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Fuzzy Attitude Control of Flexible Multi-Body Spacecraft

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1. Introduction

In order to complete the flexible multi-body spacecraft attitude control, this chapter will research on the dynamics and attitude control problems of flexible multi-body spacecraft which will be used in the future space missions.

Through investigating plentiful literatures, it is known that some important progress has been obtained in the research of flexible multi-body spacecraft dynamic modeling and fuzzy attitude control technologies. In the aspect of dynamic modeling, most models were founded according to spacecrafts with some special structures. In order to satisfy the requirement of modern project design and optimization, acquire higher efficiency and lower cost, researching on the dynamic modeling problem of flexible multi-body spacecraft with general structures and founding universal and programmable dynamic models are needed.

In the aspect of attitude control system design, the issues encountered in flexible spacecraft have increased the difficulties in attitude control system design, including the high stability and accuracy requirements of orientation, attitude control and vibration suppression, high robustness against the different kinds of uncertain disturbances. At the present time, classical control theory and modern control theory are often used in flexible multi-body spacecraft attitude control. These two methods have one common characteristic which is basing on mathematics models, including control object model and external disturbance model. It is usually considered that the models are already known or could be obtained by identification. But those two methods which are based on accurate math models both have unavoidable defects for large flexible multi-body spacecraft. Until this time, the most advanced and effective control system is the human itself. Therefore, researching on the control theory of human being and simulating the control process is an important domain of intelligent control. If we consider the brain and the nerve center system as a black box, we only investigate the relationship between the inputs and the outputs and the behavior represented from this process, that is called fuzzy control. The fuzzy control doesn’t depend on the accurate math models of the original system. It controls the complicated, nonlinear, uncertainty original system through the qualitative cognition of the system dynamic characteristics, intuitional consequence, online determination or changing the control strategies. This control method could more easily be realized and ensured its real time characteristic. It is especially becoming to the control problem of math models unknown, complicated, uncertainty nonlinear system. Accordingly, large flexible multi-body spacecraft attitude control using fuzzy control theory is a problem which is worth researching.

2. Attitude dynamic modeling of flexible multi-body spacecraft

Mathematics model is the basement of most control system design. Dynamic modelling is to describe the real system in physics world using models in mathematics world. Mathematics model provide the mapping from input to response, the coincidence extent between the response and the real object being controlled represent the quality of the model. Mathematics model world is totally different from physics system world, so a real physics object being controlled can not be constructed exactly by mathematics models. Therefore, engineers intend to establish a model which can reflect dynamic characteristics of spacecraft system, as well as the controller design based on the model can be applied into the real system.
In this section, the attitude dynamic equations of flexible multi-body spacecraft with topological tree configuration have been derived based on the Lagrange equations in terms of quasi-coordinates. The dynamic equations are universal and programmable due to the information of system configuration being introduced into the modelling process.

2.1 Description of system configuration
2.1.1 Coordinate system definition
The movement of spacecraft is always described in a reference coordinate, several coordinate systems used in the attitude dynamic modeling process are as follows:
1. Inertial coordinate system $\boldsymbol{f_i}(o_i x_i y_i z_i)$

This inertial frame is defined as its origin at the mass center of the earth, the third axis z_i perpendicular to the Earth's equatorial plane pointing to the arctic, axis x_i and y_i lying in the Earth's equatorial plane, axis x_i pointing to the direction of the vernal equinox, axis y_i forms this coordinate system as a right-handed one.
2. Orbit coordinate system $\boldsymbol{f_o}(o_o x_o y_o z_o)$

Its origin at the mass center of the spacecraft, axis z_o pointing to the geocenter, axis x_o lying in the orbit plane perpendicular to axis z_o, pointing along the direction of spacecraft velocity, axis y_o forms this coordinate system as a right-handed one.
3. Central body coordinate system $\boldsymbol{f_b}(o_b x_b y_b z_b)$

Its origin at the mass center of the spacecraft which has not been deformed, axis x_b pointing along the direction of spacecraft velocity, axis z_b pointing to the geocenter, axis y_b forms this coordinate system as a right-handed one.
4. Floating coordinate system $\boldsymbol{f_{ai}}(o_{ai} x_{ai} y_{ai} z_{ai})$, $i = 2, 3, \cdots, n$

Floating coordinate system is the body frame of the flexible body i, its origin usually at the mass center of the flexible body i which has not been deformed.
5. Gemel coordinate system $\boldsymbol{f_{ck}}(o_{ck} x_{ck} y_{ck} z_{ck})$

Gemel coordinate system is the body frame of the gemel k between the flexible bodies, its origin usually at the connection point between the gemel k and its inboard connected flexible body.

2.1.2 Description of spacecraft system
Considering a flexible multi-body spacecraft with topological tree configuration which contains a central body and $n-1$ flexible appendages, there are several accesses, objects are connected by gemels, ignoring collision and friction at gemels, access l of the system can be shown in figure 1:

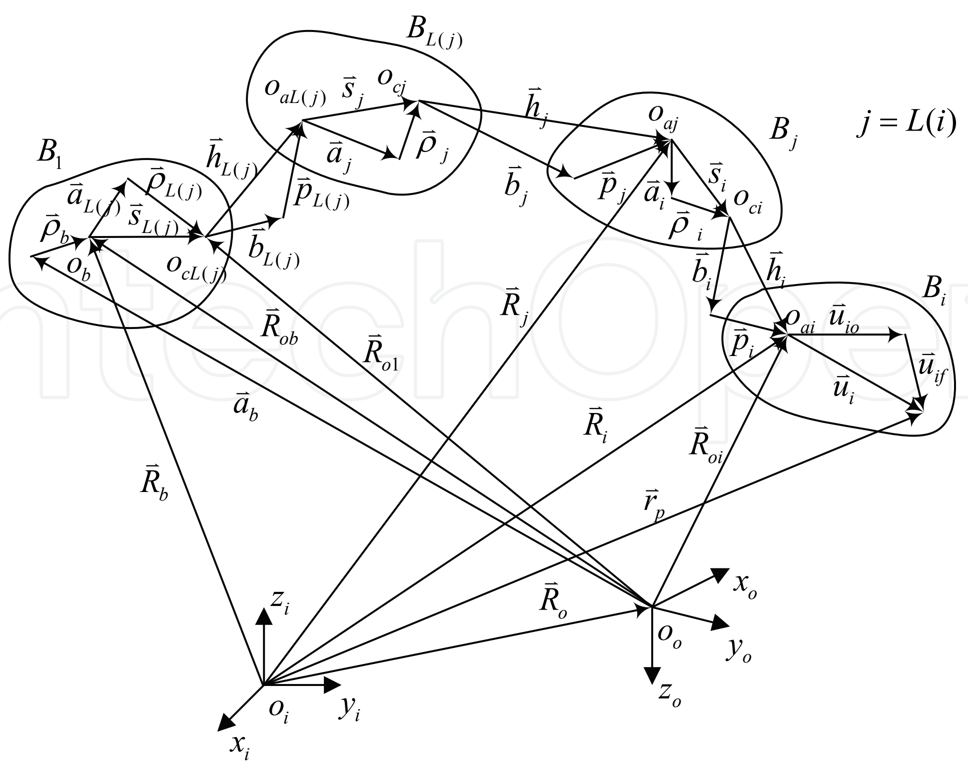


Fig. 1. Access l of flexible multi-body spacecraft system

In this figure, $o_i x_i y_i z_i$ is the inertial frame, $o_o x_o y_o z_o$ is the orbit frame. B_1 is the central body, origin of $o_b x_b y_b z_b$ is at the mass center of the spacecraft which has not been deformed, as well as axises are parallel to the principle axises of inertia. Radius vecter $\vec{a}_b$ and $\vec{\rho}_b$ respectively are rigid and flexible displacement of mass center o_b, $\vec{R}_{ob}$ is the radius vector of mass center o_b in orbit coordinate system, $\vec{R}_{ob} = \vec{a}_b + \vec{\rho}_b$. Gemel frames are founded at each gemel h_k, o_{ck} is the origin of gemel coordinate system of h_k, its radius vector in flexible body $B_{L(k)}$ is $\vec{s}_k$, $\vec{s}_k = \vec{a}_k + \vec{\rho}_k$, $\vec{a}_k$ and $\vec{\rho}_k$ respectively are rigid and flexible displacement of h_k in flexible body $B_{L(k)}$. o_{ak} is the origin of floating coordinate system of object B_k, its radius vector in gemel coordinate system of h_k is $\vec{h}_k$, $\vec{h}_k = \vec{b}_k + \vec{p}_k$, $\vec{b}_k$ and $\vec{p}_k$ respectively are rigid and flexible displacement of origin o_{ak} in gemel coordinate system of h_k, $k = 2, \cdots, n$.

If the central body of the spacecraft system is rigid, the origin of orbit coordinate system is coinsident with the origin of central body coordinate system, according to the analysis result of mass matrix (Lu, 1996), choosing frames like this can eliminate the coupling term of rigid body translation and rotation., as well as $\vec{R}_{ob} = \vec{a}_b = \vec{\rho}_b = 0$.

2.2 Description of flexible multi-body system using graph theory

Graph theory (Wittenburg & Roberson, 1977) is a useful tool to describe topological configuration, here several relative concept were given before we using it to do more research.

Oriented graph description: multi-body system can be described using gemel and its adjacent objects, if we express the objects in system using the vertex, express gemels using arc, then topological configuration of multi-body can be expressed as a oriented graph $D = \langle V, A \rangle$. There is a bijection between the collection of vertex V and the collection of objects and also a bijection between the collection of arc A and the collection of gemels. A description of multi-body system with topological tree configuration using oriented graph is shown in figure 2. H_i represent Gemels, and B_i represent objects.

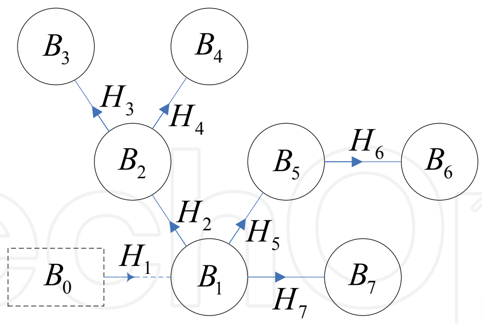


Fig. 2. Description of multi-body system using oriented graph

Regular labelling: Regular labelling approach is specified as follows:

1. The adjacency object of the root object B_0 is defined as B_1, relative gemel defined as H_1;
2. Each object has the same serial number with its inboard connected gemel;
3. Each object has a bigger serial number than its inboard connected object;
4. Each gemel has a deviated direction from the root object B_0.

Multi-body system shown in figure 2 is numbered in accordance with rules

Inboard connected object array: according to the regular labelling approach, label the N objects of spacecraft system. Define a N order one dimension integer array $L(i)$, $i = 1, \cdots, N$, i is the subscript of object B_i, $L(i)$ is the subscript of the inboard connected object of B_i. System topological configuration can be described by array $L(i)$ which is called inboard connected object array of system.

A graph can be conveniently expressed by matrix, its advantage is that structural features and character can be studied using of kinds of operation in matrix algebra.

Access matrix: Supposed that $D = \langle V, A \rangle$ is an oriented graph, $V = \{u_1, u_2, \cdots, u_n\}$, name matrix $T = (t_{ij})$ is the access matrix of the ortiented graph D, if:

$$t_{ij} = \begin{cases} 1, & \text{when there is a connectivity between } u_j \text{ and } u_i \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Access matrix of the system with topological tree configuration shown in figure 2 can be written as:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

2.3 Recursion relationship of adjacency bodies kinematics

Considering the kinematics relationship between flexible body i and flexible body j, $j = L(i)$. According to figure 2, we know that :

$$\vec{R}_i = \vec{R}_j + \vec{s}_i + \vec{h}_i \tag{3}$$

where $\vec{s}_i = \vec{a}_i + \vec{\rho}_i$, $\vec{h}_i = \vec{b}_i + \vec{p}_i$.

The time derivative of equation (3) in inertial coordinate system is obtained as follows :

$$\frac{D\vec{R}_i}{Dt} = \dot{\vec{R}}_i = \dot{\vec{R}}_j + \vec{\omega}_{aj} \times \vec{s}_i + \dot{\vec{\rho}}_i + \vec{\omega}_{ci} \times \vec{h}_i + \dot{\vec{p}}_i,\ \ i = 2,3,\cdots,N \tag{4}$$

where $\frac{D(\bullet)}{Dt}$ expresses the time derivative of vector "•"in inertial coordinate system, the" • "and" •• " above vectors respectively express the 1-order and 2-order time derivative in their own body coordinate systems. $\vec{\omega}_{aj}$ is the angular velocity vector of the floating coordinate system of flexible body B_j, $\vec{\omega}_{ci}$ is the angular velocity vector of the gemel coordinate system of gemel h_i.

