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# Stability analysis of 2-d linear discrete feedback control systems with state delays on the basis of lagrange solutions

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## 1. Introduction

Researches on the two dimensional (2d) systems back to 1950s, when the main concern was the study of stability conditions for analog networked circuits (Levenstein, 1958; Ozaki & Kasami, 1960). Then, with the advent of new technologies and developments in the digital systems engineering as well as advances in the mathematical fields, this paradigm has evolved over the last decades into a major shift to discrete systems, addressing in addition to stability issues the control systems theory problems, which have ever since called the attention of mathematicians, digital signal processing community, control systems theorists and computer scientists among others.

These investigations on the stability and control of 2d systems can be gathered into basically two approaches: the multidimensional z-transform framework (Bose, 1982; Lim, 1990) and the energy method (see for example (Du & Xie, 2002) and references therein). The z-transform formalism has contributed greatly to the stability analysis of systems expressed in terms of the transfer function representation by providing a variety of stability methods as the well known Shanks stability criteria. Due to the fact that these techniques are useful instruments to checking the bounded input bounded output (BIBO) stability of system (Lim, 1990), this philosophy has been applied to systems described by their state space model representations, and as a result many stability conditions have been established in terms of the characteristic equations and eigenvalues (Fornasini & Marchesini, 1978), which have provided helpful tools for people in the systems engineering to establish control systems design methodologies (Kaczorek, 1985). On the other hand, unlike the z-transform, the energy method consists essentially in finding a Lyapunov function that expresses the energy of the system, and then showing that this energy vanishes as the equations indices increase. Thus, since the success of this method relies fundamentally in one's ability to formulate an adequate energy function, the role and the influence of the eigenvalues of the state space matrices are in many cases left uncovered. Incidentally, the discovering of a suitable function is also inherent in the stability and design procedures based on the linear matrix inequalities (LMI) approach, which is essentially a branch of the energy method (Boyd et al., 1994). Despite this point, LMI's based

techniques have led to hands on control design tools in a 'black box' fashion; and due to this fact, they have been intensively studied in the last decades.

Nevertheless there are quite a large number of published materials on these subjects, the kind of systems concerned there are primarily systems whose state space descriptions are partial difference equations depending only on the actual values, which means that none of the equations variables are functions of variables with indices less than the current values. These kinds of systems including past indices are called systems with delays or delayed systems, and unfortunately, due to the mathematical characteristics of their partial difference equations, which define the state space models, a generalization of the theories and techniques so far to this more general case is neither straightforward nor easy. The few recent reports focusing on these systems with delays and carried out on the grounds of the LMI formalism (Izuta, 2; 1; Pazke et al., 2004) have suggested interesting procedures focusing mainly on the control design issues.

Motivated by the facts described above, this paper is concerned with the stability analysis of 2-d discrete linear feedback control systems with delays. Thus, the state space model is composed by a matrix with current indices variables and another one with past indices variables. Moreover, the main goal here is to understand the conditions for this control system to be stable. In fact, to accomplish it, a feedback scheme is applied on the original discrete systems to yield a feedback control system with at least one of the matrices diagonal. Furthermore, rather than the resulting control system with a diagonal matrix of any values, two other systems are studied here. The first one is the system with the diagonal matrix of any values replaced by another diagonal matrix, but with all entries set to the maximum value in the original diagonal matrix. Similarly, the diagonal matrix of the other system that is considered has all entries set to the minimum value. These two systems are used to draw conclusions on whether the original system is asymptotically stable or not. For this, the similarity transformation is applied on these systems in order to transform the other non-diagonal matrix composing the system into either a diagonal or a Jordan type matrix. Once done, the Lagrange method comes into play here to render solutions to the set of partial difference equations expressing the systems transformed by means of the similarity transformation, and these solutions are used to study the stability conditions of the feedback control systems

The remainder of this paper is organized as follows. In section 2, the 2-d discrete linear systems with delay terms in the state space model, the controller used to turn one of the matrices of the model into a diagonal matrix, and the definitions are presented. The basic framework for solving the problem is introduced in section 3; and the results are given in section 4, which is split into four parts. Section 4.1 handles the case in which both system matrices are diagonal, and sections 4.2 through 4.3 are concerned with the systems with matrices of dimension  $2 \times 2$  whereas section 4.4 presents the stability conditions for general systems. Examples to illustrate how the suggested procedures work are given in section 5 and a few remarks are given in the last section, 6.

## **2. Problem Formulation**

In this section, the problem statement is formalized following the definition of 2-d control systems with delays terms in their state space models, and the concept of asymptotic stability which is closely related to the Lagrange solutions fulfilling the partial difference equations describing the state space models.

**Definition 1.** 2-d control systems with state delays are systems in which the state space models are described by the set of partial difference equations

$$\begin{aligned} \begin{bmatrix} x_h(i+1, j) \\ x_v(i, j+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_h(i, j) \\ x_v(i, j) \end{bmatrix} + \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} x_h(i-\theta, j) \\ x_v(i, j-\phi) \end{bmatrix} \\ &+ \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_h(i, j) \\ u_v(i, j) \end{bmatrix}, \\ \begin{bmatrix} y_h(i, j) \\ y_v(i, j) \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_h(i, j) \\ x_v(i, j) \end{bmatrix}, \end{aligned} \quad (1)$$

where the states vectors  $x_h \in \mathbb{R}^{n_h}$ ,  $x_v \in \mathbb{R}^{n_v}$  are such that the entries  $x'_h s$ ,  $x'_v s : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ; the inputs vectors  $u_h \in \mathbb{R}^{m_h}$ ,  $u_v \in \mathbb{R}^{m_v}$  have entries  $u'_h s$ ,  $u'_v s : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , and the outputs vectors  $y_h \in \mathbb{R}^{l_h}$ ,  $y_v \in \mathbb{R}^{l_v}$  are composed by  $y'_h s$ ,  $y'_v s : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ . Moreover,  $A_{pq}$ ,  $\bar{A}_{pq}$ ,  $B_{pq}$  and  $C_{pq}$ ,  $\forall p, q$ , are real valued matrices of adequate dimensions.

**Remark 1.** Nevertheless in 2-d systems the meaning of the word 'delays' referring to the components  $\theta$  and  $\phi$  is not necessarily related to the concept of time in the common sense, this terminology is adopted here in order to be consistent with the jargon used in the ordinary 1-d control systems theory.

In order to simplify the notations, the vectors and matrices are compactly written accordingly to the following definition.

