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Coexistence of Synchronization and Anti-Synchronization for Chaotic Systems via Feedback Control

Hammami Sonia and Benrejeb Mohamed

UR LARA Automatique, École Nationale d'Ingénieurs de Tunis
BP 37, Tunis le Belvédère 1002
Tunisia

1. Introduction

The study of chaos can be introduced in several applications as: medical field, fractal theory, electrical circuits and secure communication, essentially based on synchronization techniques.

The synchronization for chaotic systems has been extended to the scope, such as generalized synchronization, phase synchronization, lag synchronization, and even anti-synchronization (Pecora & Carroll, 1990; Wu & Chua, 1993; Michael et al., 1996; Bai & Lonngsen, 1997; Yang & Duan, 1998; Taherion & Lai, 1999; Zhang & Sun, 2004; Li, 2005; Yassen, 2005; Hammami et al., 2009; Juhn et al., 2009; Hammami et al., 2010b).

For an n master chaotic system coupled to an n slave one, described respectively by $\dot{x}_m(t) = f(x_m(t))$ and by $\dot{x}_s(t) = g(x_s(t))$, where x_m and x_s are phase space variables, and $f(.)$ and $g(.)$ the corresponding nonlinear functions, the synchronization in a direct sense implies that $|x_s(t) - x_m(t)| \to 0$ as $t \to +\infty$.

The property of anti-synchronization constitutes a prevailing phenomenon in symmetrical oscillators, in which the state vectors have the same absolute values but opposite signs, which implies that $|x_s(t) + x_m(t)| \to 0$ as $t \to +\infty$.

When synchronization and anti-synchronization coexist, simultaneously, in chaotic systems, the synchronization is called hybrid synchronization (Juhn et al., 2009).

2. Chaotic hybrid synchronization and stability study

Let consider the chaotic master system (S_m) described, in the state space, by:

$$\dot{x}_m(t) = A\big(x_m(t)\big)x_m(t) \tag{1}$$

coupled with the following forced slave system (S_s):

$$\dot{x}_s(t) = A\big(x_s(t)\big)x_s(t) + B\big(x_s(t)\big)u(t) \tag{2}$$

where x_m and x_s are the n state vectors of the considered systems (S_m) and (S_s), respectively, $x_m = \begin{bmatrix} x_{m1} & \ldots & x_{mn} \end{bmatrix}^T$, $x_s = \begin{bmatrix} x_{s1} & \ldots & x_{sn} \end{bmatrix}^T$, u the h control vector, $u = \begin{bmatrix} u_1 & \ldots & u_h \end{bmatrix}^T$, $A(.)$ an $n \times n$ instantaneous characteristic matrix, and $B(.)$ an $n \times h$ instantaneous control matrix.

Let denote by e_S the state synchronous error:

$$e_S(t) = x_s(t) - x_m(t) \tag{3}$$

and e_{AS} the anti-synchronous one:

$$e_{AS}(t) = x_s(t) + x_m(t) \tag{4}$$

$e_S = \begin{bmatrix} e_{S1} & \ldots & e_{Sn} \end{bmatrix}^T$, $e_{AS} = \begin{bmatrix} e_{AS1} & \ldots & e_{ASn} \end{bmatrix}^T$, and consider that the hybrid synchronization of the two chaotic systems, (S_m) and (S_s), is achieved if the following conditions hold (Juhn et al., 2009):

$$\begin{cases} \lim\limits_{t \to +\infty} |e_{Si}(t)| = \lim\limits_{t \to +\infty} |x_{si}(t) - x_{mi}(t)| = 0, & \forall i = 1, 2, \ldots, p, \ p < n \\ \lim\limits_{t \to +\infty} |e_{ASj}(t)| = \lim\limits_{t \to +\infty} |x_{sj}(t) + x_{mj}(t)| = 0, & \forall j = p+1, \ldots, n \end{cases} \tag{5}$$

The satisfaction of the first relations of conditions (5) means that the synchronization property as far as p states of the error vector are concerned is satisfied. Nevertheless, the fulfilment of the second relations of the same system (5) guarantees the anti-synchronism relatively to the $(n - p)$ remaining states.

Then, the problem consisting in studying the convergence of both synchronous and anti-synchronous errors is equivalent to a stability study problem.

To force this property to the error system characterizing the evolution of the error vector of the coupled chaotic systems, one solution is to design a suitable feedback control law $u(t)$ of the slave system which can be chosen in the form, Fig. 1. (Kapitanialc, 2000):

$$u(t) = -K(x_s(t), x_m(t))(x_s(t) \Delta x_m(t)) \tag{6}$$

such that, the operator Δ is replaced by the sign $(-)$ in the synchronization case and by the sign $(+)$ in the anti-synchronization one; $K(.)$ is an $h \times n$ nonlinear gain matrix.
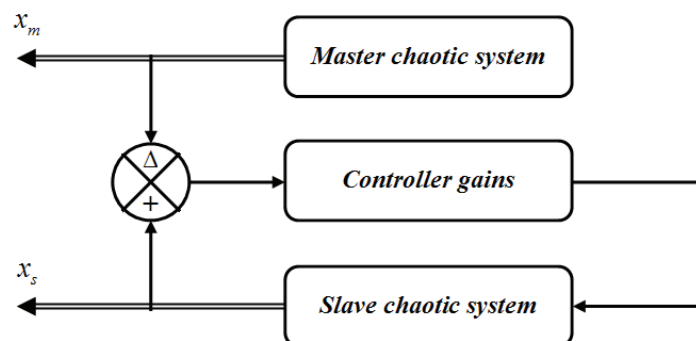

Fig. 1. Schematic representation of two chaotic systems synchronized in hybrid manner

The determination of the controller's gains, intended for the hybrid synchronization of the dynamic coupled master-slave chaotic system, is considered in the following section.

3. Analytic synchronization conditions of chaotic systems (Benrejeb & Hammami, 2008; Hammami, 2009; Hammami et al., 2010a)

To guarantee the synchronization as well as the anti-synchronization of chaotic systems, proposed approaches are based on the choice of both adapted stability method and of system description.

3.1 Case of nonlinear monovariable systems

Let consider the error dynamic system described by (7) in the monovariable case:

$$\dot{e}(t) = \hat{A}\big(x_s(t), x_m(t)\big)e(t) + \hat{B}\big(x_s(t), x_m(t)\big)u(t) \tag{7}$$

$$u(t) = -K\big(x_s(t), x_m(t)\big)e(t) \tag{8}$$

with $u \in \mathbb{R}$, $\hat{A}(.) = \left\{\hat{a}_{ij}(.)\right\}$, $\hat{B}(.) = \left\{\hat{b}_i(.)\right\}$, and $K(.) = \left\{k_j(.)\right\}$, $\forall i,j = 1,\dots,n$, and the characteristic matrix of the closed-loop system $\tilde{A}(.)$, defined by:

$$\tilde{A}(.) = \hat{A}(.) - \hat{B}(.)K(.) \tag{9}$$

Theorem 1. *The system described by (7) and (8) and verifying (10):*

$$\hat{a}_{ij}(.) - \hat{b}_i(.)k_j(.) = 0 \ \forall i,j = 1,\dots,n-1 \text{ for } i \neq j \tag{10}$$

such that:

i.   *the nonlinear elements are located in either one row or one column of the matrix $\tilde{A}(.)$,*

ii.   *the diagonal elements $\left(\hat{a}_{ii}(.) - \hat{b}_i(.)k_i(.)\right)$ of the matrix $\tilde{A}(.)$ are such that:*

$$\left(\hat{a}_{ii}(.) - \hat{b}_i(.)k_i(.)\right) < 0 \ \forall i = 1,\dots,n-1 \tag{11}$$

is asymptotically stable, if there exist $\varepsilon > 0$ such that:

$$\left(\hat{a}_{nn}(.) - \hat{b}_n(.)k_n(.)\right) - \sum_{i=1}^{n-1}\left[\left|\left(\hat{a}_{ni}(.) - \hat{b}_n(.)k_i(.)\right)\left(\hat{a}_{in}(.) - \hat{b}_i(.)k_n(.)\right)\right|\left(\hat{a}_{ii}(.) - \hat{b}_i(.)k_i(.)\right)^{-1}\right] \leq -\varepsilon \tag{12}$$