Suppose that the deformation of flexible bodies is always in the range of elastic deformation, translation and rotation modal matrix of gemel h_i respectively are $\boldsymbol{\phi}_{ci}$ and $\boldsymbol{\psi}_{ci}$, corresponding modal coordinate is $\boldsymbol{q}_{ci}$, translation modal matrix of flexible body B_i is $\boldsymbol{\phi}_{ai}$, cooresponding modal coordinate is $\boldsymbol{q}_i$, we have:

$$\vec{\rho}_i = \boldsymbol{\phi}_{ci}\vec{q}_{ci},\ \vec{p}_i = \boldsymbol{\phi}_{ai}\vec{q}_i,\ \vec{\omega}_{ci} = \vec{\omega}_{aj} + \boldsymbol{\psi}_{ci}\vec{q}_{ci} \tag{5}$$

Write equation 4 in form of matrix for convenience like:

$$\dot{\vec{R}}_i = \boldsymbol{i}^{\mathrm{T}}\dot{\boldsymbol{R}}_j - \boldsymbol{a}_j^{\mathrm{T}}\boldsymbol{s}_i^{\times}\boldsymbol{\omega}_{aj} + \boldsymbol{a}_j^{\mathrm{T}}\dot{\boldsymbol{\rho}}_i - \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{h}_i^{\times}\boldsymbol{\omega}_{ci} + \boldsymbol{c}_i^{\mathrm{T}}\dot{\boldsymbol{p}}_i \tag{6}$$

where $\boldsymbol{i}^{\mathrm{T}} = [\vec{i}_x \quad \vec{i}_y \quad \vec{i}_z]$ is the unit base vector of $\boldsymbol{f}_i$; $\boldsymbol{a}_j^T = [\vec{a}_{xj} \quad \vec{a}_{yj} \quad \vec{a}_{zj}]$, $j = 2,3,\cdots,N$, is the unit base vector of $\boldsymbol{f}_{aj}$; $\boldsymbol{c}_i^T = [\vec{c}_{xi} \quad \vec{c}_{yi} \quad \vec{c}_{zi}]$, is the unit base vector of $\boldsymbol{f}_{ci}$, $\boldsymbol{\omega}_{aj}$, $\boldsymbol{s}_i$ and $\boldsymbol{\rho}_i$ respectively are component column arrays of corresponding vector in $\boldsymbol{f}_{aj}$, $\boldsymbol{\omega}_{ci}$, $\boldsymbol{h}_i$ and $\boldsymbol{p}_i$ respectively are component column arrays of corresponding vector in $\boldsymbol{f}_{ci}$.

Vector equation (5) can be written in form of matrix as follows:

$$\vec{\rho}_i = \boldsymbol{c}_i^T\boldsymbol{\phi}_{ci}\boldsymbol{q}_{ci},\ \vec{p}_i = \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{\phi}_{ai}\boldsymbol{q}_i,\ \vec{\omega}_{ci} = \boldsymbol{a}_j^T(\boldsymbol{\omega}_{aj} + \boldsymbol{A}_{ajci}\boldsymbol{\psi}_{ci}\boldsymbol{q}_{ci}) \tag{7}$$

From Eq. (6) and Eq. (7), we obtain

$$\dot{\vec{R}}_i = \boldsymbol{i}^{\mathrm{T}}\dot{\boldsymbol{R}}_j - (\boldsymbol{a}_j^{\mathrm{T}}\boldsymbol{s}_i^{\times} + \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{h}_i^{\times}\boldsymbol{A}_{ciaj})\boldsymbol{\omega}_{aj} + \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{\phi}_{ci}\dot{\boldsymbol{q}}_{ci} - \boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{h}_i^{\times}\boldsymbol{\psi}_{ci}\boldsymbol{q}_{ci} + \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{\phi}_{ai}\dot{\boldsymbol{q}}_i \tag{8}$$

where $\boldsymbol{q}_{ci}$ is the component column array of corresponding vector in $\boldsymbol{f}_{ci}$; $\boldsymbol{q}_i$ is the component column array of corresponding vector in $\boldsymbol{f}_{ai}$; $\boldsymbol{A}_{ciaj}$ is the coordinate conversion matrix from $\boldsymbol{f}_{aj}$ to $\boldsymbol{f}_{ci}$, besides $\boldsymbol{A}_{ciaj} = \boldsymbol{A}_{ajci}^{\mathrm{T}}$.
For central body B_1, we have

$$\vec{R}_b = \vec{R}_o + \vec{R}_{ob} = \vec{R}_o + \vec{a}_b + \vec{\rho}_b \tag{9}$$

where $\vec{\omega}_o$ is the angular velocity of orbit coordinate system; $\dot{\vec{\rho}}_b = \boldsymbol{\phi}_b \dot{\vec{q}}_b$, ϕ_b is the translation modal matrix of the central body, q_b is the corresponding modal coordinate. Upon that matrix form of absolute velocity vector of the spacecraft can be written as

$$\dot{\vec{R}}_b = \boldsymbol{i}^{\mathrm{T}} \dot{\boldsymbol{R}}_o - \boldsymbol{i}_o^{\mathrm{T}} \boldsymbol{R}_{ob}^{\times} \boldsymbol{\omega}_o + \boldsymbol{b}^{\mathrm{T}} \boldsymbol{\phi}_b \dot{\boldsymbol{q}}_b = \boldsymbol{i}^{\mathrm{T}} \boldsymbol{v}_o - \boldsymbol{i}_o^{\mathrm{T}} \boldsymbol{R}_{ob}^{\times} \boldsymbol{\omega}_o + \boldsymbol{b}^{\mathrm{T}} \boldsymbol{\phi}_b \dot{\boldsymbol{q}}_b \tag{10}$$

where $\boldsymbol{i}_o^{\mathrm{T}} = [\vec{i}_{ox} \quad \vec{i}_{oy} \quad \vec{i}_{oz}]$ is the unit base vector of orbit coordinate system $\boldsymbol{f}_o$; $\boldsymbol{b}^{\mathrm{T}} = [\vec{b}_x \quad \vec{b}_y \quad \vec{b}_z]$ is the unit base vector of cantral body coordinate system $\boldsymbol{f}_b$.
From Eq.(8) and Eq. (10), we get

$$\begin{aligned} \dot{\vec{R}}_i = {} & \boldsymbol{i}^{\mathrm{T}} \boldsymbol{v}_o - \boldsymbol{i}_o^{\mathrm{T}} \boldsymbol{R}_{ob}^{\times} \boldsymbol{\omega}_o + \boldsymbol{b}^{\mathrm{T}} \boldsymbol{\phi}_b \dot{\boldsymbol{q}}_b - (\boldsymbol{b}^{\mathrm{T}} \boldsymbol{s}_k^{\times} + \boldsymbol{c}_k^{\mathrm{T}} \boldsymbol{h}_k^{\times} \boldsymbol{A}_{ckb}) \boldsymbol{\omega}_b + \boldsymbol{c}_k^{\mathrm{T}} \boldsymbol{\phi}_{ck} \dot{\boldsymbol{q}}_{ck} - \boldsymbol{c}_k^{\mathrm{T}} \boldsymbol{h}_k^{\times} \boldsymbol{\psi}_{ck} \boldsymbol{q}_{ck} + \boldsymbol{a}_k^{\mathrm{T}} \boldsymbol{\phi}_{ak} \dot{\boldsymbol{q}}_k \\ & + \sum_{i \in D} \left[-(\boldsymbol{a}_j^{\mathrm{T}} \boldsymbol{s}_i^{\times} + \boldsymbol{c}_i^{\mathrm{T}} \boldsymbol{h}_i^{\times} \boldsymbol{A}_{ciaj}) \boldsymbol{\omega}_{aj} + \boldsymbol{c}_i^{\mathrm{T}} \boldsymbol{\phi}_{ci} \dot{\boldsymbol{q}}_{ci} - \boldsymbol{c}_i^{\mathrm{T}} \boldsymbol{h}_i^{\times} \boldsymbol{\psi}_{ci} \boldsymbol{q}_{ci} + \boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{\phi}_{ai} \dot{\boldsymbol{q}}_i \right] \end{aligned} \tag{11}$$

where k is the serial number of the outboard connected object of the central body B_1 in the access from object B_1 to B_i; D is the serial number collection of the objects in the access from object B_1 to B_i except for B_1 and its outboard connected object; $j = L(i)$; $\boldsymbol{\omega}_b$ is the component column array of angular velocity of the central body B_1 in $\boldsymbol{f}_b$.
If we define that $\boldsymbol{b}^{\mathrm{T}} = \boldsymbol{a}_1^{\mathrm{T}}$, $\boldsymbol{A}_{cib} = \boldsymbol{A}_{cia1}$, $\boldsymbol{\omega}_b = \boldsymbol{\omega}_{a1}$, then the Eq.(11) can be written in more concise form like:

$$\begin{aligned} \dot{\vec{R}}_i = {} & \boldsymbol{i}^{\mathrm{T}} \boldsymbol{v}_o - \boldsymbol{i}_o^{\mathrm{T}} \boldsymbol{R}_{ob}^{\times} \boldsymbol{\omega}_o + \boldsymbol{b}^{\mathrm{T}} \boldsymbol{\phi}_b \dot{\boldsymbol{q}}_b \\ & + \sum_{i \in E} \left[-(\boldsymbol{a}_j^{\mathrm{T}} \boldsymbol{s}_i^{\times} + \boldsymbol{c}_i^{\mathrm{T}} \boldsymbol{h}_i^{\times} \boldsymbol{A}_{ciaj}) \boldsymbol{\omega}_{aj} + \boldsymbol{c}_i^{\mathrm{T}} \boldsymbol{\phi}_{ci} \dot{\boldsymbol{q}}_{ci} - \boldsymbol{c}_i^{\mathrm{T}} \boldsymbol{h}_i^{\times} \boldsymbol{\psi}_{ci} \boldsymbol{q}_{ci} + \boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{\phi}_{ai} \dot{\boldsymbol{q}}_i \right] \end{aligned} \tag{12}$$

where is the serial number collection of objects in the access from object B_1 to B_i except for the central body B_1.
The projection of the vector $\dot{\vec{R}}_i$ in the central body coordinate system can be written as:

$$\begin{aligned} (\dot{R}_i)_b = {} & \boldsymbol{A}_{bi} \boldsymbol{v}_o - \boldsymbol{A}_{bo} \boldsymbol{R}_{ob}^{\times} \boldsymbol{\omega}_o + \boldsymbol{\phi}_b \dot{\boldsymbol{q}}_b \\ & + \sum_{i \in E} \left[-(\boldsymbol{A}_{baj} \boldsymbol{s}_i^{\times} + \boldsymbol{A}_{bci} \boldsymbol{h}_i^{\times} \boldsymbol{A}_{ciaj}) \boldsymbol{\omega}_{aj} + \boldsymbol{A}_{bci} \boldsymbol{\phi}_{ci} \dot{\boldsymbol{q}}_{ci} - \boldsymbol{A}_{bci} \boldsymbol{h}_i^{\times} \boldsymbol{\psi}_{ci} \boldsymbol{q}_{ci} + \boldsymbol{A}_{bai} \boldsymbol{\phi}_{ai} \dot{\boldsymbol{q}}_i \right] \end{aligned} \tag{13}$$

where $\boldsymbol{A}_{bi}$ is the coordinate conversion matrix from $\boldsymbol{f}_i$ to $\boldsymbol{f}_b$, besides, $\boldsymbol{A}_{bi} = \boldsymbol{A}_{ib}^{\mathrm{T}}$; $\boldsymbol{A}_{bo}$ is the coordinate conversion matrix from $\boldsymbol{f}_o$ to $\boldsymbol{f}_b$, besides, $\boldsymbol{A}_{bo} = \boldsymbol{A}_{ob}^{\mathrm{T}}$; $\boldsymbol{A}_{baj}$ is the coordinate conversion matrix from $\boldsymbol{f}_{aj}$ to $\boldsymbol{f}_b$, besides, $\boldsymbol{A}_{baj} = \boldsymbol{A}_{ajb}^{\mathrm{T}}$; $\boldsymbol{A}_{bci}$ is the coordinate conversion

matrix from $\boldsymbol{f}_{ci}$ to $\boldsymbol{f}_b$, besides, $\boldsymbol{A}_{bci} = \boldsymbol{A}_{cib}^{\mathrm{T}}$; $\boldsymbol{A}_{bai}$ is the coordinate conversion matrix from $\boldsymbol{f}_{ai}$ to $\boldsymbol{f}_b$, besides, $\boldsymbol{A}_{bai} = \boldsymbol{A}_{aib}^{\mathrm{T}}$.
We make several definitions as follows:

$$\begin{aligned}
&\boldsymbol{E}_1 = \boldsymbol{A}_{bi}\boldsymbol{v}_o - \boldsymbol{A}_{bo}\boldsymbol{R}_{ob}^{\times}\boldsymbol{\omega}_o + \boldsymbol{\phi}_b\dot{\boldsymbol{q}}_b \\
&\boldsymbol{E}_2 = [-(\boldsymbol{A}_{baj}\boldsymbol{s}_i^{\times} + \boldsymbol{A}_{bci}\boldsymbol{h}_i^{\times}\boldsymbol{A}_{ciaj})\boldsymbol{\omega}_{aj} + \boldsymbol{A}_{bci}\boldsymbol{\phi}_{ci}\dot{\boldsymbol{q}}_{ci} - \boldsymbol{A}_{bci}\boldsymbol{h}_i^{\times}\boldsymbol{\psi}_{ci}\boldsymbol{q}_{ci} + \boldsymbol{A}_{bai}\boldsymbol{\phi}_{ai}\dot{\boldsymbol{q}}_i]_{i=2} \\
&\vdots \\
&\boldsymbol{E}_n = [-(\boldsymbol{A}_{baj}\boldsymbol{s}_i^{\times} + \boldsymbol{A}_{bci}\boldsymbol{h}_i^{\times}\boldsymbol{A}_{ciaj})\boldsymbol{\omega}_{aj} + \boldsymbol{A}_{bci}\boldsymbol{\phi}_{ci}\dot{\boldsymbol{q}}_{ci} - \boldsymbol{A}_{bci}\boldsymbol{h}_i^{\times}\boldsymbol{\psi}_{ci}\boldsymbol{q}_{ci} + \boldsymbol{A}_{bai}\boldsymbol{\phi}_{ai}\dot{\boldsymbol{q}}_i]_{i=n}
\end{aligned} \tag{14}$$

$$\boldsymbol{E} = [\boldsymbol{E}_1 \quad \boldsymbol{E}_2 \quad \cdots \quad \boldsymbol{E}_n]^{\mathrm{T}} \tag{15}$$

Upon that the origin velocity of arbitrary flexible body coordinate system can be expressed using access matrix as follows:

$$(\dot{\boldsymbol{R}}_i)_b = (\boldsymbol{T})_i\boldsymbol{E} \tag{16}$$

where $(\boldsymbol{T})_i$ represents the i row of access matrix $\boldsymbol{T}$.