**Definition 2.** Compact representations for the vectors and matrices are the notations

$$\begin{aligned} x(i \pm \hat{i}, j \pm \hat{j}) &= \begin{bmatrix} x_h(i \pm \hat{i}, j) \\ x_v(i, j \pm \hat{j}) \end{bmatrix}, & \begin{cases} \hat{i} = 1, 0, \theta \\ \hat{j} = 1, 0, \phi \end{cases} \\ u(i, j) &= \begin{bmatrix} u_h(i, j) \\ u_v(i, j) \end{bmatrix}, & y(i, j) = \begin{bmatrix} y_h(i, j) \\ y_v(i, j) \end{bmatrix}, \\ A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, & A_d = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \\ B &= \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, & C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}. \end{aligned} \quad (2)$$

**Remark 2.** In the sequel, when it is clear from the context and no confusion arises, the vectors and matrices will sometimes be expressed by not only their compact notations but also both the compact and the original ones will be used in a mixed fashion.

As far as the feedback control laws are concerned, the following schemes will be objects of study in this work.

**Definition 3.** A closed loop system is a feedback control system composed by (1) and the feedback law

$$u(i, j) = Kx(i, j) + K_d x(i - \theta, j - \phi), \quad (3)$$

which renders

$$x(i+1, j+1) = (A + BK)x(i, j) + (A_d + BK_d)x(i - \theta, j - \phi). \quad (4)$$

For the state feedback law

$$u(i, j) = Kx(i, j) + K_d x(i - \theta, j - \phi), \quad (5)$$

the system reads

$$x(i+1, j+1) = (A + BKC)x(i, j) + (A_d + BKC_d)x(i - \theta, j - \phi). \quad (6)$$

**Remark 3.** It is worth pointing out that, in practice, measurement limitations and restricted data storage capacity may force matrices  $K$  or  $K_d$  be null.

Next, the following concept of asymptotic stability, which relies on the solutions given by the Lagrange method, is adopted here.

**Definition 4.** A feedback control system is said to be asymptotically stable if the real valued Lagrange solutions  $x(i,j)$  given by the Lagrange method vanish as  $i, j$  tend to infinity (Jerri, 1996).

Taking these into account, the problem to be discussed is the following:

**Problem 1.** Let system (1) be such that its matrices have eigenvalues assigned at any desired points by means of the pole assignment techniques developed for 1-d control systems theory. Then, the question to be investigate here is "what are the conditions that the assigned eigenvalues have to fulfill in order to guarantee the asymptotic stability of the feedback control systems?".

The purpose here is to carry out a stability analysis by pursuing the Lagrange solutions of the partial difference equations defining the feedback control system. Hence, the controller design is basically settled by means of the assumption that pole assignment procedures can be used to place the eigenvalues of the feedback control system matrices at any points. Finally, pole assignment procedures developed for 1-d systems can be found for example in (Bachelier et al., 2006; Chen, 1999; Kailath, 1980; Syrmos et al., 1997).

### 3. Preliminaries

In this section, the basic framework for handling the problem is presented. Basically, the feedback control system is linearly transformed twice by means of the similarity transformations into a system with either diagonal matrices or a diagonal and Jordan matrices in its state space model description. The stability conditions are discussed on the basis of the Lagrange solutions of the transformed systems. Thus, the similarity transformation that is used in the sequel is provided in the following statement.

**Definition 5.** Consider system (4)-(6) and let  $J = \bar{T}T\bar{A}T^{-1}\bar{T}^{-1}$  ( $\bar{A} = A - BF$  or  $\bar{A} = A - BFC$ ), and  $y_{max}(i,j), y_{min}(i,j) = \bar{T}Tx(i,j)$ , in which  $T$  and  $\bar{T}$  are matrices composed by the eigenvectors of  $\bar{A}_d$  ( $\bar{A} = A - BF$  or  $\bar{A} = A - BFC$ ), and  $T\bar{A}T^{-1}$ , respectively. Furthermore, let  $\Lambda_{max} = \text{diag}\{\lambda_{max}, \dots, \lambda_{max}\}$  and  $\Lambda_{min} = \text{diag}\{\lambda_{min}, \dots, \lambda_{min}\}$  be diagonal matrices with maximum and minimum eigenvalues of  $\bar{A}_d$  as entries. Then, the systems obtained are the maximum and minimum doubly transformed systems and are given by

$$y(i+1, j+1) = J_y y(i, j) + \Lambda_{max} y(i-\theta, j-\phi) \quad (7)$$

and

$$y(i+1, j+1) = \bar{J}_y y(i, j) + \Lambda_{min} y(i-\theta, j-\phi), \quad (8)$$

where the  $J$ 's are Jordan matrices. Analogously, interchanging the roles of the matrices  $\bar{A}$  and  $\bar{A}_d$ , one arrives at

$$z(i+1, j+1) = \bar{\Lambda}_{max} z(i, j) + J_z z(i-\theta, j-\phi) \quad (9)$$

and

$$z(i+1, j+1) = \bar{\Lambda}_{min} z(i, j) + \bar{J}_z z(i-\theta, j-\phi). \quad (10)$$

Clearly if there are no constraints on the values of the assigned eigenvalues for at least one of the matrices of (4)-(6), then just set the all the eigenvalues of this matrix to the same value, and the stability conditions for the system can be established without being aware of the maximum and minimum eigenvalues cases. However, if it is necessary to assign eigenvalues of different values, then the asymptotic stability conditions are established by considering only these cases. To see that this procedure is in fact reasonable, focus, without loss of generality, on the simply similarity transformation of (4), which is given by

$$w(i+1, j+1) = \bar{A}w(i, j) + \Lambda w(i-\theta, j-\phi), \quad (11)$$

where  $\Lambda$  is a diagonal matrix composed by the eigenvalues of  $\bar{A}_d$ . It is easy to see that each single equation in (11) can be expressed as

$$w_1(i, j) = \sum_{k=1}^k a_{1k} w_k(i-1, j) + \lambda_1 w_1(i-\theta-1, j-\phi) \quad (12)$$

in which  $\lambda_1$  is an eigenvalue of  $\bar{A}_d$ .

Now, let us see what happens to  $w_1(i, j)$  if one replaces  $\lambda_1$  by  $\lambda_{max}$  or  $\lambda_{min}$  in (12) for the same values of  $w_k(i-1, j)$  and  $w_1(i-\theta-1, j-\phi)$ . On carrying out these operations, the following set of equations

$$\begin{aligned} w_1(i, j) &= (\text{constant value}) + \check{w}_1 w_1(i-\theta-1, j-\phi), \\ \hat{w}_1(i, j) &= (\text{constant value}) + \check{w}_{max} w_1(i-\theta-1, j-\phi), \\ \check{w}_1(i, j) &= (\text{constant value}) + \check{w}_{min} w_1(i-\theta-1, j-\phi), \end{aligned} \quad (13)$$

are yielded.

Clearly, these equations mean that the values of  $w_1(i, j)$  are in-between the ones of  $\hat{w}_1(i, j)$  and  $\check{w}_1(i, j)$ . In addition, the fact that the definition 2 of asymptotic stability adopted here is concerned only with the values of the solutions as the indices increase allows us to examine the behavior of (12) by using only the maximum and minimum eigenvalues.