Proof. Conditions (10) lead to an arrow form closed-loop characteristic matrix $\tilde{A}(.)$ (Benrejeb, 1980), called Benrejeb matrix (Borne et al., 2007):

$$\tilde{A}(.) = \begin{bmatrix} \times & & & & \times \\ & \times & & & \times \\ & & \ddots & & \vdots \\ & & & \times & \times \\ \times & \times & \dots & \times & \times \end{bmatrix} \tag{13}$$

The choice of a comparison system having a characteristic matrix $\tilde{M}(\tilde{A}(.))$, relatively to the vectorial norm $p(z) = \begin{bmatrix} |z_1| & \ldots & |z_n| \end{bmatrix}^T$, $z = \begin{bmatrix} z_1 & \ldots & z_n \end{bmatrix}^T$, leads, when the nonlinearities are isolated in either one row or one column, to the following sufficient conditions, by the use of Borne-Gentina practical stability criterion (Borne & Benrejeb, 2008):

$$(-1)^i \tilde{M}(\tilde{A}(.)) \begin{pmatrix} 1 & 2 & \ldots & i \\ 1 & 2 & \ldots & i \end{pmatrix} > 0 \quad \forall i = 1, 2, \ldots, n \tag{14}$$

This criterion, useful for the stability study of complex and large scale systems generalizes the Kotelyanski lemma for nonlinear systems and defines large classes of systems for which the linear conjecture can be verified, either for the initial system or for its comparison system.

The comparison system associated to the previous vectorial norm $p(z)$, is defined, in this case, by the following differential equations:

$$\dot{z}(t) = \tilde{M}(\tilde{A}(.)) z(t) \tag{15}$$

such that the elements $\tilde{m}_{ij}(.)$ of $\tilde{M}(\tilde{A}(.))$ are deduced from the ones of the matrix $\tilde{A}(.)$ by substituting the off-diagonal elements by their absolute values, which can be written as:

$$\begin{cases} \tilde{m}_{ii}(.) = \tilde{a}_{ii}(.) & \forall i = 1, \ldots, n \\ \tilde{m}_{ij}(.) = \left| \hat{a}_{ij}(.) - \hat{b}_i(.) k_j(.) \right| & \forall i, j = 1, \ldots, n, \ i \neq j \end{cases} \tag{16}$$

The system (7) is then stabilized by (8) if the matrix $\tilde{M}(\tilde{A}(.))$ is the opposite of an $M-$matrix, or equivalently, by application of the practical Borne-Gentina criterion:

$$\begin{cases} \tilde{a}_{ii}(.) < 0 \ \forall i = 1, 2, \ldots, n-1 \\ (-1)^n \det(\tilde{M}(\tilde{A}(.))) > 0 \end{cases} \tag{17}$$

The development of the first member of the last inequality of (17):

$$(-1)^n \det(\tilde{M}(\tilde{A}(.))) = (-1)\left(\tilde{a}_{nn}(.) - \sum_{i=1}^{n-1} \left(|\tilde{a}_{ni}(.) \tilde{a}_{in}(.)| \right) \tilde{a}_{ii}^{-1}(.) \right) (-1)^{n-1} \prod_{j=1}^{n-1} \tilde{a}_{jj}(.) \tag{18}$$

achieves easily the proof of theorem 1.

Corollary 1. *The system described by (7) and (8) and verifying (10) such that:*

i.   *the nonlinear elements are located in either one row or one column of the matrix $\tilde{A}(.)$,*

ii.   *the diagonal elements $\left(\hat{a}_{ii}(.) - \hat{b}_i(.) k_i(.) \right)$, $\forall i = 1, \ldots, n-1$, of the matrix $\tilde{A}(.)$ are strictly negative,*

iii. *the products of the off-diagonal elements of the matrix $\tilde{A}(.)$ are such that:*

$$\left(\hat{a}_{ni}(.) - \hat{b}_n(.) k_i(.) \right) \left(\hat{a}_{in}(.) - \hat{b}_i(.) k_n(.) \right) \geq 0 \ \forall i = 1, \ldots, n-1 \tag{19}$$

is asymptotically stable if there exist $\varepsilon > 0$ for which the instantaneous characteristic polynomial $P_{\tilde{A}(.)}(., \lambda)$ of $\tilde{A}(.)$ satisfies the condition:

$$P_{\tilde{A}(.)}(., 0) = (-1)^n \det\left(M\left(\tilde{A}(.)\right)\right) \geq \varepsilon \tag{20}$$

Proof. The proof of corollary 1 is inferred of theorem 1 by taking into account the new added iii. conditions, which guarantee, through a simple transformation, the identity of the matrix $\tilde{A}(.)$ and its overvaluing matrix $\tilde{M}\left(\tilde{A}(.)\right)$.

These conditions, associated to aggregation techniques based on the use of vector norms, have led to stability domains for a class of Lur'e Postnikov systems whereas, for example, Popov stability criterion use failed (Benrejeb, 1980).

3.2 Nonlinear multivariable systems case

In the case of nonlinear multivariable systems, let us consider the closed-loop error system described by:

$$\dot{e}(t) = \hat{A}(.)e(t) + \hat{B}(.)u(t) \tag{21}$$

$$u(t) = -K(.)e(t) \tag{22}$$

then by:

$$\dot{e}(t) = \tilde{A}(.)e(t) \tag{23}$$

$$\tilde{A}(.) = \hat{A}(.) - \hat{B}(.)K(.) \tag{24}$$

with $u \in \mathbb{R}^h$ the control vector, $\hat{A}(.)$ the $n \times n$ instantaneous characteristic matrix, $\hat{A}(.) = \{\hat{a}_{ij}(.)\}$, $\hat{B}(.)$ the $n \times h$ control matrix, $\hat{B}(.) = \{\hat{b}_{ij}(.)\}$ and $K(.)$ the $h \times n$ instantaneous gain matrix, $K(.) = \{k_{ij}(.)\}$.

The conditions allowing to put $\tilde{A}(.)$ under arrow form, are expressed as follows:

$$\hat{a}_{ij}(.) - \sum_{l=1}^{h} \hat{b}_{il}(.)k_{lj}(.) = 0 \ \ \forall i, j = 1, \ldots, n-1, \ i \neq j \tag{25}$$

Then, a necessary condition leading to the existence of a control law is that the number of equations to solve, $(n-1)(n-2)$, must be less than or equal to the number of unknown parameters, $n \times h$, then:

$$h \geq n-2 \tag{26}$$

Remark 1. If there exist i, $i < n$, such that $\hat{b}_{il}(.)$, $\forall l = 1, \ldots, h$, is equal to zero and by considering the conditions (25) allowing to put the matrix $\tilde{A}(.)$ under the arrow form, the system (21) must be such that:

$$\hat{a}_{ij}(.) = 0 \ \ \forall j = 1, \ldots, n-1 \text{ for } i \neq j \tag{27}$$

Remark 2. If all the elements $\hat{b}_{il}(.)$, $\forall i = 1,\ldots,n-1$, $\forall l = 1,\ldots,h$, are equal to zero, then the equations (25) cannot be satisfied only in the particular case where the matrix $\hat{A}(.)$ is under the arrow form; that is to say, that all the elements $\hat{a}_{ij}(.)$, $\forall i, j = 1,\ldots,n-1$, for $i \neq j$, are equal to zero, due to:

$$\sum_{l=1}^{h} \hat{b}_{il}(.)k_{lj}(.) = 0 \ \forall i, j = 1,\ldots,n-1, \ i \neq j \tag{28}$$