2.4 Dynamics modeling based on quasi-Lagrange equations

2.4.1 Quasi-Lagrange equations

Quasi-Lagrange equations is a kind of improvement of classical Lagrange equations, on one hand they have the advantage of normalized derivation, on the other hand they can reserve the presentation form of dynamics equations of rigid body. Therefore they are applicable for researching the dynamics problem of large spacecraft. Using quasi-Lagrange equations system dynamics can be expressed as follows:

$$\begin{cases}
\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \boldsymbol{v}_b}\right) + \boldsymbol{\omega}_b^{\times}\dfrac{\partial L}{\partial \boldsymbol{v}_b} = \boldsymbol{Q}_{bt} \\
\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \boldsymbol{\omega}_b}\right) + \boldsymbol{v}_b^{\times}\dfrac{\partial L}{\partial \boldsymbol{v}_b} + \boldsymbol{\omega}_b^{\times}\dfrac{\partial L}{\partial \boldsymbol{\omega}_b} = \boldsymbol{Q}_{br} \\
\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \boldsymbol{\omega}_{ai}}\right) + \boldsymbol{A}_{aib}\boldsymbol{v}_i^{\times}\dfrac{\partial L}{\partial \boldsymbol{v}_i} + \left(\boldsymbol{A}_{aib}\boldsymbol{\omega}_b^{\times}\boldsymbol{A}_{aib}^{T} + \boldsymbol{\omega}_{ai}^{\times}\right)\dfrac{\partial L}{\partial \boldsymbol{\omega}_{ai}} = \boldsymbol{Q}_{air} \\
\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\boldsymbol{q}}_b}\right) + \dfrac{d\Phi}{d\dot{\boldsymbol{q}}_b} - \dfrac{dL}{d\boldsymbol{q}_b} = \boldsymbol{Q}_{vb} \\
\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\right) + \dfrac{d\Phi}{d\dot{\boldsymbol{q}}_i} - \dfrac{dL}{d\boldsymbol{q}_i} = \boldsymbol{Q}_{vi}
\end{cases} \tag{17}$$

where L is the Lagrange function of system, $L = T - U$, T is the kinetic energy of system, is the potential energy of system, is the dissipated energy of system; $\boldsymbol{v}_b, \boldsymbol{\omega}_b$ respectively are the spacecraft central body coordinate system velocity and angular velocity coordinates in $o_b x_b y_b z_b$ relative to the inertial coordinate system; $\boldsymbol{v}_{ai}$ is the velocity coordinate in $o_b x_b y_b z_b$ of the floating coordinate system of flexible body i relative to the inertial coordinate system; $\boldsymbol{\omega}_{ai}$ is the angular velocity coordinate in $o_b x_b y_b z_b$ of the floating coordinate system of flexible body i relative to the inertial coordinate system; $\boldsymbol{q}_b$ is the modal coordinate of the central body B_1; $\boldsymbol{q}_i$ is the modal coordinate of the flexible body B_i; $\boldsymbol{Q}_{bt}$ is the generalized force corresponding to the translation of spacecraft central body; $\boldsymbol{Q}_{br}$ is the generalized moment corresponding to the rotation of spacecraft central body; $\boldsymbol{Q}_{air}$ is the generalized moment corresponding to the rotation of flexible body B_i; $\boldsymbol{Q}_{vb}$ is the generalized force corresponding to the modal coordinate $\boldsymbol{q}_b$; $\boldsymbol{Q}_{vi}$ is the generalized force corresponding to the modal coordinate $\boldsymbol{q}_i$; $\boldsymbol{A}_{aib}$ is the conversion matrix from $o_b x_b y_b z_b$ to $o_{ai} x_{ai} y_{ai} z_{ai}$; for an arbitrary 3×1 column array $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$, $\boldsymbol{x}^\times$ represents the skew symmetric matrix as follows:

$$\boldsymbol{x}^\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{18}$$

2.4.2 Lagrange function

Using Lagrange equations to found the system dynamics model, firstly, we should calculate the kinetic energy and potential energy of each body in the system, then add them together in order to get the total kinetic and potential energy of the system, finally obtain the Lagrange function.

2.4.2.1 Kinetic energy of system

The kinetic energy of flexible body i expressed by generalized velocity is as follows:

$$T_i = \frac{1}{2} \boldsymbol{\upsilon}_i^T \boldsymbol{M}_i \boldsymbol{\upsilon}_i \tag{19}$$

The generalized velocity $\boldsymbol{\upsilon}_i$ in the above formula is defined as:

$$\boldsymbol{\upsilon}_i = \begin{pmatrix} \dot{\boldsymbol{R}}_i^{\mathrm{T}} & \boldsymbol{\omega}_{ai}^{\mathrm{T}} & \dot{\boldsymbol{q}}_i \end{pmatrix}^{\mathrm{T}} \tag{20}$$

where $\dot{\boldsymbol{R}}_i$ is the floating coordinate system origin velocity of the flexible body i; $\boldsymbol{\omega}_{ai}$ is the floating coordinate system origin angular velocity of the flexible body i; $\boldsymbol{q}_i$ is the deformation modal coordinate of the flexible body i. $\boldsymbol{M}_i$ is the mass matrix of the flexible body i, which is defined as:

$$\boldsymbol{M}_i = \begin{bmatrix} \boldsymbol{m}_{RR}^i & \boldsymbol{m}_{R\omega}^i & \boldsymbol{m}_{Rf}^i \\ \boldsymbol{m}_{R\omega}^i & \boldsymbol{m}_{\omega\omega}^i & \boldsymbol{m}_{\omega f}^i \\ \boldsymbol{m}_{Rf}^i & \boldsymbol{m}_{\omega f}^i & \boldsymbol{m}_{ff}^i \end{bmatrix} \tag{21}$$

The specific expression of each sub-matrix can be found in reference written by Lu.
Upon that the kinetic energy of the flexible body i can be written as follows:

$$T_i = \frac{1}{2}(\dot{\boldsymbol{R}}_i^T \boldsymbol{m}_{RR}^i \dot{\boldsymbol{R}}_i + 2\dot{\boldsymbol{R}}_i^T \boldsymbol{m}_{R\omega}^i \boldsymbol{\omega}_{ai} + 2\dot{\boldsymbol{R}}_i^T \boldsymbol{m}_{Rf}^i \dot{\boldsymbol{q}}_i + \boldsymbol{\omega}_{ai}^T \boldsymbol{m}_{\omega\omega}^i \boldsymbol{\omega}_{ai} + 2\boldsymbol{\omega}_{ai}^T \boldsymbol{m}_{\omega f}^i \dot{\boldsymbol{q}}_i + \dot{\boldsymbol{q}}_i^T \boldsymbol{m}_{ff}^i \dot{\boldsymbol{q}}_i) \tag{22}$$

The entire kinetic energy of the system can be expressd as:

$$\begin{aligned} T &= \frac{1}{2}\sum_{i=1}^{n}(\dot{\boldsymbol{R}}_i^T \boldsymbol{m}_{RR}^i \dot{\boldsymbol{R}}_i + 2\dot{\boldsymbol{R}}_i^T \boldsymbol{m}_{R\omega}^i \boldsymbol{\omega}_{ai} + 2\dot{\boldsymbol{R}}_i^T \boldsymbol{m}_{Rf}^i \dot{\boldsymbol{q}}_i + \boldsymbol{\omega}_{ai}^T \boldsymbol{m}_{\omega\omega}^i \boldsymbol{\omega}_{ai} + 2\boldsymbol{\omega}_{ai}^T \boldsymbol{m}_{\omega f}^i \dot{\boldsymbol{q}}_i + \dot{\boldsymbol{q}}_i^T \boldsymbol{m}_{ff}^i \dot{\boldsymbol{q}}_i) \\ &= \frac{1}{2}(\dot{\boldsymbol{R}}_b^T \boldsymbol{m}_{RR}^b \dot{\boldsymbol{R}}_b + 2\dot{\boldsymbol{R}}_b^T \boldsymbol{m}_{R\omega}^b \boldsymbol{\omega}_b + 2\dot{\boldsymbol{R}}_b^T \boldsymbol{m}_{Rf}^b \dot{\boldsymbol{q}}_b + \boldsymbol{\omega}_b^T \boldsymbol{m}_{\omega\omega}^b \boldsymbol{\omega}_b + 2\boldsymbol{\omega}_b^T \boldsymbol{m}_{\omega f}^b \dot{\boldsymbol{q}}_b + \dot{\boldsymbol{q}}_b^T \boldsymbol{m}_{ff}^b \dot{\boldsymbol{q}}_b) \\ &\quad + \frac{1}{2}\sum_{i=2}^{n}(\dot{\boldsymbol{R}}_i^T \boldsymbol{m}_{RR}^i \dot{\boldsymbol{R}}_i + 2\dot{\boldsymbol{R}}_i^T \boldsymbol{m}_{R\omega}^i \boldsymbol{\omega}_{ai} + 2\dot{\boldsymbol{R}}_i^T \boldsymbol{m}_{Rf}^i \dot{\boldsymbol{q}}_i + \boldsymbol{\omega}_{ai}^T \boldsymbol{m}_{\omega\omega}^i \boldsymbol{\omega}_{ai} + 2\boldsymbol{\omega}_{ai}^T \boldsymbol{m}_{\omega f}^i \dot{\boldsymbol{q}}_i + \dot{\boldsymbol{q}}_i^T \boldsymbol{m}_{ff}^i \dot{\boldsymbol{q}}_i) \end{aligned} \tag{23}$$

2.4.2.2 *Potential and dissipated energy of system*

Define p_{bm} is the m^{th} order natural frequency of the central body, then the elastic energy of the central body can be expressed as:

$$U_b = \frac{1}{2}\boldsymbol{q}_b^{\mathrm{T}} \boldsymbol{K}_b \boldsymbol{q}_b \tag{24}$$

where $\boldsymbol{K}_b$ is the rigidity matrix of the central body, $\boldsymbol{K}_b = \mathrm{diag}(p_{b1}^2, p_{b2}^2, \cdots, p_{bM}^2)$.
Define p_{im} is the m^{th} order natural frequency of the flexible body B_i, then the elastic energy of the flexible body i can be expressed as:

$$U_i = \frac{1}{2}\boldsymbol{q}_i^{\mathrm{T}} \boldsymbol{K}_i \boldsymbol{q}_i \tag{25}$$

where $\boldsymbol{K}_i$ is the rigidity matrix of the flexible body i, $\boldsymbol{K}_i = \mathrm{diag}(p_{i1}^2, p_{i2}^2, \cdots, p_{iM}^2)$.
If we only consider about the deformation energy of the spacecrfta structure, the potential energy of the system can be expressed as:

$$U = \frac{1}{2}\boldsymbol{q}_b^{\mathrm{T}} \boldsymbol{K}_b \boldsymbol{q}_b + \sum_{i=2}^{n}\frac{1}{2}\boldsymbol{q}_i^{\mathrm{T}} \boldsymbol{K}_i \boldsymbol{q}_i = \frac{1}{2}\boldsymbol{q}_b^{\mathrm{T}} \boldsymbol{K}_b \boldsymbol{q}_b + \frac{1}{2}\boldsymbol{q}_a^{\mathrm{T}} \boldsymbol{K}_a \boldsymbol{q}_a \tag{26}$$

where

$$\boldsymbol{K}_a = \begin{bmatrix} \boldsymbol{K}_2 & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \cdots & \vdots \\ \mathbf{0} & \cdots & \boldsymbol{K}_i & \cdots & \mathbf{0} \\ \vdots & \cdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \boldsymbol{K}_n \end{bmatrix} \tag{27}$$

$$\boldsymbol{q}_a = \begin{bmatrix} \boldsymbol{q}_2^{\mathrm{T}} & \boldsymbol{q}_3^{\mathrm{T}} & \cdots & \boldsymbol{q}_n^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tag{28}$$

Define ξ_{bm} is the damping coefficient corresponding to p_{bm} of the central body B_1, then the dissipated energy of the central body can be expressed as:

$$\Phi_b = \frac{1}{2}\dot{\boldsymbol{q}}_b^{\mathrm{T}} \boldsymbol{D}_b \dot{\boldsymbol{q}}_b \tag{29}$$

where $\boldsymbol{D}_b$ is the damping matrix of the central body, $\boldsymbol{D}_b = \mathrm{diag}(2\xi_{b1}p_{b1}, 2\xi_{b2}p_{b2}, \cdots, 2\xi_{bM}p_{bM})$. Define ξ_{im} is the damping coefficient corresponding to p_{im} of the flexible body B_i, then the dissipated energy of the flexible body B_i can be expressed as:

$$\Phi_i = \frac{1}{2}\dot{\boldsymbol{q}}_i^{\mathrm{T}} \boldsymbol{D}_i \dot{\boldsymbol{q}}_i \tag{30}$$

where $\boldsymbol{D}_i$ is the damping matrix of the flexible body B_i, $\boldsymbol{D}_i = \mathrm{diag}(2\xi_{i1}p_{i1}, 2\xi_{i2}p_{i2}, \cdots, 2\xi_{iM}p_{iM})$. Upon that the dissipated energy of the system can be expressed as:

$$\Phi = \frac{1}{2}\dot{\boldsymbol{q}}_b^{\mathrm{T}} \boldsymbol{D}_b \dot{\boldsymbol{q}}_b + \sum_{i=2}^{n} \frac{1}{2}\dot{\boldsymbol{q}}_i^{\mathrm{T}} \boldsymbol{D}_i \dot{\boldsymbol{q}}_i = \frac{1}{2}\dot{\boldsymbol{q}}_b^{\mathrm{T}} \boldsymbol{D}_b \dot{\boldsymbol{q}}_b + \frac{1}{2}\dot{\boldsymbol{q}}_a^{T} \boldsymbol{D}_a \dot{\boldsymbol{q}}_a \tag{31}$$

where

$$\boldsymbol{D}_a = \begin{bmatrix} \boldsymbol{D}_2 & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & \cdots & \vdots \\ \mathbf{0} & \cdots & \boldsymbol{D}_i & \cdots & \mathbf{0} \\ \vdots & \cdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \boldsymbol{D}_n \end{bmatrix} \tag{32}$$