Thus, due to the fact that the theory on the similarity transformation of systems (Gantmacher, 1959; Kawamata & Higuchi, 1995) guarantees that the original feedback control system in terms of the vector  $x(i, j)$  is stable if and only if either the simply similarity transformed system in terms of  $w(i, j)$  or doubly similarity transformed system  $y(i, j)$ 's ( $z(i, j)$ 's) is also stable, hereafter the subscripts *max* and *min* are dropped. The variable without the subscripts will implicitly mean that it is referred to both cases treated separately each time.

It is worth noting that in some cases, a singular similarity transformation will be enough to analyze the stability of the system. In what follows, no matter whether the systems are doubly or simply transformed, the feedback system in terms of the variables  $z(i, j)$ 's mean transformed systems.

In the sequel, in order to keep track of the overall picture of the work, the Lagrange solutions are determined only for the transformed systems. To see what the solutions for the original feedback control systems look like, for example in the doubly similarity transformation case, simply compute  $x(i, j)$  from the relation  $z(i, j) = \bar{T}Tx(i, j)$ .

Finally, a very useful notation from the combinatory mathematics is written down here for future use.

**Definition 6.** Let  $C_\theta(p, q)$  be a set of selections of  $q$  elements from the set  $\{\theta_1, \dots, \theta_p\}$  (for example,  $C_\theta(n, n) = \{\theta_1\theta_2 \dots \theta_n\}$ ). Then,  $S(p, q)$  is defined to be a set with same cardinality as  $C_\theta(p, q)$  equipped with elements that are the sum of the  $\theta$ 's constituting the elements of  $C_\theta(p, q)$  (for example,  $S_\theta(n, n) = \{\theta_1 + \theta_2 + \dots + \theta_n\}$ ). Moreover,  $S_{\theta_i}(p, q)$  stands for an element in  $S_\theta(p, q)$ .

## 4. Results

For the sake of clarity, the results are divided into four parts. The stability conditions for systems with both diagonal matrices of any dimensions in the state space representation are dealt with in the first section. These very simple and ideal systems allow us to figure out the basic computations procedures to pursue the results for general systems as well as to shed some light onto the relationships between the matrices eigenvalues and the stability of the systems. The following two sections present stability conditions in a more general framework in the sense that these basic ideas are extended to systems with  $2 \times 2$  matrices. In the last section, the previous results are further generalized to systems with matrices of any sizes.

### 4.1 State space models with both $n \times n$ matrices diagonal

This section gives the results for systems equipped with both matrices diagonal, which can be of any size greater than dimension  $2 \times 2$ . Let us firstly focus on the doubly similarity transformation of (4)-(6) yielding diagonal matrices, for which the following claim holds.

**Theorem 1.** *Let the doubly similarity transformation of (4)-(6) be*

$$\begin{bmatrix} z_h(i+1, j) \\ z_v(i, j+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_h(i, j) \\ z_v(i, j) \end{bmatrix} + \begin{bmatrix} \lambda_\theta & 0 \\ 0 & \lambda_\phi \end{bmatrix} \begin{bmatrix} z_h(i-\theta, j) \\ z_v(i, j-\phi) \end{bmatrix}, \quad (14)$$

for some given scalars  $\lambda_1, \lambda_2, \lambda_\theta, \lambda_\phi$ . Furthermore, let the Lagrange solutions to (14) be the expressions

$$\begin{aligned} z_h(i, j) &= \alpha^i, \quad \alpha \neq 0, \\ z_v(i, j) &= \beta^j, \quad \beta \neq 0. \end{aligned} \quad (15)$$

Then the asymptotic stability is guaranteed with  $\alpha$ 's,  $|\alpha| < 1$ , and  $\beta$ 's,  $|\beta| < 1$ , fulfilling the characteristic equations of the system described by

$$\begin{cases} \alpha^{\theta+1} - \lambda_1 \alpha^\theta - \lambda_\theta = 0 \\ \beta^{\phi+1} - \lambda_2 \beta^\phi - \lambda_\phi = 0 \end{cases} \quad (16)$$

*Proof.* Since (14) is a set of partial difference equations, in which the first one is a function of only index  $i$  whereas the second one depends only on  $j$ , it is natural to expect that the Lagrange solutions  $z_h(i, j)$  and  $z_v(i, j)$  are such that  $z_h(i, j) = z_h(i)$  and  $z_v(i, j) = z_v(j)$ , respectively. Thus, (15) are in fact candidate solutions to the transformed system (14).

Now, the substitution of (15) into (14) yields the set of partial difference equations described by

$$\begin{cases} \alpha^i (\alpha^{\theta+1} - \lambda_1 \alpha^\theta - \lambda_\theta) = 0, \\ \beta^j (\beta^{\phi+1} - \lambda_2 \beta^\phi - \lambda_\phi) = 0, \end{cases} \quad (17)$$

which means that the candidate solutions are indeed Lagrange solutions (14) if (16) is satisfied. On the other hand, it is not difficult to verify from (15) that, for given  $\lambda_1, \lambda_2, \lambda_\theta$  and  $\lambda_\phi$ , the asymptotic stability conditions for the feedback control system translate into the existences of  $\alpha$  with  $|\alpha| < 1$ , and  $\beta$  with  $|\beta| < 1$  as claimed.  $\square$

**Remark 4.** *The system (15) is in general reached only in special cases. Nevertheless, as pointed out in the previous section, the similarity transformation leads to (15) with either  $\lambda_1 = \lambda_2$  or  $\lambda_\theta = \lambda_\phi$ . Here*

these values are taken as different numbers in order to include all the cases. Thus, if  $\lambda_\theta = \lambda_\phi$  for given  $\lambda_1, \lambda_2, \lambda_\theta$  and  $\lambda_\phi$ , then from (16), the equality

$$\alpha^{\theta+1} - \lambda_1 \alpha^\theta = \beta^{\phi+1} - \lambda_2 \beta^\phi, \quad (18)$$

holds. Thus, the solutions  $\alpha$ 's and  $\beta$ 's to the characteristic equations are related by means of (18), which means that  $\beta$ 's are determined by  $\alpha$ 's, and vice-versa.

The above result means that the matrices of the feedback control system must be such that the eigenvalues  $\lambda_1, \lambda_2, \lambda_\theta$  and  $\lambda_\phi$  lead to characteristic equations (16) provided with real and norm less than unit polynomial roots.

It is also interesting to recall that researches on the 2-d systems with delays on the grounds of the Lyapunov methods (Izuta, 2) tend to handle these systems by separating into delay dependent and independent cases; each one with its specific methods for analyzing the stability. Here, since the 'delay terms'  $\theta$  and  $\phi$  turn to be the order of the characteristic polynomials, the splitting into delay dependent and independent cases is not a concern.