Theorem 2. *The system described by (21) and (22) and verifying (25) and (26), is stabilized by the control law (22) if the instantaneous characteristic matrix $\tilde{A}(.)$, defined by (24), is such that:*

i.   *the nonlinear elements are located in either one row or one column of the matrix $\tilde{A}(.)$,*

ii.   *the first $(n-1)$ diagonal elements of the matrix $\tilde{A}(.)$ are such that:*

$$\left(\hat{a}_{ii}(.) - \sum_{l=1}^{h} \hat{b}_{il}(.)k_{li}(.) \right) < 0 \ \forall i = 1, 2, \ldots, n-1 \tag{29}$$

iii. *there exist $\varepsilon > 0$, such that:*

$$\left(\hat{a}_{nn}(.) - \sum_{l=1}^{h} \hat{b}_{nl}(.)k_{ln}(.) \right) - \sum_{i=1}^{n-1} \left[\begin{array}{c} \left\| \left(\hat{a}_{ni}(.) - \sum_{l=1}^{h} \hat{b}_{nl}(.)k_{li}(.) \right) \left(\hat{a}_{in}(.) - \sum_{l=1}^{h} \hat{b}_{il}(.)k_{ln}(.) \right) \right\| \\ \times \left(\hat{a}_{ii}(.) - \sum_{l=1}^{h} \hat{b}_{il}(.)k_{li}(.) \right)^{-1} \end{array} \right] \leq -\varepsilon \tag{30}$$

Proof. The proof of theorem 2 is similar to the theorem 1 relatively to the dynamic nonlinear monovariable continuous system; the conditions (29) and (30) are inferred from the conditions (11) and (12), by replacing the elements $\tilde{a}_{ij}(.)$ by their expressions according to the elements of $\hat{A}(.)$, $\hat{B}(.)$ and $K(.)$ matrices.

Remark 3. The previous results, well adapted for multivariable systems, can be applied only for monovariable ones which are, at most, of third order, by respect to the above-mentioned necessary condition (26).

Although, according to the class of systems described by differential scalar equation, the arrow form matrix can be advantageously used in a different way.

In such a view, let us consider the closed-loop nonlinear system assumed to be described by the following differential scalar equation:

$$s^{(n)}(t) + \sum_{i=0}^{n-1} a_i \left(s(t), \ s'(t), \ \ldots, \ s^{(n-1)}(t) \right) s^{(i)}(t) = 0 \tag{31}$$

s is the output, $s \in \mathbb{R}$, y the state vector $y = \begin{bmatrix} s & s' & \ldots & s^{(n-1)} \end{bmatrix}^T$, $y \in \mathbb{R}^n$, and $a_i(.)$, $\forall i = 0, 1, \ldots, n-1$, coefficients of the instantaneous characteristic polynomial $P_{A(.)}(.,\lambda)$ of the matrix $A(.)$, such that:

$$P_{A(.)}(.,\lambda) = \det(\lambda \mathbb{I} - A(.)) = \lambda^n + \sum_{i=0}^{n-1} a_i(.)\lambda^i \tag{32}$$

The system (31) can be rewritten as:

$$\dot{y}(t) = A(.)y(t) \tag{33}$$

$$A(.) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & 0 \\ 0 & & \dots & 0 & 1 \\ -a_0(.) & & \dots & & -a_{n-1}(.) \end{bmatrix} \tag{34}$$

The change of base defined by (Benrejeb, 1980):

$$y = Px \tag{35}$$

$$P = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 \\ \alpha_1 & \alpha_2 & \dots & \alpha_{n-1} & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_1^{n-2} & \alpha_2^{n-2} & \dots & \alpha_{n-1}^{n-2} & 0 \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_{n-1}^{n-1} & 1 \end{bmatrix} \tag{36}$$

with α_i, $\forall i = 1, 2, \dots, n-1$, distinct arbitrary constant parameters, $\alpha_i \neq \alpha_j$, $\forall i \neq j$, leads to the following description:

$$\dot{x}(t) = \breve{A}(.)x(t) \tag{37}$$

where the new instantaneous characteristic matrix, denoted by $\breve{A}(.)$ is in the Benrejeb arrow form:

$$\breve{A}(.) = P^{-1}A(.)P \tag{38}$$

$$\breve{A}(.) = \begin{bmatrix} \alpha_1 & & & \beta_1 \\ & \ddots & & \vdots \\ & & \alpha_{n-1} & \beta_{n-1} \\ \gamma_1(.) & \dots & \gamma_{n-1}(.) & \gamma_n(.) \end{bmatrix} \tag{39}$$

with:

$$\beta_i = \prod_{\substack{i,j=1 \\ i \neq j}}^{n-1} \left(\alpha_i - \alpha_j \right)^{-1} \tag{40}$$

$$\gamma_i(.) = -P_{\breve{A}(.)}\left(., \alpha_i \right) \ \forall i = 1, 2, \dots, n-1 \tag{41}$$

$$\gamma_n(.) = -a_{n-1}(.) - \sum_{i=1}^{n-1} \alpha_i \tag{42}$$

For the system described by (37) and (39), the application of Borne-Gentina criterion can lead to the stability conditions of the studied nonlinear system as shown in the following.

The equilibrium state of the nonlinear system (37) is asymptotically stable if the conditions:

i.
$$\alpha_i < 0, \ i = 1,\ldots,n-1, \ \alpha_i \neq \alpha_j, \ \forall i \neq j \tag{43}$$

ii.   there exist a positive parameter ε, such that:

$$\gamma_n(.) - \sum_{i=0}^{n-1} \left| \beta_i \gamma_i(.) \right| \alpha_i^{-1} \leq -\varepsilon \tag{44}$$

are satisfied.

When the $(n-1)$ products $\beta_i \gamma_i(.), \ \forall i = 1,\ldots,n-1,$ are non-negative, the condition (44) can be reduced and stated, by means of the instantaneous characteristic polynomial of the matrix $\breve{A}(.),$ in the following manner:

$$P_{\breve{A}(.)}(.,0) \geq \varepsilon \tag{45}$$

which constitutes a verification case of the validity of the linear Aizerman conjecture.

4. Synchronization and anti-synchronization cases

4.1 Synchronization of two identical coupled chaotic Chen systems

In this part, let focus on the problem of synchronization process of two identical coupled chaotic Chen dynamical systems.

The studied system is described by the following differential equations (Fallahi et al., 2008):

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\alpha & \alpha & 0 \\ \gamma - \alpha & \gamma & 0 \\ 0 & 0 & -\beta \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -x_1(t)x_3(t) \\ x_1(t)x_2(t) \end{bmatrix} \tag{46}$$

where α, β and γ are three positive parameters.

The system (46) exhibits a chaotic attractor at the parameter values: $\alpha = 35$, $\beta = 3$ and $\gamma = 28$, starting at the initial value of the state vector $x(0) = \begin{bmatrix} 1 & 1 & 0.5 \end{bmatrix}^T$, Fig. 2.

Obviously, the Chen nonlinear model can also be presented by respect to the following model:

$$\dot{x}(t) = A(.)x(t) \tag{47}$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^T$ is the state vector and $A(.)$ the instantaneous characteristic matrix such that:

$$A(.) = \begin{bmatrix} -\alpha & \alpha & 0 \\ \gamma - \alpha & \gamma & -x_1(t) \\ 0 & x_1(t) & -\beta \end{bmatrix} \tag{48}$$

At this stage, we choose a master Chen system given by:

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$$\begin{cases} \dot{x}_{m1}(t) = \alpha\left(x_{m2}(t) - x_{m1}(t)\right) \\ \dot{x}_{m2}(t) = \left(\gamma - \alpha - x_{m3}(t)\right)x_{m1}(t) + \gamma x_{m2}(t) \\ \dot{x}_{m3}(t) = x_{m1}(t)x_{m2}(t) - \beta x_{m3}(t) \end{cases} \qquad (49)$$

which drives a slave Chen system described by:

$$\begin{cases} \dot{x}_{s1}(t) = \alpha\left(x_{s2}(t) - x_{s1}(t)\right) \\ \dot{x}_{s2}(t) = \left(\gamma - \alpha - x_{s3}(t)\right)x_{s1}(t) + \gamma x_{s2}(t) + u_1(t) \\ \dot{x}_{s3}(t) = x_{s1}(t)x_{s2}(t) - \beta x_{s3}(t) + u_2(t) \end{cases} \qquad (50)$$

$u_i(t),\ i = 1,2,$ are the appropriate control functions to be determined.

