in \mathcal{S}_\beta$ ,  $\forall t \in \mathfrak{T}$ , (Debeljkovic et al. 1997.b, 1997.c).

**Definition 2.1.2.6** System (1) with initial function (2), is *finite time stable* with respect to  $\{t_0, \mathfrak{T}, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$ , iff  $\|\mathbf{x}(t_0)\|^2 = \|\mathbf{x}_0\|^2 < \alpha$ , implies  $\|\mathbf{x}(t)\|^2 < \beta$ ,  $\forall t \in \mathfrak{T}$ , (Debeljkovic et al. 2010).

**Definition 2.1.2.7** System (1) with initial function (2), is *attractive practically stable* with respect to  $\{t_0, \mathfrak{T}, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$ , iff  $\|\mathbf{x}(t_0)\|_p^2 = \|\mathbf{x}_0\|_p^2 < \alpha$ , implies:  $\|\mathbf{x}(t)\|_p^2 < \beta$ ,  $\forall t \in \mathfrak{T}$ , with property

that:  $\lim_{k \rightarrow \infty} \|\mathbf{x}(t)\|_p^2 \rightarrow 0$ , (Debeljkovic et al. 2010).

### STABILITY THEOREMS - Dependent delay stability conditions

**Theorem 2.1.2.1** System (1) with the initial function (2) is finite time stable with respect to  $\{\alpha, \beta, \tau, \mathfrak{I}\}$  if the following condition is satisfied

$$\|\Phi(t)\|_2 < \frac{\sqrt{\beta/\alpha}}{1 + \tau\|A_1\|_2}, \quad \forall t \in [0, T] \quad (3)$$

$\|(\cdot)\|$  is Euclidean norm and  $\Phi(t)$  is fundamental matrix of system (1), (Nenadic et al. 1997), (Debeljkovic et al. 1997.a).

When  $\tau = 0$  or  $\|A_1\| = 0$ , the problem is reduced to the case of the ordinary linear systems, (Angelo 1974).

**Theorem 2.1.2.2** System (1) with initial function (2) is finite time stable w.r.t.  $\{\alpha, \beta, \tau, T\}$  if the following condition is satisfied:

$$e^{\mu(A_0)t} < \frac{\sqrt{\beta/\alpha}}{1 + \tau\|A_1\|_2}, \quad \forall t \in [0, T], \quad (4)$$

where  $\|(\cdot)\|$  denotes Euclidean norm, (Debeljkovic et al. 1997.b).

**Theorem 2.1.2.3** System (1) with the initial function (2) is finite time stable with respect to  $\{\alpha, \beta, \tau, T, \mu_2(A_0) \neq 0\}$  if the following condition is satisfied:

$$e^{\mu(A_0)t} < \frac{\beta/\alpha}{1 + \mu_2^{-1}(A_0) \cdot \|A_1\|_2 \cdot (1 - e^{-\mu_2(A_0)\tau})}, \quad \forall t \in [0, T], \quad (5)$$

(Debeljkovic et al. 1997.c, 1997.d).

**Theorem 2.1.2.4** System (1) with the initial function (2) is finite time stable with respect to  $\{\sqrt{\alpha}, \sqrt{\beta}, \tau, T, \mu(A_0) = 0\}$  if the following condition is satisfied:

$$1 + \tau\|A_1\|_2 < \sqrt{\beta/\alpha}, \quad \forall t \in [0, T], \quad (6)$$

(Debeljkovic et al. 1997.d).

Results that will be presented in the sequel enable to check finite time stability of the systems to be considered, namely the system given by (1) and (2), without finding the fundamental matrix or corresponding matrix measure.

Equation (2) can be rewritten in it's general form as:

$$\mathbf{x}(t_0 + \vartheta) = \Psi_x(\vartheta), \quad \Psi_x(\vartheta) \in \mathcal{C}[-\tau, 0], \quad -\tau \leq \vartheta \leq 0, \quad (7)$$

where  $t_0$  is the initial time of observation of the system (1) and  $\mathcal{C}[-\tau, 0]$  is a Banach space of continuous functions over a time interval of length  $\tau$ , mapping the interval  $[(t-\tau), t]$  into  $\mathbb{R}^n$  with the norm defined in the following manner:

$$\|\Psi\|_{\mathcal{C}} = \max_{-\tau \leq \vartheta \leq 0} \|\Psi(\vartheta)\|. \quad (8)$$

It is assumed that the usual smoothness conditions are present so that there is no difficulty with questions of existence, uniqueness, and continuity of solutions with respect to initial data. Moreover one can write:

$$\mathbf{x}(t_0 + \vartheta) = \Psi_x(\vartheta), \quad (9)$$

as well as:

$$\dot{\mathbf{x}}(t_0) = \mathbf{f}(t_0, \Psi_x(\vartheta)). \quad (10)$$

**Theorem 2.1.2.5** System given by (1) with initial function (2) is finite time stable w.r.t.  $\{\alpha, \beta, t_0, \mathfrak{I}\}$  if the following condition is satisfied:

$$\left(1 + (t - t_0)\sigma_{\max}\right)^2 e^{2(t-t_0)\sigma_{\max}} < \frac{\beta}{\alpha}, \quad \forall t \in \mathfrak{I}, \quad (11)$$

$\sigma_{\max}(\cdot)$  being the largest singular value of matrix  $(\cdot)$ , namely

$$\sigma_{\max} = \sigma_{\max}(A_0) + \sigma_{\max}(A_1). \quad (12)$$

(Debeljkovic et al. 1998.c) and (Lazarevic et al. 2000).

**Remark 2.1.2.1** In the case when in the Theorem 2.1.2.5  $A_1 = 0$ , e.g.  $A_1$  is null matrix, we have the result similar to that presented in (Angelo 1974).

Before presenting our crucial result, we need some discussion and explanations, as well some additional results.

For the sake of completeness, we present the following result (Lee & Dianat 1981).

**Lemma 2.1.2.1** Let us consider the system (1) and let  $P_1(t)$  be characteristic matrix of dimension  $(n \times n)$ , continuous and differentiable over time interval  $[0, \tau]$  and 0 elsewhere, and a set:

$$V(\mathbf{x}_t, \tau) = \left( \mathbf{x}(t) + \int_0^h P_1(\tau) \mathbf{x}(t-\tau) d\tau \right) P_0 \left( \mathbf{x}(t) + \int_0^h P_1(\tau) \mathbf{x}(t-\tau) d\tau \right), \quad (13)$$

where  $P_0 = P_0^* > 0$  is Hermitian matrix and  $\mathbf{x}_t(\vartheta) = \mathbf{x}(t + \vartheta)$ ,  $\vartheta \in [-\tau, 0]$ .

$$\text{If:} \quad P_0(A_0 + P_1(0)) + (A_0 + P_1(0))^* P_0 = -Q, \quad (14)$$

$$\dot{P}_1(\kappa) = (A_0 + P_1(0))P_1(\kappa), \quad 0 \leq \kappa \leq \tau, \quad (15)$$

where  $P_1(\tau) = A_1$  and  $Q = Q^* > 0$  is Hermitian matrix, then (Lee & Dianat 1981):

$$\dot{V}(\mathbf{x}_t, \tau) = \frac{d}{dt} V(\mathbf{x}_t, \tau) < 0. \quad (16)$$



Equation (13) defines Lyapunov's function for the system (1) and  $*$  denotes conjugate transpose of matrix.

In the paper (Lee, Dianat 1981) it is emphasized that the key to the success in the construction of a Lyapunov function corresponding to the system (1) is the existence of **at least one solution**  $P_1(t)$  of (15) with boundary condition  $P_1(\tau) = A_1$ .

In other words, it is required that the nonlinear algebraic matrix equation:

$$e^{(A_0 + P_1(0))\tau} P_1(0) = A_1, \quad (17)$$

has **at least one** solution for  $P_1(0)$ .

**Theorem 2.1.2.6** Let the system be described by (1). If for *any* given positive definite Hermitian matrix  $Q$  there exists a positive definite Hermitian matrix  $P_0$ , such that:

$$P_0(A_0 + P_1(0)) + (A_0 + P_1(0))^* P_0 + Q = 0, \quad (18)$$

where for  $\vartheta \in [0, \tau]$  and  $P_1(\vartheta)$  satisfies:

$$\dot{P}_1(\vartheta) = (A_0 + P_1(0))P_1(\vartheta), \quad (19)$$

with boundary condition  $P_1(\tau) = A_1$  and  $P_1(\tau) = 0$  elsewhere, then the system is asymptotically stable, (Lee, Dianat 1981).

**Theorem 2.1.2.7** Let the system be described by (1) and furthermore, let (17) have solution for  $P_1(0)$ , which is nonsingular. Then, system (1) is asymptotically stable if (19) of Theorem 2.1.2.6 is satisfied, (Lee, Dianat 1981).

Necessary and sufficient conditions for the stability of the system are derived by Lyapunov's direct method through construction of the corresponding "energy" function. This function is known to exist if a solution  $P_1(0)$  of the algebraic nonlinear matrix equation  $A_1 = \exp \tau(A_0 + P_1(0)) \cdot P_1(0)$  can be determined.

It is asserted, (Lee, Dianat 1981), that derivative sign of a Lyapunov function (Lemma 2.1.2.1) and thereby asymptotic stability of the system (Theorem 2.1.2.6 and Theorem 2.1.2.7) can be determined based on the knowledge of **only one or any**, solution of the particular nonlinear matrix equation.

We now demonstrate that Lemma 2.1.2.1 should be improved since it does not take into account all possible solutions for (17). The counterexample, based on original approach and supported by the Lambert function application, is given in (Stojanovic & Debeljkovic 2006), (Debeljkovic & Stojanovic 2008).

The final results, that we need in the sequel, should be:

**Lemma 2.1.2.2** Suppose that there exist(s) the solution(s)  $P_1(0)$  of (19) and let the Lyapunov's function be (13). Then,  $\dot{V}(\mathbf{x}_t, \tau) < 0$  **if and only if** for any matrix  $Q = Q^* > 0$  there exists matrix  $P_0 = P_0^* > 0$  such that (5) holds for **all** solution(s)  $P_1(0)$ , (Stojanovic & Debeljkovic 2006) and (Debeljkovic & Stojanovic 2008).

**Remark 2.1.2.1** The necessary condition of Lemma 2.1.2.2. follows directly from the proof of Theorem 2 in (Lee & Dianat 1981) and (Stojanovic & Debeljkovic 2006).

**Theorem 2.1.2.8** Suppose that there exist(s) the solution(s) of  $P_1(0)$  of (17). Then, the system (1) is asymptotically stable if for any matrix  $Q = Q^* > 0$  there exists matrix  $P_0 = P_0^* > 0$  such that (14) holds for **all** solutions  $P_1(0)$  of (17), (Stojanovic & Debeljkovic 2006) and (Debeljkovic & Stojanovic 2008).

**Remark 2.1.2.2** Statements Lemma 2.1.2.2. and Theorems 2.1.2.7 and Theorems 2.1.2.8 require that corresponding conditions are fulfilled for any solution  $P_1(0)$  of (17).

These matrix conditions are analogous to the following known scalar condition of asymptotic stability.

System (1) is asymptotically stable iff the condition  $\text{Re}(s) < 0$  holds for **all** solutions  $s$  of :

$$f(s) = \det(sI - A_0 - e^{-s\tau} A_1) = 0. \quad (20)$$

Now, we can present our main result, concerning practical stability of system (1).

**Theorem 2.1.2.9** System (1) with initial function (2), is attractive practically stable with respect to  $\{t_0, \mathfrak{T}, \alpha, \beta, \|(\cdot)\|^2\}$ ,  $\alpha < \beta$ , if there exist a positive real number  $q$ ,  $q > 1$ , such that:

$$\|\mathbf{x}(t+\tau)\|_{P_0} \leq \sup_{\vartheta \in [-\tau, 0]} \|\mathbf{x}(t+\vartheta)\|_{P_0} < q \|\mathbf{x}(t)\|_{P_0}, \quad q > 1, t \geq t_0, \forall t \in \mathfrak{T}, \forall \mathbf{x}(t) \in \mathcal{S}_\beta, \quad (21)$$

and if for any matrix  $Q = Q^* > 0$  there exists matrix  $P_0 = P_0^* > 0$  such that (14) holds for **all** solutions  $P_1(0)$  of (17) and if the following conditions are satisfied (Debeljkovic et al. 2011.b):

$$e^{\bar{\lambda}_{\max}(\bar{\Upsilon})(t-t_0)} < \frac{\beta}{\alpha}, \quad \forall t \in \mathfrak{T}, \quad (22)$$

where:

$$\bar{\lambda}_{\max}(\bar{\Upsilon}) = \bar{\lambda}_{\max} \left( \mathbf{x}^T(t) \left( P_0 A_1 P_0^{-1} A_1^T P_0 + q^2 P_0 \right) \mathbf{x}(t) : \mathbf{x}^T(t) P_0 \mathbf{x}(t) = 1 \right), \quad (23)$$

**Proof.** Define tentative aggregation function, as:

$$\begin{aligned} V(\mathbf{x}_t, \tau) = & \mathbf{x}^T(t) P_0 \mathbf{x}(t) + \int_0^\tau \int_0^\tau \mathbf{x}^T(t-\nu) P_1^T(\nu) P_0 P_1(\eta) \mathbf{x}(t-\eta) d\nu d\eta \\ & + \mathbf{x}^T(t) P_0 \int_0^\tau P_1(\eta) \mathbf{x}(t-\eta) d\eta + \int_0^\tau \mathbf{x}^T(t-\eta) P_1(\eta) d\eta \end{aligned} \quad (24)$$

The total derivative  $\dot{V}(t, \mathbf{x}(t))$  along the trajectories of the system, yields<sup>1</sup>

<sup>1</sup> Under conditions of Lemma 2.1.2.1.



$$\dot{V}(\mathbf{x}_t, \tau) = \left( \mathbf{x}(t) + \int_0^\tau P_1(\eta) \mathbf{x}(t-\eta) d\eta \right)^T \times (-Q) \times \left( \mathbf{x}(t) + \int_0^\tau P_1(\eta) \mathbf{x}(t-\eta) d\eta \right), \quad (25)$$

and since,  $(-Q)$  is negative definite and obviously  $\dot{V}(\mathbf{x}_t, \tau) < 0$ , time delay system (1) possesses attractivity property.

Furthermore, it is obvious that

$$\begin{aligned} \frac{dV(\mathbf{x}_t, \tau)}{dt} &= \frac{d(\mathbf{x}^T(t) P_0 \mathbf{x}(t))}{dt} + \frac{d}{dt} \left( \int_0^\tau \int_0^\tau \mathbf{x}^T(t-\nu) P_1^T(\nu) P_0 P_1(\eta) \mathbf{x}(t-\eta) d\nu d\eta \right. \\ &\quad \left. + \mathbf{x}^T(t) P_0 \int_0^\tau P_1(\eta) \mathbf{x}(t-\eta) d\eta + \int_0^\tau \mathbf{x}^T(t-\eta) P_1(\eta) d\eta \right) \end{aligned} \quad (26)$$

so, the standard procedure, leads to:

$$\frac{d}{dt}(\mathbf{x}^T(t) P_0 \mathbf{x}(t)) = \mathbf{x}^T(t) (A_0^T P_0 + P_0 A_0) \mathbf{x}(t) + 2\mathbf{x}^T(t) P_0 A_1 \mathbf{x}(t-\tau), \quad \text{or} \quad (27)$$

$$\frac{d}{dt}(\mathbf{x}^T(t) P_0 \mathbf{x}(t)) = \mathbf{x}^T(t) (A_0^T P_0 + P_0 A_0 + Q) \mathbf{x}(t) + 2\mathbf{x}^T(t) P_0 A_1 \mathbf{x}(t-\tau) - \mathbf{x}^T(t) Q \mathbf{x}(t) \quad (28)$$

From the fact that the time delay system under consideration, upon the statement of the *Theorem*, is asymptotically stable<sup>2</sup>, follows:

$$\frac{d}{dt}(\mathbf{x}^T(t) P_0 \mathbf{x}(t)) = -\mathbf{x}^T(t) Q \mathbf{x}(t) + 2\mathbf{x}^T(t) P_0 A_1 \mathbf{x}(t-\tau), \quad (29)$$

and using very well known inequality<sup>3</sup>, with particular choice:

$$\mathbf{x}^T(t) \Gamma \mathbf{x}^T(t) = \mathbf{x}^T(t) P_0 \mathbf{x}^T(t) > 0, \quad \forall t \in \mathfrak{I}, \quad (30)$$

and the fact that:

$$\mathbf{x}^T(t) Q \mathbf{x}(t) > 0, \quad \forall t \in \mathfrak{I}, \quad (31)$$

is positive definite quadratic form, one can get:

$$\begin{aligned} \frac{d}{dt}(\mathbf{x}^T(t) P_0 \mathbf{x}(t)) &= 2\mathbf{x}^T(t) P_0 A_1 \mathbf{x}(t-\tau) \\ &\leq \mathbf{x}^T(t) P_0 A_1 P_0^{-1} A_1^T P_0 \mathbf{x}(t) + \mathbf{x}^T(t-\tau) P_0 \mathbf{x}(t-\tau) \end{aligned} \quad (32)$$

and using (21), (Su & Huang 1992), (Xu & Liu 1994) and (Mao 1997), clearly (32) reduces to:

<sup>2</sup> Clarify *Theorem 2.1.2.8*.