**Remark 5.** Note that since the Lagrange solutions are composed by the solutions of the characteristic equations, the number of solutions in terms of, for example,  $\alpha$  is equal to the degree of the polynomial representing the characteristic equation; however, for the sake of simplicity, equations in (16) refer loosely to only a single solution. Hence, when solving them one has to be aware that  $z_h(i, j)$  and  $z_v(i, j)$  are linear combinations of the solutions  $\alpha$ 's and  $\beta$ 's, respectively.

**Remark 6.** Although the initial values and boundary conditions problems play key roles in the studies of the solutions to the partial difference equations, this work concentrates only on the system stability problem and leave these issues to be discussed elsewhere.

Now, before making it clear the  $\lambda_1, \lambda_2, \lambda_\theta, \lambda_\phi$  that solve the problem, another way to interpret the solutions of (14) is introduced at this point in order to help us to understand the roles of  $\lambda$ 's in the characteristic equations.

**Theorem 2.** Consider the characteristic equation described by

$$\alpha^{\theta+1} - \lambda_1 \alpha^\theta - \lambda_\theta = 0, \quad (19)$$

and let the functions  $f(x_\alpha)$  and  $g(x_\alpha)$  be expressed as

$$\begin{aligned} f(x_\alpha) &= -\lambda_1 - x_\alpha, \\ g(x_\alpha) &= \frac{(-1)^\theta \lambda_\theta}{x_\alpha^\theta}, \end{aligned} \quad (20)$$

with a finite number of points fulfilling the equality  $f(x_\alpha) = g(x_\alpha)$ . Then, these points with opposite signals provide the set of solutions to (19).

*Proof.* Firstly, note that from basic polynomial algebra, equation (19) can be written as

$$(\alpha - \alpha_1) \cdots (\alpha - \alpha_{\theta+1}) = 0. \quad (21)$$



On the other hand, the combinatorial notation as stated in definition 6, allows one to express the coefficients of (19) with respect to the terms  $\alpha'_i$ s as

$$\left\{ \begin{array}{l} \alpha_1 + \sum_{p=2}^{\theta+1} \alpha_p = -\lambda_1 \\ \alpha_1 \sum_{p=2}^{\theta+1} \alpha_p + C_{\{\alpha_2, \dots, \alpha_{\theta+1}\}}^2 = 0 \\ \alpha_1 C_{\{\alpha_2, \dots, \alpha_{\theta+1}\}}^2 + C_{\{\alpha_2, \dots, \alpha_{\theta+1}\}}^3 = 0 \\ \vdots \\ \alpha_1 C_{\{\alpha_2, \dots, \alpha_{\theta+1}\}}^{\theta-1} + C_{\{\alpha_2, \dots, \alpha_{\theta+1}\}}^{\theta} = 0 \\ \alpha_1 C_{\{\alpha_2, \dots, \alpha_{\theta+1}\}}^{\theta} = -\lambda_{\theta} \end{array} \right. \quad (22)$$

From the last equation in (22),  $C_{\{\alpha_2, \dots, \alpha_{\theta+1}\}}^{\theta}$  can be determined in terms of  $\lambda_{\theta}$  and  $\alpha_1$ . Taking this value and substituting into the upper equation and continuing this computation process up to the second equation in (22), the following equations are obtained.

$$\left\{ \begin{array}{l} \sum_{p=2}^{\theta+1} \alpha_p = -\lambda_1 - \alpha_1, \\ \sum_{p=2}^{\theta+1} \alpha_p = \frac{(-1)^{\theta} \lambda_{\theta}}{\alpha_1^{\theta}}, \end{array} \right. \quad (23)$$

which mean that the set of solution to the problem, when exists, is composed by  $\theta + 1$  points that fulfill both the polynomials in (20) simultaneously. In addition, once  $\alpha_1$  is computed, the computation steps above are carried out  $\theta$  times to establish the remaining  $\alpha'$ s. However, due to the pattern of the polynomials, it turns out that the  $(\theta + 1)$ 's  $\alpha_1$  computed at the very beginning are the solutions to (19) unless the signal.  $\square$

**Remark 7.** Similar result can be established for the characteristic equation expressed in terms of  $\beta$ 's and, throughout the text, their solutions are written  $x_{\beta}$  to distinguish from the solutions  $x_{\alpha}$ 's relative to  $\alpha$ 's. However, as far as the stability of the system is concerned, the solutions to both characteristic equations (16) play indistinctly the same role, and must be analyzed individually.

Hence, taken into account these standpoints, theorem 1 can alternatively be rewritten making explicit requirements on  $\lambda_{\theta}$  and  $\lambda_{\phi}$ .

**Theorem 3.** The stability of the feedback control is guaranteed if and only if there exist  $\lambda_1, \lambda_2, \lambda_{\theta}$  ( $|\lambda_{\theta}| < 1$ ) and  $\lambda_{\phi}$  ( $|\lambda_{\phi}| < 1$ ) yielding solvable  $f(x_{\alpha}) = g(x_{\alpha})$  and  $f(x_{\beta}) = g(x_{\beta})$  as in (20), for which the solutions  $x_{\alpha}$ 's and  $x_{\beta}$ 's have non null absolute values less than unit.

*Proof.* Firstly, it is clear from (21) that asymptotic stability implies  $x_{\alpha}$ 's and  $x_{\beta}$ 's with non null absolute values less than unit. Hence, by means of the proof to theorem 2, the claim holds. On the other hand, beginning with non null and absolute values less than unit  $x_{\alpha}$ 's and  $x_{\beta}$ 's, the arguments of the same proof straightforwardly yield an asymptotically stable system.  $\square$

**Remark 8.** If the feedback control systems are such that they are devoided of delay components; i.e.,  $\theta = 0$  and  $\phi = 0$  then  $\lambda_{\theta} = 0, \lambda_{\phi} = 0$ . Furthermore, the equation (17) becomes

$$\alpha^{\theta}(\alpha - \lambda_1) = 0, \quad (24)$$

which means that  $\lambda_1$  and  $\lambda_2$  have to be less than unit to assure the asymptotic stability of the feedback control system.

Thus, to establish the values of  $\lambda_\theta$ ,  $\lambda_\phi$ ,  $\lambda_1$  and  $\lambda_2$  that provide a feasible solution to the problem, start out by setting  $\lambda_\theta$  ( $|\lambda_\theta| < 1$ ) and  $\lambda_\phi$  ( $|\lambda_\phi| < 1$ ) and then seek for  $\lambda_1$  and  $\lambda_2$  that leads to  $f(x_\alpha) = g(x_\alpha)$  and  $f(x_\beta) = g(x_\beta)$  with all the solutions with non null absolute values less than unit. Once, a solution is settled, apply the feedback laws in order to generate matrices with the above eigenvalues characteristics, and finally establish an asymptotically stable feedback control system.