2.4.3 Dynamics equations of the system

2.4.3.1 Translation equations of the central body

$$\frac{\partial L}{\partial \boldsymbol{v}_b} = (\boldsymbol{m}_{RR}^b + \sum_{i=2}^{n} \boldsymbol{m}_{RR}^i)\boldsymbol{v}_b + \boldsymbol{m}_{R\omega}^b \boldsymbol{\omega}_b + \boldsymbol{m}_{Rf}^b \dot{\boldsymbol{q}}_b + \sum_{i=2}^{n}\left(\boldsymbol{m}_{R\omega}^i \boldsymbol{A}_{bai} \boldsymbol{\omega}_{ai} + \boldsymbol{m}_{Rf}^i \boldsymbol{A}_{bai} \dot{\boldsymbol{q}}_i\right) \tag{33}$$

$$\boldsymbol{\omega}_b^{\times} \frac{\partial L}{\partial \mathbf{v}_b} = \boldsymbol{\omega}_b^{\times}\left[(\boldsymbol{m}_{RR}^b + \sum_{i=2}^{n} \boldsymbol{m}_{RR}^i)\boldsymbol{v}_b + \boldsymbol{m}_{R\omega}^b \boldsymbol{\omega}_b + \boldsymbol{m}_{Rf}^b \dot{\boldsymbol{q}}_b + \sum_{i=2}^{n}\left(\boldsymbol{m}_{R\omega}^i \boldsymbol{A}_{bai} \boldsymbol{\omega}_{ai} + \boldsymbol{m}_{Rf}^i \boldsymbol{A}_{bai} \dot{\boldsymbol{q}}_i\right)\right] \tag{34}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}_b}\right)=(\boldsymbol{m}_{RR}^{b}+\sum_{i=2}^{n}\boldsymbol{m}_{RR}^{i})\dot{\boldsymbol{v}}_b+\boldsymbol{m}_{R\omega}^{b}\dot{\boldsymbol{\omega}}_b+\boldsymbol{m}_{Rf}^{b}\ddot{\boldsymbol{q}}_b$$
$$+\sum_{i=2}^{n}\left(\boldsymbol{m}_{R\omega}^{i}\dot{\boldsymbol{A}}_{bai}\boldsymbol{\omega}_{ai}+\boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{\omega}}_{ai}+\boldsymbol{m}_{Rf}^{i}\dot{\boldsymbol{A}}_{bai}\dot{\boldsymbol{q}}_i+\boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\ddot{\boldsymbol{q}}_i\right) \tag{35}$$

Substitute Eq. (33)-Eq. (35) into the first formula of Eq. (17), the translation equations of the central body is obtained as follows:

$$(\boldsymbol{m}_{RR}^{b}+\sum_{i=2}^{n}\boldsymbol{m}_{RR}^{i})\dot{\boldsymbol{v}}_b+\boldsymbol{m}_{R\omega}^{b}\dot{\boldsymbol{\omega}}_b+\boldsymbol{m}_{Rf}^{b}\ddot{\boldsymbol{q}}_b$$
$$+\sum_{i=2}^{n}\left(\boldsymbol{m}_{R\omega}^{i}\dot{\boldsymbol{A}}_{bai}\boldsymbol{\omega}_{ai}+\boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{\omega}}_{ai}+\boldsymbol{m}_{Rf}^{i}\dot{\boldsymbol{A}}_{bai}\dot{\boldsymbol{q}}_i+\boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\ddot{\boldsymbol{q}}_i\right) \tag{36}$$
$$+\boldsymbol{\omega}_b^{\times}\left[(\boldsymbol{m}_{RR}^{b}+\sum_{i=2}^{n}\boldsymbol{m}_{RR}^{i})\boldsymbol{v}_b+\boldsymbol{m}_{R\omega}^{b}\boldsymbol{\omega}_b+\boldsymbol{m}_{Rf}^{b}\dot{\boldsymbol{q}}_b+\sum_{i=2}^{n}\left(\boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\boldsymbol{\omega}_{ai}+\boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{q}}_i\right)\right]=F_b$$

2.4.3.2 Rotation equations of the central body

$$\frac{\partial L}{\partial \boldsymbol{\omega}_b}=\left(\boldsymbol{A}_{bi}\boldsymbol{v}_o-\boldsymbol{A}_{bo}\boldsymbol{R}_{ob}^{\times}\boldsymbol{\omega}_o+\boldsymbol{\phi}_b\dot{\boldsymbol{q}}_b\right)^{\mathrm{T}}\boldsymbol{m}_{R\omega}^{b}+\boldsymbol{m}_{\omega\omega}^{b}\boldsymbol{\omega}_b+\boldsymbol{m}_{\omega f}^{b}\dot{\boldsymbol{q}}_b$$
$$+\sum_{i=2}^{N}\left[\boldsymbol{m}_{RR}^{i}(\boldsymbol{T})_i\boldsymbol{E}(\boldsymbol{T})_i\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}+\boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\boldsymbol{\omega}_{ai}(\boldsymbol{T})_i\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}+\boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{q}}_i(\boldsymbol{T})_i\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}\right] \tag{37}$$

$$\boldsymbol{v}_b^{\times}\frac{\partial L}{\partial \boldsymbol{v}_b}=\boldsymbol{v}_b^{\times}\left[(\boldsymbol{m}_{RR}^{b}+\sum_{i=2}^{n}\boldsymbol{m}_{RR}^{i})\boldsymbol{v}_b+\boldsymbol{m}_{R\omega}^{b}\boldsymbol{\omega}_b+\boldsymbol{m}_{Rf}^{b}\dot{\boldsymbol{q}}_b+\sum_{i=2}^{n}\left(\boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\boldsymbol{\omega}_{ai}+\boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{q}}_i\right)\right] \tag{38}$$

$$\boldsymbol{\omega}_b^{\times}\frac{\partial L}{\partial \boldsymbol{\omega}_b}=\boldsymbol{\omega}_b^{\times}\left[\left(\boldsymbol{A}_{bi}\boldsymbol{v}_o-\boldsymbol{A}_{bo}\boldsymbol{R}_{ob}^{\times}\boldsymbol{\omega}_o+\boldsymbol{\phi}_b\dot{\boldsymbol{q}}_b\right)^{\mathrm{T}}\boldsymbol{m}_{R\omega}^{b}+\boldsymbol{m}_{\omega\omega}^{b}\boldsymbol{\omega}_b+\boldsymbol{m}_{\omega f}^{b}\dot{\boldsymbol{q}}_b\right]$$
$$+\boldsymbol{\omega}_b^{\times}\left\{\sum_{i=2}^{N}\left[\boldsymbol{m}_{RR}^{i}(\boldsymbol{T})_i\boldsymbol{E}(\boldsymbol{T})_i\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}+\boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\boldsymbol{\omega}_{ai}(\boldsymbol{T})_i\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}+\boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{q}}_i(\boldsymbol{T})_i\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}\right]\right\} \tag{39}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\omega}_b}\right)=\left(\dot{\boldsymbol{A}}_{bi}\boldsymbol{v}_o+\boldsymbol{A}_{bi}\dot{\boldsymbol{v}}_o-\dot{\boldsymbol{A}}_{bo}\boldsymbol{R}_{ob}^{\times}\boldsymbol{\omega}_o-\boldsymbol{A}_{bo}\dot{\boldsymbol{R}}_{ob}^{\times}\boldsymbol{\omega}_o-\boldsymbol{A}_{bo}\boldsymbol{R}_{ob}^{\times}\dot{\boldsymbol{\omega}}_o+\boldsymbol{\phi}_b\ddot{\boldsymbol{q}}_b\right)^{\mathrm{T}}\boldsymbol{m}_{R\omega}^{b}$$
$$+\boldsymbol{m}_{\omega\omega}^{b}\dot{\boldsymbol{\omega}}_b+\boldsymbol{m}_{\omega f}^{b}\ddot{\boldsymbol{q}}_b \tag{40a}$$

$$+\sum_{i=2}^{N}\begin{bmatrix} \boldsymbol{m}_{RR}^{i}(\boldsymbol{T})_i\dot{\boldsymbol{E}}(\boldsymbol{T})_i\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}+\boldsymbol{m}_{RR}^{i}(\boldsymbol{T})_i\boldsymbol{E}(\boldsymbol{T})_i\dfrac{d}{dt}\left(\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}\right)+\boldsymbol{m}_{R\omega}^{i}\dot{\boldsymbol{A}}_{bai}\boldsymbol{\omega}_{ai}(\boldsymbol{T})_i\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b} \\ +\boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{\omega}}_{ai}(\boldsymbol{T})_i\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}+\boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\boldsymbol{\omega}_{ai}(\boldsymbol{T})_i\dfrac{d}{dt}\left(\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}\right)+\boldsymbol{m}_{Rf}^{i}\dot{\boldsymbol{A}}_{bai}\dot{\boldsymbol{q}}_i(\boldsymbol{T})_i\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b} \\ +\boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\ddot{\boldsymbol{q}}_i(\boldsymbol{T})_i\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}+\boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{q}}_i(\boldsymbol{T})_i\dfrac{d}{dt}\left(\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}\right) \end{bmatrix} \tag{40b}$$

Substitute Eq. (37)-Eq. (40) into the second formula of Eq. (17), the rotation equations of the central body is obtained as follows:

$$\begin{aligned}
&\left(\dot{\boldsymbol{A}}_{bi}\boldsymbol{v}_o+\boldsymbol{A}_{bi}\dot{\boldsymbol{v}}_o-\dot{\boldsymbol{A}}_{bo}\boldsymbol{R}_{ob}^{\times}\boldsymbol{\omega}_o-\boldsymbol{A}_{bo}\dot{\boldsymbol{R}}_{ob}^{\times}\boldsymbol{\omega}_o-\boldsymbol{A}_{bo}\boldsymbol{R}_{ob}^{\times}\dot{\boldsymbol{\omega}}_o+\boldsymbol{\phi}_b\ddot{\boldsymbol{q}}_b\right)^{\mathrm{T}}\boldsymbol{m}_{R\omega}^{b}+\boldsymbol{m}_{\omega\omega}^{b}\dot{\boldsymbol{\omega}}_b+\boldsymbol{m}_{\omega f}^{b}\ddot{\boldsymbol{q}}_b \\
&+\sum_{i=2}^{N}\begin{bmatrix} \boldsymbol{m}_{RR}^{i}(\boldsymbol{T})_i\dot{\boldsymbol{E}}(\boldsymbol{T})_i\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}+\boldsymbol{m}_{RR}^{i}(\boldsymbol{T})_i\boldsymbol{E}(\boldsymbol{T})_i\dfrac{d}{dt}\left(\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}\right)+\boldsymbol{m}_{R\omega}^{i}\dot{\boldsymbol{A}}_{bai}\boldsymbol{\omega}_{ai}(\boldsymbol{T})_i\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b} \\ +\boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{\omega}}_{ai}(\boldsymbol{T})_i\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}+\boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\boldsymbol{\omega}_{ai}(\boldsymbol{T})_i\dfrac{d}{dt}\left(\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}\right)+\boldsymbol{m}_{Rf}^{i}\dot{\boldsymbol{A}}_{bai}\dot{\boldsymbol{q}}_i(\boldsymbol{T})_i\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b} \\ +\boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\ddot{\boldsymbol{q}}_i(\boldsymbol{T})_i\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}+\boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{q}}_i(\boldsymbol{T})_i\dfrac{d}{dt}\left(\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}\right) \end{bmatrix} \\
&+\boldsymbol{v}_b^{\times}\left[(\boldsymbol{m}_{RR}^{b}+\sum_{i=2}^{n}\boldsymbol{m}_{RR}^{i})\boldsymbol{v}_b+\boldsymbol{m}_{R\omega}^{b}\boldsymbol{\omega}_b+\boldsymbol{m}_{Rf}^{b}\dot{\boldsymbol{q}}_b+\sum_{i=2}^{n}\left(\boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\boldsymbol{\omega}_{ai}+\boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{q}}_i\right)\right] \\
&+\boldsymbol{\omega}_b^{\times}\left[\left(\boldsymbol{A}_{bi}\boldsymbol{v}_o-\boldsymbol{A}_{bo}\boldsymbol{R}_{ob}^{\times}\boldsymbol{\omega}_o+\boldsymbol{\phi}_b\dot{\boldsymbol{q}}_b\right)^{\mathrm{T}}\boldsymbol{m}_{R\omega}^{b}+\boldsymbol{m}_{\omega\omega}^{b}\boldsymbol{\omega}_b+\boldsymbol{m}_{\omega f}^{b}\dot{\boldsymbol{q}}_b\right] \\
&+\boldsymbol{\omega}_b^{\times}\left\{\sum_{i=2}^{N}\left[\boldsymbol{m}_{RR}^{i}(\boldsymbol{T})_i\boldsymbol{E}(\boldsymbol{T})_i\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}+\boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\boldsymbol{\omega}_{ai}(\boldsymbol{T})_i\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}+\boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{q}}_i(\boldsymbol{T})_i\dfrac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_b}\right]\right\}=\boldsymbol{Q}_{br}
\end{aligned} \tag{41}$$