<sup>3</sup>  $2\mathbf{u}^T(t) \mathbf{v}(t-\tau) \leq \mathbf{u}^T(t) \Gamma^{-1} \mathbf{u}(t) + \mathbf{v}^T(t-\tau) \Gamma \mathbf{v}(t-\tau)$ ,  $\Gamma = \Gamma^T > 0$ .

$$\frac{d}{dt}(\mathbf{x}^T(t)P_0\mathbf{x}(t)) < \mathbf{x}^T(t)(P_0A_1P_0^{-1}A_1^TP_0 + q^2P)\mathbf{x}(t), \quad (33)$$

or, using (22), one can get:

$$\frac{d}{dt}(\mathbf{x}^T(t)P_0\mathbf{x}(t)) < \bar{\lambda}_{\max}(\bar{\Upsilon})\mathbf{x}^T(t)P_0\mathbf{x}(t), \quad (34)$$

or:

$$\int_{t_0}^t \frac{d(\mathbf{x}^T(t)P_0\mathbf{x}(t))}{\mathbf{x}^T(t)P_0\mathbf{x}(t)} < \int_{t_0}^t \bar{\lambda}_{\max}(\bar{\Upsilon}) dt, \quad (35)$$

and:

$$\mathbf{x}^T(t)P_0\mathbf{x}(t) < \mathbf{x}^T(t_0)P_0\mathbf{x}(t_0)e^{\bar{\lambda}_{\max}(\bar{\Upsilon})(t-t_0)}. \quad (36)$$

Finally, if one applies the first condition, given in *Definition 2.1.2.7*, and then:

$$\mathbf{x}^T(t)P_0\mathbf{x}(t) < \alpha \cdot e^{\bar{\lambda}_{\max}(\bar{\Upsilon})(t-t_0)}, \quad (37)$$

and by applying the basic condition (22) of the *Theorem 2.1.2.9*, one can get

$$\mathbf{x}^T(t)P_0\mathbf{x}(t) < \alpha \cdot \frac{\beta}{\alpha} < \beta, \quad \forall t \in \mathfrak{T}. \quad Q.E.D. \quad (38)$$

### STABILITY THEOREMS - Independent delay stability conditions

**Theorem 2.1.2.10** Time delayed system (1), is *finite time stable* w.r.t.  $\{t_0, \mathfrak{T}, \alpha, \beta, \|(\cdot)\|^2\}$ ,  $\alpha < \beta$ , if there exist a positive real number  $q$ ,  $q > 1$ , such that:

$$\|\mathbf{x}(t+\tau)\| \leq \sup_{\vartheta \in [-\tau, 0]} \|\mathbf{x}(t+\vartheta)\| < q\|\mathbf{x}(t)\|, \quad q > 1, t \geq t_0, \forall t \in \mathfrak{T}, \forall \mathbf{x}(t) \in \mathcal{S}_\beta, \quad (39)$$

if the following condition is satisfied (*Debeljkovic et al. 2010*):

$$e^{\bar{\lambda}_{\max}(\Psi)(t-t_0)} < \frac{\beta}{\alpha}, \quad \forall t \in \mathfrak{T}, \quad (40)$$

where:

$$\lambda_{\max}(\Pi) = \lambda_{\max}(A_0^T + A_0 + A_1A_1^T + q^2I). \quad (41)$$

**Proof.** Define tentative aggregation function as:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{x}(t) + \int_{t-\tau}^t \mathbf{x}^T(\vartheta)\mathbf{x}(\vartheta)d\vartheta. \quad (42)$$

The total derivative  $\dot{V}(t, \mathbf{x}(t))$  along the trajectories of the system, yields:

$$\begin{aligned} \dot{V}(t, \mathbf{x}(t)) &= \frac{d}{dt}(\mathbf{x}^T(t)\mathbf{x}(t)) + \frac{d}{dt} \int_{t-\tau}^t \mathbf{x}^T(\vartheta)\mathbf{x}(\vartheta)d\vartheta \\ &= \mathbf{x}^T(t)(A_0^T + A_0)\mathbf{x}(t) + 2\mathbf{x}^T(t)A_1\mathbf{x}(t-\tau) + \mathbf{x}^T(t)\mathbf{x}(t) + \mathbf{x}^T(t-\tau)\mathbf{x}(t-\tau). \end{aligned} \quad (43)$$

From (43), it is obvious:

$$\frac{d}{dt}(\mathbf{x}^T(t)\mathbf{x}(t)) = \mathbf{x}^T(t)(A_0^T + A_0)\mathbf{x}(t) + 2\mathbf{x}^T(t)A_1\mathbf{x}(t-\tau), \quad (44)$$

and based on the previous inequality and with the particular choice:

$$\mathbf{x}^T(t)\Gamma\mathbf{x}(t) = \mathbf{x}^T(t)\mathbf{x}(t) > 0, \quad \forall t \in \mathfrak{S}, \quad \text{so that} \quad (45)$$

$$\frac{d}{dt}(\mathbf{x}^T(t)\mathbf{x}(t)) \leq \mathbf{x}^T(t)(A_0^T + A_0)\mathbf{x}(t) + \mathbf{x}^T(t)A_1A_1^T\mathbf{x}(t) + \mathbf{x}^T(t-\tau)I\mathbf{x}(t-\tau), \quad (46)$$

Based on (39), (Su & Huang 1992), (Xu & Liu 1994) and (Mao 1997), it is clear that (46) reduces to:

$$\frac{d}{dt}(\mathbf{x}^T(t)\mathbf{x}(t)) < \mathbf{x}^T(t)(A_0^T + A_0 + A_1A_1^T + q^2I)\mathbf{x}(t) < \lambda_{\max}(\Pi)\mathbf{x}^T(t)\mathbf{x}(t), \quad (47)$$

where matrix  $\Pi$  is defined by (41). From (47) one can get:

$$\int_{t_0}^t \frac{d(\mathbf{x}^T(t)\mathbf{x}(t))}{\mathbf{x}^T(t)\mathbf{x}(t)} < \int_{t_0}^t \lambda_{\max}(\Pi)dt, \quad \text{and:} \quad (48)$$

$$\mathbf{x}^T(t)\mathbf{x}(t) < \mathbf{x}^T(t_0)\mathbf{x}(t_0)e^{\lambda_{\max}(\Pi)(t-t_0)} < \alpha \cdot e^{\lambda_{\max}(\Pi)(t-t_0)} < \alpha \cdot \frac{\beta}{\alpha} < \beta, \quad \forall t \in \mathfrak{S}. \quad (49)$$

under the identical technique from the previous proof of *Theorem 2.1.2.9*. *Q.E.D.*

## 2.2 Discrete time delay systems

### 2.2.1 Discrete time delay systems – stability in the sense of Lyapunov

#### ASYMPTOTIC STABILITY-APPROACH BASED ON THE RESULTS OF TISSIR AND HMAMED<sup>4</sup>

In particular case we are concerned with a linear, autonomous, multivariable discrete time delay system in the form:

$$\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1), \quad (50)$$

<sup>4</sup> (Tissir & Hmamed 1996).

The equation (50) is referred to as homogenous or the unforced state equation,  $\mathbf{x}(k)$  is the state vector,  $A_0$  and  $A_1$  are constant system matrices of appropriate dimensions.

**Theorem 2.2.1.1.** System (50) is asymptotically stable if:

$$\|A_0\| + \|A_1\| < 1, \quad (51)$$

holds, (Mori et al. 1981).

**Theorem 2.2.1.2.** System (50) is asymptotically stable, independent of delay, if:

$$\|A_1\| < \frac{\sigma_{\min}\left(Q^{-\frac{1}{2}}\right)}{\sigma_{\max}\left(Q^{-\frac{1}{2}}A_0^T P\right)}, \quad (52)$$

where  $P$  is the solution of the *discrete Lyapunov matrix equation*:

$$A_0^T P A_0 - P = -(2Q + A_1^T P A_1), \quad (53)$$

where  $\sigma_{\max}(\cdot)$  and  $\sigma_{\min}(\cdot)$  are the maximum and minimum singular values of the matrix  $(\cdot)$ , (Debeljkovic et al. 2004.a, 2004.b, 2004.d, 2005.a).

**Theorem 2.2.1.3** Suppose the matrix  $(Q - A_1^T P A_1)$  is regular. System (50) is asymptotically stable, independent of delay, if:

$$\|A_1\| < \frac{\sigma_{\min}\left((Q - A_1^T P A_1)^{-\frac{1}{2}}\right)}{\sigma_{\max}\left(Q^{-\frac{1}{2}}A_0^T P\right)}, \quad (54)$$

where  $P$  is the solution of the *discrete Lyapunov matrix equation*:

$$A_0^T P A_0 - P = -2Q, \quad (55)$$

where  $\sigma_{\max}(\cdot)$  and  $\sigma_{\min}(\cdot)$  are the maximum and minimum singular values of the matrix  $(\cdot)$ , (Debeljkovic et al. 2004.c, 2004.d, 2005.a, 2005.b).

### ASYMPTOTIC STABILTY- LYAPUNOV BASED APPROACH

A linear, autonomous, multivariable linear discrete time-delay system can be represented by the difference equation:

$$\mathbf{x}(k+1) = \sum_{j=0}^N A_j \mathbf{x}(k-h_j), \quad \mathbf{x}(\vartheta) = \boldsymbol{\psi}(\vartheta), \quad \vartheta \in \{-h_N, -h_N+1, \dots, 0\} \triangleq \Delta, \quad (56)$$

where  $\mathbf{x}(k) \in \mathbb{R}^n$ ,  $A_j \in \mathbb{R}^{n \times n}$ ,  $0 = h_0 < h_1 < h_2 < \dots < h_N$  - are integers and represent the systems time delays. Let  $V(\mathbf{x}(k)): \mathbb{R}^n \rightarrow \mathbb{R}$ , so that  $V(\mathbf{x}(k))$  is bounded for, and for which  $\|\mathbf{x}(k)\|$  is also bounded.

**Lemma 2.2.1.1** For any two matrices of the same dimensions  $F$  and  $G$  and for some positive constant  $\varepsilon$  the following statement is true (Wang & Mau 1997):

$$(F + G)^T (F + G) \leq (1 + \varepsilon)F^T F + (1 + \varepsilon^{-1})G^T G. \quad (57)$$

**Theorem 2.2.1.4** Suppose that  $A_0$  is not null matrix. If for any given matrix  $Q = Q^T > 0$  there exists matrix  $P = P^T > 0$  such that the following matrix equation is fulfilled:

$$(1 + \varepsilon_{\min})A_0^T P A_0 + (1 + \varepsilon_{\min}^{-1})A_1^T P A_1 - P = -Q, \quad (58)$$

where:

$$\varepsilon_{\min} = \frac{\|A_1\|_2}{\|A_0\|_2}, \quad (59)$$

then, system (56) is asymptotically stable, (Stojanovic & Debeljkovic 2005.b).

**Corollary 2.2.1.1** If for any given matrix  $Q = Q^T > 0$  there exists matrix  $P = P^T > 0$  being the solution of the following Lyapunov matrix equation:

$$A_0^T P A_0 - P = -\frac{\varepsilon_{\min}}{1 + \varepsilon_{\min}} Q, \quad (60)$$

where  $\varepsilon_{\min}$  is defined by (59) and if the following condition is satisfied:

$$\sigma_{\max}(A_0) + \sigma_{\max}(A_1) < \frac{\lambda_{\min}(Q - P)}{\sigma_{\max}(A_0)\lambda_{\max}(P)}, \quad (61)$$

then, system (59) is asymptotically stable, (Stojanovic & Debeljkovic 2005.b).

**Corollary 2.2.1.2** If for any given matrix  $Q = Q^T > 0$  there exists matrix  $P = P^T > 0$  being solution of the following matrix equation:

$$(1 + \varepsilon_{\min})A_0^T P A_0 - P = -\varepsilon_{\min} Q, \quad (62)$$

where  $\varepsilon_{\min}$  is defined by (59), and if the following condition is satisfied, too:

$$\sigma_{\max}(A_0) + \sigma_{\max}(A_1) < \frac{\lambda_{\min}(Q)}{\sigma_{\max}(A_0)\lambda_{\max}(P)}, \quad (63)$$

then, system (56) is asymptotically stable, (Stojanovic & Debeljkovic 2005.b).

**Theorem 2.2.1.5** If for any given matrix  $Q = Q^T > 0$  there exists matrix  $P = P^T > 0$  such that the following matrix equation is fulfilled:

$$2A_0^T P A_0 + 2A_1^T P A_1 - P = -Q, \quad (64)$$

then, system (56) is asymptotically stable, (Stojanovic & Debeljkovic 2006.a).

**Corollary 2.2.1.3** System (56) is asymptotically stable, independent of delay, if :

$$\sigma_{\max}^2(A_1) < \frac{\lambda_{\min}(2Q - P)}{2\sigma_{\max}^2(P^{\frac{1}{2}})}, \quad (65)$$

where, for any given matrix  $Q = Q^T > 0$  there exists matrix  $P = P^T > 0$  being the solution of the following *Lyapunov matrix equation* (Stojanovic & Debeljkovic 2006.a):

$$A_0^T P A_0 - P = -Q. \quad (66)$$

**Corollary 2.2.1.4** System (56) is asymptotically stable, independent of delay, if:

$$\sigma_{\max}^2(A_1) < \frac{\lambda_{\min}(Q)}{2\sigma_{\max}^2(P^{\frac{1}{2}})}, \quad (67)$$

where, for any given matrix  $Q = Q^T > 0$  there exists matrix  $P = P^T > 0$  being the solution of the following *Lyapunov matrix equation* (Stojanovic & Debeljkovic 2006.a):

$$2A_0^T P A_0 - P = -Q. \quad (68)$$

## 2.2.2 Discrete time delay systems – Stability over finite time interval

As far as we know the only result, considering and investigating the problem of non-Lyapunov analysis of linear discrete time delay systems, is one that has been mentioned in the introduction, e.g. (Debeljkovic & Aleksendric 2003), where this problem has been considered for the first time.

Investigating the system stability throughout the discrete fundamental matrix is very cumbersome, so there is a need to find some more efficient expressions that should be based on calculation appropriate eigenvalues or norm of appropriate systems matrices as it has been done in continuous case.

### SYSTEM DESCRIPTION

Consider a linear discrete system with state delay, described by:

$$\mathbf{x}(k+1) = A_0 \mathbf{x}(k) + A_1 \mathbf{x}(k-1), \quad (69)$$

with known vector valued function of initial conditions:

$$\mathbf{x}(k_0) = \boldsymbol{\psi}(k_0), \quad -1 \leq k_0 \leq 0, \quad (70)$$

where  $\mathbf{x}(k) \in \mathbb{R}^n$  is a state vector and with constant matrices  $A_0$  and  $A_1$  of appropriate dimensions. Time delay is constant and equals one. For some other purposes, the state delay equation can be represented in the following way:

$$\mathbf{x}(k+1) = A_0 \mathbf{x}(k) + \sum_{j=1}^M A_j \mathbf{x}(k-h_j), \quad (71)$$



$$\mathbf{x}(\vartheta) = \boldsymbol{\Psi}(\vartheta), \quad \vartheta \in \{-h, -h+1, \dots, 0\}, \quad (72)$$

where  $\mathbf{x}(k) \in \mathbb{R}^n$ ,  $A_j \in \mathbb{R}^{n \times n}$ ,  $j=1,2$ ,  $h$  - is integer representing system time delay and  $\boldsymbol{\Psi}(\cdot)$  is a priori known vector function of initial conditions, as well.

### STABILITY DEFINITIONS

**Definition 2.2.2.1** System, given by (69), is *attractive practically stable* with respect to

$\{k_0, \mathcal{K}_N, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$ , iff  $\|\mathbf{x}(k_0)\|_{A_0^T P A_0}^2 = \|\mathbf{x}_0\|_{A_0^T P A_0}^2 < \alpha$ , implies:

$$\|\mathbf{x}(k)\|_{A_0^T P A_0}^2 < \beta, \quad \forall k \in \mathcal{K}_N$$

with property that  $\lim_{k \rightarrow \infty} \|\mathbf{x}(k)\|_{A_0^T P A_0}^2 \rightarrow 0$ , (Nestorovic et al. 2011).

**Definition 2.2.2.2** System, given by (69), is *practically stable* with respect to  $\{k_0, \mathcal{K}_N, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$ ,

if and only if:  $\|\mathbf{x}_0\|^2 < \alpha$ , implies  $\|\mathbf{x}(k)\|^2 < \beta$ ,  $\forall k \in \mathcal{K}_N$ .

**Definition 2.2.2.3** System given by (69), is *attractive practically unstable* with respect

$\{k_0, \mathcal{K}_N, \alpha, \beta, \|\cdot\|^2\}$ ,  $\alpha < \beta$ , if for  $\|\mathbf{x}_0\|_{A_0^T P A_0}^2 < \alpha$ , there exist a moment:  $k = k^* \in \mathcal{K}_N$ , so that

the next condition is fulfilled  $\|\mathbf{x}(k^*)\|_{A_0^T P A_0}^2 \geq \beta$  with property that  $\lim_{k \rightarrow \infty} \|\mathbf{x}(k)\|_{A_0^T P A_0}^2 \rightarrow 0$ ,

(Nestorovic et al. 2011).

**Definition 2.2.2.4** System given by (69), is *practically unstable* with respect

$\{k_0, \mathcal{K}_N, \alpha, \beta, \|\cdot\|^2\}$ ,  $\alpha < \beta$ , if for  $\|\mathbf{x}_0\|^2 < \alpha$  there exist a moment:  $k = k^* \in \mathcal{K}_N$ , such that the

next condition is fulfilled  $\|\mathbf{x}(k^*)\|^2 \geq \beta$  for some  $k = k^* \in \mathcal{K}_N$ .

**Definition 2.2.2.5** Linear discrete time delay system (69) is *finite time stable* with respect to

$\{\alpha, \beta, k_0, k_N, \|\cdot\|\}$ ,  $\alpha \leq \beta$ , if and only if for every trajectory  $\mathbf{x}(k)$  satisfying initial function,

(70) such that  $\|\mathbf{x}(k)\| < \alpha$ ,  $k = 0, -1, -2, \dots, -N$  imply  $\|\mathbf{x}(k)\|^2 < \beta$ ,  $k \in \mathcal{K}_N$ , (Aleksendric 2002), (Aleksendric & Debeljkovic 2002), (Debeljkovic & Aleksendric 2003).

This Definition is analogous to that presented, for the first time, in (Debeljković et al. 1997.a, 1997.b, 1997.c, 1997.d) and (Nenadic et al. 1997).

### SOME PREVIOUS RESULTS

**Theorem 2.2.2.1** Linear discrete time delay system (69), is *finite time stable* with respect to

$\{\alpha, \beta, M, N, \|\cdot\|^2\}$ ,  $\alpha < \beta$ ,  $\alpha, \beta \in \mathbb{R}_+$ , if the following sufficient condition is fulfilled:

$$\|\Phi(k)\| < \frac{\beta}{\alpha} \cdot \frac{1}{1 + \sum_{j=1}^M \|A_j\|}, \quad \forall k = 0, 1, \dots, N, \quad (73)$$

$\Phi(k)$  being fundamental matrix, (Aleksendric 2002), (Aleksendric & Debeljkovic 2002), (Debeljkovic & Aleksendric 2003).