Note that theorem 3 makes explicit allusion only to the possible values constraints that  $\lambda_\theta$  and  $\lambda_\phi$  have to bound, and there is no reference related to the values of  $\lambda_1$  and  $\lambda_2$  as far as they exist. Thus, it is interesting to characterize  $\lambda_1$  ( $\lambda_2$ ) in terms of  $\lambda_\theta$  ( $\lambda_\phi$ ) and some kind of constraints as  $|\lambda_1| < c$  ( $|\lambda_2| < c$ ) for a given constant positive number  $c$ .

**Proposition 1.** *Let the feedback control system be as in theorem 2. Then  $|\lambda_1| < c$  for  $c \in \Re > 0$  if*

$$|\lambda_\theta| < c(|x^\theta| - |x^{\theta+1}|). \quad (25)$$

*Proof.* Equation (25) can be arranged as

$$\frac{|\lambda_\theta|}{|x^\theta|} + |x| < c. \quad (26)$$

On applying the inequalities rules

$$\frac{|\lambda_\theta|}{|x^\theta|} + |x| \geq \left| \frac{\lambda_\theta}{x^\theta} + x \right| \quad (27)$$

holds. Consequently, the following inequality is valid.

$$\left| \frac{\lambda_\theta}{x^\theta} + x \right| < c. \quad (28)$$

On recalling equation (20), the expression

$$|-\lambda_1| = \left| \frac{(-1)^\theta \lambda_\theta}{x^\theta} + x \right| \quad (29)$$

comes up. Hence, comparing (29) with (28) and back tracking the calculations up to (26), the hypothesis is reached.  $\square$

#### 4.2 State space models with a single $2 \times 2$ diagonal matrix - case 1

In what follows, transformed systems with only one  $2 \times 2$  diagonal matrix are studied. Since the non diagonal matrix can be of any type, in general, the transformed system is likely to be the result of a single transformation.

**Lemma 1.** *Let the system transformed via similarity transformation be*

$$\begin{bmatrix} z_h(i+1, j) \\ z_v(i, j+1) \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} z_h(i, j) \\ z_v(i, j) \end{bmatrix} + \begin{bmatrix} \lambda_\theta & 0 \\ 0 & \lambda_\phi \end{bmatrix} \begin{bmatrix} z_h(i-\theta, j) \\ z_v(i, j-\phi) \end{bmatrix} \quad (30)$$

and its Lagrange candidate solutions be expressed by

$$\begin{aligned} z_h(i, j) &= \alpha^i \beta^j, \\ z_v(i, j) &= \gamma^i \delta^j, \\ \alpha, \beta, \gamma, \delta &\neq 0. \end{aligned} \quad (31)$$

Then (31) are solutions of (30) if

$$\beta^{\phi+1} - \lambda_\beta(\alpha) \beta^\phi - \lambda_\phi = 0 \quad (32)$$

is satisfied. Here  $\lambda_\beta(\alpha)$  is a polynomial in terms of variable  $\alpha$  given by

$$\begin{aligned} \lambda_\beta(\alpha) &= t_{22} \frac{\lambda_n(\alpha)}{\lambda_d(\alpha)}, \\ \lambda_n(\alpha) &= \alpha^{\theta+1} - \frac{\det(T)}{t_{22}} \alpha^\theta - \lambda_\theta, \\ \lambda_d(\alpha) &= \alpha^{\theta+1} - t_{11} \alpha^\theta - \lambda_\theta, \\ \det(T) &= \begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix}. \end{aligned} \quad (33)$$

*Proof.* On substituting (31) into (30), the following system of partial difference equations is yielded.

$$\begin{cases} \alpha^{i+1} \beta^j - t_{11} \alpha^i \beta^j - \lambda_\theta \alpha^{i-\theta} \beta^j = t_{12} \gamma^i \delta^j \\ \gamma^i \delta^{j+1} - t_{22} \gamma^i \delta^j - \lambda_\phi \gamma^i \delta^{j-\phi} = t_{21} \alpha^i \beta^j. \end{cases} \quad (34)$$

Thus, from the first equation in (34)

$$\gamma^i \delta^j = \frac{\alpha^{i+1} \beta^j - t_{11} \alpha^i \beta^j - \lambda_\theta \alpha^{i-\theta} \beta^j}{t_{12}} \quad (35)$$

is computed. Now, plugging (35) into the second equation in (34) produces

$$\beta^{\phi+1} - \lambda_\beta(\alpha) \beta^\phi - \lambda_\phi = 0, \quad (36)$$

in which  $\lambda_\beta(\alpha)$  is the fractional polynomial defined in (33). Hence, (31) are the solutions to the partial difference equations defining the transformed feedback control system as claimed.  $\square$

**Remark 9.** Note that (33) allows one to write (35) as

$$\gamma^i \delta^j = \frac{\lambda_n(\alpha)}{t_{12}} \alpha^{i+\theta} \beta^j, \quad (37)$$

which says that the solutions (31) are basically a function of the solutions  $\alpha$ 's and  $\beta$ 's. In addition, if  $\lambda_n(\alpha)$  is written as

$$\lambda_n(\alpha) = \lambda_d(\alpha) + \frac{t_{12} t_{21}}{t_{22}} \alpha^\theta \quad (38)$$

the roots of the polynomials  $\lambda_n(\alpha)$  and  $\lambda_d(\alpha)$  are distinct from each other as far as the roots are non null and the off diagonal entries of the matrix are non null.

On gathering all the details discussed so far, the following asymptotic stability conditions for systems with only one diagonal matrix are settled.

**Theorem 4.** Let the feedback control system transformed by means of the similarity transformation be as in lemma 1, and let  $\alpha_1, \dots, \alpha_{\theta+1}$  be the roots of the polynomial  $\lambda_n(\alpha)$  in (33). Then the system is asymptotically stable at  $\alpha_i$  ( $i, \dots, \theta + 1$ ) and  $\beta$ 's fulfilling

$$\begin{aligned} \beta^{\phi+1} &= \lambda_\phi, \\ |\lambda_\phi| &< 1. \end{aligned} \quad (39)$$

if there exists  $\lambda_\theta$  with  $|\lambda_\theta| < 1$  such that the absolute values of the roots of  $\lambda_n(\alpha)$  are all non null and less than unit.