It comes the error dynamics equations below:

$$\begin{cases} \dot{e}_{S1}(t) = \alpha\left(e_{S2}(t) - e_{S1}(t)\right) \\ \dot{e}_{S2}(t) = (\gamma - \alpha)e_{S1}(t) + \gamma e_{S2}(t) + \left(x_{m1}(t)x_{m3}(t) - x_{s1}(t)x_{s3}(t)\right) + u_1(t) \\ \dot{e}_{S3}(t) = -\beta e_{S3}(t) - \left(x_{m1}(t)x_{m2}(t) - x_{s1}(t)x_{s2}(t)\right) + u_2(t) \end{cases} \qquad (51)$$

which can be rewritten in the form:

$$\begin{bmatrix} \dot{e}_{S1}(t) \\ \dot{e}_{S2}(t) \\ \dot{e}_{S3}(t) \end{bmatrix} = \begin{bmatrix} \alpha\left(e_{S2}(t) - e_{S1}(t)\right) \\ (\gamma - \alpha)e_{S1}(t) + \gamma e_{S2}(t) + \left(x_{m1}(t)x_{m3}(t) - x_{s1}(t)x_{s3}(t)\right) \\ -\beta e_{S3}(t) - \left(x_{m1}(t)x_{m2}(t) - x_{s1}(t)x_{s2}(t)\right) \end{bmatrix} + Bu(t) \qquad (52)$$
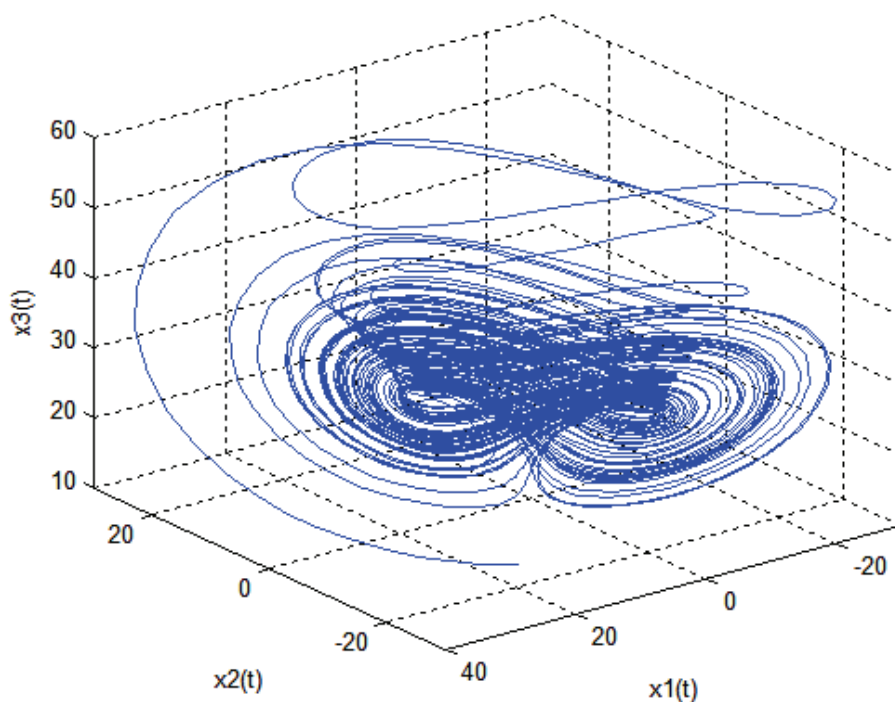

Fig. 2. Three-dimensional attractor of Chen dynamical chaotic system

with:

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{53}$$

The synchronization problem for coupled Chen chaotic dynamical systems is to achieve the asymptotic stability of this error system (52), with the following active control functions $u_i(t)$, $i = 1, 2$, defined by:

$$u_i(t) = -\sum_{j=1}^{2} k_{ij}(.)x_j(t) - f_i(.) \ \forall i = 1, 2 \tag{54}$$

Then, for an instantaneous 2×2 gain matrix $K(.)$, $K(.) = \{k_{ij}(.)\}$, we obtain:

$$\begin{cases} \dot{e}_{S1}(t) = \alpha\left(e_{S2}(t) - e_{S1}(t)\right) \\ \dot{e}_{S2}(t) = \left(\gamma - \alpha - k_{11}(.)\right)e_{S1}(t) + \left(\gamma - k_{12}(.)\right)e_{S2}(t) + \left(x_{m1}(t)x_{m3}(t) - x_{s1}(t)x_{s3}(t)\right) - f_1(.) \\ \dot{e}_{S3}(t) = -k_{21}(.)e_{S1}(t) - k_{22}(.)e_{S2}(t) - \beta e_{S3}(t) - \left(x_{m1}(t)x_{m2}(t) - x_{s1}(t)x_{s2}(t)\right) - f_2(.) \end{cases} \tag{55}$$

The characterization of the closed-loop system by an arrow form matrix is easily checked, by choosing the correction parameter $k_{22}(.)$ such that:

$$k_{22} = 0 \tag{56}$$

Then, to satisfy the constraints (29) as well as the condition (30) of the theorem 2, the two following inequalities must be, respectively, fulfilled:

$$\gamma - k_{12}(.) < 0 \tag{57}$$

$$-\alpha - \left(\left|\alpha\left(\gamma - \alpha - k_{11}(.)\right)\right|\left(\gamma - k_{12}(.)\right)^{-1}\right) < 0 \tag{58}$$

So, $\forall k_{21}(.)$, possible choices of the other parameters are given by:

$$\begin{cases} k_{12} = \alpha \\ k_{11} = \gamma - \alpha \end{cases} \tag{59}$$

Finally, it remains to study the stability of a linear controlled system, in the case where the following possible choices are adopted:

$$\begin{cases} f_1(.) = x_{m1}(t)x_{m3}(t) - x_{s1}(t)x_{s3}(t) \\ f_2(.) = -x_{m1}(t)x_{m2}(t) + x_{s1}(t)x_{s2}(t) \end{cases} \tag{60}$$

Then, the constant gain matrix K, can be chosen as:

$$K = \begin{bmatrix} \gamma - \alpha & \alpha \\ 2 & 0 \end{bmatrix} \tag{61}$$

Thus, when stabilized by the above-mentioned feedback $u_i(t)$, $i = 1, 2$, the error system (55) will converge to zero as $t \rightarrow +\infty$ implying that system (50) will globally synchronize with system (49).