This result is analogous to that, for the first time derived, in (Debeljkovic et al. 1997.a) for continuous time delay systems.

**Remark 2.2.2.1** The matrix measure is widely used when continuous time delay system are investigated, (Coppel 1965), (Desoer & Vidysagar 1975). The nature of discrete time delay enables one to use this approach as well as Bellman's principle, so the problem must be attack from the point of view which is based only on norms.

### STABILITY THEOREMS: PRACTICAL AND FINITE TIME STABILITY

**Theorem 2.2.2.2** System given by (71), with  $\det A_1 \neq 0$ , is *attractive practically stable* with respect to  $\left\{k_0, \mathcal{K}_N, \alpha, \beta, \|\cdot\|^2\right\}$ ,  $\alpha < \beta$ , if there exist  $P = P^T > 0$ , being the solution of:

$$2A_0^T P A_0 - P = -Q, \quad (74)$$

where  $Q = Q^T > 0$  and if the following conditions are satisfied (Nestorovic et al. 2011):

$$\|A_1\| < \sigma_{\min} \left( \left( Q - A_1^T P A_1 \right)^{\frac{1}{2}} \right) \sigma_{\max}^{-1} \left( Q^{\frac{1}{2}} A_0^T P \right), \quad (75)$$

$$\bar{\lambda}_{\max}^{\frac{1}{2}k}(\cdot) < \frac{\beta}{\alpha}, \quad \forall k \in \mathcal{K}_N, \quad (76)$$

where:

$$\bar{\lambda}_{\max}(\cdot) = \max \left\{ \mathbf{x}^T(k) A_1^T P A_1 \mathbf{x}(k) : \mathbf{x}^T(k) A_0^T P A_0 \mathbf{x}(k) = 1 \right\}. \quad (77)$$

**Proof.** Let us use a functional, as a possible aggregation function, for the system to be considered:

$$V(\mathbf{x}(k)) = \mathbf{x}^T(k) P \mathbf{x}(k) + \mathbf{x}^T(k-1) Q \mathbf{x}(k-1), \quad (78)$$

with matrices  $P = P^T > 0$  and  $Q = Q^T > 0$ .

Clearly, using the equation of motion of (69), we have:

$$\Delta V(\mathbf{x}(k)) = V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)), \quad (79)$$

or:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= \mathbf{x}^T(k+1) P \mathbf{x}(k+1) - \mathbf{x}^T(k) P \mathbf{x}(k) \\ &\quad + \mathbf{x}^T(k) Q \mathbf{x}(k) - \mathbf{x}^T(k-1) Q \mathbf{x}(k-1) \\ &= \mathbf{x}^T(k) \left( A_0^T P A_0 + Q - P \right) \mathbf{x}(k) + 2 \mathbf{x}^T(k) A_0^T P A_1 \mathbf{x}(k-1) \\ &\quad - \mathbf{x}^T(k-1) \left( Q - A_1^T P A_1 \right) \mathbf{x}(k-1) \end{aligned} \quad (80)$$

It has been shown, (Debeljković *et al.* 2004, 2008), that if:

$$2A_0^T P A_0 - P = -Q, \quad (81)$$

where  $P = P^T > 0$  and  $Q = Q^T > 0$ , then for:

$$V(\mathbf{x}(k)) = \mathbf{x}^T(k) P \mathbf{x}(k) + \mathbf{x}^T(k-1) Q \mathbf{x}(k-1), \quad (82)$$

the backward difference along the trajectories of the systems is:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \\ &= \mathbf{x}^T(k) (A_0^T P A_0 - P + Q) \mathbf{x}(k) + \mathbf{x}^T(k-1) (A_1^T P A_1 - Q) \mathbf{x}(k-1) \\ &\quad + \mathbf{x}^T(k) A_0^T P A_1 \mathbf{x}(k-1) + \mathbf{x}^T(k-1) A_1^T P A_0 \mathbf{x}(k) \end{aligned} \quad (83)$$

or:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= \mathbf{x}^T(k) (2A_0^T P A_0 - P + Q) \mathbf{x}(k) \\ &\quad + \mathbf{x}^T(k-1) (2A_1^T P A_1 - Q) \mathbf{x}(k-1) + \mathbf{x}^T(k) A_0^T P A_1 \mathbf{x}(k-1) \\ &\quad + \mathbf{x}^T(k-1) A_1^T P A_0 \mathbf{x}(k) - \mathbf{x}^T(k) A_0^T P A_0 \mathbf{x}(k) - \mathbf{x}^T(k-1) A_1^T P A_1 \mathbf{x}(k-1) \end{aligned} \quad (84)$$

and since we have to take into account (80), one can get:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= \mathbf{x}^T(k-1) (2A_1^T P A_1 - Q) \mathbf{x}(k-1) \\ &\quad - [A_0 \mathbf{x}(k) - A_1 \mathbf{x}(k-1)]^T P [A_0 \mathbf{x}(k) - A_1 \mathbf{x}(k-1)]. \end{aligned} \quad (85)$$

Since the matrix  $P = P^T > 0$ , it is more than obvious, that:

$$\Delta V(\mathbf{x}(k)) < \mathbf{x}^T(k-1) (2A_1^T P A_1 - Q) \mathbf{x}(k-1). \quad (86)$$

Combining the right sides of (80) and (86), yields:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= \mathbf{x}^T(k) (A_0^T P A_0 + Q - P) \mathbf{x}(k) + 2\mathbf{x}^T(k) A_0^T P A_1 \mathbf{x}(k-1) \\ &< \mathbf{x}^T(k-1) (A_1^T P A_1) \mathbf{x}(k-1) \end{aligned} \quad (87)$$

Using the very well known inequality, with particular choice:

$$\Gamma = \frac{1}{2} (A_1^T P A_1), \quad (88)$$

it can be obtained:

$$\begin{aligned} &\mathbf{x}^T(k) \left( A_0^T P A_0 + Q - P + A_0^T P A_1 \left( \frac{1}{2} A_1^T P A_1 \right)^{-1} A_1^T P A_0 \right) \mathbf{x}(k) \\ &+ \frac{1}{2} \mathbf{x}^T(k-1) (A_1^T P A_1) \mathbf{x}(k-1) < \mathbf{x}^T(k-1) (A_1^T P A_1) \mathbf{x}(k-1) \end{aligned} \quad (89)$$

$$\mathbf{x}^T(k) \left( 2A_0^T P A_0 + Q - P + A_0^T P A_0 \right) \mathbf{x}(k) < \frac{1}{2} \mathbf{x}^T(k-1) \left( A_1^T P A_1 \right) \mathbf{x}(k-1). \quad (90)$$

Since: 
$$2A_0^T P A_0 + Q - P = 0, \quad (91)$$

it is finally obtained:

$$\mathbf{x}^T(k) A_0^T P A_0 \mathbf{x}(k) < \frac{1}{2} \mathbf{x}^T(k-1) \left( A_1^T P A_1 \right) \mathbf{x}(k-1), \quad (92)$$

or: 
$$\mathbf{x}^T(k) A_0^T P A_0 \mathbf{x}(k) < \frac{1}{2} \bar{\lambda}_{\max}(\cdot) \mathbf{x}^T(k-1) A_0^T P A_0 \mathbf{x}(k-1), \quad (93)$$

where:

$$\bar{\lambda}_{\max}(\cdot) = \max \left\{ \mathbf{x}^T(k) A_1^T P A_1 \mathbf{x}(k) : \left( 2A_0^T P A_0 - P \right) = -Q, \mathbf{x}^T(k) A_0^T P A_0 \mathbf{x}(k) = 1 \right\}. \quad (94)$$

Since this manipulation is independent of  $k$ , it can be written:

$$\mathbf{x}^T(k+1) A_0^T P A_0 \mathbf{x}(k+1) < \frac{1}{2} \bar{\lambda}_{\max}(\cdot) \mathbf{x}^T(k) A_0^T P A_0 \mathbf{x}(k), \quad (95)$$

or:

$$\begin{aligned} \ln \mathbf{x}^T(k+1) A_0^T P A_0 \mathbf{x}(k+1) &< \ln \frac{1}{2} \bar{\lambda}_{\max}(\cdot) \mathbf{x}^T(k) A_0^T P A_0 \mathbf{x}(k) \\ &< \ln \frac{1}{2} \bar{\lambda}_{\max}(\cdot) + \ln \mathbf{x}^T(k) A_0^T P A_0 \mathbf{x}(k) \end{aligned} \quad (96)$$

and:

$$\ln \mathbf{x}^T(k+1) A_0^T P A_0 \mathbf{x}(k+1) - \ln \mathbf{x}^T(k) A_0^T P A_0 \mathbf{x}(k) < \ln \frac{1}{2} \bar{\lambda}_{\max}(\cdot). \quad (97)$$

It can be shown that:

$$\begin{aligned} &\sum_{j=k_0}^{k_0+k-1} \left( \ln \mathbf{x}^T(j+1) \mathbf{x}(j+1) - \ln \mathbf{x}^T(j) \mathbf{x}(j) \right) = \\ &= \ln \mathbf{x}^T(k_0+1) \mathbf{x}(k_0+1) + \ln \mathbf{x}^T(k_0+2) \mathbf{x}(k_0+2) + \dots + \\ &+ \ln \mathbf{x}^T(k_0+k-2+1) \mathbf{x}(k_0+k-2+1) + \ln \mathbf{x}^T(k_0+k-1+1) \mathbf{x}(k_0+k-1+1) \\ &- \left( \ln \mathbf{x}^T(k_0) \mathbf{x}(k_0) + \ln \mathbf{x}^T(k_0+1) \mathbf{x}(k_0+1) + \dots + \ln \mathbf{x}^T(k_0+k-1) \mathbf{x}(k_0+k-1) \right) \\ &= \ln \mathbf{x}^T(k_0+k) \mathbf{x}(k_0+k) - \ln \mathbf{x}^T(k_0) \mathbf{x}(k_0) \end{aligned} \quad (98)$$

If the summing  $\sum_{j=k_0}^{k_0+k-1}$  is applied to both sides of (97) for  $\forall k \in \mathcal{K}_N$ , one can obtain:

$$\begin{aligned}
& \sum_{j=k_0}^{k_0+k-1} \ln \mathbf{x}^T(k+1) A_0^T P A_0 \mathbf{x}(k+1) - \ln \mathbf{x}^T(k) A_0^T P A_0 \mathbf{x}(k) \\
& \leq \sum_{j=k_0}^{k_0+k-1} \ln \lambda_{\max}^2(\cdot) \leq \ln \prod_{j=k_0}^{k_0+k-1} \lambda_{\max}^2(\cdot)
\end{aligned} \tag{99}$$

so that, for (99), it seems to be:

$$\begin{aligned}
& \ln \mathbf{x}^T(k_0+k) A_0^T P A_0 \mathbf{x}(k_0+k) - \ln \mathbf{x}^T(k_0) A_0^T P A_0 \mathbf{x}(k_0) \\
& < \ln \prod_{j=k_0}^{k_0+k-1} \lambda_{\max}^2(\cdot) < \ln \bar{\lambda}_{\max}^2(\cdot), \quad \forall k \in \mathcal{K}_N
\end{aligned} \tag{100}$$

as well as:

$$\begin{aligned}
& \ln \mathbf{x}^T(k_0+k) A_0^T P A_0 \mathbf{x}(k_0+k) \leq \ln \prod_{j=k_0}^{k_0+k-1} \lambda_{\max}^2(\cdot) \\
& \leq \ln \bar{\lambda}_{\max}^2(\cdot) + \ln \mathbf{x}^T(k_0) A_0^T P A_0 \mathbf{x}(k_0) \quad \forall k \in \mathcal{K}_N
\end{aligned} \tag{101}$$

Taking into account fact that  $\|\mathbf{x}_0\|_{A_0^T P A_0}^2 < \alpha$  and basic condition of *Theorem 2.2.2.2*, (76), one can get:

$$\begin{aligned}
& \ln \mathbf{x}^T(k_0+k) A_0^T P A_0 \mathbf{x}(k_0+k) < \ln \bar{\lambda}_{\max}^2(\cdot) + \ln \mathbf{x}^T(k_0) A_0^T P A_0 \mathbf{x}(k_0) \\
& < \ln \alpha \cdot \bar{\lambda}_{\max}^2(\cdot) < \ln \alpha \cdot \frac{\beta}{\alpha} < \ln \beta, \quad \forall k \in \mathcal{K}_N. \quad Q.E.D.
\end{aligned} \tag{102}$$

**Remark 2.2.2.2** Assumption  $\det A_1 \neq 0$  do not reduce the generality of this result, since this condition is not crucial when discrete time systems are considered.

**Remark 2.2.2.3** Lyapunov asymptotic stability and finite time stability are independent concepts: a system that is finite time stable may not be Lyapunov asymptotically stable, conversely Lyapunov asymptotically stable system could not be finite time stable if, during the transients, its motion exceeds the pre-specified bounds ( $\beta$ ). Attraction property is guaranteed by (74) and (75), (*Debeljković et al. 2004*) and system motion within pre-specified boundaries is well provided by (76).

**Remark 2.2.2.4** For the numerical treatment of this problem  $\bar{\lambda}_{\max}(\cdot)$  can be calculated in the following way (*Kalman, Bertram 1960*):

$$\bar{\lambda}_{\max}(\cdot) = \max_{\mathbf{x}} \{ \cdot \} = \bar{\lambda}_{\max} \left( A_1^T P A_1 \left( A_0^T P A_0 \right)^{-1} \right). \tag{103}$$

**Remark 2.2.2.5** These results are in some sense analogous to those given in (*Amato et al. 2003*), although results presented there are derived for continuous time varying systems.

Now we proceed to develop delay independent criteria, for finite time stability of system under consideration, not to be necessarily asymptotic stable, e.g. so we reduce previous demand that basic system matrix  $A_0$  should be discrete stable matrix.

**Theorem 2.2.2.3** Suppose the matrix  $(I - A_1^T A_1) > 0$ . System given by (69), is finite time stable with respect to  $\{k_0, \mathcal{K}_N, \alpha, \beta, \|\cdot\|^2\}$ ,  $\alpha < \beta$ , if there exist a positive real number  $p$ ,  $p > 1$ , such that:

$$\|\mathbf{x}(k-1)\|^2 < p^2 \|\mathbf{x}(k)\|^2, \quad \forall k \in \mathcal{K}_N, \quad \forall \mathbf{x}(k) \in \mathcal{S}_\beta, \quad (104)$$

and if the following condition is satisfied (Nestorovic et al. 2011):

$$\lambda_{\max}^k(\cdot) < \frac{\beta}{\alpha}, \quad \forall k \in \mathcal{K}_N, \quad (105)$$

where: 
$$\lambda_{\max}(\cdot) = \lambda_{\max}\left(A_0^T (I - A_1^T A_1) A_0 + p^2 I\right). \quad (106)$$

**Proof.** Now we consider, again, system given by (69). Define:

$$V(\mathbf{x}(k)) = \mathbf{x}^T(k) \mathbf{x}(k) + \mathbf{x}^T(k-1) \mathbf{x}(k-1), \quad (107)$$

as a tentative Lyapunov-like function for the system, given by (69).

Then, the  $\Delta V(\mathbf{x}(k))$  along the trajectory, is obtained as:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) = \mathbf{x}^T(k+1) \mathbf{x}(k+1) - \mathbf{x}^T(k-1) \mathbf{x}(k-1) \\ &= \mathbf{x}^T(k) A_0^T A_0 \mathbf{x}(k) + 2 \mathbf{x}^T(k) A_0^T A_1 \mathbf{x}(k-1) \\ &\quad + \mathbf{x}^T(k-1) A_1^T A_1 \mathbf{x}(k-1) - \mathbf{x}^T(k-1) \mathbf{x}(k-1) \end{aligned} \quad (108)$$

From (108), one can get:

$$\begin{aligned} \mathbf{x}^T(k+1) \mathbf{x}(k+1) &= \mathbf{x}^T(k) A_0^T A_0 \mathbf{x}(k) \\ &\quad + 2 \mathbf{x}^T(k) A_0^T A_1 \mathbf{x}(k-1) + \mathbf{x}^T(k-1) A_1^T A_1 \mathbf{x}(k-1) \end{aligned} \quad (109)$$

Using the very well known inequality, with choice:

$$\Gamma = (I - A_1^T A_1) > 0, \quad (110)$$

$I$  being the identity matrix, it can be obtained:

$$\begin{aligned} \mathbf{x}^T(k+1) \mathbf{x}(k+1) &\leq \mathbf{x}^T(k) A_0^T A_0 \mathbf{x}(k) + \\ &\quad \mathbf{x}^T(k) A_1 (I - A_1^T A_1)^{-1} A_1^T \mathbf{x}(k) + \mathbf{x}^T(k-1) \mathbf{x}(k-1) \end{aligned} \quad (111)$$

and using assumption (104), it is clear that (111) reduces to:



$$\begin{aligned} \mathbf{x}^T(k+1)\mathbf{x}(k+1) &< \mathbf{x}^T(k)A_0^T\left(\left(I-A_1A_1^T\right)^{-1}+p^2I\right)A_0\mathbf{x}(k) \\ &< \lambda_{\max}(A_0, A_1, p)\mathbf{x}^T(k)\mathbf{x}(k) \end{aligned} \quad (112)$$

where: 
$$\lambda_{\max}(A_0, A_1, p) = \lambda_{\max}\left(A_0^T(I-A_1A_1^T)^{-1}A_0 + p^2I\right) \quad (113)$$

with obvious property, that gives the natural sense to this problem:  $\lambda_{\max}(A_0, A_1, p) \geq 0$  when  $(I - A_1A_1^T) \geq 0$ .