*Proof.* The hypothesis implies that  $\lambda_\beta(\alpha) = 0$ . Hence, (33) reduces to (39), which is endowed with  $(\phi + 1)$  roots at  $\lambda_\phi$ . By imposing  $\lambda_\phi$  to assume values  $|\lambda_\phi| < 1$ , the solutions  $z_h(i, j)$  in (31) of the partial difference equations will vanish as the indices increase. On the other hand, (39) assures that  $z_v(i, j)$  in (31) decreases as the indices tend to infinity. Finally, the second equation in (33) and the last equation in (22) imply that a condition to have  $\alpha$  less than unit is the constraint  $|\lambda_\theta| < 1$ .  $\square$

**Remark 10.** If the non-diagonal matrix is triangular, then the solutions are quite much simpler. In fact, since  $\lambda_\beta(\alpha) = t_{22}$  holds, the solutions are functions of elements as described by  $z_h(i, j) = z_h(i, j, \beta)$ ,  $z_v(i, j) = z_v(i, j, \gamma, \delta)$  for lower triangular matrix case, and  $z_h(i, j) = z_h(i, j, \alpha, \beta)$ ,  $z_v(i, j) = z_v(i, j, \delta)$  for upper triangular matrix case.

If the non diagonal matrix of the transformed system is non singular, then from (33), the stability condition depends only on  $\lambda_\theta$  and  $\lambda_\phi$ .

Finally, it is interesting to note that the value of  $\det(T)$  is restricted by the values of  $\alpha_i$  ( $i = 1, \dots, \theta + 1$ ),  $\lambda_\theta$  and  $t_{22}$  as stated next.

**Corollary 1.** Let the system be as in theorem 4. Then

$$|\det(T)| \geq |t_{22}| \frac{|\alpha_i^{\theta+1}| - |\lambda_\theta|}{|\alpha_i^\theta|}, \quad \forall i, \quad (40)$$

holds.

*Proof.* It is settled straightforwardly by just applying the inequality rules on  $\lambda_n(\alpha_i)$  in (33).  $\square$

### 4.3 State space models with a single $2 \times 2$ diagonal matrix - case 2

This section parallels the previous one. The difference is that here the first matrix in (4)-(6) is a  $2 \times 2$  diagonal matrix, and the second one can be anything else. Thus, since the analogous reasoning applies here, the details are left out.

**Lemma 2.** Let the similarity transformed system be

$$\begin{bmatrix} z_h(i+1, j) \\ z_v(i, j+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_h(i, j) \\ z_v(i, j) \end{bmatrix} + \begin{bmatrix} \bar{t}_{11} & \bar{t}_{12} \\ \bar{t}_{21} & \bar{t}_{22} \end{bmatrix} \begin{bmatrix} z_h(i-\theta, j) \\ z_v(i, j-\phi) \end{bmatrix}. \quad (41)$$

and its Lagrange candidate solutions be expressed by

$$\begin{aligned} z_h(i, j) &= \alpha^i \beta^j, \\ z_v(i, j) &= \gamma^i \delta^j, \\ \alpha, \beta, \gamma, \delta &\neq 0. \end{aligned} \quad (42)$$

Then (42) are solutions of (41) if

$$\beta^{\phi+1} - \lambda_2 \beta^\phi - \lambda_\theta(\alpha) = 0 \quad (43)$$

is satisfied. Here  $\lambda_\theta(\alpha)$  is a polynomial in terms of variable  $\alpha$  given by

$$\begin{aligned} \lambda_\theta(\alpha) &= \bar{t}_{22} \frac{\lambda_n(\alpha)}{\lambda_d(\alpha)}, \\ \lambda_n(\alpha) &= \alpha^{\theta+1} - \lambda_1 \alpha^\theta + \frac{\det(\bar{T})}{\bar{t}_{22}}, \\ \lambda_d(\alpha) &= \alpha^{\theta+1} - \lambda_1 \alpha^\theta - \bar{t}_{11}, \\ \det(\bar{T}) &= \begin{vmatrix} \bar{t}_{11} & \bar{t}_{12} \\ \bar{t}_{21} & \bar{t}_{22} \end{vmatrix}. \end{aligned} \quad (44)$$

*Proof.* Note that the following system of partial difference equations are yielded by substituting (42) into (41).

$$\begin{cases} \alpha^{i+1} \beta^j = \lambda_1 \alpha^i \beta^j + \bar{t}_{11} \alpha^{i-\theta} \beta^j + \bar{t}_{12} \gamma^i \delta^{j-\phi} \\ \gamma^i \delta^{j+1} = \lambda_2 \gamma^i \delta^j + \bar{t}_{21} \alpha^{i-\theta} \beta^j + \bar{t}_{22} \gamma^i \delta^{j-\phi}. \end{cases} \quad (45)$$

Thus, by using the first equation in (41) and substituting  $\gamma^i \delta^j$  into the second equation renders

$$\beta^{\phi+1} - \lambda_2 \beta^\phi - \lambda_\theta(\alpha) = 0, \quad (46)$$

where  $\lambda_\theta(\alpha)$  is the fractional polynomial as defined in (44). Hence the claim follows.  $\square$

Now, the asymptotic stability conditions are as in the following theorem.

**Theorem 5.** Consider the similarity transformed system as in lemma 2 and let  $\alpha_1, \dots, \alpha_{\theta+1}$  be the roots of  $\lambda_n(\alpha)$ . Then the asymptotic stability at the points  $\alpha_i$  ( $i, \dots, \theta + 1$ ) for all  $\lambda_2$  ( $|\lambda_2| < 1$ ) is guaranteed as far as  $\frac{\det(\bar{A})}{\bar{a}_{22}} < 1$  and there exists  $\lambda_1$  such that the absolute values of the roots of  $\lambda_n(\alpha)$  are all non null and less than unit.

*Proof.* It basically parallels the reasoning of the proof to theorem 4.  $\square$

Theorem 5 can be stated without making it explicit the condition  $\frac{\det(\bar{A})}{\bar{a}_{22}} < 1$  as in the following paragraph.

**Theorem 6.** Consider the similarity transformed system as in lemma 2 and let  $\alpha_p$ , ( $p = 1 \dots, \theta + 1$ ) be the roots of  $\lambda_n(\alpha)$ , and  $\beta_q$  ( $q = 1 \dots, \phi + 1$ ) the roots of (46). Then the asymptotic stability at the points  $\alpha_p$  and  $\beta_q$  (for  $\forall p, q$ ) is guaranteed if there exist  $\lambda_1$  and  $\lambda_2$  such that the absolute values of the roots of  $\lambda_n(\alpha)$  are all non null and less than unit.

#### 4.4 State space models with a single $n \times n$ diagonal matrix

Let us begin by looking at the Lagrange solutions for a set of equations. As a matter of fact, these equations are sub-structures of the first type of systems transformed by means of similarity transformation that shall be considered hereafter.