Fig. 3. shows the error dynamics in the uncontrolled state, while both Fig. 4. and Fig. 5. illustrate the error dynamics when controller is switched on. Obviously, the two chaotic Chen systems evolve in the same direction as well as the same amplitude; they are globally asymptotically synchronized by means of the proposed controller.
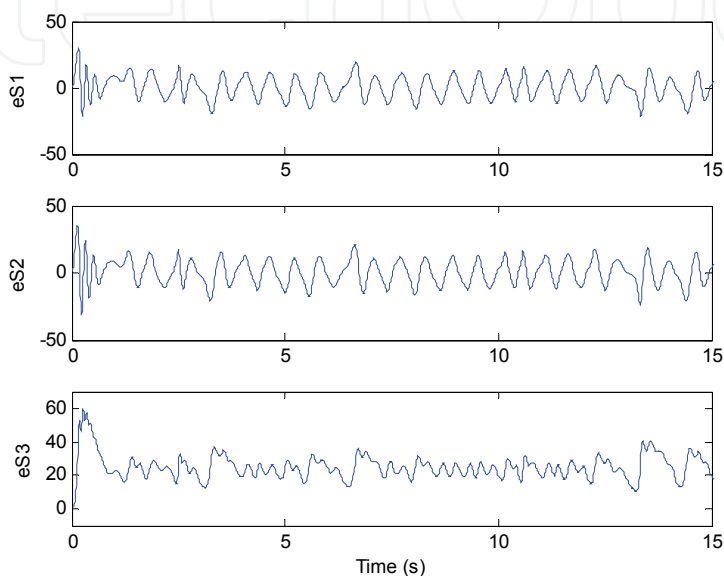
Fig. 3. Error dynamics of the coupled master-slave Chen system
when controller is deactivated
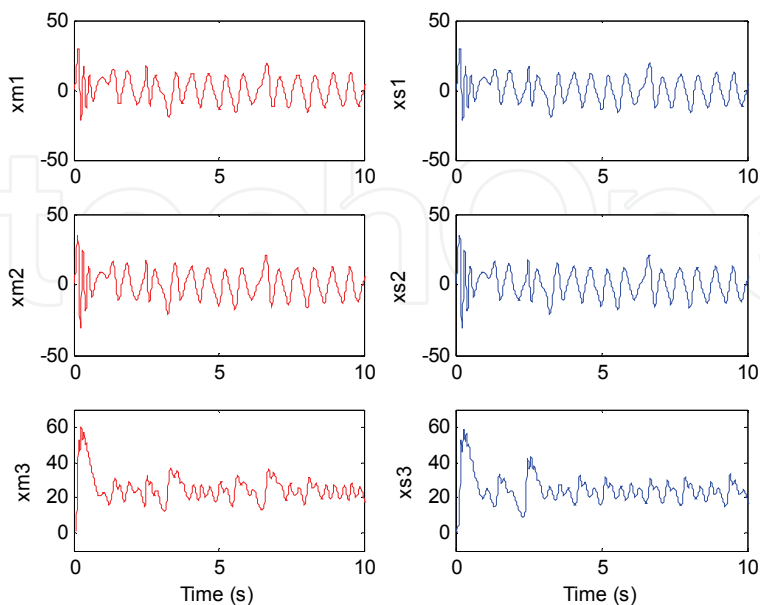
Fig. 4. Synchronization dynamics between the coupled master-slave Chen system
when controller is activated
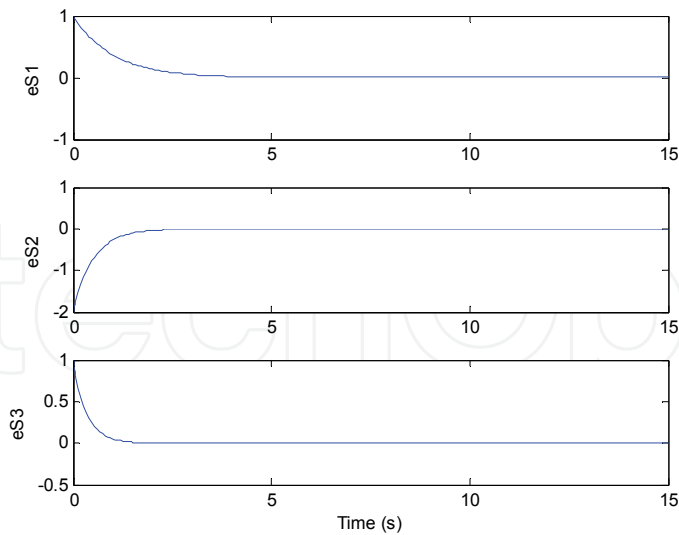
Fig. 5. Error dynamics of the coupled master-slave Chen system
when controller is switched on

4.2 Anti-synchronization of two identical coupled chaotic Lee systems

The studied chaotic Lee system is described by the following differential equations (Juhn et al., 2009):

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 40 & 0 & -x_1(t) \\ 4x_1(t) & 0 & -2.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{62}$$

It exhibits a chaotic attractor, starting at the initial value of the state vector $x(0) = \begin{bmatrix} 2 & 3 & 2 \end{bmatrix}^T$, Fig. 6.
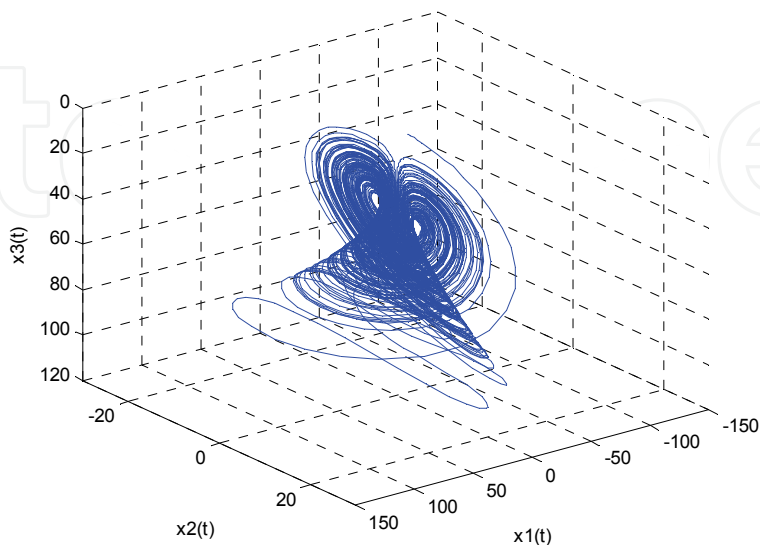
Fig. 6. Three-dimensional view of the Lee chaotic attractor

Let us consider the master Lee system $\left(S_m\right)$ given by (63):

$$\begin{bmatrix} \dot{x}_{m1}(t) \\ \dot{x}_{m2}(t) \\ \dot{x}_{m3}(t) \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 40 & 0 & -x_{m1}(t) \\ 4x_{m1}(t) & 0 & -2.5 \end{bmatrix} \begin{bmatrix} x_{m1}(t) \\ x_{m2}(t) \\ x_{m3}(t) \end{bmatrix} \tag{63}$$

driving a similar controlled slave Lee system $\left(S_s\right)$ described by (64):

$$\begin{bmatrix} \dot{x}_{s1}(t) \\ \dot{x}_{s2}(t) \\ \dot{x}_{s3}(t) \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 40 & 0 & -x_{s1}(t) \\ 4x_{s1}(t) & 0 & -2.5 \end{bmatrix} \begin{bmatrix} x_{s1}(t) \\ x_{s2}(t) \\ x_{s3}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) \tag{64}$$

$u(t)$ is the scalar active control.