Following the procedure from the previous section, it can be written:

$$\ln \mathbf{x}^T(k+1)\mathbf{x}(k+1) - \ln \mathbf{x}^T(k)\mathbf{x}(k) < \ln \lambda_{\max}(\quad). \quad (114)$$

By applying the sum  $\sum_{j=k_0}^{k_0+k-1}$  on both sides of (112) for  $\forall k \in \mathcal{K}_N$ , one can obtain:

$$\ln \mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k) \leq \ln \prod_{j=k_0}^{k_0+k-1} \lambda_{\max}(\quad) \leq \ln \lambda_{\max}^k(\quad) + \ln \mathbf{x}^T(k_0)\mathbf{x}(k_0), \quad \forall k \in \mathcal{K}_N \quad (115)$$

Taking into account the fact that  $\|\mathbf{x}_0\|^2 < \alpha$  and condition of *Theorem 2.2.2.3*, (105), one can get:

$$\begin{aligned} \ln \mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k) &< \ln \lambda_{\max}^k(A_0, A_1, p) + \ln \mathbf{x}^T(k_0)\mathbf{x}(k_0) \\ &< \ln \alpha \cdot \lambda_{\max}^k(A_0, A_1, p) < \ln \alpha \cdot \frac{\beta}{\alpha} < \ln \beta, \quad \forall k \in \mathcal{K}_N \end{aligned} \quad (116)$$

**Remark 2.2.2.6** In the case when  $A_1$  is null matrix and  $p=0$  result, given by (106), reduces to that given in (*Debeljkovic 2001*) earlier developed for ordinary discrete time systems.

**Theorem 2.2.2.4** Suppose the matrix  $(I - A_1^T A_1) > 0$ . System, given by (69), is *practically unstable* with respect to  $\left\{k_0, \mathcal{K}_N, \alpha, \beta, \left\|\left(\cdot\right)\right\|^2\right\}$ ,  $\alpha < \beta$ , if there exist a positive real number  $p$ ,  $p > 1$ , such that:

$$\left\|\mathbf{x}(k-1)\right\|^2 < p^2 \left\|\mathbf{x}(k)\right\|^2, \quad \forall k \in \mathcal{K}_N, \quad \forall \mathbf{x}(k) \in \mathcal{S}_\beta, \quad (117)$$

and if there exist: real, positive number  $\delta$ ,  $\delta \in ]0, \alpha[$  and time instant  $k$ ,  $k = k^* : \exists!(k^* > k_0) \in \mathcal{K}_N$  for which the next condition is fulfilled:

$$\lambda_{\min}^{k^*} > \frac{\beta}{\delta}, \quad k^* \in \mathcal{K}_N. \quad (118)$$

**Proof.** Let:

$$V(\mathbf{x}(k)) = \mathbf{x}^T(k)\mathbf{x}(k) + \mathbf{x}^T(k-1)\mathbf{x}(k-1) \quad (119)$$

Then following the identical procedure as in the previous *Theorem*, one can get:

$$\ln \mathbf{x}^T(k+1)\mathbf{x}(k+1) - \ln \mathbf{x}^T(k)\mathbf{x}(k) > \ln \lambda_{\min}(\cdot), \quad (120)$$

where:

$$\lambda_{\min}(A_0, A_1, p) = \lambda_{\min}\left(A_0^T(I - A_1 A_1^T)^{-1} A_0 + p^2 I\right). \quad (121)$$

If we apply the summing  $\sum_{j=k_0}^{k_0+k-1}$  on both sides of (120) for  $\forall k \in \mathcal{K}_N$ , one can obtain:

$$\ln \mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k) \geq \ln \prod_{j=k_0}^{k_0+k-1} \lambda_{\min}(\cdot) \geq \ln \lambda_{\min}^k(\cdot) + \ln \mathbf{x}^T(k_0)\mathbf{x}(k_0), \quad \forall k \in \mathcal{K}_N. \quad (122)$$

It is clear that for any  $\mathbf{x}_0$  follows:  $\delta < \|\mathbf{x}_0\|^2 < \alpha$  and for some  $k^* \in \mathcal{K}_N$  and with (118), one can get:

$$\begin{aligned} \ln \mathbf{x}^T(k_0+k^*)\mathbf{x}(k_0+k^*) &> \ln \lambda_{\min}^{k^*}(A_0, A_1, p) + \ln \mathbf{x}^T(k_0)\mathbf{x}(k_0) \\ &> \ln \delta \cdot \lambda_{\min}^{k^*}(A_0, A_1, p) > \ln \delta \cdot \frac{\beta}{\delta} > \ln \beta, \quad \exists! k^* \in \mathcal{K}_N. \quad Q.E.D. \end{aligned} \quad (123)$$

### 3. Singular and descriptive time delay systems

Singular and descriptive systems represent very important classes of systems. Their stability was considered in detail in the previous chapter. Time delay phenomena, which often occur in real systems, may introduce instability, which must not be neglected. Therefore a special attention is paid to stability of singular and descriptive time delay systems, which are considered in detail in this section.

#### 3.1 Continuous singular time delayed systems

##### 3.1.1 Continuous singular time delayed systems – Stability in the sense of Lyapunov

Consider a linear continuous singular system with state delay, described by:

$$E\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau), \quad (124)$$

with known compatible vector valued function of initial conditions:

$$\mathbf{x}(t) = \boldsymbol{\psi}(t), \quad -\tau \leq t \leq 0, \quad (125)$$

where  $A_0$  and  $A_1$  are constant matrices of appropriate dimensions.

Time delay is constant, e.g.  $\tau \in \mathbb{R}_+$ . Moreover we shall assume that  $\text{rank } E = r < n$ .

**Definition 3.1.1.1** The matrix pair  $(E, A_0)$  is regular if  $\det(sE - A_0)$  is not identically zero, (Xu et al. 2002.a).

**Definition 3.1.1.2** The matrix pair  $(E, A_0)$  is impulse free if  $\text{degree } \det(sE - A) = \text{rank } E$ , (Xu et al. 2002.a).

The linear continuous singular time delay system (124) may have an impulsive solution, however, the regularity and the absence of impulses of the matrix pair  $(E, A_0)$  ensure the existence and uniqueness of an impulse free solution to the system under consideration, which is defined in the following Lemma.

**Lemma 3.1.1.1** Suppose that the matrix pair  $(E, A_0)$  is regular and impulsive free and unique on  $[0, \infty[$ , (Xu et al. 2002).

Necessity for system stability investigation makes need for establishing a proper stability definition. So one can has:

**Definition 3.1.1.3** Linear continuous singular time delay system (124) is said to be regular and impulsive free if the matrix pair  $(E, A_0)$  is regular and impulsive free, (Xu et al. 2002.a).

### STABILITY DEFINITIONS

**Definition 3.1.1.4** If  $\forall t_0 \in T$  and  $\forall \varepsilon > 0$ , there always exists  $\delta(t_0, \varepsilon)$ , such that  $\forall \psi \in \mathbb{S}_\delta(0, \delta) \cap \mathbb{S}(t_0, t^*)$ , the solution  $\mathbf{x}(t, t_0, \psi)$  to (124) satisfies that  $\|\mathbf{q}(t, \mathbf{x}(t))\| \leq \varepsilon$ ,  $\forall t \in (t_0, t^*)$ , then the zero solution to (124) is said to be stable on  $\{\mathbf{q}(t, \mathbf{x}(t)), T\}$ , where  $T = [0, +t^*]$ ,  $0 < t^* \leq +\infty$  and  $\mathbb{S}_\delta(0, \delta) = \{\psi \in \mathcal{C}([- \tau, 0], \mathbb{R}^n), \|\psi\| < \delta, \delta > 0\}$ .  $\mathbb{S}_*(t_0, t^*)$  is a set of all consistency initial functions and for  $\forall \psi \in \mathbb{S}_*(t_0, t^*)$ , there exists a continuous solution to (122) in  $[t_0 - \tau, t^*)$  through  $(t_0, \psi)$  at least, (Li & Liu 1997, 1998).

**Definition 3.1.1.5** If  $\delta$  is only related to  $\varepsilon$  and has nothing to do with  $t_0$ , then the zero solution is said to be uniformly stable on  $\{\mathbf{q}(t, \mathbf{x}(t)), T\}$ , (Li & Liu 1997, 1998).

**Definition 3.1.1.6** Linear continuous singular time delay system (124) is said to be stable if for any  $\varepsilon > 0$  there exist a scalar  $\delta(\varepsilon) > 0$  such that, for any compatible initial conditions  $\psi(t)$ , satisfying condition:  $\sup_{-\tau \leq t \leq 0} \|\psi(t)\| \leq \delta(\varepsilon)$ , the solution  $\mathbf{x}(t)$  of system (2) satisfies

$$\|\mathbf{x}(t)\| \leq \varepsilon, \forall t \geq 0.$$

Moreover if  $\lim_{t \rightarrow \infty} \|\mathbf{x}(t)\| \rightarrow 0$ , system is said to be asymptotically stable, (Xu et al. 2002.a).

### STABILITY THEOREMS

**Theorem 3.1.1.1** Suppose that the matrix pair  $(E, A_0)$  is regular with system matrix  $A_0$  being nonsingular, e.i.  $\det A_0 \neq 0$ . System (124) is asymptotically stable, independent of delay, if there exist a symmetric positive definite matrix  $P = P^T > 0$ , being the solution of Lyapunov matrix equation

$$A_0^T P E + E^T P A_0 = -2(S + Q), \quad (126)$$

with matrices  $Q = Q^T > 0$  and  $S = S^T$ , such that:

$$\mathbf{x}^T(t)(S + Q)\mathbf{x}(t) > 0, \quad \forall \mathbf{x}(t) \in \mathcal{W}_{k^*} \setminus \{0\}, \quad (127)$$

is positive definite quadratic form on  $\mathcal{W}_{k^*} \setminus \{0\}$ ,  $\mathcal{W}_{k^*}$  being the subspace of consistent initial conditions, and if the following condition is satisfied:

$$\|A_1\| < \sigma_{\min} \left( Q^{\frac{1}{2}} \right) \sigma_{\max}^{-1} \left( Q^{-\frac{1}{2}} E^T P \right), \quad (128)$$

Here  $\sigma_{\max}(\cdot)$  and  $\sigma_{\min}(\cdot)$  are maximum and minimum singular values of matrix  $(\cdot)$ , respectively, (Debeljkovic et al. 2003, 2004.c, 2006, 2007).

**Proof.** Let us consider the functional:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t) E^T P E \mathbf{x}(t) + \int_{t-\tau}^t \mathbf{x}^T(\vartheta) Q \mathbf{x}(\vartheta) d\kappa. \quad (129)$$

Note that (Owens, Debeljković 1985) indicates that:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t) E^T P E \mathbf{x}(t), \quad (130)$$

is positive quadratic form on  $\mathcal{W}_{k^*}$ , and it is obvious that all smooth solutions  $\mathbf{x}(t)$  evolve in  $\mathcal{W}_{k^*}$ , so  $V(\mathbf{x}(t))$  can be used as a *Lyapunov function* for the system under consideration, (Owens, Debeljkovic 1985). It will be shown that the same argument can be used to declare the same property of another quadratic form present in (129).

Clearly, using the equation of motion of (124), we have:

$$\begin{aligned} \dot{V}(\mathbf{x}(t)) = & \mathbf{x}^T(t) (A_0^T P E + E^T P A_0 + Q) \mathbf{x}(t) \\ & + 2\mathbf{x}^T(t) (E^T P A_1) \mathbf{x}(t-\tau) - \mathbf{x}^T(t-\tau) Q \mathbf{x}(t-\tau) \end{aligned} \quad (131)$$

and after some manipulations, to the following expression is obtained:

$$\begin{aligned} \dot{V}(\mathbf{x}(t)) = & \mathbf{x}^T (A_0^T P E + E^T P A_0 + 2Q + 2S) \mathbf{x} + 2\mathbf{x}^T(t) (E^T P A_1) \mathbf{x}(t-\tau) \\ & - \mathbf{x}^T(t) Q \mathbf{x}(t) - 2\mathbf{x}^T(t) S \mathbf{x}(t) - \mathbf{x}^T(t-\tau) Q \mathbf{x}(t-\tau) \end{aligned} \quad (132)$$

From (126) and the fact that the choice of matrix  $S$ , can be done, such that:

$$\mathbf{x}^T(t) S \mathbf{x}(t) \geq 0, \quad \forall \mathbf{x}(t) \in \mathcal{W}_{k^*} \setminus \{0\}, \quad (133)$$

one obtains the following result:

$$\dot{V}(\mathbf{x}(t)) \leq 2\mathbf{x}^T(t) \left( E^T P A_1 \right) \mathbf{x}(t-\tau) - \mathbf{x}^T(t) Q \mathbf{x}(t) - \mathbf{x}^T(t-\tau) Q \mathbf{x}(t-\tau), \quad (134)$$

and based on well known inequality:

$$\begin{aligned} 2\mathbf{x}^T(t) E^T P A_1 \mathbf{x}(t-\tau) &= 2\mathbf{x}^T(t) E^T P A_1 Q^{-\frac{1}{2}} Q^{\frac{1}{2}} \mathbf{x}(t-\tau) \\ &\leq \mathbf{x}^T(t) E^T P A_1 Q^{-1} A_1^T P E^T \mathbf{x}(t) + \mathbf{x}^T(t-\tau) Q \mathbf{x}(t-\tau) \end{aligned} \quad (135)$$

and by substituting into (134), it yields:

$$\dot{V}(\mathbf{x}(t)) \leq -\mathbf{x}^T(t) Q \mathbf{x}(t) + \mathbf{x}^T(t) E^T P A_1 Q^{-1} A_1^T P E \mathbf{x}(t) \leq -\mathbf{x}^T(t) Q^{\frac{1}{2}} \Gamma Q^{\frac{1}{2}} \mathbf{x}(t), \quad (136)$$

with matrix  $\Gamma$  defined by:

$$\Gamma = \left( I - Q^{-\frac{1}{2}} E^T P A_1 Q^{-\frac{1}{2}} Q^{-\frac{1}{2}} A_1^T P E Q^{-\frac{1}{2}} \right) \quad (137)$$

$\dot{V}(\mathbf{x}(t))$  is negative definite, if:

$$1 - \lambda_{\max} \left( Q^{-\frac{1}{2}} E^T P A_1 Q^{-\frac{1}{2}} Q^{-\frac{1}{2}} A_1^T P E Q^{-\frac{1}{2}} \right) > 0, \quad (138)$$

which is satisfied, if: 
$$1 - \sigma_{\max}^2 \left( Q^{-\frac{1}{2}} E^T P A_1 Q^{-\frac{1}{2}} \right) > 0. \quad (139)$$

Using the properties of the singular matrix values, (Amir - Moez 1956), the condition (139), holds if:

$$1 - \sigma_{\max}^2 \left( Q^{-\frac{1}{2}} E^T P \right) \sigma_{\max}^2 \left( A_1 Q^{-\frac{1}{2}} \right) > 0, \quad (140)$$

which is satisfied if:

$$1 - \sigma_{\min}^{-2} \left( Q^{\frac{1}{2}} \right) \left( \|A_1\|^2 \sigma_{\max}^2 \left( Q^{-\frac{1}{2}} E^T P \right) \right) > 0. \quad Q.E.D. \quad (141)$$

**Remark 3.1.1.1** (126-127) are, in modified form, taken from (Owens & Debeljkovic 1985).

**Remark 3.1.1.2** If the system under consideration is just ordinary time delay, e.g.  $E = I$ , we have result identical to that presented in (Tissir & Hmamed 1996).

**Remark 3.1.1.3** Let us discuss first the case when the time delay is absent.

Then the singular (weak) Lyapunov matrix (126) is natural generalization of classical Lyapunov theory. In particular:

- a. If  $E$  is nonsingular matrix, then the system is asymptotically stable if and only if  $A = E^{-1}A_0$  Hurwitz matrix. (126) can be written in the form:

$$A^T E^T P E + E^T P E A = -(Q + S), \quad (142)$$

with matrix  $Q$  being symmetric and positive definite, in whole state space, since then  $\mathcal{W}_{k^*} = \mathfrak{R}(E^{k^*}) = \mathbb{R}^n$ . In this circumstances  $E^T P E$  is a Lyapunov function for the system.

- b. The matrix  $A_0$  by necessity is nonsingular and hence the system has the form:

$$E_0 \dot{\mathbf{x}}(t) = \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0. \quad (143)$$

Then for this system to be stable (143) must hold also, and has familiar Lyapunov structure:

$$E_0^T P + P E_0 = -Q, \quad (144)$$

where  $Q$  is symmetric matrix but only required to be positive definite on  $\mathcal{W}_{k^*}$ .

**Remark 3.1.1.4** There is no need for the system, under consideration, to possess properties given in Definition 3.1.1.2, since this is obviously guaranteed by demand that all smooth solutions  $\mathbf{x}(t)$  evolve in  $\mathcal{W}_{k^*}$ .

**Remark 3.1.1.5** Idea and approach is based upon the papers of (Owens & Debeljkovic 1985) and (Tissir & Hmamed 1996).

**Theorem 3.1.1.2** Suppose that the system matrix  $A_0$  is nonsingular, e.i.  $\det A_0 \neq 0$ . Then we can consider system (124) with known compatible vector valued function of initial conditions and we shall assume that  $\text{rank } E_0 = r < n$ .

Matrix  $E_0$  is defined in the following way  $E_0 = A_0^{-1}E$ . System (124) is asymptotically stable, independent of delay, if :

$$\|A_1\| < \sigma_{\min} \left( Q^{\frac{1}{2}} \right) \sigma_{\max}^{-1} \left( Q^{-\frac{1}{2}} E_0^T P \right), \quad (145)$$

and if there exist  $(n \times n)$  matrix  $P$ , being the solution of Lyapunov matrix:

$$E_0^T P + P E_0 = -2I_{\mathcal{W}_k}, \quad (146)$$

with the properties given by (3)–(7).

Moreover matrix  $P$  is symmetric and positive definite on the subspace of consistent initial conditions. Here  $\sigma_{\max}(\cdot)$  and  $\sigma_{\min}(\cdot)$  are maximum and minimum singular values of matrix  $(\cdot)$ , respectively (Debeljkovic et al. 2005.b, 2005.c, 2006.a).

For the sake of brevity the proof is here omitted and is completely identical to that of preceding Theorem.

**Remark 3.1.1.6** Basic idea and approach is based upon the paper of (Pandolfi 1980) and (Tissir, Hmamed 1996).



### 3.1.2 Continuous singular time delayed systems – stability over finite time interval

Let us consider the case when the subspace of consistent initial conditions for *singular time delay* and *singular nondelay system* coincide.

#### STABILITY DEFINITIONS

**Definition 3.1.2.1** Regular and impulsive free singular time delayed system (124), is *finite time stable* with respect to  $\{t_0, \mathfrak{T}, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$ , if and only if  $\forall \mathbf{x}_0 \in \mathcal{W}_k^*$  satisfying  $\|\mathbf{x}(t_0)\|_{E^T E}^2 = \|\mathbf{x}_0\|_{E^T E}^2 < \alpha$ , implies  $\|\mathbf{x}(t)\|_{E^T E}^2 < \beta$ ,  $\forall t \in \mathfrak{T}$ .