**Lemma 3.** Consider the set of equations given by

$$\begin{cases} w_1(i+1, j) &= \lambda_1 w_1(i, j) + w_2(i, j) + \lambda w_1(i - \theta_1, j) \\ &\vdots \\ w_{n-1}(i+1, j) &= \lambda_1 w_{n-1}(i, j) + w_n(i, j) + \lambda w_{n-1}(i - \theta_{n-1}, j) \\ w_n(i+1, j) &= \lambda_1 w_n(i, j) + \lambda w_n(i - \theta_n, j) \end{cases} \quad (47)$$

or

$$\begin{cases} w_1(i+1, j) &= \lambda w_1(i, j) + \lambda_1 w_1(i - \theta_1, j) + w_2(i - \theta_2, j) \\ &\vdots \\ w_{n-1}(i+1, j) &= \lambda w_{n-1}(i, j) + \lambda_1 w_{n-1}(i - \theta_{n-1}, j) + w_n(i - \theta_n, j) \\ w_n(i+1, j) &= \lambda_1 w_n(i, j) + \lambda w_n(i - \theta_n, j). \end{cases} \quad (48)$$

Then, for given  $\lambda$  and  $\lambda_1$ , the Lagrange solutions

$$\begin{aligned} w_1(i, j) &= \alpha_1^i, \quad \dots, \quad w_n(i, j) = \alpha_n^i, \\ \alpha_i &\neq 0, \quad |\alpha_i| < 1 \end{aligned} \quad (49)$$

to (47) or (48) satisfy

$$\begin{aligned} &(-1)^{n+1}(\alpha_1 - \lambda_1)^n (\alpha_1^{S_{\theta_1}(n,n)}) + \\ &(-1)^n \lambda (\alpha_1 - \lambda_1)^{n-1} (\sum_{\text{over } i} \alpha_1^{S_{\theta_i}(n,n-1)}) + \\ &\quad \vdots \\ &(-1)^2 \lambda^{n-1} (\alpha_1 - \lambda_1) (\sum_{\text{over } i} \alpha_1^{S_{\theta_i}(n,1)}) + \\ &(-1)^1 \lambda^n \\ &= 0. \end{aligned} \quad (50)$$

*Proof.* The claim is shown by means of the mathematical induction on  $n$ . Due to the lengthy computations required to get the final result for large  $n$ , it is here presented only an outline of the operations. Firstly, consider the set of equations (47) with  $n = 2$ . Thus

$$\begin{aligned} \alpha_1^{i+1} &= \lambda_1 \alpha_1^i + \alpha_2 + \lambda \alpha_1^{i-\theta_1} \\ \alpha_2^{i+1} &= \lambda_1 \alpha_2^i + \lambda \alpha_2^{i-\theta_2} \end{aligned} \quad (51)$$

hold. Now, substituting the first equation in (51) into the second one leads to

$$\begin{aligned} &\lambda_1 (\alpha_1^{i+1} - \lambda_1 \alpha_1^i - \lambda \alpha_1^{i-\theta_1}) + \lambda \alpha_1^{i+1+\theta_2} - \lambda_1 \lambda \alpha_1^{i-\theta_2} \\ &- \lambda^2 \alpha_1^{i-\theta_1-\theta_2} - \alpha_1^{i+2} + \lambda_1 \alpha_1^{i+1} + \lambda \alpha_1^{i-\theta_1+1} = 0 \end{aligned} \quad (52)$$

and hence

$$-(\alpha_1 - \lambda_1)^2 \alpha_1^{\theta_1+\theta_2} + \lambda (\alpha_1 - \lambda) \alpha_1^{\theta_1-\theta_2} - \lambda^2 = 0, \quad (53)$$

which is in accordance with (50).

For the case  $n = 3$ , the following set of equations are obtained.

$$\begin{aligned} \alpha_2^i &= \alpha_1^{i+1} - \lambda_1 \alpha_1^i - \lambda \alpha_1^{i-\theta_1} \\ \alpha_3^i &= \alpha_2^{i+1} - \lambda_1 \alpha_2^i - \lambda \alpha_2^{i-\theta_2} \\ \lambda_1 \alpha_3^i + \lambda \alpha_3^{i-\theta_3} - \alpha_3^{i+1} &= 0. \end{aligned} \quad (54)$$

Now, on substituting the first equation in (54) into the second one and further substituting this result into the third equation give

$$(\alpha - \lambda_1)^3 \alpha^{\theta_1 + \theta_2 + \theta_3} - \lambda(\alpha - \lambda_1)^2 (\alpha^{\theta_1 + \theta_2} + \alpha^{\theta_1 + \theta_3} + \alpha^{\theta_2 + \theta_3}) + \lambda^2 (\alpha - \lambda_1) (\alpha^{\theta_1} + \alpha^{\theta_2} + \alpha^{\theta_3}) - \lambda^3 = 0. \quad (55)$$

Finally, continuing this process mechanically for higher values of  $n$ , clearly one establishes the claim of the theorem.  $\square$

**Remark 11.** Once  $\alpha_1$  is determined by means of (53),  $\alpha_2$  is computed by inserting  $\alpha_1$  into the first equation in (51); and this is the procedure to completely solve the set of difference equations.

In fact, the results collected in the following claim.

**Theorem 7.** Consider the system

$$\begin{bmatrix} w(i+1, j) \\ v(i, j+1) \end{bmatrix} = \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix} \begin{bmatrix} w(i, j) \\ v(i, j) \end{bmatrix} + \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} w(i-\theta, j) \\ v(i, j-\phi) \end{bmatrix}, \quad (56)$$

or

$$\begin{bmatrix} w(i+1, j) \\ v(i, j+1) \end{bmatrix} = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} w(i, j) \\ v(i, j) \end{bmatrix} + \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix} \begin{bmatrix} w(i-\theta, j) \\ v(i, j-\phi) \end{bmatrix} \quad (57)$$

with Jordan matrices  $J_{11}$  and  $J_{22}$  such that the vectors  $w(i, j)$  and  $v(i, j)$  are composed by equations as in (47) (or (48)). Then system (56) (or (57)) is asymptotically stable if and only if there exist Lagrange solutions

$$\begin{aligned} w_s(i, j) &= \alpha_s^i, & v_t(i, j) &= \beta_t^i, & \forall s, t. \\ 0 < |\alpha| < 1, & & 0 < |\beta| < 1, & & \forall \alpha, \beta. \end{aligned} \quad (58)$$

to the set of equations

$$\begin{aligned} &(-1)^{n+1} (\alpha_1 - \lambda_1)^n (\sum_{\text{over } i} \alpha_1^{S_{\theta i}(n, n)}) + \\ &\quad + \dots + \\ &(-1)^2 \lambda^{n-1} (\alpha_1 - \lambda_1) (\sum_{\text{over } i} \alpha_1^{S_{\theta i}(n, 1)}) + \\ &(-1)^1 \lambda^n \\ &= 0 \end{aligned} \quad (59)$$

and

$$\begin{aligned} &(-1)^{n+1} (\beta_1 - \lambda_1)^n (\sum_{\text{over } i} \beta_1^{S_{\phi i}(n, n)}) + \\ &\quad + \dots + \\ &(-1)^2 \lambda^{n-1} (\beta_1 - \lambda_1) (\sum_{\text{over } i} \beta_1^{S_{\phi i}(n, 1)}) + \\ &(-1)^1 \lambda^n \\ &= 0 \end{aligned} \quad (60)$$

for given  $\lambda_1, \lambda_2$  and  $\lambda$

*Proof.* It follows from lemma 3.  $\square$

Now, let us investigate a more general type of state space models, which have sub-structures of the following type.