For the following state error vector components, defined relatively to anti-synchronization study:

$$\begin{cases} e_{AS1}(t) = x_{s1}(t) + x_{m1}(t) \\ e_{AS2}(t) = x_{s2}(t) + x_{m2}(t) \\ e_{AS3}(t) = x_{s3}(t) + x_{m3}(t) \end{cases} \tag{65}$$

the error system can be defined by the following differential equations:

$$\begin{cases} \dot{e}_{AS1}(t) = -10e_{AS1}(t) + 10e_{AS2}(t) \\ \dot{e}_{AS2}(t) = 40e_{AS1}(t) - \left(x_{s1}(t)x_{s3}(t) + x_{m1}(t)x_{m3}(t)\right) + u(t) \\ \dot{e}_{AS3}(t) = -2.5e_{AS3}(t) + 4\left(x_{s1}^2(t) + x_{m1}^2(t)\right) \end{cases} \tag{66}$$

The problem of chaos anti-synchronization between two identical Lee chaotic dynamical systems is solved here by the design of a state feedback structure $k_i(.)$, $\forall i = 1,2,3$, and the choice of nonlinear functions $f_i(.)$, $\forall i = 1,2,3$, such that:

$$u(t) = -\sum_{i=1}^{3} k_i(.)x_i(t) - f_j(.) \ \forall j = 1,2,3 \tag{67}$$

It comes the following closed-loop dynamical error system:

$$\begin{bmatrix} \dot{e}_{AS1}(t) \\ \dot{e}_{AS2}(t) \\ \dot{e}_{AS3}(t) \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 40 - k_1(.) & -k_2(.) & -k_3(.) \\ -2.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{AS1}(t) \\ e_{AS2}(t) \\ e_{AS3}(t) \end{bmatrix} - \begin{bmatrix} f_1(.) \\ \left(x_{s1}(t)x_{s3}(t) + x_{m1}(t)x_{m3}(t)\right) + f_2(.) \\ 4\left(x_{s1}^2(t) + x_{m1}^2(t)\right) + f_3(.) \end{bmatrix} \tag{68}$$

The nonlinear elements $f_i(.)$ and $k_i(.)$ have to be chosen to make the instantaneous characteristic matrix of the closed-loop system in the arrow form and the closed-loop error system asymptotically stable.

From the possible solutions, allowing to put the instantaneous characteristic matrix of (68) under the arrow form, let consider the following:

$$k_3 = 0 \tag{69}$$

and:

$$\begin{cases} f_1(.) = 0 \\ f_2(.) = -\left(x_{s1}(t)x_{s3}(t) + x_{m1}(t)x_{m3}(t)\right) \\ f_3(.) = e_{AS3}(t) - 4\left(x_{s1}^2(t) + x_{m1}^2(t)\right) \end{cases} \tag{70}$$

For the vector norm $p(e_{AS})$, $e_{AS} = \begin{bmatrix} e_{AS1} & e_{AS2} & e_{AS3} \end{bmatrix}^T$:

$$p(e_{AS}) = \begin{bmatrix} |e_{AS1}| & |e_{AS2}| & |e_{AS3}| \end{bmatrix}^T \tag{71}$$

the overvaluing matrix is in arrow form and has non negative off-diagonal elements and nonlinearities isolated in either one row or one column.

By the use of the proposed theorem 2, stability and anti-synchronization properties are satisfied for the both following sufficient conditions (72) and (73):

$$-k_2(.) < 0 \tag{72}$$

$$\left(-10 - \left(-10\left(40 - k_1(.)\right)k_2^{-1}(.)\right)\right) < 0 \tag{73}$$

Various choices of the gain vector $K(.)$, $K(.) = \begin{bmatrix} k_1(.) & k_2(.) & 0 \end{bmatrix}$, are possible, such as the following linear one:

$$K = \{k_i\} = \begin{bmatrix} 4 & 2 & 0 \end{bmatrix} \tag{74}$$

By considering the initial condition $e_{AS}(0) = \begin{bmatrix} 2 & 3 & 2 \end{bmatrix}^T$, for the Lee error system (66) when the active controller is deactivated, it is obvious that the error states grow with time chaotically, as shown in Fig. 7., and after activating the controller, Fig. 8. shows three parametrically harmonically excited 3D systems evolve in the opposite direction. The trajectories of error system (68) imply that the asymptotical anti-synchronization has been, successfully, achieved, Fig. 9.

5. Hybrid synchronization by a nonlinear state feedback controller – Application to the Chen–Lee chaotic system (Hammami, 2009)

Let consider two coupled chaotic Chen and Lee systems (Juhn et al., 2009). The following nonlinear differential equations, of the form (1), correspond to a master system (Tam & Si Tou, 2008):

$$\dot{x}_m(t) = A\left(x_m(t)\right)x_m(t)$$

with:

$$A\left(x_m(t)\right) = \begin{bmatrix} a & -x_{m3}(t) & 0 \\ 0 & b & x_{m1}(t) \\ cx_{m2}(t) & 0 & d \end{bmatrix} \qquad (75)$$

x_{m1}, x_{m2} and x_{m3} are state variables, and a, b, c and d four system parameters.

For the following parameters $\left(a,b,c,d\right) = \left(5,-10,0.3,-3.8\right)$, and initial condition $\left[x_{m1}(0) \quad x_{m2}(0) \quad x_{m3}(0)\right]^T = \left[1.5 \quad -32 \quad 13\right]^T$, the drive system, described by (1) and (75) is a chaotic attractor, as shown in Fig. 10.
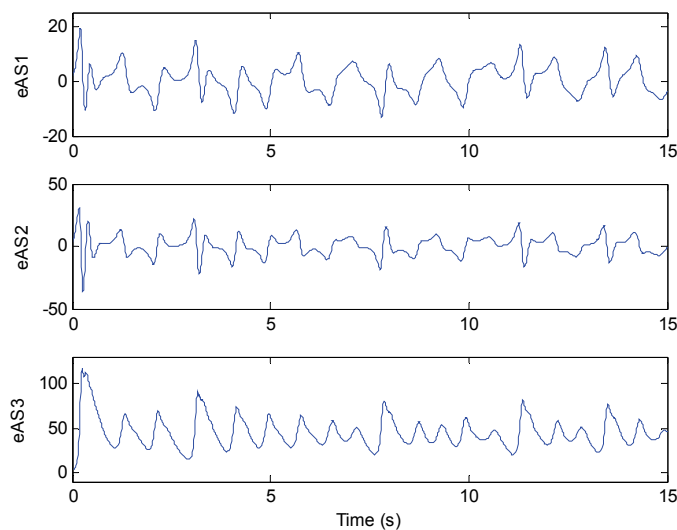
Fig. 7. Error dynamics $\left(e_{AS1}, e_{AS2}, e_{AS3}\right)$ of the coupled master-slave Lee system when the active controller is deactivated
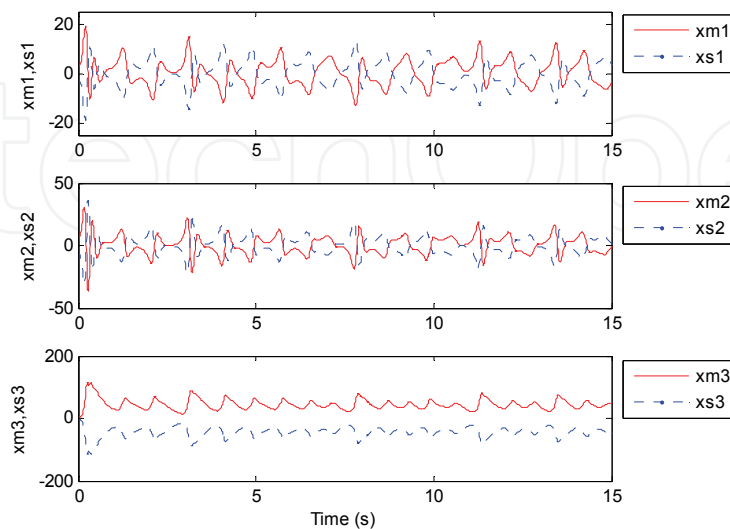
Fig. 8. Partial time series of anti-synchronization for Lee chaotic system when the active controller is switched on
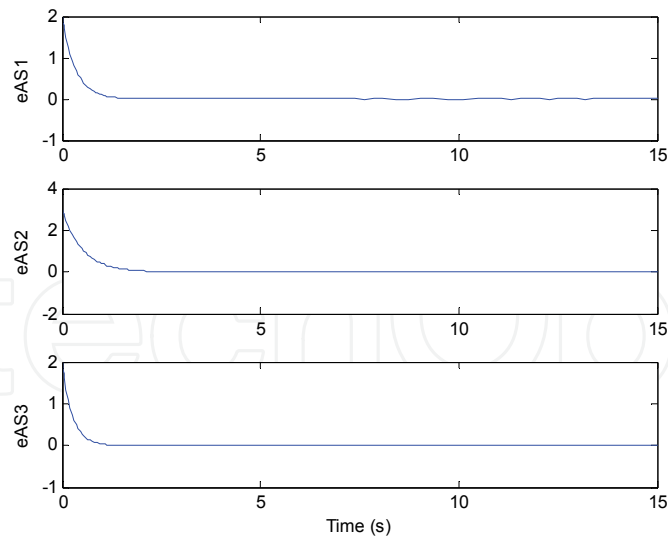
Fig. 9. Error dynamics of the coupled master-slave Lee system when control is activated
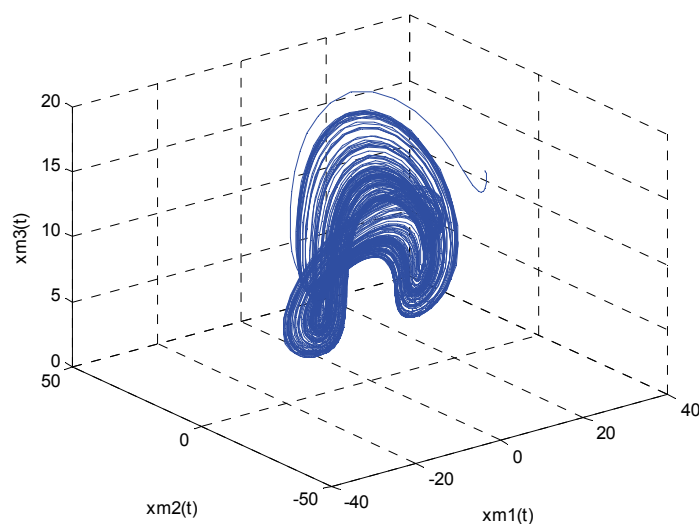
Fig. 10. The 3-dimensional strange attractor of the chaotic Chen–Lee master system