**Definition 3.1.2.2** . Regular and impulsive free singular time delayed system (124), is *attractive practically stable* with respect to  $\{t_0, \mathfrak{T}, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$ , if and only if  $\forall \mathbf{x}_0 \in \mathcal{W}_k^*$  satisfying

$\|\mathbf{x}(t_0)\|_{G=E^T P E}^2 = \|\mathbf{x}_0\|_{G=E^T P E}^2 < \alpha$  implies:  $\|\mathbf{x}(t)\|_{G=E^T P E}^2 < \beta$ ,  $\forall t \in \mathfrak{T}$ , with property that

$\lim_{k \rightarrow \infty} \|\mathbf{x}(t)\|_{G=E^T P E}^2 \rightarrow 0$ ,  $\mathcal{W}_k^*$  being the subspace of consistent initial conditions, (Debeljkovic et al. 2011.b).

**Remark 3.1.2.1** The singularity of matrix  $E$  will ensure that solutions to (6) exist for only special choice of  $\mathbf{x}_0$ .

In (Owens, Debeljković 1985) the subspace of  $\mathcal{W}_k^*$  of consistent initial conditions is shown to be the limit of the nested subspace algorithm (12)–(14).

#### STABILITY THEOREMS

**Theorem 3.1.2.1** Suppose that  $(I - E^T E) > 0$ . Singular time delayed system (124), is *finite time stable* with respect to  $\{t_0, \mathfrak{T}, \alpha, \beta, \|\cdot\|^2\}$ ,  $\alpha < \beta$ , if there exist a positive real number  $q$ ,  $q > 1$ , such that:

$$\|\mathbf{x}(t + \vartheta)\|^2 < q^2 \|\mathbf{x}(t)\|^2, \quad \vartheta \in [-\tau, 0], \quad \forall t \in \mathfrak{T}, \quad \mathbf{x}(t) \in \mathcal{W}_k^*, \quad \forall \mathbf{x}(t) \in \mathcal{S}_\beta, \quad (147)$$

and if the following condition is satisfied:

$$e^{\bar{\lambda}_{\max}(\Xi)(t-t_0)} < \frac{\beta}{\alpha}, \quad \forall t \in \mathfrak{T}, \quad (148)$$

where:

$$\begin{aligned} \bar{\lambda}_{\max}(\Xi) = \bar{\lambda}_{\max} \{ & \mathbf{x}^T(t)(A_0^T E + E^T A_0 + E^T A_1 (I - E^T E)^{-1} A_1^T E \\ & + q^2 I) \mathbf{x}(t), \mathbf{x}(t) \in \mathcal{W}_k^*, \mathbf{x}^T(t) E^T E \mathbf{x}(t) = 1 \}. \end{aligned} \quad (149)$$

**Proof.** Define tentative aggregation function as:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t) E^T E \mathbf{x}(t) + \int_{t-\tau}^t \mathbf{x}^T(\vartheta) \mathbf{x}(\vartheta) d\vartheta. \quad (150)$$

Let  $\mathbf{x}_0$  be an arbitrary consistent initial condition and  $\mathbf{x}(t)$  resulting system trajectory.

The total derivative  $\dot{V}(t, \mathbf{x}(t))$  along the trajectories of the system, yields:

$$\begin{aligned} \dot{V}(t, \mathbf{x}(t)) &= \frac{d}{dt}(\mathbf{x}^T(t)E^T E \mathbf{x}(t)) + \frac{d}{dt} \int_{t-\tau}^t \mathbf{x}^T(\vartheta) \mathbf{x}(\vartheta) d\vartheta \\ &= \mathbf{x}^T(t)(A_0^T E + E^T A_0) \mathbf{x}(t) + 2\mathbf{x}^T(t)E^T A_1 \mathbf{x}(t-\tau) + \mathbf{x}^T(t) \mathbf{x}(t) - \mathbf{x}^T(t-\tau) \mathbf{x}(t-\tau) \end{aligned} \quad (151)$$

From (148) it is obvious:

$$\frac{d}{dt}(\mathbf{x}^T(t)E^T E \mathbf{x}(t)) = \mathbf{x}^T(t)(A_0^T E + E^T A_0) \mathbf{x}(t) + 2\mathbf{x}^T(t)E^T A_1 \mathbf{x}(t-\tau), \quad (152)$$

and based on well known inequality and with the particular choice:

$$\mathbf{x}^T(t) \Gamma \mathbf{x}(t) = \mathbf{x}^T(t)(I - E^T E) \mathbf{x}(t) > 0, \quad \forall \mathbf{x}(t) \in \mathcal{W}_k^* \setminus \{0\}, \quad (153)$$

so:

$$\begin{aligned} \frac{d}{dt}(\mathbf{x}^T(t)E^T E \mathbf{x}(t)) &\leq \mathbf{x}^T(t)(A_0^T E + E^T A_0) \mathbf{x}(t) \\ &+ \mathbf{x}^T(t)E^T A_1 (I - E^T E)^{-1} A_1^T E \mathbf{x}(t) + \mathbf{x}^T(t-\tau)(I - E^T E) \mathbf{x}(t-\tau). \end{aligned} \quad (154)$$

Moreover, since:

$$\|\mathbf{x}(t-\tau)\|_{E^T E}^2 \geq 0, \quad \forall \mathbf{x}(t) \in \mathcal{W}_k^* \setminus \{0\}, \quad (155)$$

and using assumption (147), it is clear that (154) reduces to:

$$\begin{aligned} \frac{d}{dt}(\mathbf{x}^T(t)E^T E \mathbf{x}(t)) &< \mathbf{x}^T(t) \left( A_0^T E + E^T A_0 + E^T A_1 (E^T E - I)^{-1} A_1^T E + q^2 I \right) \mathbf{x}(t) \\ &< \bar{\lambda}_{\max}(\Xi) \mathbf{x}^T(t)E^T E \mathbf{x}(t) \end{aligned} \quad (156)$$

**Remark 3.1.2.2** Note that *Lemma 2.2.1.1* and *Theorem 2.2.1.1* indicates that:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)E^T E \mathbf{x}(t), \quad (157)$$

is positive quadratic form on  $\mathcal{W}_k^*$ , and it is obvious that all smooth solutions  $\mathbf{x}(t)$  evolve in  $\mathcal{W}_k^*$ , so  $V(\mathbf{x}(t))$  can be used as a *Lyapunov function* for the system under consideration, (Owens, Debeljkovic 1985).

Using (149) one can get (Debeljkovic et al. 2011.b):

$$\int_{t_0}^t \frac{d(\mathbf{x}^T(t)E^T E \mathbf{x}(t))}{\mathbf{x}^T(t)E^T E \mathbf{x}(t)} < \int_{t_0}^t \bar{\lambda}_{\max}(\Xi) dt, \quad (158)$$

and:

$$\begin{aligned} \mathbf{x}^T(t)E^T E\mathbf{x}(t) &< \mathbf{x}^T(t_0)E^T E\mathbf{x}(t_0)e^{\bar{\lambda}_{\max}(\Xi)(t-t_0)} \\ &< \alpha \cdot e^{\bar{\lambda}_{\max}(\Xi)(t-t_0)} < \alpha \cdot \frac{\beta}{\alpha} < \beta, \forall t \in \mathfrak{I}. \quad Q.E.D. \end{aligned} \quad (159)$$

**Remark 3.1.2.3** In the case on non-delay system, e.g.  $A_1 \equiv 0$ , (148) reduces to basic result, (Debeljkovic, Owens 1985).

**Theorem 3.1.2.2** Suppose that  $(Q - E^T E) > 0$ . Singular time delayed system (124), with system matrix  $A_0$  being nonsingular, is attractive practically stable with respect to  $\left\{t_0, \mathfrak{I}, \alpha, \beta, \left\|(\cdot)\right\|_{G=E^T P E}^2\right\}$ ,  $\alpha < \beta$ , if there exist matrix  $P = P^T > 0$ , being solution of:

$$A_0^T P E + E^T P A_0 = -Q, \quad (160)$$

with matrices  $Q = Q^T > 0 \wedge S = S^T$ , such that:

$$\mathbf{x}^T(t)(S + Q)\mathbf{x}(t) > 0, \quad \forall \mathbf{x}(t) \in \mathcal{W}_k^* \setminus \{0\}, \quad (161)$$

is positive definite quadratic form on  $\mathcal{W}_k^* \setminus \{0\}$ ,  $\mathcal{W}_k^*$  being the subspace of consistent initial conditions, if there exist a positive real number  $q$ ,  $q > 1$ , such that:

$$\left\|\mathbf{x}(t-\tau)\right\|_Q^2 < q^2 \left\|\mathbf{x}(t)\right\|_Q^2, \quad \forall t \in \mathfrak{I}, \quad \forall \mathbf{x}(t) \in \mathcal{S}_\beta, \quad \forall \mathbf{x}(t) \in \mathcal{W}_k^* \setminus \{0\}, \quad (162)$$

and if the following conditions are satisfied (Debeljkovic et al. 2011.b):

$$\left\|A_1\right\| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}A_0^T P\right), \quad (163)$$

and:

$$e^{\bar{\lambda}_{\max}(\Psi)(t-t_0)} < \frac{\beta}{\alpha}, \quad \forall t \in \mathfrak{I}, \quad (164)$$

where:

$$\begin{aligned} \bar{\lambda}_{\max}(\Psi) &= \max\{\mathbf{x}^T(t)(E^T P A_1(Q - E^T P E)^{-1} A_1^T P E + q^2 Q)\mathbf{x}(t), \\ &\quad \mathbf{x}(t) \in \mathcal{W}_k^*, \mathbf{x}^T(t)E^T P E\mathbf{x}(t) = 1\}. \end{aligned} \quad (165)$$

**Proof.** Define tentative aggregation function as:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)E^T P E\mathbf{x}(t) + \int_{t-\tau}^t \mathbf{x}^T(\vartheta)Q\mathbf{x}(\vartheta)d\vartheta. \quad (166)$$

The total derivative  $\dot{V}(t, \mathbf{x}(t))$  along the trajectories of the system, yields:

$$\begin{aligned}\dot{V}(t, \mathbf{x}(t)) &= \frac{d}{dt}(\mathbf{x}^T(t)E^TPE\mathbf{x}(t)) + \frac{d}{dt} \int_{t-\tau}^t \mathbf{x}^T(\vartheta)Q\mathbf{x}(\vartheta)d\vartheta \\ &= \mathbf{x}^T(t)(A_0^TPE + E^TPA_0)\mathbf{x}(t) + 2\mathbf{x}^T(t)E^TPA_1\mathbf{x}(t-\tau) \\ &\quad + \mathbf{x}^T(t)Q\mathbf{x}(t) - \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau).\end{aligned}\quad (167)$$

From (162), it is obvious:

$$\frac{d}{dt}(\mathbf{x}^T(t)E^TPE\mathbf{x}(t)) = \mathbf{x}^T(t)(A_0^TPE + E^TPA_0)\mathbf{x}(t) + 2\mathbf{x}^T(t)E^TPA_1\mathbf{x}(t-\tau), \quad (168)$$

or:

$$\begin{aligned}\frac{d}{dt}(\mathbf{x}^T(t)E^TPE\mathbf{x}(t)) &= \mathbf{x}^T(t)(A_0^TPE + E^TPA_0 + Q + S)\mathbf{x}(t) \\ &\quad + 2\mathbf{x}^T(t)E^TPA_1\mathbf{x}(t-\tau) - \mathbf{x}^T(t)(Q + S)\mathbf{x}(t).\end{aligned}\quad (169)$$

From (160), it follows:

$$\frac{d}{dt}(\mathbf{x}^T(t)E^TPE\mathbf{x}(t)) = -\mathbf{x}^T(t)(Q + S)Q\mathbf{x}(t) + 2\mathbf{x}^T(t)E^TPA_1\mathbf{x}(t-\tau), \quad (170)$$

as well, using before mentioned inequality, with particular choice:

$$\mathbf{x}^T(t)\Gamma\mathbf{x}(t) = \mathbf{x}^T(t)(Q - E^TPE)\mathbf{x}(t) > 0, \quad \forall \mathbf{x}(t) \in \mathcal{W}_k^* \setminus \{0\}, \quad (171)$$

and fact that:

$$\mathbf{x}^T(t)(Q + S)\mathbf{x}(t) > 0, \quad \forall \mathbf{x}(t) \in \mathcal{W}_k^* \setminus \{0\}, \quad (172)$$

is positive definite quadratic form on  $\mathcal{W}_k^* \setminus \{0\}$ , one can get :

$$\begin{aligned}\frac{d}{dt}(\mathbf{x}^T(t)E^TPE\mathbf{x}(t)) &= 2\mathbf{x}^T(t)E^TPA_1\mathbf{x}(t-\tau) \\ &\leq \mathbf{x}^T(t)E^TPA_1(Q - E^TPE)^{-1}A_1^TPE\mathbf{x}(t) + \mathbf{x}^T(t-\tau)(Q - E^TPE)\mathbf{x}(t-\tau)\end{aligned}\quad (173)$$

Moreover, since:

$$\|\mathbf{x}(t-\tau)\|_{E^TPE}^2 \geq 0, \quad \forall \mathbf{x}(t) \in \mathcal{W}_k^* \setminus \{0\}, \quad (174)$$

and using assumption (162) it is clear that (173), reduces to:

$$\frac{d}{dt}(\mathbf{x}^T(t)E^TPE\mathbf{x}(t)) < \mathbf{x}^T(t)\left(E^TPA_1(E^TPE - Q)^{-1}A_1^TPE + q^2Q\right)\mathbf{x}(t), \quad (175)$$

or using (169), one can get:

$$\begin{aligned}\frac{d}{dt}(\mathbf{x}^T(t)E^TPE\mathbf{x}(t)) &< \mathbf{x}^T(t)\left(E^TPA_1(E^TPE - Q)^{-1}A_1^TPE + q^2Q\right)\mathbf{x}(t) \\ &< \bar{\lambda}_{\max}(\Psi)\mathbf{x}^T(t)E^TPE\mathbf{x}(t)\end{aligned}\quad (176)$$

or finally:

$$\begin{aligned} \mathbf{x}^T(t)E^TPE\mathbf{x}(t) &< \mathbf{x}^T(t_0)E^TPE\mathbf{x}(t_0)e^{\bar{\lambda}_{\max}(\Psi)(t-t_0)} \\ &< \alpha \cdot e^{\bar{\lambda}_{\max}(\Psi)(t-t_0)} < \alpha \cdot \frac{\beta}{\alpha} < \beta, \forall t \in \mathfrak{I}. \quad Q.E.D. \end{aligned} \quad (177)$$

### 3.2 Discrete descriptor time delayed systems

#### 3.2.1 Discrete descriptor time delayed systems – Stability in the sense of Lyapunov

Consider a linear discrete descriptor system with state delay, described by:

$$E\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1), \quad (178)$$

$$\mathbf{x}(k_0) = \boldsymbol{\varphi}(k_0), \quad -1 \leq k_0 \leq 0, \quad (179)$$

where  $\mathbf{x}(k) \in \mathbb{R}^n$  is a state vector. The matrix  $E \in \mathbb{R}^{n \times n}$  is a necessarily singular matrix, with property  $\text{rank } E = r < n$  and with matrices  $A_0$  and  $A_1$  of appropriate dimensions.

For a (DDTDS), (178), we present the following definitions taken from, (Xu et al. 2002.b).

**Definition 3.2.1.1** The (DDTDS) is said to be *regular* if  $\det(z^2E - zA_0 - A_1)$ , is not identically zero.

**Definition 3.2.1.2** The (DDTDS) is said to be *causal* if it is *regular* and  $\deg(z^n \det(zE - A_0 - z^{-1}A_1)) = n + \text{rang } E$ .

**Definition 3.2.1.3** The (DDTDS) is said to be *stable* if it is *regular* and  $\rho(E, A_0, A_1) \subset D(0, 1)$ ,

where  $\rho(E, A_0, A_1) = \{z \mid \det(z^2E - zA_0 - A_1) = 0\}$ .

**Definition 3.2.1.4** The (DDTDS) is said to be *admissible* if it is *regular*, *causal* and *stable*.

#### STABILITY DEFINITIONS

**Definition 3.2.1.5** System (178) is *E-stable* if for any  $\varepsilon > 0$ , there always exists a positive  $\delta$  such that  $\|E\mathbf{x}(k)\| < \varepsilon$ , when  $\|E\mathbf{x}_0\| < \delta$ , (Liang 2000).

**Definition 3.2.1.6** System (178) is *E-asymptotically stable* if (178) is *E-stable* and  $\lim_{k \rightarrow +\infty} E\mathbf{x}(k) \rightarrow \mathbf{0}$ , (Liang 2000).

#### STABILITY THEOREMS

**Theorem 3.2.1.1** Suppose that system (173) is *regular* and *causal* with system matrix  $A_0$  being nonsingular, i.e.  $\det A_0 \neq 0$ . System (178) is *asymptotically stable*, independent of delay, if

$$\|A_1\| < \frac{\sigma_{\min}\left(Q^{\frac{1}{2}}\right)}{\sigma_{\max}\left(Q^{-\frac{1}{2}}A_0^T P\right)}, \quad (180)$$

and if there exist a symmetric positive definite matrix  $P$  on the whole state space, being the solution of *discrete Lyapunov matrix equation* :

$$A_0^T P A_0 - E^T P E = -2(S + Q), \quad (181)$$

with matrices  $Q = Q^T > 0$  and  $S = S^T$ , such that:

$$\mathbf{x}^T(k)(S + Q)\mathbf{x}(k) > 0, \quad \forall \mathbf{x}(k) \in \mathcal{W}_{d,k^*} \setminus \{0\}, \quad (182)$$

is positive definite quadratic form on  $\mathcal{W}_{d,k^*} \setminus \{0\}$ ,  $\mathcal{W}_{d,k^*}$  being the subspace of consistent initial conditions. Here  $\sigma_{\max}(\cdot)$  and  $\sigma_{\min}(\cdot)$  are maximum and minimum singular values of matrix  $(\cdot)$ , respectively, (Debeljkovic et al. 2004).