2.4.3.3 *Rotation equations of the flexible body* i

$$\begin{aligned}
\frac{\partial L}{\partial \boldsymbol{\omega}_{ai}} &= \boldsymbol{A}_{aib}[(\boldsymbol{T})_i\boldsymbol{E}]^{\mathrm{T}}\boldsymbol{m}_{R\omega}^{i}+\boldsymbol{m}_{\omega\omega}^{i}\boldsymbol{\omega}_{ai}+\boldsymbol{m}_{\omega f}^{i}\dot{\boldsymbol{q}}_i \\
&+\boldsymbol{A}_{aib}\sum_{j=2}^{n}\left[\boldsymbol{m}_{RR}^{j}(\boldsymbol{T})_j\boldsymbol{E}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}}+\boldsymbol{m}_{R\omega}^{j}\boldsymbol{A}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}}+\boldsymbol{m}_{Rf}^{j}\boldsymbol{A}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}}\right]
\end{aligned} \tag{42}$$

$$\frac{\partial L}{\partial \boldsymbol{v}_i}=\boldsymbol{A}_{aib}\boldsymbol{m}_{RR}^{i}(\boldsymbol{T})_i\boldsymbol{E}+\boldsymbol{m}_{R\omega}^{i}\boldsymbol{\omega}_{ai}+\boldsymbol{m}_{Rf}^{i}\dot{\boldsymbol{q}}_i \tag{43}$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\omega}_{ai}}\right) = {} & \dot{\boldsymbol{A}}_{aib}\left[(\boldsymbol{T})_i \boldsymbol{E}\right]^{\mathrm{T}} \boldsymbol{m}^{i}_{R\omega} + \boldsymbol{A}_{aib}\left[(\boldsymbol{T})_i \dot{\boldsymbol{E}}\right]^{\mathrm{T}} \boldsymbol{m}^{i}_{R\omega} + \boldsymbol{m}^{i}_{\omega\omega}\dot{\boldsymbol{\omega}}_{ai} + \boldsymbol{m}^{i}_{\omega f}\ddot{\boldsymbol{q}}_i \\
& + \dot{\boldsymbol{A}}_{aib}\sum_{j=2}^{n}\left[\boldsymbol{m}^{j}_{RR}(\boldsymbol{T})_j \boldsymbol{E}(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} + \boldsymbol{m}^{j}_{R\omega}\boldsymbol{A}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} + \boldsymbol{m}^{j}_{Rf}\boldsymbol{A}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}}\right] \\
& + \boldsymbol{A}_{aib}\sum_{j=2}^{n}\left[\begin{array}{l}
\boldsymbol{m}^{j}_{RR}(\boldsymbol{T})_j \dot{\boldsymbol{E}}(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} + \boldsymbol{m}^{j}_{RR}(\boldsymbol{T})_j \boldsymbol{E}(\boldsymbol{T})_j \frac{d}{dt}\left(\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}}\right) + \boldsymbol{m}^{j}_{R\omega}\dot{\boldsymbol{A}}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} \\
+\boldsymbol{m}^{j}_{R\omega}\boldsymbol{A}_{baj}\dot{\boldsymbol{\omega}}_{aj}(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} + \boldsymbol{m}^{j}_{R\omega}\boldsymbol{A}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j \frac{d}{dt}\left(\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}}\right) + \boldsymbol{m}^{j}_{Rf}\dot{\boldsymbol{A}}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} \\
+\boldsymbol{m}^{j}_{Rf}\boldsymbol{A}_{baj}\ddot{\boldsymbol{q}}_j(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} + \boldsymbol{m}^{j}_{Rf}\boldsymbol{A}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j \frac{d}{dt}\left(\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}}\right)
\end{array}\right]
\end{aligned}
\tag{44}
$$

Substitute Eq. (42)-Eq. (44) into the third formula of Eq. (17), the rotation equations of the flexible body $\boldsymbol{i}$ is obtained as follows:

$$
\begin{aligned}
& \dot{\boldsymbol{A}}_{aib}\left[(\boldsymbol{T})_i \boldsymbol{E}\right]^{\mathrm{T}} \boldsymbol{m}^{i}_{R\omega} + \boldsymbol{A}_{aib}\left[(\boldsymbol{T})_i \dot{\boldsymbol{E}}\right]^{\mathrm{T}} \boldsymbol{m}^{i}_{R\omega} + \boldsymbol{m}^{i}_{\omega\omega}\dot{\boldsymbol{\omega}}_{ai} + \boldsymbol{m}^{i}_{\omega f}\ddot{\boldsymbol{q}}_i \\
& + \dot{\boldsymbol{A}}_{aib}\sum_{j=2}^{n}\left[\boldsymbol{m}^{j}_{RR}(\boldsymbol{T})_j \boldsymbol{E}(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} + \boldsymbol{m}^{j}_{R\omega}\boldsymbol{A}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} + \boldsymbol{m}^{j}_{Rf}\boldsymbol{A}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}}\right] \\
& + \boldsymbol{A}_{aib}\sum_{j=2}^{n}\left[\begin{array}{l}
\boldsymbol{m}^{j}_{RR}(\boldsymbol{T})_j \dot{\boldsymbol{E}}(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} + \boldsymbol{m}^{j}_{RR}(\boldsymbol{T})_j \boldsymbol{E}(\boldsymbol{T})_j \frac{d}{dt}\left(\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}}\right) + \boldsymbol{m}^{j}_{R\omega}\dot{\boldsymbol{A}}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} \\
+\boldsymbol{m}^{j}_{R\omega}\boldsymbol{A}_{baj}\dot{\boldsymbol{\omega}}_{aj}(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} + \boldsymbol{m}^{j}_{R\omega}\boldsymbol{A}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j \frac{d}{dt}\left(\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}}\right) + \boldsymbol{m}^{j}_{Rf}\dot{\boldsymbol{A}}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} \\
+\boldsymbol{m}^{j}_{Rf}\boldsymbol{A}_{baj}\ddot{\boldsymbol{q}}_j(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} + \boldsymbol{m}^{j}_{Rf}\boldsymbol{A}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j \frac{d}{dt}\left(\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}}\right)
\end{array}\right] \\
& + \boldsymbol{A}_{aib}\boldsymbol{v}_i^{\times}\left\{\boldsymbol{m}^{i}_{RR}(\boldsymbol{T})_i \boldsymbol{E} + \boldsymbol{m}^{i}_{R\omega}\boldsymbol{A}_{bai}\boldsymbol{\omega}_{ai} + \boldsymbol{m}^{i}_{Rf}\boldsymbol{A}_{bai}\dot{\boldsymbol{q}}_i\right\} \\
& + \left(\boldsymbol{A}_{aib}\boldsymbol{\omega}_b^{\times}\boldsymbol{A}_{aib}^{T} + \boldsymbol{\omega}_{ai}^{\times}\right)\left\{\begin{array}{l}
\boldsymbol{A}_{aib}[(\boldsymbol{T})_i \boldsymbol{E}]^{\mathrm{T}}\boldsymbol{m}^{i}_{R\omega} + \boldsymbol{m}^{i}_{\omega\omega}\boldsymbol{\omega}_{ai} + \boldsymbol{m}^{i}_{\omega f}\dot{\boldsymbol{q}}_i \\
+\boldsymbol{A}_{aib}\sum_{j=2}^{n}\left[\begin{array}{l}
\boldsymbol{m}^{j}_{RR}(\boldsymbol{T})_j \boldsymbol{E}(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} + \boldsymbol{m}^{j}_{R\omega}\boldsymbol{A}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}} \\
+\boldsymbol{m}^{j}_{Rf}\boldsymbol{A}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\omega}_{ai}}
\end{array}\right]
\end{array}\right\} = \boldsymbol{Q}_{air}
\end{aligned}
\tag{45}
$$

2.4.3.4 Vibration equations of the central body

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\boldsymbol{q}}_b} &= \boldsymbol{m}_{RR}^{b}(\boldsymbol{A}_{bi}\boldsymbol{v}_o - \boldsymbol{A}_{bo}\boldsymbol{R}_{ob}^{\times}\boldsymbol{\omega}_o + \boldsymbol{\phi}_b\dot{\boldsymbol{q}}_b)\boldsymbol{\phi}_b + \boldsymbol{m}_{R\omega}^{b}\boldsymbol{\omega}_b\boldsymbol{\phi}_b + \boldsymbol{m}_{Rf}^{b}\dot{\boldsymbol{q}}_b\boldsymbol{\phi}_b \\
&+(\boldsymbol{A}_{bi}\boldsymbol{v}_o - \boldsymbol{A}_{bo}\boldsymbol{R}_{ob}^{\times}\boldsymbol{\omega}_o + \boldsymbol{\phi}_b\dot{\boldsymbol{q}}_b)^{\mathrm{T}}\boldsymbol{m}_{Rf}^{b} + \boldsymbol{\omega}_b^{\mathrm{T}}\boldsymbol{m}_{\omega f}^{b} + \boldsymbol{m}_{ff}^{b}\dot{\boldsymbol{q}}_b \\
&+\sum_{i=2}^{n}\left[\boldsymbol{m}_{RR}^{i}(\boldsymbol{T})_i\boldsymbol{E}\boldsymbol{\phi}_b + \boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\boldsymbol{\omega}_{ai}\boldsymbol{\phi}_b + \boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{q}}_i\boldsymbol{\phi}_b\right]
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}_b}\right) &= \boldsymbol{m}_{RR}^{b}(\dot{\boldsymbol{A}}_{bi}\boldsymbol{v}_o + \boldsymbol{A}_{bi}\dot{\boldsymbol{v}}_o - \dot{\boldsymbol{A}}_{bo}\boldsymbol{R}_{ob}^{\times}\boldsymbol{\omega}_o - \boldsymbol{A}_{bo}\dot{\boldsymbol{R}}_{ob}^{\times}\boldsymbol{\omega}_o - \boldsymbol{A}_{bo}\boldsymbol{R}_{ob}^{\times}\dot{\boldsymbol{\omega}}_o + \boldsymbol{\phi}_b\ddot{\boldsymbol{q}}_b)\boldsymbol{\phi}_b \\
&+(\dot{\boldsymbol{A}}_{bi}\boldsymbol{v}_o + \boldsymbol{A}_{bi}\dot{\boldsymbol{v}}_o - \dot{\boldsymbol{A}}_{bo}\boldsymbol{R}_{ob}^{\times}\boldsymbol{\omega}_o - \boldsymbol{A}_{bo}\dot{\boldsymbol{R}}_{ob}^{\times}\boldsymbol{\omega}_o - \boldsymbol{A}_{bo}\boldsymbol{R}_{ob}^{\times}\dot{\boldsymbol{\omega}}_o + \boldsymbol{\phi}_b\ddot{\boldsymbol{q}}_b)^{\mathrm{T}}\boldsymbol{m}_{Rf}^{b} \\
&+\sum_{i=2}^{n}\left[\boldsymbol{m}_{RR}^{i}(\boldsymbol{T})_i\dot{\boldsymbol{E}}\boldsymbol{\phi}_b + \boldsymbol{m}_{R\omega}^{i}\dot{\boldsymbol{A}}_{bai}\boldsymbol{\omega}_{ai}\boldsymbol{\phi}_b + \boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{\omega}}_{ai}\boldsymbol{\phi}_b + \boldsymbol{m}_{Rf}^{i}\dot{\boldsymbol{A}}_{bai}\dot{\boldsymbol{q}}_i\boldsymbol{\phi}_b + \boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\ddot{\boldsymbol{q}}_i\boldsymbol{\phi}_b\right] \\
&+\boldsymbol{m}_{R\omega}^{b}\dot{\boldsymbol{\omega}}_b\boldsymbol{\phi}_b + \boldsymbol{m}_{Rf}^{b}\ddot{\boldsymbol{q}}_b\boldsymbol{\phi}_b + \dot{\boldsymbol{\omega}}_b^{\mathrm{T}}\boldsymbol{m}_{\omega f}^{b} + \boldsymbol{m}_{ff}^{b}\ddot{\boldsymbol{q}}_b
\end{aligned}
\tag{47}
$$

$$
\frac{d\Phi}{d\dot{\boldsymbol{q}}_b} = \boldsymbol{D}_b\dot{\boldsymbol{q}}_b \tag{48}
$$

$$
\frac{dL}{d\boldsymbol{q}_b} = -\boldsymbol{K}_b\boldsymbol{q}_b \tag{49}
$$

Substitute Eq. (46)-Eq. (49) into the fourth formula of Eq. (17), the vibration equations of the central body is obtained as follows:

$$
\begin{aligned}
&\boldsymbol{m}_{RR}^{b}(\dot{\boldsymbol{A}}_{bi}\boldsymbol{v}_o + \boldsymbol{A}_{bi}\dot{\boldsymbol{v}}_o - \dot{\boldsymbol{A}}_{bo}\boldsymbol{R}_{ob}^{\times}\boldsymbol{\omega}_o - \boldsymbol{A}_{bo}\dot{\boldsymbol{R}}_{ob}^{\times}\boldsymbol{\omega}_o - \boldsymbol{A}_{bo}\boldsymbol{R}_{ob}^{\times}\dot{\boldsymbol{\omega}}_o + \boldsymbol{\phi}_b\ddot{\boldsymbol{q}}_b)\boldsymbol{\phi}_b \\
&+(\dot{\boldsymbol{A}}_{bi}\boldsymbol{v}_o + \boldsymbol{A}_{bi}\dot{\boldsymbol{v}}_o - \dot{\boldsymbol{A}}_{bo}\boldsymbol{R}_{ob}^{\times}\boldsymbol{\omega}_o - \boldsymbol{A}_{bo}\dot{\boldsymbol{R}}_{ob}^{\times}\boldsymbol{\omega}_o - \boldsymbol{A}_{bo}\boldsymbol{R}_{ob}^{\times}\dot{\boldsymbol{\omega}}_o + \boldsymbol{\phi}_b\ddot{\boldsymbol{q}}_b)^{\mathrm{T}}\boldsymbol{m}_{Rf}^{b} \\
&+\sum_{i=2}^{n}\left[\boldsymbol{m}_{RR}^{i}(\boldsymbol{T})_i\dot{\boldsymbol{E}}\boldsymbol{\phi}_b + \boldsymbol{m}_{R\omega}^{i}\dot{\boldsymbol{A}}_{bai}\boldsymbol{\omega}_{ai}\boldsymbol{\phi}_b + \boldsymbol{m}_{R\omega}^{i}\boldsymbol{A}_{bai}\dot{\boldsymbol{\omega}}_{ai}\boldsymbol{\phi}_b + \boldsymbol{m}_{Rf}^{i}\dot{\boldsymbol{A}}_{bai}\dot{\boldsymbol{q}}_i\boldsymbol{\phi}_b + \boldsymbol{m}_{Rf}^{i}\boldsymbol{A}_{bai}\ddot{\boldsymbol{q}}_i\boldsymbol{\phi}_b\right] \\
&+\boldsymbol{m}_{R\omega}^{b}\dot{\boldsymbol{\omega}}_b\boldsymbol{\phi}_b + \boldsymbol{m}_{Rf}^{b}\ddot{\boldsymbol{q}}_b\boldsymbol{\phi}_b + \dot{\boldsymbol{\omega}}_b^{\mathrm{T}}\boldsymbol{m}_{\omega f}^{b} + \boldsymbol{m}_{ff}^{b}\ddot{\boldsymbol{q}}_b + \boldsymbol{D}_b\dot{\boldsymbol{q}}_b + \boldsymbol{K}_b\boldsymbol{q}_b = \boldsymbol{Q}_{vb}
\end{aligned}
\tag{50}
$$

2.4.3.5 Vibration equations of the flexible body i

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\boldsymbol{q}}_i} &= \boldsymbol{A}_{aib}\sum_{j=2}^{n}\left[\boldsymbol{m}_{RR}^{j}(\boldsymbol{T})_j\boldsymbol{E}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i} + \boldsymbol{m}_{R\omega}^{j}\boldsymbol{A}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i} + \boldsymbol{m}_{Rf}^{j}\boldsymbol{A}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}\right] \\
&+\boldsymbol{A}_{aib}[(\boldsymbol{T})_i\boldsymbol{E}]^{\mathrm{T}}\boldsymbol{m}_{Rf}^{i} + \boldsymbol{\omega}_{ai}^{\mathrm{T}}\boldsymbol{m}_{\omega f}^{i} + \boldsymbol{m}_{ff}^{i}\dot{\boldsymbol{q}}_i
\end{aligned}
\tag{51}
$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\right) = \dot{\boldsymbol{A}}_{aib}\sum_{j=2}^{n}\left[\boldsymbol{m}_{RR}^{j}(\boldsymbol{T})_j\boldsymbol{E}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}+\boldsymbol{m}_{R\omega}^{j}\boldsymbol{A}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}+\boldsymbol{m}_{Rf}^{j}\boldsymbol{A}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}\right]$$
$$+\boldsymbol{A}_{aib}\sum_{j=2}^{n}\left[\begin{array}{l}\boldsymbol{m}_{RR}^{j}(\boldsymbol{T})_j\dot{\boldsymbol{E}}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}+\boldsymbol{m}_{RR}^{j}(\boldsymbol{T})_j\boldsymbol{E}(\boldsymbol{T})_j\frac{d}{dt}\left(\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}\right)+\boldsymbol{m}_{R\omega}^{j}\dot{\boldsymbol{A}}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}\\+\boldsymbol{m}_{R\omega}^{j}\boldsymbol{A}_{baj}\dot{\boldsymbol{\omega}}_{aj}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}+\boldsymbol{m}_{R\omega}^{j}\boldsymbol{A}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j\frac{d}{dt}\left(\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}\right)+\boldsymbol{m}_{Rf}^{j}\dot{\boldsymbol{A}}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}\\+\boldsymbol{m}_{Rf}^{j}\boldsymbol{A}_{baj}\ddot{\boldsymbol{q}}_j(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}+\boldsymbol{m}_{Rf}^{j}\boldsymbol{A}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j\frac{d}{dt}\left(\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}\right)\end{array}\right] \tag{52}$$
$$+\dot{\boldsymbol{A}}_{aib}[(\boldsymbol{T})_i\boldsymbol{E}]^{\mathrm{T}}\boldsymbol{m}_{Rf}^{i}+\boldsymbol{A}_{aib}[(\boldsymbol{T})_i\dot{\boldsymbol{E}}]^{\mathrm{T}}\boldsymbol{m}_{Rf}^{i}+\dot{\boldsymbol{\omega}}_{ai}^{\mathrm{T}}\boldsymbol{m}_{\omega f}^{i}+\boldsymbol{m}_{ff}^{i}\ddot{\boldsymbol{q}}_i$$

$$\frac{d\Phi}{d\dot{\boldsymbol{q}}_i}=\boldsymbol{D}_i\dot{\boldsymbol{q}}_i \tag{53}$$

$$\frac{dL}{d\boldsymbol{q}_i}=-\boldsymbol{K}_i\boldsymbol{q}_i \tag{54}$$

Substitute Eq. (51)-Eq. (54) into the fifth formula of Eq. (17), the vibration equations of the flexible body i is obtained as follows:

$$\dot{\boldsymbol{A}}_{aib}\sum_{j=2}^{n}\left[\boldsymbol{m}_{RR}^{j}(\boldsymbol{T})_j\boldsymbol{E}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}+\boldsymbol{m}_{R\omega}^{j}\boldsymbol{A}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}+\boldsymbol{m}_{Rf}^{j}\boldsymbol{A}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}\right]$$
$$+\boldsymbol{A}_{aib}\sum_{j=2}^{n}\left[\begin{array}{l}\boldsymbol{m}_{RR}^{j}(\boldsymbol{T})_j\dot{\boldsymbol{E}}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}+\boldsymbol{m}_{RR}^{j}(\boldsymbol{T})_j\boldsymbol{E}(\boldsymbol{T})_j\frac{d}{dt}\left(\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}\right)+\boldsymbol{m}_{R\omega}^{j}\dot{\boldsymbol{A}}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}\\+\boldsymbol{m}_{R\omega}^{j}\boldsymbol{A}_{baj}\dot{\boldsymbol{\omega}}_{aj}(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}+\boldsymbol{m}_{R\omega}^{j}\boldsymbol{A}_{baj}\boldsymbol{\omega}_{aj}(\boldsymbol{T})_j\frac{d}{dt}\left(\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}\right)+\boldsymbol{m}_{Rf}^{j}\dot{\boldsymbol{A}}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}\\+\boldsymbol{m}_{Rf}^{j}\boldsymbol{A}_{baj}\ddot{\boldsymbol{q}}_j(\boldsymbol{T})_j\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}+\boldsymbol{m}_{Rf}^{j}\boldsymbol{A}_{baj}\dot{\boldsymbol{q}}_j(\boldsymbol{T})_j\frac{d}{dt}\left(\frac{\partial \boldsymbol{E}}{\partial \dot{\boldsymbol{q}}_i}\right)\end{array}\right] \tag{55}$$
$$+\dot{\boldsymbol{A}}_{aib}[(\boldsymbol{T})_i\boldsymbol{E}]^{\mathrm{T}}\boldsymbol{m}_{Rf}^{i}+\boldsymbol{A}_{aib}[(\boldsymbol{T})_i\dot{\boldsymbol{E}}]^{\mathrm{T}}\boldsymbol{m}_{Rf}^{i}+\dot{\boldsymbol{\omega}}_{ai}^{\mathrm{T}}\boldsymbol{m}_{\omega f}^{i}+\boldsymbol{m}_{ff}^{i}\ddot{\boldsymbol{q}}_i+\boldsymbol{D}_i\dot{\boldsymbol{q}}_i+\boldsymbol{K}_i\boldsymbol{q}_i=\boldsymbol{Q}_{vi}$$

2.4.4 Acquisition of generalized extraneous force

In order to acquire the explicit equations of the system dynamics, we need to express the generalized force in Lagrange equations (17) by actual force and moment. Assuming that the column array in $\boldsymbol{f}_b$ of actual force and moment on the central body respectively are $\boldsymbol{F}_b$ and $\boldsymbol{T}_b$, the column array in $\boldsymbol{f}_{ai}$ of the driving moment on the gemel is $\boldsymbol{T}_{ai}$, the column array in $\boldsymbol{f}_{ai}$ of the distributed force on the flexible appendages is $\boldsymbol{F}_{ai}$. Calculate the generalized extraneous force according to the priciple of virtual work, we firstly obtain the virtual displacement in $\boldsymbol{f}_b$ of an arbitrary volum differential element dV_{ai} in flexible body B_i as follows:

$$\begin{aligned}\delta\boldsymbol{R}_{ai} &= \boldsymbol{A}_{bi}\delta\boldsymbol{R}_b - (\boldsymbol{R}_{bai}^{\times} + \boldsymbol{A}_{bai}\boldsymbol{R}_{ai}^{\times}\boldsymbol{A}_{aib} + \boldsymbol{A}_{bai}\boldsymbol{u}_{ai}^{\times}\boldsymbol{A}_{aib})\delta\boldsymbol{\varphi}_b \\ &- \boldsymbol{A}_{bai}(\boldsymbol{R}_{ai}^{\times} + \boldsymbol{u}_{ai}^{\times})\delta\boldsymbol{\varphi}_{ai} + \boldsymbol{\phi}_b\delta\boldsymbol{q}_b + \boldsymbol{A}_{bak}\boldsymbol{\phi}_{ak}\delta\boldsymbol{q}_{ak}\end{aligned} \tag{56}$$

where $\boldsymbol{R}_{bai}$ is the component column array in $\boldsymbol{f}_b$ of the position vector from o_b to o_{ai}; $\boldsymbol{R}_{ai}$ is the component column array in $\boldsymbol{f}_{ai}$ of the position vector from o_{ai} when the flexible body B_i has not been deformed to the mass differential element dm_{ai}; $\boldsymbol{u}_{ai}$ is the component column array in $\boldsymbol{f}_{ak}$ of the elastic displacement vector of the mass differential element dm_{ai}; $\delta\boldsymbol{R}_b$, $\delta\boldsymbol{\varphi}_b$, $\delta\boldsymbol{\varphi}_{ai}$, $\delta\boldsymbol{q}_b$ and $\delta\boldsymbol{q}_{ai}$ are the virtual displacements of the generalized coordinates. The virtual work of the system expressed by actual force and moment can be written as:

$$\begin{aligned}\delta W &= \boldsymbol{F}_b^{\mathrm{T}}\boldsymbol{A}_{bi}\delta\boldsymbol{R}_b + \boldsymbol{T}_b^{\mathrm{T}}\delta\boldsymbol{\varphi}_b + \sum_{i=2}^{n}\boldsymbol{T}_{ai}^{\mathrm{T}}\delta\boldsymbol{\varphi}_{ai} + \boldsymbol{F}_b^{\mathrm{T}}\boldsymbol{\phi}_b\delta\boldsymbol{q}_b + \sum_{i=2}^{n}\int_{ai}\left(\boldsymbol{A}_{bai}\boldsymbol{F}_{ai}\right)^{\mathrm{T}}\delta\boldsymbol{R}_{ai}dV_{ai} \\ &= \boldsymbol{F}_b^{\mathrm{T}}\boldsymbol{A}_{bi}\delta\boldsymbol{R}_b + \boldsymbol{T}_b^{\mathrm{T}}\delta\boldsymbol{\varphi}_b + \sum_{i=2}^{n}\boldsymbol{T}_{ai}^{\mathrm{T}}\delta\boldsymbol{\varphi}_{ai} + \boldsymbol{F}_b^{\mathrm{T}}\boldsymbol{\phi}_b\delta\boldsymbol{q}_b + \sum_{i=2}^{n}\int_{ai}\left(\boldsymbol{A}_{bai}\boldsymbol{F}_{ai}\right)^{\mathrm{T}}[\boldsymbol{A}_{bi}\delta\boldsymbol{R}_b \\ &- (\boldsymbol{R}_{bai}^{\times} + \boldsymbol{A}_{bai}\boldsymbol{R}_{ai}^{\times}\boldsymbol{A}_{aib} + \boldsymbol{A}_{bai}\boldsymbol{u}_{ai}^{\times}\boldsymbol{A}_{aib})\delta\boldsymbol{\varphi}_b - \boldsymbol{A}_{bai}(\boldsymbol{R}_{ai}^{\times} + \boldsymbol{u}_{ai}^{\times})\delta\boldsymbol{\varphi}_{ai} + \boldsymbol{\phi}_b\delta\boldsymbol{q}_b + \boldsymbol{A}_{bai}\boldsymbol{\phi}_{ai}\delta\boldsymbol{q}_{ai}]dV_{ai} \\ &= \boldsymbol{Q}_{bt}^{\mathrm{T}}\delta\boldsymbol{R}_b + \boldsymbol{Q}_{br}^{\mathrm{T}}\delta\boldsymbol{\varphi}_b + \sum_{i=1}^{N}\boldsymbol{Q}_{air}^{\mathrm{T}}\delta\boldsymbol{\varphi}_{ak} + \boldsymbol{Q}_{vb}^{\mathrm{T}}\delta\boldsymbol{q}_b + \sum_{i=1}^{N}\boldsymbol{Q}_{vi}^{\mathrm{T}}\delta\boldsymbol{q}_{ak}\end{aligned} \tag{57}$$