For the Chen–Lee slave system, described in the state space by:

$$\dot{x}_s(t) = A(x_s(t))x_s(t) + u(t) \tag{76}$$

with:

$$A(x_s(t)) = \begin{bmatrix} a & -x_{s3}(t) & 0 \\ 0 & b & x_{s1}(t) \\ cx_{s2}(t) & 0 & d \end{bmatrix} \tag{77}$$

we have selected the anti-synchronization state variables x_{m1} and x_{m3} facing to x_{s1} and x_{s3}, and the synchronization state variable x_{m2} facing to x_{s2}.

Then, the hybrid synchronization errors between the master and the slave systems $e(t) = \begin{bmatrix} e_{AS1}(t) & e_{S2}(t) & e_{AS3}(t) \end{bmatrix}^T$, are such as:

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$$\begin{cases} e_{AS1}(t) = x_{s1}(t) + x_{m1}(t) \\ e_{S2}(t) \ = x_{s2}(t) - x_{m2}(t) \\ e_{AS3}(t) = x_{s3}(t) + x_{m3}(t) \end{cases} \tag{78}$$

Let compute the following continuous state feedback controller's structure:

$$u(t) = -K\big(x_m(t), x_s(t)\big)e(t) \tag{79}$$

to guarantee the asymptotic stability of the error states defined by (78), $e = \begin{bmatrix} e_{AS1} & e_{S2} & e_{AS3} \end{bmatrix}^T$, so that the slave system, characterized by (76) and (77), synchronizes and anti-synchronizes, simultaneously, to the master one, described by (1) and (75), by assuring that the synchronization error e_{S2} and the anti-synchronization errors e_{AS1} and e_{AS3} decay to zero, within a finite time.

Thus, for a state feedback controller of the form (79), $K(.)$, $K(.) = \{k_{ij}(.)\}$, $\forall i, j = 1,2,3$, and by considering the differential systems (1), (75), (76), (77) and (78), we obtain the following state space description of the error resulting system:

$$\dot{e}(t) = A(.)e(t) \tag{80}$$

with:

$$A(.) = \begin{bmatrix} a - k_{11}(.) & -x_{m3}(t) - k_{12}(.) & -k_{13}(.) \\ -k_{21}(.) & b - k_{22}(.) & x_{m1}(t) - k_{23}(.) \\ cx_{m2}(t) - k_{31}(.) & -k_{32}(.) & d - k_{33}(.) \end{bmatrix} \tag{81}$$

By respect to the stabilisability conditions announced in the above-mentioned theorem 2, the dynamic error system (80) is first characterized by an instantaneous arrow form matrix $A(.)$, that is to say, the main requirements concerning the choice of the feedback gains $k_{12}(.)$ and $k_{21}(.)$ are given by:

$$\begin{cases} k_{12}(.) = -x_{m3}(t) \\ k_{21} = 0 \end{cases} \tag{82}$$

To satisfy that the two first diagonal elements of the characteristic matrix $A(.)$ are strictly negative:

$$\begin{cases} a - k_{11}(.) < 0 \\ b - k_{22}(.) < 0 \end{cases} \tag{83}$$

a possible solution is:

$$\begin{cases} k_{11} = 7 \\ k_{22} = -6 \end{cases} \tag{84}$$

Besides, to annihilate the nonlinearities in system (80), a solution is:

$$\begin{cases} k_{23}(.) = x_{m1}(t) \\ k_{31}(.) = cx_{m2}(t) \end{cases} \tag{85}$$

Finally, by considering the fixed values of k_{11}, $k_{12}(.)$, k_{21}, k_{22}, $k_{23}(.)$ and $k_{31}(.)$, it is relevant to denote that to satisfy the sufficient condition (30) of theorem 2, for any arbitrary chosen parameters of correction $k_{13}(.)$ and $k_{32}(.)$, it is necessary to tune the remaining design parameter $k_{33}(.)$, guaranteeing the hybrid synchronization of the coupled chaotic studied system such that:

$$d - k_{33}(.) < 0 \tag{86}$$

Then, for the following instantaneous gain matrix $K(.)$, easily obtained:

$$K(.) = \begin{bmatrix} 7 & -x_{m3}(t) & 0 \\ 0 & -6 & x_{m1}(t) \\ cx_{m2}(t) & 0 & -1 \end{bmatrix} \tag{87}$$

the studied dynamic error system (80) is asymptotically stable.

For the following initial master and slave systems conditions, $x_m(0) = \begin{bmatrix} 1.5 & -52 & 13 \end{bmatrix}^T$, $x_s(0) = \begin{bmatrix} -2 & 10 & -15 \end{bmatrix}^T$, and without activation of the designed controller, the numerical simulation results of the above master-slave system are shown in Fig. 11.

It is obvious, from Fig. 12., that the error states grow with time chaotically.

Therefore, by designing an adequate nonlinear controlled slave system and under mild conditions, the hybrid synchronization is achieved within a shorter time, as it is shown in Fig. 13., with an exponentially decaying error, Fig. 14.