**Remark 3.2.1.1** (181 - 182) are, in modify form, taken from (Owens, Debeljkovic 1985).

**Remark 3.2.1.2** If the system under consideration is just ordinary time delay, e.g.  $E = I$ , we have result identical to that presented in Debeljkovic et al. (2004.a - 2004.d, 2005.a, 2005.b).

**Remark 3.2.1.3** Idea and approach is based upon the papers of (Owens, Debeljkovic 1985) and (Tissir, Hmamed 1996).

**Theorem 3.2.1.2** Suppose that system (178) is *regular* and *causal*. Moreover, suppose matrix  $(Q_\lambda - A_1^T P_\lambda A_1)$  is regular, with  $Q_\lambda = Q_\lambda^T > 0$ .

System (178) is *asymptotically stable*, independent of delay, if:

$$\|A_1\| < \frac{\sigma_{\min}\left(\left(Q_\lambda - A_1^T P_\lambda A_1\right)^{\frac{1}{2}}\right)}{\sigma_{\max}\left(Q_\lambda^{-\frac{1}{2}}\left(A_0 - \lambda E\right)^T P_\lambda\right)}, \quad (183)$$

and if there exist real positive scalar  $\lambda^* > 0$  such that for all  $\lambda$  within the range  $0 < |\lambda| < \lambda^*$  there exist symmetric positive definite matrix  $P_\lambda$ , being the solution of *discrete Lyapunov matrix equation*:

$$\left(A_0 - \lambda E\right)^T P_\lambda \left(A_0 - \lambda E\right) - E^T P_\lambda E = -2(S_\lambda + Q_\lambda) \quad (184)$$

with matrix  $S_\lambda = S_\lambda^T$ , such that:

$$\mathbf{x}^T(k)(S_\lambda + Q_\lambda)\mathbf{x}(k) > 0, \quad \forall \mathbf{x}(k) \in \mathcal{W}_{d,k^*} \setminus \{0\} \quad (185)$$

is positive definite quadratic form on  $\mathcal{W}_{d,k^*} \setminus \{0\}$ ,  $\mathcal{W}_{d,k^*}$  being the subspace of consistent initial conditions for both *time delay* and *non-time delay* discrete descriptor system. Such conditions we call *compatible consistent initial conditions*. Here  $\sigma_{\max}(\cdot)$  and  $\sigma_{\min}(\cdot)$  are maximum and minimum singular values of matrix  $(\cdot)$  respectively, (Debeljkovic et al. 2007).

### 3.2.2 Discrete descriptor time delayed systems – stability over finite time interval

To the best knowledge of the authors, there is not any paper treating the problem of finite time stability for *discrete descriptor* time delay systems. Only one paper has been written in



context of practical and finite time stability for *continuous* singular time delay systems, see (Yang et al. 2006).

**Definition 3.2.2.1** Causal system, given by (178), is *finite time stable* with respect to  $\{k_0, \mathcal{K}_N, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$ , if and only if  $\forall \mathbf{x}_0 \in \mathcal{W}_{d,k^*}$  satisfying  $\|\mathbf{x}_0\|_{E^T E}^2 < \alpha$ , implies:  $\|\mathbf{x}(k)\|_{E^T E}^2 < \beta, \forall k \in \mathcal{K}_N$ .

**Definition 3.2.2.2** Causal system given by (178), is *practically unstable* with respect to  $\{k_0, \mathcal{K}_N, \alpha, \beta, \|\cdot\|^2\}$ ,  $\alpha < \beta$ , if and only if  $\exists \mathbf{x}_0 \in \mathcal{W}_{d,k^*}$  such that  $\|\mathbf{x}_0\|_{E^T E}^2 < \alpha$ , there exist some  $k^* \in \mathcal{K}_N$ , such that the following condition is fulfilled  $\|\mathbf{x}(k^*)\|_{E^T E}^2 \geq \beta$ , for some  $k^* \in \mathcal{K}_N$ .

**Definition 3.2.2.3** Causal system, given by (178), is *attractive practically stable* with respect to  $\{k_0, \mathcal{K}_N, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$ , if and only if  $\forall \mathbf{x}_0 \in \mathcal{W}_{d,k^*}$  satisfying  $\|\mathbf{x}(k_0)\|_{G=E^T P E}^2 = \|\mathbf{x}_0\|_{G=E^T P E}^2 < \alpha$ , implies  $\|\mathbf{x}(k)\|_{G=E^T P E}^2 < \beta, \forall k \in \mathcal{K}_N$ , with property that  $\lim_{k \rightarrow \infty} \|\mathbf{x}(k)\|_{G=E^T P E}^2 \rightarrow 0$ , (Nestorovic & Debeljkovic 2011).

**Remark 3.2.2.1** We shall also need the following *Definitions* of the smallest and the largest eigenvalues, respectively, of the matrix  $R = R^T$ , with respect to subspace of consistent initial conditions  $\mathcal{W}_{d,k^*}$  and matrix  $G$ .

**Proposition 3.2.2.1** If  $\mathbf{x}^T(t)R\mathbf{x}(t)$  is quadratic form on  $\mathbb{R}^n$ , then it follows that there exist numbers  $\lambda_{\min}(R)$  and  $\lambda_{\max}(R)$  satisfying:  $-\infty \leq \lambda_{\min}(R) \leq \lambda_{\max}(R) \leq +\infty$ , such that:

$$\lambda_{\min}(\Xi) \leq \frac{\mathbf{x}^T(k)R\mathbf{x}(k)}{\mathbf{x}^T(k)G\mathbf{x}(k)} \leq \lambda_{\max}(\Xi), \forall \mathbf{x}(k) \in \mathcal{W}_{d,k^*} \setminus \{0\}, \quad (186)$$

with matrix  $R = R^T$  and corresponding eigenvalues:

$$\lambda_{\min}(R, G, \mathcal{W}_{d,k^*}) = \min\{\mathbf{x}^T(k)R\mathbf{x}(k) : \mathbf{x}(k) \in \mathcal{W}_{d,k^*} \setminus \{0\}, \mathbf{x}^T(k)G\mathbf{x}(k) = 1\}, \quad (187)$$

$$\lambda_{\max}(R, G, \mathcal{W}_{d,k^*}) = \max\{\mathbf{x}^T(k)R\mathbf{x}(k) : \mathbf{x}(k) \in \mathcal{W}_{d,k^*} \setminus \{0\}, \mathbf{x}^T(k)G\mathbf{x}(k) = 1\}. \quad (188)$$

Note that  $\lambda_{\min} > 0$  if  $R = R^T > 0$ .

Let us consider the case when the subspace of consistent initial conditions for *discrete descriptor time delay* and *discrete descriptor nondelay system* coincide.

### STABILITY THEOREMS

**Theorem 3.2.2.1** Suppose matrix  $(A_1^T A_1 - E^T E) > 0$ . Causal system given by (178), is *finite time stable* with respect to  $\{k_0, \mathcal{K}_N, \alpha, \beta, \|\cdot\|^2\}$ ,  $\alpha < \beta$ , if there exist a positive real number  $p$ ,  $p > 1$ , such that:

$$\|\mathbf{x}(k-1)\|_{A_1^T A_1}^2 < p^2 \|\mathbf{x}(k)\|_{A_1^T A_1}^2, \quad \forall k \in \mathcal{K}_N, \quad \forall \mathbf{x}(k) \in \mathcal{S}_\beta, \quad \forall \mathbf{x}(k) \in \mathcal{W}_{d,k}^* \setminus \{0\} \quad (189)$$

and if the following condition is satisfied (Nestorovic & Debeljkovic 2011):

$$\bar{\lambda}_{\max}^k(\cdot) < \frac{\beta}{\alpha}, \quad \forall k \in \mathcal{K}_N, \quad (190)$$

where:

$$\begin{aligned} \bar{\lambda}_{\max}(\cdot) = & \bar{\lambda}_{\max} \{ \mathbf{x}^T(k) A_0^T (I - A_1 (A_1^T A_1 - E^T E)^{-1} A_1^T \\ & + p^2 A_1^T A_1) A_0 \mathbf{x}(k), \mathbf{x}(k) \in \mathcal{W}_{d,k}^*, \mathbf{x}^T(k) E^T E \mathbf{x}(k) = 1 \}. \end{aligned} \quad (191)$$

**Proof.** Define:

$$V(\mathbf{x}(k)) = \mathbf{x}^T(k) \mathbf{x}(k) + \mathbf{x}^T(k-1) \mathbf{x}(k-1). \quad (192)$$

Let  $\mathbf{x}_0$  be an arbitrary consistent initial condition and  $\mathbf{x}(k)$  the resulting system trajectory.

The backward difference  $\Delta V(\mathbf{x}(k))$  along the trajectories of the system, yields:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) = & \mathbf{x}^T(k) (A_0^T A_0 - E^T E + I) \mathbf{x}(k) \\ & + 2\mathbf{x}^T(k) (A_0^T A_1) \mathbf{x}(k-1) + \mathbf{x}^T(k-1) (A_1^T A_1 - I) \mathbf{x}(k-1) \end{aligned} \quad (193)$$

From (192) one can get:

$$\begin{aligned} \mathbf{x}^T(k+1) E^T E \mathbf{x}(k+1) = & \mathbf{x}^T(k) (A_0^T A_0) \mathbf{x}(k) \\ & + 2\mathbf{x}^T(k) (A_0^T A_1) \mathbf{x}(k-1) + \mathbf{x}^T(k-1) (A_1^T A_1) \mathbf{x}(k-1) \end{aligned} \quad (194)$$

Using the very well known inequality, with particular choice:

$$\mathbf{x}^T(k) \Gamma \mathbf{x}(k) = \mathbf{x}^T(k) (A_1^T A_1 - E^T E) \mathbf{x}(k) \geq 0, \quad \mathbf{x}(k) \in \mathcal{W}_{d,k}^*, \quad \forall \mathbf{x}(k) \in \mathcal{S}_\beta, \quad \forall k \in \mathcal{K}_N, \quad (195)$$

it can be obtained:

$$\begin{aligned} \mathbf{x}^T(k+1) E^T E \mathbf{x}(k+1) \leq & \mathbf{x}^T(k) A_0^T A_0 \mathbf{x}(k) \\ & - \mathbf{x}^T(k) A_0^T A_1 (A_1^T A_1 - E^T E)^{-1} A_1^T A_0 \mathbf{x}(k) + \mathbf{x}^T(k-1) (2A_1^T A_1 - E^T E) \mathbf{x}(k-1) \end{aligned} \quad (196)$$

Moreover, since:

$$\|\mathbf{x}(k-1)\|_{E^T E}^2 \geq 0, \quad \forall k \in \mathcal{K}_N, \quad \forall \mathbf{x}(k) \in \mathcal{W}_{d,k}^* \setminus \{0\} \quad (197)$$

and using assumption (189) it is clear that (196), reduces to:

$$\begin{aligned} \mathbf{x}^T(k+1) E^T E \mathbf{x}(k+1) < & \mathbf{x}^T(k) A_0^T \left( I - A_1 (A_1^T A_1 - E^T E)^{-1} A_1^T + 2p^2 I \right) A_0 \mathbf{x}(k) \\ < & \bar{\lambda}_{\max}(\cdot) \mathbf{x}^T(k) E^T E \mathbf{x}(k) \end{aligned} \quad (198)$$

where: 
$$\bar{\lambda}_{\max}(\cdot) = \{\mathbf{x}^T(k) A_0^T \left( I - A_1 (A_1^T A_1 - E^T E)^{-1} A_1^T + 2p^2 I \right) A_0 \mathbf{x}(k), \quad (199)$$

$$\mathbf{x}(k) \in \mathcal{W}_{dis}^*, \mathbf{x}^T(k) E^T E \mathbf{x}(k) = 1\}.$$

Following the procedure from the previous section, it can be written:

$$\ln \mathbf{x}^T(k+1) E^T E \mathbf{x}(k+1) - \ln \mathbf{x}^T(k) E^T E \mathbf{x}(k) < \ln \bar{\lambda}_{\max}(\cdot). \quad (200)$$

By applying the summing  $\sum_{j=k_0}^{k_0+k-1}$  on both sides of (200) for  $\forall k \in \mathcal{K}_N$ , one can obtain:

$$\begin{aligned} \ln \mathbf{x}^T(k_0+k) E^T E \mathbf{x}(k_0+k) &\leq \ln \prod_{j=k_0}^{k_0+k-1} \bar{\lambda}_{\max}(\cdot) \\ &\leq \ln \bar{\lambda}_{\max}^k(\cdot) + \ln \mathbf{x}^T(k_0) E^T E \mathbf{x}(k_0), \quad \forall k \in \mathcal{K}_N \end{aligned} \quad (201)$$

Taking into account the fact that  $\|\mathbf{x}_0\|_{E^T E}^2 < \alpha$  and the condition of *Theorem 3.2.2.1*, eq. (190), one can get:

$$\begin{aligned} \ln \mathbf{x}^T(k_0+k) E^T E \mathbf{x}(k_0+k) &< \ln \bar{\lambda}_{\max}^k(\cdot) + \ln \mathbf{x}^T(k_0) E^T E \mathbf{x}(k_0) \\ &< \ln \alpha \cdot \bar{\lambda}_{\max}^k(\cdot) < \ln \alpha \cdot \frac{\beta}{\alpha} < \ln \beta, \quad \forall k \in \mathcal{K}_N. \quad Q.E.D. \end{aligned} \quad (202)$$

**Theorem 3.2.2.2** Suppose matrix  $(A_1^T A_1 - E^T E) > 0$ . Causal system (178), is finite time unstable with respect to  $\{k_0, \mathcal{K}_N, \alpha, \beta, \|(\cdot)\|^2\}$ ,  $\alpha < \beta$ , if there exist a positive real number  $p$ ,  $p > 1$ , such that:

$$\|\mathbf{x}(k-1)\|_{A_1^T A_1}^2 < p^2 \|\mathbf{x}(k)\|_{A_1^T A_1}^2, \quad \forall k \in \mathcal{K}_N, \forall \mathbf{x}(k) \in \mathcal{S}_\beta, \forall \mathbf{x}(k) \in \mathcal{W}_{d,k}^* \setminus \{0\} \quad (203)$$

and if for  $\forall \mathbf{x}_0 \in \mathcal{W}_{d,k^*}$  and  $\|\mathbf{x}_0\|_{G=E^T E}^2 < \alpha$  there exist: real, positive number  $\delta$ ,  $\delta \in ]0, \alpha[$  and time instant  $k, k = k^* : \exists!(k^* > k_0) \in \mathcal{K}_N$ , for which the next condition is fulfilled (*Nestorovic & Debeljkovic 2011*):

$$\bar{\lambda}_{\min}^{k^*}(\cdot) > \frac{\beta}{\delta}, \quad k^* \in \mathcal{K}_N \quad (204)$$

where:

$$\begin{aligned} \bar{\lambda}_{\min}(\cdot) &= \bar{\lambda}_{\min} \{ \mathbf{x}^T(k) A_0^T \left( I - A_1 (A_1^T A_1 - E^T E)^{-1} A_1^T + 2\phi(k) I \right) A_0 \mathbf{x}(k), \\ &\mathbf{x}(k) \in \mathcal{W}_{k^*}^d, \mathbf{x}^T(k) E^T E \mathbf{x}(k) = 1 \}. \end{aligned} \quad (205)$$

**Proof.** Following the identical procedure as in the previous *Theorem*, with the same aggregation function, one can get:

$$\begin{aligned} \ln \mathbf{x}^T(k_0 + k^*) E^T E \mathbf{x}(k_0 + k^*) &> \ln \bar{\lambda}_{\min}^{k^*}(\cdot) + \ln \mathbf{x}^T(k_0) E^T E \mathbf{x}(k_0) \\ &> \ln \delta \cdot \bar{\lambda}_{\min}^{k^*}(\cdot) > \ln \delta \cdot \frac{\beta}{\delta} > \ln \beta, \text{ for some } k^* \in \mathcal{K}_N, \end{aligned} \quad (206)$$

where  $\bar{\lambda}_{\min}(\cdot)$  is given by (187). Q.E.D.

**Theorem 3.2.2.3** Suppose matrix  $(A_1^T P A_1 - E^T P E) \geq 0$ . Causal system given by (178), with  $\det A_0 \neq 0$ , is attractive practically stable with respect to  $\{k_0, \mathcal{K}_N, \alpha, \beta, \|\cdot\|^2\}$ ,  $\alpha < \beta$ , if there exists a matrix  $P = P^T > 0$ , being the solution of:

$$A_0^T P A_0 - E^T P E = -2(Q + S), \quad (207)$$

with matrices  $Q = Q^T > 0$  and  $S = S^T$ , such that:

$$\mathbf{x}^T(k)(Q + S)\mathbf{x}(k) > 0, \quad \forall \mathbf{x}(k) \in \mathcal{W}_{d,k^*} \setminus \{0\} \quad (208)$$

is positive definite quadratic form on  $\mathcal{W}_{d,k^*} \setminus \{0\}$ ,  $p$  real number,  $p > 1$ , such that:

$$\|\mathbf{x}(k-1)\|_{A_1^T P A_1}^2 < p^2 \|\mathbf{x}(k)\|_{A_1^T P A_1}^2, \quad \forall k \in \mathcal{K}_N, \quad \forall \mathbf{x}(k) \in \mathcal{S}_\beta, \quad \forall \mathbf{x}(k) \in \mathcal{W}_{k,d}^* \setminus \{0\} \quad (209)$$

and if the following conditions are satisfied (Nestorovic & Debeljkovic 2011):

$$\|A_1\| < \sigma_{\min} \left( Q^{\frac{1}{2}} \right) \sigma_{\max}^{-1} \left( Q^{-\frac{1}{2}} E^T P \right), \quad (210)$$

and

$$\bar{\lambda}_{\max}^k(\cdot) < \frac{\beta}{\alpha}, \quad \forall k \in \mathcal{K}_N, \quad (211)$$

where:

$$\begin{aligned} \bar{\lambda}_{\max}(\cdot) &= \max \{ \mathbf{x}^T(k) A_0^T P^{\frac{1}{2}} \left( I - A_1 (A_1^T P A_1 - E^T P E)^{-1} A_1^T + p^2 I \right) P^{\frac{1}{2}} A_0 \mathbf{x}(k) : \\ &\quad \mathbf{x}(k) \in \mathcal{W}_{d,k^*}, \mathbf{x}^T(k) E^T P E \mathbf{x}(k) = 1 \}. \end{aligned} \quad (212)$$

**Proof.** Let us consider the functional:

$$V(\mathbf{x}(k)) = \mathbf{x}^T(k) E^T P E \mathbf{x}(k) + \mathbf{x}^T(k-1) Q \mathbf{x}(k-1) \quad (213)$$

with matrices  $P = P^T > 0$  and  $Q = Q^T > 0$ .