From Eq. (57), we can obtain the expression of the generalized extraneous force as follows:

$$\boldsymbol{Q}_{bt} = \boldsymbol{A}_{ib}\boldsymbol{F}_b + \sum_{i=2}^{n}\int_{ai}\boldsymbol{A}_{iai}\boldsymbol{F}_{ai}dV_{ai} \tag{58}$$

$$\boldsymbol{Q}_{br} = \boldsymbol{T}_b + \sum_{i=2}^{n}\int_{ai}(\boldsymbol{R}_{bai}^{\times}\boldsymbol{A}_{bai} + \boldsymbol{A}_{bai}\boldsymbol{R}_{ai}^{\times} + \boldsymbol{A}_{bai}\boldsymbol{u}_{ai}^{\times})\boldsymbol{F}_{ai}dV_{ai} \tag{59}$$

$$\boldsymbol{Q}_{air} = \boldsymbol{T}_{ai} + \int_{ai}(\boldsymbol{R}_{ai}^{\times} + \boldsymbol{u}_{ai}^{\times})\boldsymbol{F}_{ai}dV_{ai} \tag{60}$$

$$\boldsymbol{Q}_{vb} = \boldsymbol{F}_b^{\mathrm{T}}\boldsymbol{\phi}_b + \sum_{i=2}^{n}\int_{ai}\boldsymbol{\phi}_b^{\mathrm{T}}\boldsymbol{F}_{ai}dV_{ai} \tag{61}$$

$$\boldsymbol{Q}_{vi} = \int_{ai}\boldsymbol{\phi}_{ai}^{\mathrm{T}}\boldsymbol{F}_{ai}dV_{ai} \tag{62}$$

3. Design and analysis of variable universe fractal fuzzy controller

After several decades of effort, we have achieved great success in terms of the research on the attitude control of flexible multi-body spacecraft. However, it still need to base on the precise mathematical model. On one hand, this kind of spacecraft has complicated dynamics characteristics including low rigidity, high flexibility, weak damping, low first order and intensive modal due to the launch weight limit and the configuration symmetry; On the other hand, it is difficult to establish the precise mathematical of flexible multi-body spacecraft. All about these factors challenge the classical and modern control theory which depends on precise mathematical model. However, the fuzzy control theory does not need the accurate model of system, which is suitable for the control problem of complicated large system. Nevertheless, the main disadvantages of general fuzzy control are the limited control accuracy and adaptive ability. Upon that, fuzzy control theory only has a few applications in astrospace fields.

Variable universe fuzzy control is a primary method for improving the performance of the fuzzy controller (Li, 1995). Input and output variables values change in rationally in the variable universe fuzzy control system. Adaptive variable universe fuzzy control problems have already been researched on (Si & Li, 2007). In that research real-time calculating shrinkage parameters are applied. However, real-time calculation of the shrinkage parameters will lead to the real-time shrinkage of the universe, consequently it can not constrain the future input signals by rules, which practicality requires future research.

Aiming at this problem, variable universe fractal fuzzy control method is introduced into the fuzzy control system, which could avoid the real-time calculating of the shrinkage parameters, make the contracted universe practical.

3.1 Attitude dynamics simulation model of spacecraft system

The dynamics equations of flexible multi-body spacecraft with topological tree configuration obtained in section 2 is strongly nonlinear. In this section we only research on the spacecraft attitude control problems. In order to design the attitude controller conveniently, we usually form such hypotheses as follows:

1. Consider the central body of large complicated configuration flexible spacecraft as rigid.
2. The central body coordinate system has its origin at the mass centre, so the displacement and velocity of the mass centre has little effect on attitude of spacecraft.
3. There is not any distributed control force on the flexible appendages usually in project.
4. The angular velocity of the central body, the angular velocity of the flexible appendages relative to the central body and the vibration velocity of the flexible appendages usually are very small, so we could ignore the high order nonlinear coupling item caused by them.

Through all above simplification, we obtain the finally spacecraft dynamics equations with uncertain moment of inertial as follows:

$$\begin{aligned} &\left(I+\Delta I\right)\dot{\omega}+\omega^{\times}\left[\left(I+\Delta I\right)\omega+C\dot{\eta}\right]+C\ddot{\eta}=u+w \\ &\ddot{\eta}+D\dot{\eta}+K\eta+C^{T}\dot{\omega}=0 \end{aligned} \tag{63}$$

where I is the moment of inertial matrix of spacecraft; ΔI is the uncertain increment of moment of inertial caused by rotation of the solar panel; C is the coupling coefficient between the central body and the flexible appendages; u is the control torque of three axes; η is the disturbing torque; is the flexible modal coordinate; $D = 2\xi\Lambda$, $K = \Lambda^2$, ξ is the modal damping coefficient matrix of flexible appendages, Λ is the modal frequency matrix of flexible appendages, assume that D and K are both positive.

In order to avoid the large angle singular problem caused by using Euler angle, we adopt Modified Rodrigues Parameters (Crassidis & Markley, 1996) to describe the spacecraft kinematics as follows:

$$\dot{p} = \frac{1}{4}\left\{\left(1 - p^T p\right) I_{3\times3} + 2\left(p^\times + pp^T\right)\right\}\omega = F\left(p\right)\omega \tag{64}$$

where $\omega = \left[\omega_1 \quad \omega_2 \quad \omega_3\right]^T$ is the angular velocity of spacecraft; $p = \left[p_1 \quad p_2 \quad p_3\right]^T$ is Modified Rodrigues Parameters of spacecraft relative to inertial space, $p^\times$ represent the skew-symmetric matrix of vector p .

From the flexible spacecraft dynamics and kinematics equations we know that the rigid attitude motion and the flexible vibration interact with each other. On one hand, extraneous force makes the attitude changed, at the same time, it also cause the flexible deformation. On the other hand, any deformation of the flexible body could cause the attitude angular changes. Otherwise, there is also some disturbing torque directly influence the rigid attitude motion, such as gravity gradient moments, atmosphere resistance moments, solar pressure moments and geomagnetic moments. Upon that, in order to ensure the attitude control accuracy, the designed controller is supposed to have the ability to suppress the disturbance efficiently and have the adaptation in the interaction between the rigid and flexible bodies.

3.2 Variable universe fractal attitude fuzzy controller

3.2.1 Variable universe fuzzy controller

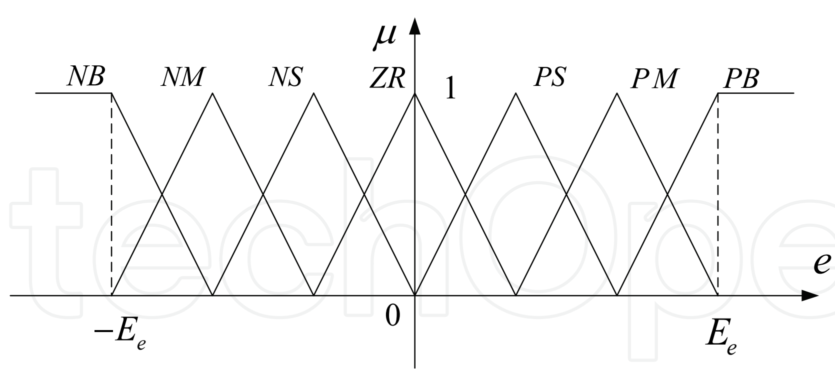


Fig. 3. Initial universe and fuzzy division

Variable universe ideology is proposed by Professor Li H. X. firstly. The control effect could be improved by changing the input and output universe values reasonably. Take the two inputs and one output fuzzy control system as an example. Assuming that the input variable is $\vec{e} = [e \quad \dot{e}]^{\mathrm{T}}$, which has the initial universe as $\left[-E_e, E_e\right]$ and $\left[-E_{ec}, E_{ec}\right]$, E_e, E_{ec} are real numbers. Usually we use seven rules to divide the universe as figure 3, variable universe means that the input universe $\vec{E} = (E_e, E_{ec})$ and the output universe U of the fuzzy controller could adjust reasonably with the changing of the input $\vec{e}$ and the output u. Among recent researches, this kind of universe adjustment is realised by using shrinkage parameters:

$$\begin{aligned} E_i(\vec{e}) &= [-\alpha_i(\vec{e})E_i^0, \alpha_i(\vec{e})E_i^0] \\ U(u) &= [-\beta(u)U_0, \beta(u)U_0] \end{aligned} \tag{65}$$

where $\alpha_i(\vec{e})$ and $\beta(u)$ are called shrinkage parameters. By the effect of the shrinkage parameters, the expansion and compression of the universe can be shown as figure 4:

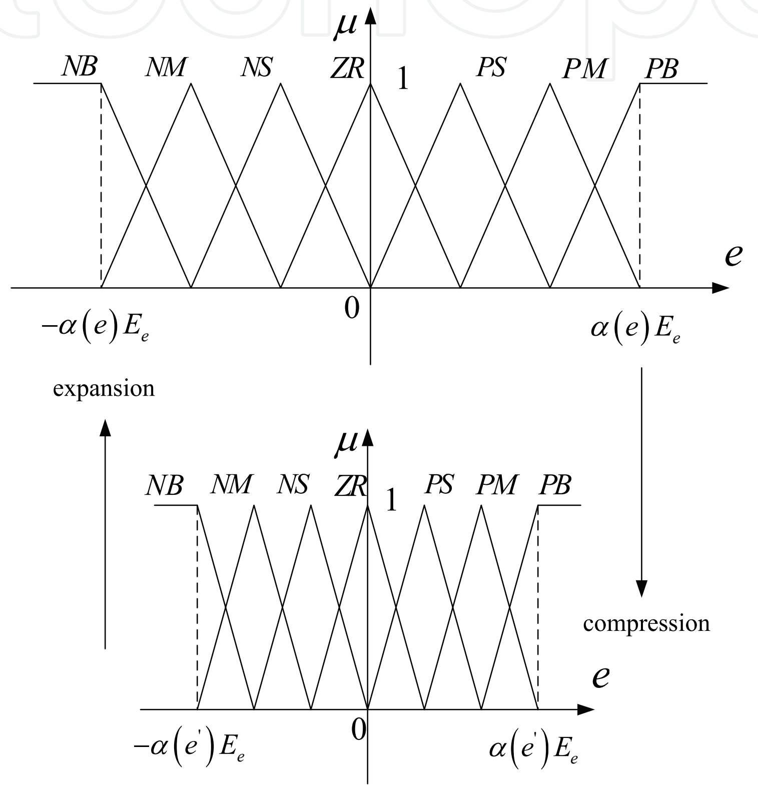


Fig. 4. The expansion and compression of the universe

Several common shrinkage parameters of variable universe control are given as follows:

1. Proportion index form

$$\begin{aligned} &\alpha(e) = (|e|/E_e)^\tau, (0<\tau<1) \\ &\beta(e,\dot{e}) = \left(\frac{|e|}{E_e}\right)^{\tau_1}\left(\frac{|\dot{e}|}{E_{ec}}\right)^{\tau_2} \text{ 或 } \beta(e,\dot{e}) = \frac{1}{2}\left[\left(\frac{|e|}{E_e}\right)^{\tau_1} + \left(\frac{|\dot{e}|}{E_{ec}}\right)^{\tau_2}\right], (0<\tau_1,\tau_2<1) \end{aligned} \tag{66}$$

2. Natural index form

$$\begin{aligned} \alpha(e) &= 1-\exp\left(-k_1 e^2\right) \\ \beta(\dot{e}) &= 1-\exp\left(-k_2 \dot{e}^2\right) \end{aligned} \tag{67}$$

3. Modified natural index form

$$\alpha(e) = 1 - c_1 \exp\left(-k_1 e^2\right), (0 < c_1 < 1)$$
$$\beta(\dot{e}) = 1 - c_2 \exp\left(-k_2 \dot{e}^2\right), (0 < c_2 < 1) \quad (68)$$

There is some research on the efficiency of several kinds of shrinkage parameters in the literature written by Pan X. F. The conclusion is that these several kinds of common shrinkage parameters can not improve the fuzzy controller performance efficiently. In order to avoid the real-time contracting problem of universe, we try to introduce the fractal control strategy into the variable universe fuzzy control system.

3.2.2 Fractal control strategy

A kind of fractal control strategy has been proposed by literature written by Xu J. B., which is formulated beforehand, and has only finite times fractal. Therefore, it can not obtain high interpolation accuracy.