The obtained phase trajectories of the Fig. 15., show that the Chen–Lee slave chaotic attractor is synchronized in a hybrid manner with the master one.
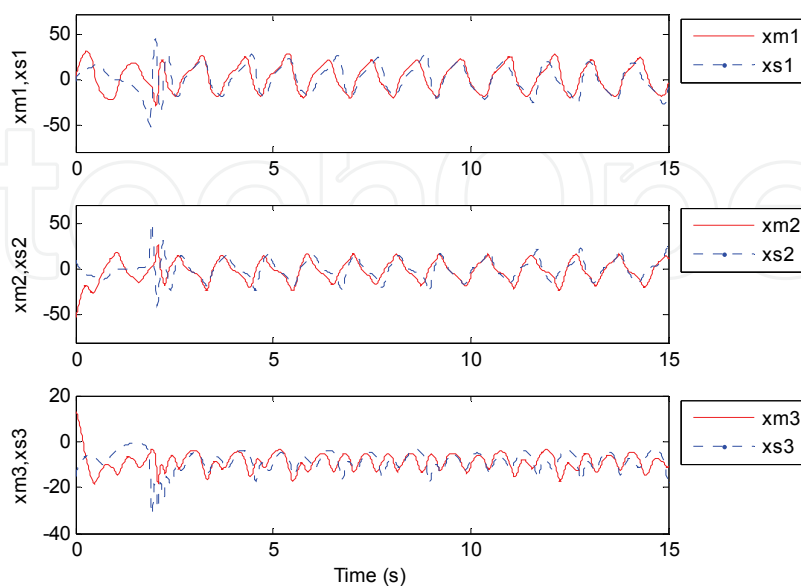

Fig. 11. Error dynamics between the master Chen–Lee system
and its corresponding slave system before their hybrid synchronization
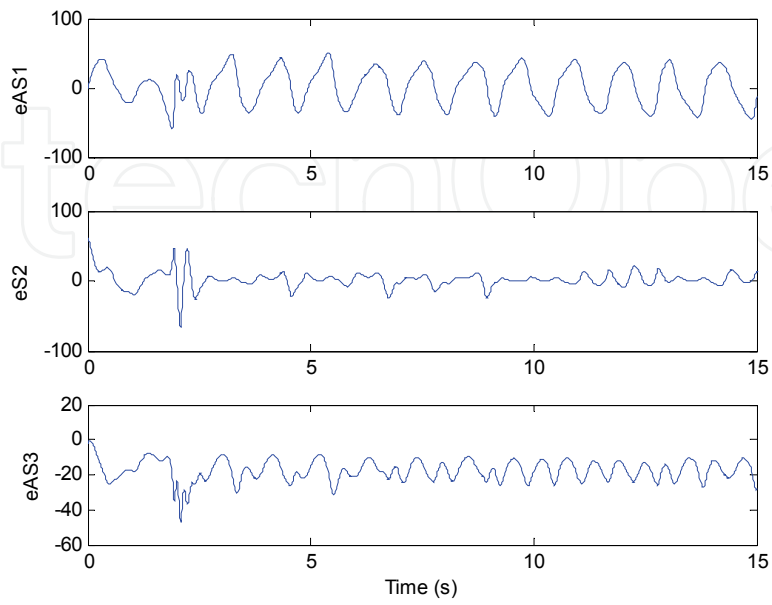
Fig. 12. Evolutions of the hybrid synchronization errors versus time
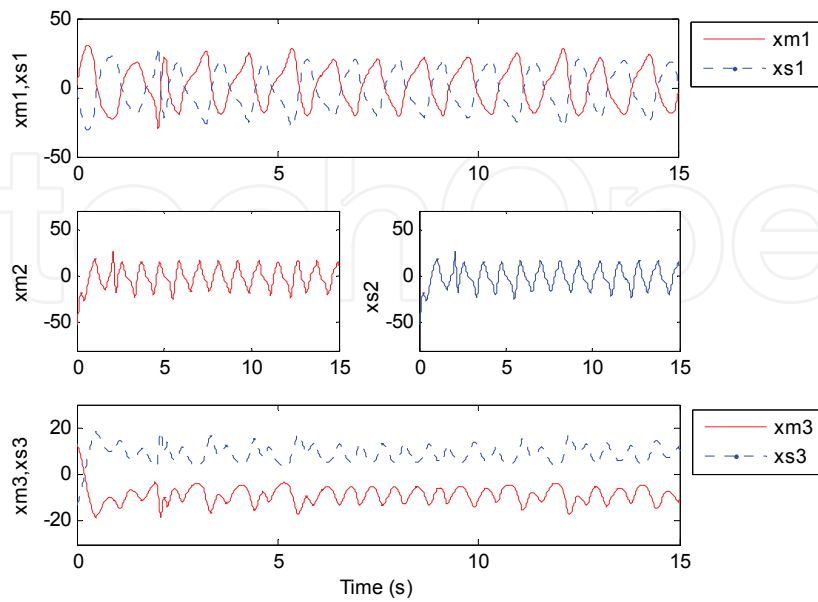when the proposed controller is turned off

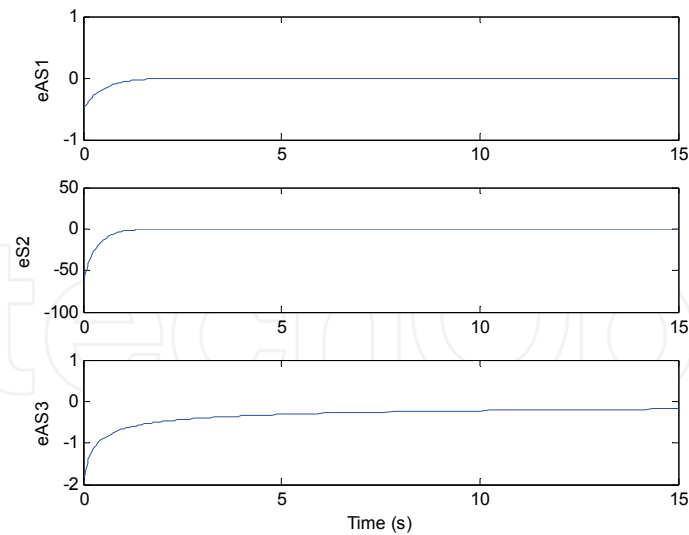Fig. 13. Hybrid synchronization of the master-slave Chen–lee chaotic system

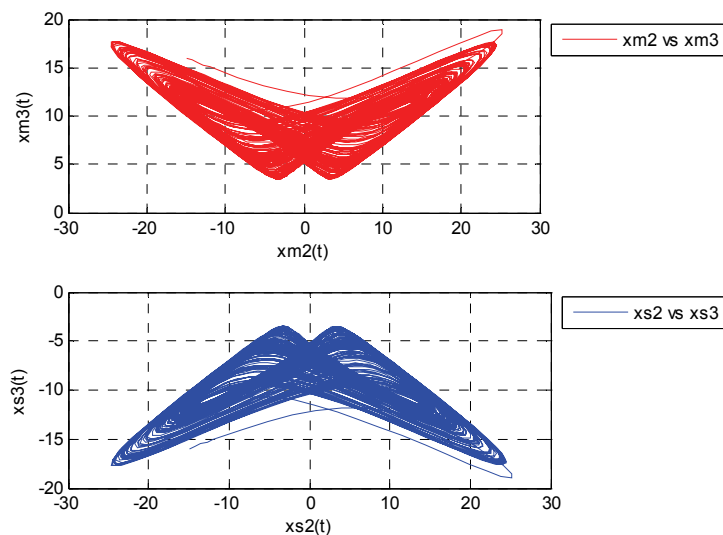Fig. 14. Exponential convergence of the error dynamics

Fig. 15. 2-D projection of the hybrid synchronized attractors
associated to the Chen–Lee chaotic system

6. Conclusion

Appropriate feedback controllers are designed, in this chapter, for the chosen slave system states to be synchronized, anti-synchronized as well as synchronized in a hybrid manner with the target master system states. It is shown that by applying a proposed control scheme, the variance of both synchronization and anti-synchronization errors can converge to zero. The synchronisation of two identical Chen chaotic systems, the anti-synchronization of two identical Lee chaotic systems and, finally, the coexistence of both synchronization and anti-synchronization for two identical Chen–Lee chaotic systems, considered as a coupled master-slave systems, are guaranteed by using the practical stability criterion of Borne and Gentina, associated to the specific matrix description, namely the arrow form matrix.

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Chaotic Systems

Edited by Prof. Esteban Tlelo-Cuautle

This book presents a collection of major developments in chaos systems covering aspects on chaotic behavioral modeling and simulation, control and synchronization of chaos systems, and applications like secure communications. It is a good source to acquire recent knowledge and ideas for future research on chaos systems and to develop experiments applied to real life problems. That way, this book is very interesting for students, academia and industry since the collected chapters provide a rich cocktail while balancing theory and applications.

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