**Remark 3.2.2.2** (208 - 209) are, in modified form, taken from (Owens, Debeljkovic 1985). For given (213), general backward difference is:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) = \mathbf{x}^T(k+1)E^TPE\mathbf{x}(k+1) \\ &+ \mathbf{x}^T(k)Q\mathbf{x}(k) - \mathbf{x}^T(k)E^TPE\mathbf{x}(k) - \mathbf{x}^T(k-1)Q\mathbf{x}(k-1). \end{aligned} \quad (214)$$

Clearly, using the equation of motion (178), we have:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= \mathbf{x}^T(k)(A_0^TPA_0 - E^TPE + Q)\mathbf{x}(k) \\ &+ 2\mathbf{x}^T(k)(A_0^TPA_1)\mathbf{x}(k-1) - \mathbf{x}^T(k-1)(Q - A_1^TPA_1)\mathbf{x}(k-1), \end{aligned} \quad (215)$$

or

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= \mathbf{x}^T(k)(A_0^TPA_0 - E^TPE + 2Q + 2S)\mathbf{x}(k) - \mathbf{x}^T(k)Q\mathbf{x}(k) \\ &- 2\mathbf{x}^T(k)S\mathbf{x}(k) + 2\mathbf{x}^T(k)(A_0^TPA_1)\mathbf{x}(k-1) - \mathbf{x}^T(k-1)(Q - A_1^TPA_1)\mathbf{x}(k-1). \end{aligned} \quad (216)$$

Using (208) and (209) yields:

$$\begin{aligned} \mathbf{x}^T(k+1)E^TPE\mathbf{x}(k+1) &= \mathbf{x}^T(k)A_0^TPA_0\mathbf{x}(k) \\ &+ 2\mathbf{x}^T(k)A_0^TPA_1\mathbf{x}(k-1) + \mathbf{x}^T(k-1)A_1^TPA_1\mathbf{x}(k-1). \end{aligned} \quad (217)$$

Using the very well known inequality, with particular choice:

$$\begin{aligned} \mathbf{x}^T(k)\Gamma\mathbf{x}(k) &= \mathbf{x}^T(k)(A_1^TPA_1 - E^TPE)\mathbf{x}(k) \geq 0, \\ \mathbf{x}(k) &\in \mathcal{W}_{k^*}^d, \forall \mathbf{x}(k) \in \mathcal{S}_\beta, \forall k \in \mathcal{K}_N \end{aligned} \quad (218)$$

one can get:

$$\begin{aligned} \mathbf{x}^T(k+1)E^TPE\mathbf{x}(k+1) &\leq \mathbf{x}^T(k)A_0^TPA_0\mathbf{x}(k) - \mathbf{x}^T(k)A_0^TPA_1(A_1^TPA_1 - \\ &- E^TPE)^{-1}A_1^TPA_0\mathbf{x}(k) + \mathbf{x}^T(k-1)(2A_1^TPA_1 - E^TPE)\mathbf{x}(k-1). \end{aligned} \quad (219)$$

Moreover, since:

$$\|\mathbf{x}(k-1)\|_{E^TPE}^2 \geq 0, \quad \forall k \in \mathcal{K}_N, \quad \forall \mathbf{x}(k) \in \mathcal{W}_{d,k}^* \setminus \{0\} \quad (220)$$

and using assumption (209) it is clear that (219), reduces to:

$$\mathbf{x}^T(k+1)E^TPE\mathbf{x}(k+1) \leq \mathbf{x}^T(k)A_0^TP^2 \left( I - A_1(A_1^TPA_1 - E^TPE)^{-1}A_1^T + 2p^2I \right) P^2 A_0 \mathbf{x}(k) \quad (221)$$

Using very well known the property of quadratic form, one can get:

$$\mathbf{x}^T(k+1)E^TPE\mathbf{x}(k+1) \leq \bar{\lambda}_{\max}(\cdot) \mathbf{x}^T(k)E^TPE\mathbf{x}(k) \quad (222)$$

where:

$$\begin{aligned} \bar{\lambda}_{\max}(\cdot) &= \{\mathbf{x}^T(k)A_0^TP^2(I - A_1(A_1^TPA_1 - E^TPE)^{-1}A_1^T + 2p^2I)P^2A_0\mathbf{x}(k), \\ &\mathbf{x}(k) \in \mathcal{W}_{d,k}^* \setminus \{0\}, \mathbf{x}^T(k)E^TPE\mathbf{x}(k) = 1\} \end{aligned} \quad (223)$$

Then following the identical procedure as in the *Theorem 3.2.2.1*, one can get:

$$\ln \mathbf{x}^T(k+1)E^T P E \mathbf{x}(k+1) - \ln \mathbf{x}^T(k)E^T P E \mathbf{x}(k) < \ln \bar{\lambda}_{\max}(\cdot) \quad (224)$$

where  $\bar{\lambda}_{\max}(\cdot)$  is given by (223).

If the summing  $\sum_{j=k_0}^{k_0+k-1}$  is applied to both sides of (224) for  $\forall k \in \mathcal{K}_N$ , one can obtain:

$$\begin{aligned} \ln \mathbf{x}^T(k_0+k)E^T P E \mathbf{x}(k_0+k) &\leq \ln \prod_{j=k_0}^{k_0+k-1} \bar{\lambda}_{\max}(\cdot) \\ &\leq \ln \bar{\lambda}_{\max}^k(\cdot) + \ln \mathbf{x}^T(k_0)E^T P E \mathbf{x}(k_0), \quad \forall k \in \mathcal{K}_N \end{aligned} \quad (225)$$

Taking into account the fact that  $\|\mathbf{x}_0\|_{E^T P E}^2 < \alpha$  and the basic condition of *Theorem 3.2.2.3*, (211), one can get:

$$\begin{aligned} \ln \mathbf{x}^T(k_0+k)E^T P E \mathbf{x}(k_0+k) &< \ln \bar{\lambda}_{\max}^k(\cdot) + \ln \mathbf{x}^T(k_0)E^T P E \mathbf{x}(k_0) \\ &< \ln \alpha \cdot \bar{\lambda}_{\max}^k(\cdot) < \ln \alpha \cdot \frac{\beta}{\alpha} < \ln \beta, \quad \forall k \in \mathcal{K}_N. \quad Q.E.D. \end{aligned} \quad (226)$$

#### 4. Conclusion

The first part of this chapter is devoted to the stability of particular classes of linear continuous and discrete time delayed systems. Here, we present a number of new results concerning stability properties of this class of systems *in the sense of Lyapunov and non-Lyapunov* and analyze the relationship between them. Some open question can arise when particular choice of parameters  $p$  and  $q$  is needed, see (Su & Huang 1992), (Xu & Liu 1994) and (Su 1994).

The geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all solutions. This fact makes possible the construction of Lyapunov stability theory even for linear continuous singular time delayed systems (LCSTDS) and linear discrete descriptor time delayed systems (LDDTDS) in that sense that asymptotic stability is equivalent to the existence of symmetric, positive definite solutions to a *weak* form of Lyapunov continuous (discrete) algebraic matrix equation (Owens, Debeljkovic 1985) respectively, incorporating condition which refers to time delay term.

To assure *asymptotical stability* for (LCSTDS) it is not only enough to have the eigenvalues of the matrix pair  $(E, A)$  in the left half complex plane or within the unit circle, respectively, but also to provide an impulse-free motion and some other certain conditions to be fulfilled for the systems under consideration. The idea and the approach, in this exposure, are based upon the papers by (Owens, Debeljkovic 1985) and (Tissir, Hmamed 1996).

Some different approaches have been shown in order to construct *Lyapunov* stability theory for a particular class of autonomous (LCSTDS) and (LDDTDS).

The second part of the chapter is concerned with the stability of particular classes of (LCSTDS) and (LDDTDS). There, we present a number of *new results* concerning stability properties of this class of systems *in the sense of non-Lyapunov (finite time stability, practical stability, attractive practical stability, etc.)* and analyze the relationship between them.

And finally this chapter extends some of the basic results in the area of non-Lyapunov to *linear, continuous singular time invariant time-delay systems* (LCSTDS) and (LDDTDS). In that sense the part of this result is hence a geometric counterpart of the algebraic theory of Campbell (1980) charged with appropriate criteria to cover the need for system stability in the presence of actual time delay term. To assure *practical stability* for (LCSTDS) it is not enough only to have the eigenvalues of matrix pair  $(E, A)$  somewhere in the complex plane, but also to provide an impulse-free motion and certain conditions to be fulfilled for the system under consideration.

Some different approaches have been shown in order to construct *non-Lyapunov* stability theory for a particular class of autonomous (LDDTDS). The geometric description of consistent initial conditions that generate tractable solutions to such problems and the construction of non-Lyapunov stability theory to bound rates of decay of such solutions are also investigated. Results are based on existing Lyapunov-like functions and their properties on sub-space of consistent initial functions (conditions). In particular, these functions need not to have: *a*) Properties of positivity in the whole state space and *b*) negative derivatives along the system trajectories.

And finally a quite new approach leads to the sufficient delay-independent criteria for finite and attractive practical stability of (LCSTDS) and (LDDTDS).

Stability issues, as well as time delay and singularity phenomena play a significant role in modeling of real systems. A need for their consideration arises from growing interest and extensive application possibilities in different areas such as large-scale systems, flexible light-weight structures and their vibration and noise control, optimization of smart structures (Nestorovic et al. 2005, 2006, 2008) etc. Development of reliable models plays a crucial role especially in early development phases, which enables performance testing, design review, optimization and controller design (Nestorovic & Trajkov 2010.a). Assumptions introduced along with model development, especially e.g. reduction of large numerical models of smart structures require consideration of many important questions from the control theory point of view, whereby the stability and singularity phenomena count among some of the most important. Therefore they represent the focus of the authors' ongoing and further research activities (Debeljkovic et al. 2011.b, Nestorovic & Trajkov 2010.b).

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## 6. References

- Aleksendric, M., (2002) *On stability of particular class of continual and discrete time delay systems on finite and infinite time interval*, Diploma Work, School of Mechanical Eng., University of Belgrade, Department of Control Eng., Belgrade.
- Aleksendric, M. & D, Lj. Debeljkovic, (2002) Finite time stability of linear discrete time delayed systems, *Proc. HIPNEF 2002*, Nis (Serbia), October, 2-5, pp. 333-340.
- Amato, F., M. Ariola, C. Cosentino, C. Abdallah, P. Dorato, (2003) Necessary and sufficient conditions for finite-time stability of linear systems, *Proc. of the 2003 American Control Conference*, Denver (Colorado), 5, pp. 4452-4456.



- Amir-Moez, A., (1956) Extreme properties of a hermitian transformations and singular values of sum and product of linear transformations, *Duke Math J.*, 23, 463-476.
- Angelo, H., (1974) *Linear time varying systems*, Allyn and Bacon, Boston.
- Campbell, S. L., (1980) *Singular systems of differential equation*, Pitman, London.
- Coppel, W. A., (1965) *Stability and asymptotic behavior of differential equations*, Boston D.C. Heath.
- Debeljkovic, D. Lj., (2001) On practical stability of discrete time control systems, *Proc. 3<sup>rd</sup> International Conference on Control and Applications*, Pretoria (South Africa), December 2001, pp. 197-201.
- Debeljkovic, D. Lj., V. B. Bajic, A. U. Grgic, S. A. Milinkovic, (1995) Non-Lyapunov stability and instability robustness consideration for linear singular systems, *Proc. 3<sup>rd</sup> ECC*, Roma (Italy), September 5 - 8, pp. 1373-1379.
- Debeljkovic, D. Lj., Z. Lj. Nenadic, S. A. Milinkovic, M. B. Jovanovic, (1997.a) On Practical and Finite-Time Stability of Time-Delay Systems, *Proc. ECC 97*, Brussels (Belgium), pp. 307-311.
- Debeljkovic, D. Lj., Z. Lj. Nenadic, Dj. Koruga, S. A. Milinkovic, M. B. Jovanovic, (1997.b) On practical stability of time-delay systems new results, *Proc. 2<sup>nd</sup> ASCC 97*, 22 - 25 July, Seoul (Korea), pp. 543-546.
- Debeljkovic, D. Lj., M. P. Lazarevic, Dj. Koruga, S. Tomašević, (1997.c) On practical stability of time delay system under perturbing forces, *Proc. AMSE 97*, Melbourne (Australia), pp. 442-446.
- Debeljkovic, D. Lj., Z. Lj. Nenadic, S. A. Milinkovic, M. B. Jovanovic, (1997.d) On the stability of linear systems with delayed state defined over finite time interval, *Proc. CDC 97*, San Diego, California (USA), pp. 2771-2772.
- Debeljkovic, D. Lj., M. P. Lazarevic, Dj. Koruga, (1997.e) Finite time stability for the metal strips cold rolling, *Proc. ASIAN international workshop on automation in steel industry*, Kyongju (Korea), July 16 - 18, pp. 233-238.
- Debeljkovic, D. Lj., Dj. Koruga, S. A. Milinkovic, M. B. Jovanovic, Lj. A. Jacic (1998.a) Further Results on Non-Lyapunov Stability of Time Delay Systems, *Proc. MELECON 98*, Tel-Aviv (Israel), May 18-20, Vol.1, pp. 509 - 512.
- Debeljkovic, D. Lj., Dj. Koruga, S. A. Milinkovic, M. B. Jovanovic, (1998.b) Non-Lyapunov stability analysis of linear time delay systems, *Preprints DYCOPS 5*, 5<sup>th</sup> IFAC Symposium on Dynamics and Process Systems, Corfu (Greece), pp. 549-553.
- Debeljkovic, D. Lj., M. P. Lazarevic, S. A. Milinkovic, M. B. Jovanovic, (1998.c) Finite time stability analysis of linear time delay systems Bellman-Gronwall approach, *Proc. 1st IFAC Workshop on Linear Time Delay Systems*, Grenoble (France) July 6 - 7, pp. 171-175.
- Debeljkovic, D. Lj., S. A. Milinkovic, M. B. Jovanovic, Lj. A. Jacic, Dj. Koruga, (1998.d), Further results on non - Lyapunov stability of time delay systems, *Prepr. 5<sup>th</sup> IFAC Symp. on (LCA)*, Shenyang (China), September 8 - 10 , TS13, pp. 6-10.
- Debeljkovic, D. Lj., Dj. Koruga, S. A. Milinkovic, M. B. Jovanovic, Lj. A. Jacic, (1998.e) Further results on non-Lyapunov stability of linear systems with delayed state, *Proc. XII CBA - Brazilian Automatic Control Conference*, Uberlandia (Brazil), September 14 - 18 Vol. IV, pp. 1229-1233.
- Debeljkovic, Lj. D. & S. A. Milinkovic, (1999) *Finite Time Stability of Time Delay Systems*, GIP Kultura, ID 72600076, Belgrade.

- Debeljkovic, D. Lj., M. P. Lazarevic, Z. Lj. Nenadic, S. A. Milinkovic, (1999) Finite Time Stability of Time Delay Systems, *IMA J. Math. Control and Info.* 16 (3) pp. 101-109, ISSN 0265-0754.
- Debeljkovic, D. Lj., M. P. Lazarevic, Dj. Koruga, S. A. Milinkovic, M. B. Jovanovic, (2000.a) Further results on the stability of linear nonautonomous systems with delayed state defined over finite time interval, *Proc. ACC 2000*, Chicago (USA), pp. 1450-1451.
- Debeljkovic, D. Lj., M. P. Lazarevic, Dj. Koruga, S. A. Milinkovic, M. B. Jovanovic, (2000.b) Further results on the stability of linear nonautonomous systems with delayed state defined over finite time interval, *Proc. APCCM (The 4<sup>th</sup> Asia-Pacific Conference on Control and Measurements)*, 9-12 July Guilin (China), D. pp.9.
- Debeljkovic, D. Lj., M. P. Lazarevic, Dj. Koruga, S. A. Milinkovic, M. B. Jovanovic, (2000.c) Further results on the stability of linear nonautonomous systems with delayed state defined over finite time interval and application to different chemical processes, *CHISA 2000*, 27-31 Avgust, Praha (Czech Republic), CD-Rom.
- Debeljkovic, D. Lj. & M. Aleksendric, (2003.a) Lyapunov and non-Lyapunov stability of linear discrete time delay systems, *Proc. ACC*, 4-7 June 2003, Denver USA, pp. 4450-4451.
- Debeljkovic, D. Lj., M. Aleksendric, Y. Y. Nie, Q. L. Zhang, (2003.b) Lyapunov and non-Lyapunov stability of linear discrete time delay systems, *Proc. The Fourth Inter. Conference on Control and Automation*, 9-12 June 2003, Montreal (Canada), pp. 296-300.
- Debeljkovic, D. Lj., S. B. Stojanovic S. A. Milinkovic, M. B. Jovanovic, (2003.c) Descriptor Time Delayed System Stability Theory in the sense of Lyapunov New Results, *Facta Universitatis, Ser. Mech. Eng.*, Vol. 1, No. 10, pp. 1311-1315.
- Debeljkovic D. Lj. & Stojanovic S. B., (2004.a) Necessary and Sufficient Conditions for Delay-Dependent Asymptotic Stability of Linear Discrete Large Scale Time Delay Autonomous System, *7<sup>th</sup> Biennial ASME Conference Eng. Systems Design and Analysis, ESDA 2004*, Manchester, UK, July 19-22, CD-Rom.
- Debeljkovic, D. Lj., S. B. Stojanovic, M. B. Jovanovic, S. A. Milinkovic, (2004.b) A Short Note to the Lyapunov Stability of  $\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1)$  New Results, *Proc. CSCSC 04 Shenyang (China)*, Avgust 08-10, pp. 11-14.
- Debeljkovic, D. Lj., S. B. Stojanovic, M. P. Lazarevic, M. B. Jovanovic, S. A. Milinkovic, (2004.c) Discrete time delayed system stability theory in the sense of Lyapunov application to the chemical engineering and process technology, *Proc. CHISA 2004*, Prague (Czech Republic), Avgust 28 - 31, CD-Rom.
- Debeljkovic, Lj. D., M. P. Lazarevic, S. B. Stojanovic, M. B. Jovanovic, S. A. Milinkovic, (2004) "Discrete Time Delayed System Stability Theory in the sense of Lyapunov: New Results", *Proc. ISIC 04 (International Symposium on Intelligent Control)*, September 01 - 04, Taipei (Taiwan) 511 - 515.
- Debeljkovic, D. Lj., S. B. Stojanovic, M. B. Jovanovic, S. A. Milinkovic, (2004.g) Singular time delayed system stability theory approach in the sense of Lyapunov, *Preprints of IFAC Workshop on Time delay systems*, Leuven, (Belgium), September 8-10, CD-Rom.
- Debeljkovic, Lj. D., A. Jacic, M. Medenica, (2005) *Time Delay Systems Stability and Robustness*, Faculty of Mechanical Engineering, University of Belgrade, ISBN 86-7083-504-5
- Debeljkovic, D. Lj., M. P. Lazarevic, S. B. Stojanovic, M. B. Jovanovic, S. A. Milinkovic, (2005.a) Discrete time delayed system stability theory in the sense of Lyapunov:

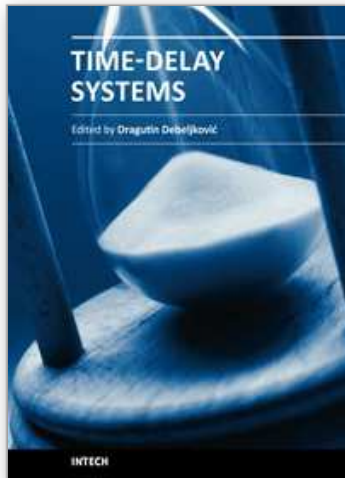
- New results, *Dynamics of Continuous, Discrete and Impulsive Systems*, (Canada), Vol. 12, Series B Numerical. Analysis, Vol. 12.b – Suppl., pp. 433 – 442, ISSN 1492 – 8760.
- Debeljkovic, Lj. D. S. B. Stojanovic, M. B. Jovanovic, S. A. Milinkovic, (2005.b) Further results on singular time delayed system stability, *Proc. ACC 05*, June, 8-10 2005, Portland (Oregon) , CD-Rom.
- Debeljkovic, Lj. D., S. B. Stojanovic, M. B. Jovanovic, S. A. Milinkovic, (2005.c) Further results on descriptor time delayed system stability theory in the sense of Lyapunov: Pandolfi based approach, *The fifth International Conference on Control and Automation, ICCA 05*, June 26-29, Budapest (Hungary), CD-Rom.
- Debeljkovic, D. Lj., S. B. Stojanovic, M. B. Jovanovic, Lj. A. Jacic, (2005.d) Further results on asymptotic stability of linear discrete descriptor time delayed systems, *Proc. 1st International Workshop on Advanced Control Circuits and Systems, ACCS 05*, Cairo (Egypt), March 06 – 10, 2005, ISBN 0-154-6310-7933-2, CD-Rom.
- Debeljkovic, D. Lj., S. B. Stojanovic, M. B. Jovanovic, S. A. Milinkovic, (2006.a) Further Results on Descriptor Time Delayed System Stability Theory in the sense of Lyapunov Pandolfi based approach, *International Journal of Information & System Science*, (Canada), Vol. 2, No. 1, pp. 1-11, ISSN 1708-296x.
- Debeljkovic, Lj. D. S. B. Stojanovic, N. S. Visnjic, S. A. Milinkovic, (2006.b) Lyapunov Stability Theory A Quite New Approach – Discrete Descriptive Time Delayed Systems, *Proc. American Control Conference 06*, June, 14-16, Minneapolis (USA), pp. 5091-5096.
- Debeljkovic D. Lj., Lj. A. Jacic., N. S. Visnjic, M. Pjescic, (2007.a) Asymptotic Stability of Generalized Discrete Descriptive Time Delayed Systems, *Proc. The 5<sup>th</sup> Edition of IFAC Know. and Tech. Transfer Conference Series on Automation for Buildings the in the Infra structure, DECOM 2007*, May 17-19, Cesme - Izmir (Turkey), pp. 369-374.
- Debeljkovic D. Lj., S. B. Stojanovic., N. S. Visnjic, S. A. Milinkovic, (2007.b) Singular Time Delayed System Stability Theory in the sense of Lyapunov A Quite New Approach, *Proc. American Control Conference - ACC 2007*, July 11-13, New York, USA, CD Rom.
- Debeljkovic, D. Lj.,& S. B. Stojanovic, (2008) *Systems, Structure and Control* - Editor Petr Husek, – Chapter *Asymptotic Stability Analysis of Linear Time Delay Systems Delay Dependent Approach*, pp. 029 – 060, I – Tech, ISBN 978-7619-05-3, Vienna.
- Debeljkovic, D. Lj., S. B. Stojanovic, N. S. Visnjic, S. A. Milinkovic, (2008) A Quite New Approach to the Asymptotic Stability Theory: Discrete Descriptive Time Delayed System, *Dynamics of Continuous, Discrete and Impulsive Systems*, (Canada), Vol. 15, Series A: Mathematical Analysis, No. 15 , pp. 469-480, ISSN 1201-3390.
- Debeljkovic, D. Lj., T. Nestorovic, I. M. Buzurovic, N. J. Dimitrijevic, (2010) A New Approach to the Stability of Time-Delay Systems in the Sense of Non-Lyapunov Delay-Independent and Delay-Dependent Criteria, *Proc. SISY 2010 (IEEE 8<sup>th</sup> International Symposium on Intelligent Systems and Informatics)*, Sept. 10-11, Subotica (Serbia), pp. 213-218
- Debeljkovic, D. Lj., T. Nestorovic, I. M. Buzurovic, G. V. Simeunovic, (2011.a) On finite time and practical stability of singular time delayed systems, *Proc. (MSC) IEEE Multi Conference on Systems and Control*, Denver (Colorado), September 28-30, submitted.
- Debeljkovic, D. Lj., T. Nestorovic, I. M. Buzurovic, G. V. Simeunovic, (2011.b) On non-Lyapunov delay-independent and delay-dependent criteria for particular class of

- continuous time delay systems, *Proc. CDC*, December 3-5, Orlando (Florida), to be submitted.
- Desoer, C. A. & M. Vidysagar, *Feedback Systems Input - Output Properties*, Academic Press, New York, 1975.
- Fang, H. H., H. J. Hunsarg, (1996) Stabilization of nonlinear singularly perturbed multiple time delay systems by dither, *Journal of Dynamic Systems, Measurements and Control*, March, Vol. 118, pp. 177-181.
- Kablar, A. N. & D. Lj. Debeljkovic, (1999) Finite time instability of time varying linear singular systems, *Proc. ACC 99*, San Diego (USA), June 2 - 4, pp. 1796-1800.
- Kalman R. E. & J. E. Bertram, (1960) Control system analysis and design via the second method of Lyapunov - Part II, Discrete-Time Systems, *Trans. of ASME, Ser. D.*, June, pp. 394-400.
- Lazarevic, M. P., D. Lj. Debeljkovic, Z. Lj. Nenadic, S. A. Milinkovic, (2000) Finite time stability of time delay systems, *IMA Journal of Mathematical Control and Information*, Vol.17, No.3, pp. 101-109, ISSN 0265-0754.
- Lazarevic, M. P. & D. Lj. Debeljkovic, (2003) Finite time stability analysis of linear autonomous fractional order systems with delayed state, *Preprints of IFAC Workshop on Time delay systems*, INRIA, Rocquencourt, Paris, (France), September 8 - 10, CD-Rom.
- Lazarevic, M. P., D. Lj. Debeljkovic, (2005) Finite time stability analysis of linear autonomous fractional order systems with delayed state, *Asian J Contr*, Vol.7, No.4, 440-447.
- Lee, T. N., S. Diant, (1981) Stability of time delay systems, *IEEE Trans. Automat. Control* AC-26 (4), 951-953, ISSN 0018-9286.
- Liang J. R., (2000) Analysis of stability for descriptor discrete systems with time-delay, *Journal of Guangxi University* (Nat. Sc. Ed.), 25(3), pp. 249-251.
- Liang J. R., (2001) The asymptotic stability and stabilization for singular systems with time delay, *System Engineering and Electronics*, 23 (2), pp. 62-64.
- Liu Y. Q. & Y. Q. Li, (1997) Stability of solutions of a class of time-varying singular systems with delay, *Proc. Control and Decision Conference*, 12 (3), pp. 193-197.
- Li Y. Q. & Y. Q. Liu, (1998) Stability of solution of singular systems with delay, *Control Theory and Application*, 15(4), 542-550.
- Mao, X., (1997) Comments on "Improved Razumikhin-Type Theorem and its Applications", *IEEE Trans. Automat. Control*, 42 (3) 429 - 430, ISSN 0018 - 9286
- Mori, T., N. Fukuma, M. Kuwahara, (1981) Simple stability criteria for single and composite linear systems with time delay, *International Journal of Control*, 34, (6), 1175-1184, ISSN 0020-7179.
- Muller, P. C., (1983) Stability of linear mechanical systems with holonomic constraints, *Appl. Mechanical Review* (11), Part II, November, 160-164, ISSN 978019514.
- Nenadic, Z. Lj., D. Lj. Debeljkovic, S. A. Milinkovic, (1997) On practical stability of time delay systems, *Proc. ACC 97*, Albuquerque, New Mexico (USA), pp. 3235-3236.
- Nestorovic-Trajkov T., Koeppe H., Gabbert U., (2005) Active Vibration Control Using Optimal LQ Tracking System with Additional Dynamics, *International Journal of Control*, 78(15), 1182-1197
- Nestorovic Trajkov T., Koeppe H., Gabbert U., (2006) Vibration control of a funnel-shaped shell structure with distributed piezoelectric actuators and sensors, *Smart Materials and Structures*, 15, 1119-1132



- Nestorovic Trajkov T., Koeppe H., Gabbert U., (2008) Direct model reference adaptive control (MRAC) design and simulation for the vibration suppression of piezoelectric smart structures, *Communications in Nonlinear Science and Numerical Simulation*, 13(9), 1896-1909
- Nestorovic T. & Trajkov M., (2010.a) Active Control of Smart Structures - An Overall Approach, *Facta Universitatis, Series Architecture and Civil Engineering*, 8(1), 35-44, ISSN 0354-4605, DOI: 10.2298/FUACE1001035N
- Nestorovic T. & Trajkov M., (2010.b) Development of Reliable Models for the Controller Design of Smart Structures and Systems, International Conference "Mechanical Engineering in the XXI Century", November 25-26, 2010, University of Niš, Serbia, pp. 263-266, ISBN: 978-86-6055-008-0
- Nestorovic, T., D. Lj. Debeljkovic, I. M. Buzurovic, S. B. Stojanovic, (2011.a) Further Results on Stability of Linear Discrete Time Delay Systems over a Finite Time Interval: Novel Delay-Independent Conditions, American Control Conference - ACC 2011, June 29 - July 1, 2011, San Francisco (USA), submission Nr. 89
- Nestorovic, T. & D. Lj. Debeljkovic, (2011.b) The stability of linear descriptive discrete time delayed systems over the finite time interval: New results and quite new approach, *Proc. CDC*, December 3 - 5, Orlando (Florida), should be submitted.
- Nie, Y. Y. & D. Lj. Debeljkovic, (2004) Non-Lyapunov Stability of Linear Singular Systems: A Quite new Approach in Time Domain, *Dynamics of Continuous, Discrete and Impulsive Systems*, (Canada), Vol. 11, Series A: Math. Analysis, No. 5-6, pp. 751-760, ISSN 1201-3390.
- Owens, H. D. & D. Lj. Debeljkovic, (1985) Consistency and Lyapunov Stability of Linear Descriptor Systems A Geometric Analysis, *IMA Journal of Mathematical Control and Information*, (2), pp. 139-151, ISSN 0265 - 0754.
- Owens, H. D. & D. Lj. Debeljkovic, (1986) On non-Lyapunov stability of discrete descriptor systems, *Proc. CDC*, Athens (Greece), December, pp. 2138-2139.
- Pandolfi L., (1980) Controllability and stabilization for linear system of algebraic and differential equations, *Jota* 30 (4) pp. 601 - 620, ISSN 0363-0129.
- Pjescic, R M., V. Chistyakov, D. Lj. Debeljkovic, (2008) *On Dynamical Analysis of Particular Class of Linear Singular Time Delayed Systems: Stability and Robustness*, Faculty of Mechanical Engineering Belgrade, ISBN 978-86-7083-631-0, Belgrade
- Stojanovic, S. B. & D. Lj. Debeljkovic, (2005) Necessary and sufficient conditions for delay-dependent asymptotic stability of linear continuous large scale time delay systems, *Asian Journal of Control*, (Taiwan) Vol. 7, No. 4, pp. 414 - 418.
- Stojanovic, S. B. & D. Lj. Debeljkovic, (2005.a) The sufficient conditions for stability of continuous and discrete large - scale time delay interval systems, *Internat. Journal of Infor. & System Science*, (Canada), Vol. 1, No. 1, pp. 61 - 74, ISSN 1708-296x.
- Stojanovic, S. B. & D. Lj. Debeljkovic, (2005.b) On the asymptotic stability of linear discrete time delay autonomous systems, *International Journal of Information & System Science*, (Canada), Vol. 1, No. 3 - 4, pp. 413 - 420, ISSN 1708-296x.
- Stojanovic, S. B. & D. Lj. Debeljkovic, (2005.c) Necessary and sufficient conditions for delay-dependent asymptotic stability of linear continuous large scale time delay autonomous systems, *Asian J Contr (Taiwan)*, Vol. 7., No. 4, 414-418, ISSN 1561-8625.
- Stojanovic, S. B. & D. Lj. Debeljkovic, (2006) Comments on stability of time-delay systems, *IEEE Trans. Automat. Contr.* (submitted), ISSN 0018-9286.

- Stojanovic, S. B. & D. Lj. Debeljkovic, (2006.a) Further results on asymptotic stability of linear discrete time delay autonomous systems, *International Journal of Information & System Science*, (Canada), Vol. 2, No. 1, 117-123, ISSN 1708-296x.
- Stojanovic, S. B. & D. Lj. Debeljkovic, (2006.b) Exponential stability of discrete time delay systems with nonlinear perturbations, *International Journal of Information & System Science*, (Canada), Vol. 2, No. 3, pp. 428 - 435, ISSN 1708-296x.
- Stojanovic, S. B. & D. Lj. Debeljkovic, (2008.a) Delay - dependent stability of linear discrete large scale time delay systems necessary and sufficient conditions, *International Journal of Information & System Science*, (Canada), Vol. 4(2), 241-250, ISSN 1708-296x.
- Stojanovic, S. B. & D. Lj. Debeljkovic, (2008.b) Necessary and sufficient conditions for delay-dependent asymptotic stability of linear discrete time delay autonomous systems, *Proc. of 17th IFAC World Congress*, Seoul, Korea, July 6-10, pp. 2613-2618, ISSN 978-1-1234-7890-2 08.
- Stojanovic, S. B. & D. Lj. Debeljkovic, (2008) Quadratic stability and stabilization of uncertain linear discrete - time systems with state delay: A LMI approach, *Dynamics of Continuous, Discrete and Impulsive Systems*, (Canada), Vol. 15, Series B: Applications and Algorithms, No. 2, pp. 195-206, ISSN 1492-8760.
- Stojanovic, S. B. & D. Lj. Debeljkovic, Necessary and sufficient conditions for delay-dependent asymptotic stability of linear discrete time delay autonomous systems, *Dynamics of Continuous, Discrete and Impulsive Systems*, (Canada), Vol. 15, Series B: Applications and Algorithms, Vol. 16, No.6, (2009), pp. 887-900, ISSN 1492-8760.
- Su, J. H., (1994) Further results on the robust stability of linear systems with single time delay, *Systems & Control Letters* (23), pp. 375 - 379, ISSN: 0167-6911
- Su, J. H. & C. G. Huang, (1992) Robust stability of delay dependence for linear uncertain systems, *IEEE Trans. Automat. Control* AC- 37 (10), pp. 1656-1659, ISSN 0018-9286.
- Syrmos V. L., P. Misra, R. Aripirala, (1995) On the discrete generalized Lyapunov equation, *Automatica*, 31(2) 297- 301, ISSN 0005-1098.
- Tissir, E. & A. Hmamed, (1996) Further Results on Stability of  $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t - \tau)$ , *Automatica*, 32 (12), 1723-1726, ISSN 0005-1098.
- Xu, B. & Y. Liu, (1994) Improved Razumikhin-Type Theorem and its Applications, *IEEE Trans. Automat. Control* AC- 39 (4), pp. 839 - 841, ISSN 0018-9286.
- Xu, S. & C. Yang, (1999) Stabilization of discrete - time singular systems: A matrix inequality approach, *Automatica*, Vol. 35, pp. 1613 - 1617, ISSN 0005 - 1098.
- Xu, S., P. V. Dooren, R. Stefan, J. Lam, (2002.a) Robust stability and stabilization for singular systems with state delay and parameter uncertainty, *IEEE Trans. Automat. Control* AC- 47 (7), pp. 1122-1128, ISSN 0018-9286.
- Xu, S., J. Lam, C. Yang, (2002.b) Robust  $H_\infty$  control for discrete singular systems with state delay and parameter uncertainty, *Dynamics Continuous, Discrete and Impulsive Systems*, Vol. 9, No.4, pp. 539 - 554, ISSN 1201- 3390.
- Yang D. M., Q. L. Zhang, B. Yao., (2004) *Descriptor systems*, Science Publisher, Beijing.
- Yang C., Q. Zhang, L. Zhou, (2006) Practical stability of descriptor systems with time delays in terms of two measurements, *J Frankl Inst*, 343, 635-646, ISSN 0016-0032.
- Wang, W. & L. Mau, (1997) Stabilization and estimation for perturbed discrete time-delay large-scale systems, *IEEE Trans. Automat. Contr.*, 42(9), 1277-1282, ISSN 0018-9286.



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Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, robotics, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability. Consequently, the problem of controllability, observability, robustness, optimization, adaptive control, pole placement and particularly stability and robustness stabilization for this class of systems, has been one of the main interests for many scientists and researchers during the last five decades.

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