The improved fractal control strategy in our research is proposed as follows:

First of all, define the initial universe of the inputs $e, \dot{e}$ and output u as $[-E_{e0}, E_{e0}]$, $[-E_{ec0}, E_{ec0}]$ and $[-U_0, U_0]$ according to experience. Through the process of the program operation, when $\dot{e} = 0$, namely when the error e achieves extreme value, system has one time fractal automatically, make the absolute value of present error e_1 as the present universe of input variable e, written as:

$$[-E_{e1}, E_{e1}] = [-|e_1|, |e_1|] \quad (69)$$

The universe of error derivation can be written as:

$$[-E_{ec1}, E_{ec1}] = [-\frac{c_{ec}|e_1|E_{ec0}}{E_{e0}}, \frac{c_{ec}|e_1|E_{ec0}}{E_{e0}}] \quad (70)$$

The universe of output variable can be written as:

$$[-U_1, U_1] = [-\frac{c_u|e_1|U_0}{E_{e0}}, \frac{c_u|e_1|U_0}{E_{e0}}] \quad (71)$$

The subscript 1 in above formulas represent the first time fractal, c_{ec} and c_u are adjustable design parameters. After the fractal the program continue running until $\dot{e} = 0$ happens again, system has the second time fractal. Write the present error as e_2, the universe of input and output variable respectively adjust as follows:

$$[-E_{e2}, E_{e2}] = [-|e_2|, |e_2|] \tag{72}$$

$$[-E_{ec2}, E_{ec2}] = [-\frac{c_{ec}|e_2|E_{ec1}}{E_{e1}}, \frac{c_{ec}|e_2|E_{ec1}}{E_{e1}}] \tag{73}$$

$$[-U_2, U_2] = [-\frac{c_u|e_2|U_1}{E_{e1}}, \frac{c_u|e_2|U_1}{E_{e1}}] \tag{74}$$

To this analogize, through the process of the program operation, system will has infinite times fractal until the accuracy satisfies the requirement.
This fractal strategy has advantages in following aspects:

1. The time to have a fractal and the expansion and compression scale of universe are determined according to the variation of input error value, not chosen beforehand by person, so it has a certain degree of universality to different control sysems.
2. Infinite times of fratal makes the distance between the interpolation points of the interpolator as the mathematical essence of fuzzy controller enough small, therefore the interpolation precision could meet an arbitrary given $\varepsilon > 0$, achieve the effect of dynamic pointwise convergent interpolator, suitable for any high precision control preblems.
3. The universe variation of the error derivation and the output variable are related to the universe expansion and compression of the error. Under the same control rules, multistage microform of the overall control information is realised, truly achieve the effect of combining the multistage coarse control with the fine control. Avoiding the complicated derivation of adaptive law and provement of stability in adaptive controllers, the stable high precision control with a certain degree of robustness can also be accomplished by using this approach.

3.3 Numerical simulation

In order to demonstrate the effectiveness of the proposed control strategy, we discuss a large angle attitude maneuver problem of a flexible multi-body spacecraft. We supposed that the spacecraft initial attitude is $p_0 = [0.020 \quad 0.322 \quad 0.288]^{\mathrm{T}}$, and the target attitude is $p_t = [0 \quad 0 \quad 0]^{\mathrm{T}}$. If the attitude is expressed by Euler angles with the transform order as 3-2-1, the initial roll is $\varphi_0 = 35^\circ$, the initial pitch is $\theta_0 = 60^\circ$, the initial yaw is $\psi_0 = 50^\circ$, the target attitude is $\varphi_t = \theta_t = \psi_t = 0^\circ$. The initial angular velocity is chosen as $\omega_0 = [0.03 \quad 0.02 \quad 0.04]^\circ / s$. The initial modal coordinate and its derivation are suppoosed to be zero. Spacefrat moment of inertial I, coupling coefficient matrix between the central body and the flexible appendages C, modal damping coefficient matrix of flexible appendages ξ, modal frequency matrix of flexible appendages Λ are given in Table 1 as below:

Moment of inertial I $(kg{\bullet}m^2)$	$\begin{bmatrix} 1070000 & -25000 & -1700 \\ -25000 & 29200 & -3100 \\ -1700 & -3100 & 1080000 \end{bmatrix}$
Coupling parameter C $(kg{\bullet}m^2)^{1/2}$	$\begin{bmatrix} 64.8 & 0 & -0.007 \\ 0 & -69.7 & 0 \\ -1.3 & -1.7 & -22.3 \end{bmatrix}$
Modal damping ratio ξ	$\begin{bmatrix} 0.046 & 0 & 0 \\ 0 & 0.031 & 0 \\ 0 & 0 & 0.019 \end{bmatrix}$
Modal frequency Λ (rad/s)	$\begin{bmatrix} 0.542 & 0 & 0 \\ 0 & 0.798 & 0 \\ 0 & 0 & 1.3 \end{bmatrix}$

Table 1. Related parameters in the spacecraft system simulation

The input and output of the controller both has 7 fuzzy subsets, respectively expressed by NB, NM, NS, ZR, PS, PM, PB. Because choosing linear or nonlinear membership functions has little influence on the fuzzy control effect, meanwhile using triangle has advantage of convenient and quick for calculating (Kruse, 1994), we adapt triangle membership function in our simulation, coincidence degree of neighbour fuzzy subsets is 0.5, Table 2 is the classical fuzzy control rule base used in the system.

input		Derivation of error						
		NB	NM	NS	ZR	PS	PM	PB
error	NB	PB	PB	PB	PB	PM	ZR	ZR
	NM	PB	PB	PB	PB	PM	ZR	ZR
	NS	PM	PM	PM	PM	ZR	NS	NS
	ZR	PM	PM	PS	ZR	NS	NM	NM
	PS	PS	PS	ZR	NM	NM	NM	NM
	PM	ZR	ZR	NM	NB	NB	NB	NB
	PB	ZR	ZR	NM	NB	NB	NB	NB

Table 2. Fuzzy control rule base

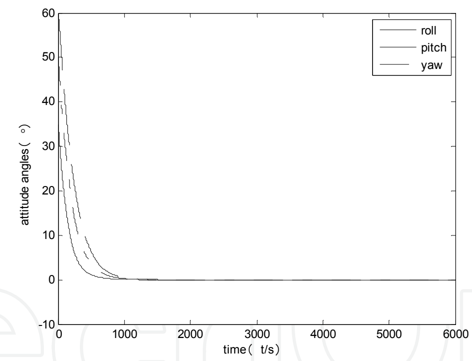
Simulation results are shown in follow figures, in order to make a comparison, the control effect of the fixed universe controller is given at the same time. Fig. 5 and Fig. 6 give the variation of the spacecraft attitude angles and modal coordinates of flexible appendages with time, and the control effect when the spacecraft moment of inertial increases 20 percent under the action of variable universe fractal fuzzy controller. Fig. 7 and Fig. 8 give the variation of the spacecraft attitude angles and modal coordinates of flexible appendages with time, and the control effect when the spacecraft moment of inertial increases 20 percent under the action of fixed universe fuzzy controller.

Take the actual attitude angles and attitude angular velocity of 2000-6000 seconds when the system has been stable as statistical data, calculate the 3σ value of attitude control accuracy and attitude stability under the action of two kinds of fuzzy controller both in the normal condition and in the condition when the moment of inertial increases 20 percent as shown in Table 3.

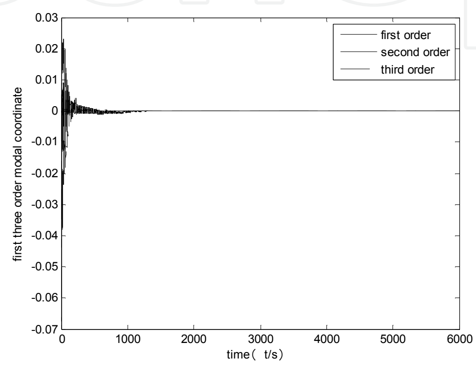
(3σ)		Variable universe fractal fuzzy control	General fuzzy control
I	e	$[0.0032 \quad 0.0404 \quad 0.0543]^{\circ}$	$[0.5309 \quad 0.0311 \quad 1.1964]^{\circ}$
	ec	$[0.0141 \quad 0.1335 \quad 0.0982] \times 10^{-4}\ {}^{\circ}/s$	$[1.5028 \quad 0.0553 \quad 1.2592] \times 10^{-3}\ {}^{\circ}/s$
	eta	$[0.1271 \quad 0.0399 \quad 0.0034] \times 10^{-4}$	$[0.4487 \quad 0.0458 \quad 0.0182] \times 10^{-4}$
II	e	$[0.0030 \quad 0.0476 \quad 0.0452]^{\circ}$	$[1.4163 \quad 0.0652 \quad 2.1949]^{\circ}$
	ec	$[0.0157 \quad 0.1210 \quad 0.0859] \times 10^{-4}\ {}^{\circ}/s$	$[7.1558 \quad 0.1637 \quad 3.0126] \times 10^{-3}\ {}^{\circ}/s$
	eta	$[0.0947 \quad 0.0502 \quad 0.0031] \times 10^{-4}$	$[1.1131 \quad 0.0647 \quad 0.0512] \times 10^{-4}$

Table 3. The control accuracy of the attitude angle, attitude angular velocity and vibration modal (I represent the normal condition, II represent the condition of moment of inertial increasing 20 percent)

From the simulation results, we know that the variable universe fractal fuzzy control has shorter dynamic adjusting time, faster response, smaller overshoot and higher static precision compared to the fixed universe fuzzy control. Meanwhile, it is insensitive to the variation of moment of inertial parameters, in other words, it has good robustness and adaptability to the model uncertainty of spacecraft. Besides, it could suppress the vibration of flexible appendages due to the attitude maneuver efficiently, then assure the attitude control precision of spacecraft.

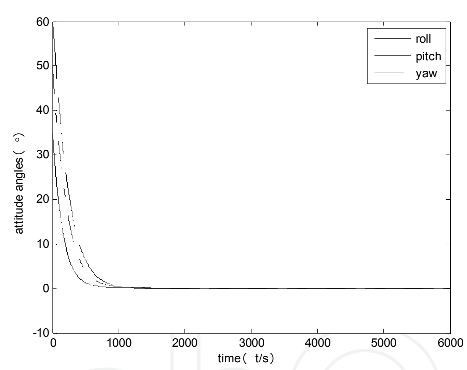


(a) Simulation results of the attitude angles

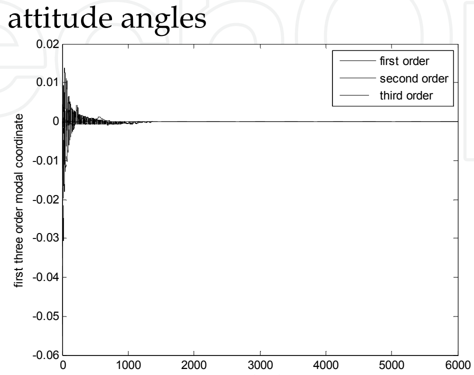


(b) Simulation results of the modal coordinates

Fig. 5. The control effect of the variable universe fractal controller

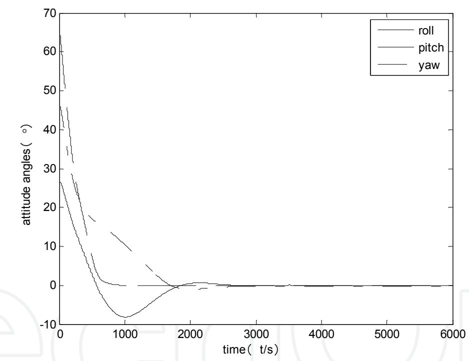


(a) Simulation results of the attitude angles

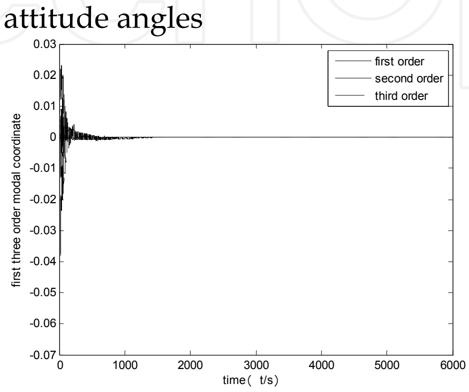


(b) Simulation results of the modal coordinates

Fig. 6. The control effect of the variable universe fractal controller when the moment of inertia of the spacecraft increases 20 percent

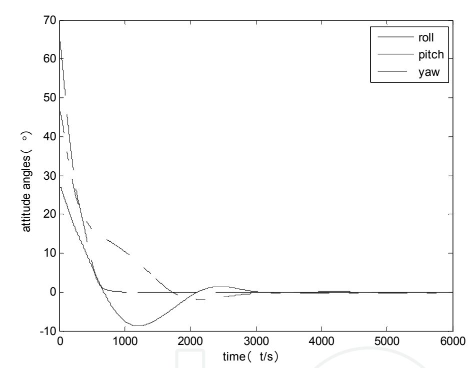


(a) Simulation results of the attitude angles

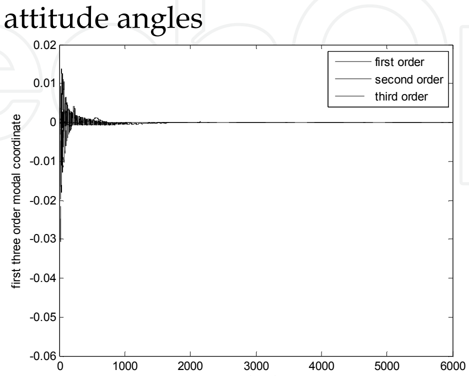


(b) Simulation results of the modal coordinates

Fig. 7. The control effect of the fixed universe controller



(a) Simulation results of the attitude angles



(b) Simulation results of the modal coordinates

Fig. 8. The control effect of the fixed universe controller when the moment of inertia of the spacecraft increases 20 percent

4. Conclusion

In this chapter, we research on the flexible multi-body spacecraft attitude dynamics and control problem. By using quasi-Lagrange equations, we have established the attitude dynamic equations of flexible multi-body spacecraft with topological tree configuration in section 2. The information of the system configuration has been introduced into the process of modelling, therefore the proposed attitude dynamic equations are universal and programmable. Then variable universe fractal fuzzy control method is developed in section3. The strongly nonlinear attitude dynamics equations are simplified under several reasonable hypothesizes. In order to avoid the disadvantage of using shrinkage factor, fractal control strategy is introduced into variable universe fuzzy control system. Finally we have demonstrated the effectiveness of the proposed control method through numerical simulation. The simulation results show that variable universe fractal fuzzy controller could accomplish the flexible multi-body spacecraft attitude control mission with good dynamic performance and high static precision.

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