**Lemma 4.** Consider the set of equations described by

$$\begin{cases} w_1(i+1, j) = \lambda_1 w_1(i, j) + w_2(i, j) + \lambda w_1(i - \theta_1, j) \\ \vdots \\ w_n(i+1, j) = \lambda_1 w_n(i, j) + v_1(i, j) + \lambda w_n(i - \theta_2, j) \\ v_1(i, j+1) = \lambda_1 v_1(i, j) + v_2(i, j) + \lambda v_1(i, j - \phi_1) \\ \vdots \\ v_m(i, j+1) = \lambda_1 v_m(i, j) + \lambda v_m(i, j - \phi_2) \end{cases} \quad (61)$$

or

$$\begin{cases} w_1(i+1, j) = \lambda w_1(i, j) + \lambda_1 w_1(i - \theta_1, j) + w_2(i - \theta_2, j) \\ \vdots \\ w_n(i+1, j) = \lambda w_n(i, j) + \lambda_1 w_n(i - \theta_n, j) + v_1(i, j - \phi_1) \\ v_1(i, j+1) = \lambda v_1(i, j) + \lambda_1 v_1(i, j - \phi_1) + v_2(i, j - \phi_2) \\ \vdots \\ v_m(i, j+1) = \lambda v_m(i, j) + \lambda_1 v_m(i, j - \phi_m). \end{cases} \quad (62)$$

Then the Lagrange solutions

$$\begin{aligned} w_1(i, j) &= \alpha_1^i \beta_1^j, & w_2(i, j) &= \alpha_2^i \beta_2^j, \\ v_1(i, j) &= \gamma_1^i \delta_1^j, & v_2(i, j) &= \gamma_2^i \delta_2^j. \end{aligned} \quad (63)$$

to either (61) or (63) satisfy

$$\begin{aligned} \mathcal{A}(\alpha, \lambda, \lambda_1) &= \\ &(-1)^n (\alpha_1 - \lambda_1)^n (\sum_{\text{over } i} \alpha_1^{S_{\theta_i}(n,n)}) + \\ &\quad \vdots \\ &(-1)^1 \lambda^{n-1} (\alpha_1 - \lambda_1) (\sum_{\text{over } i} \alpha_1^{S_{\theta_i}(n,1)}) + \\ &(-1) \lambda^n \\ &= 0, \quad \text{for } n > 1 \\ \mathcal{A}(\alpha, \lambda, \lambda_1) &= 1, \quad \text{for } n > 1 \end{aligned} \quad (64)$$

and

$$\begin{aligned} \mathcal{B}(\beta, \lambda, \lambda_1) &= \\ &(-1)^m (\beta_1 - \lambda_1)^m (\sum_{\text{over } i} \beta_1^{S_{\phi_i}(m,m)}) + \\ &\quad \vdots \\ &(-1)^1 \lambda^{m-1} (\beta_1 - \lambda_1) (\sum_{\text{over } i} \beta_1^{S_{\phi_i}(m,1)}) + \\ &(-1) \lambda^m \\ &= 0 \end{aligned} \quad (65)$$

which yield  $\alpha_1$ 's and  $\beta_1$ 's, and from which the other solutions are derived.



*Proof.* The result is obtained by means of the mathematical induction. As in the previous case, only a rough sketch of the computations is presented here.

Thus, in the very simple case for (61) with  $n = m = 1$ , the following system of difference equations

$$\begin{cases} \alpha^{i+1}\beta^j - \lambda_1\alpha^i\beta^j - \lambda\alpha^{i-\theta}\beta^j = \gamma^i\delta^j \\ \gamma^i\delta^{j+1} - \lambda_1\gamma^i\delta^j - \lambda\gamma^i\delta^{j-\phi} = 0 \end{cases} \quad (66)$$

render

$$\gamma^i\delta^j = \alpha^{i+1}\beta^j - \lambda_1\alpha^i\beta^j - \lambda\alpha^{i-\theta}\beta^j \quad (67)$$

and

$$\beta^{\phi+1} - \lambda_1\beta^\phi - \lambda = 0, \quad (68)$$

which give the assertion of the theorem.

Furthermore, for the case  $n = m = 2$ , the following set of equations holds.

$$\begin{aligned} \alpha_2^i\beta_2^j &= \alpha_1^{i+1}\beta_1^j - \lambda_1\alpha_1^i\beta_1^j - \lambda\alpha_1^{i-\theta_1}\beta_1^j, \\ \gamma_1^i\delta_1^j &= \alpha_2^{i+1}\beta_2^j - \lambda_1\alpha_2^i\beta_2^j - \lambda\alpha_2^{i-\theta_2}\beta_2^j, \\ \gamma_2^i\delta_2^j &= \gamma_1^i\delta_1^{j+1} - \lambda_1\gamma_1^i\delta_1^j - \lambda\gamma_1^i\delta_1^{j-\phi_1}, \\ \lambda_1\gamma_2^i\delta_2^j + \lambda\gamma_2^i\delta_2^{j-\phi_2} - \gamma_2^i\delta_2^{j+1} &= 0. \end{aligned} \quad (69)$$

Thus, the first two equations in (69) yield

$$\begin{aligned} \gamma_1^i\delta_1^j &= \alpha_2^{i+2}\beta_2^j - (\lambda + \lambda_1)\alpha_2^{i+1}\beta_2^j - \lambda\alpha_2^{i-\theta_1+1}\beta_2^j \\ &\quad + \lambda_1^2\alpha_2^i\beta_2^j + \lambda_1\lambda\alpha_2^{i-\theta_1}\beta_2^j - \lambda\alpha_2^{i-\theta_2+1}\beta_2^j \\ &\quad + \lambda\lambda_1\alpha_2^{i-\theta_2}\beta_2^j + \lambda^2\alpha_2^{i-\theta_1-\theta_2}\beta_2^j. \end{aligned} \quad (70)$$

Hence, on substituting this into the third equation in (69), and this result into the fourth equation in (69) produce

$$\begin{aligned} &\alpha_1^{\theta_1+\theta_1+2}\mathcal{B}(\beta_1, \lambda, \lambda_1) - \alpha_1^{\theta_1+\theta_2+1}\{\lambda\mathcal{B}(\beta_1, \lambda, \lambda_1) - \lambda_1\mathcal{B}(\beta_1, \lambda, \lambda_1)\} \\ &+ \alpha_1^{\theta_1+\theta_2}\lambda_1^2\mathcal{B}(\beta_1, \lambda, \lambda_1) - \alpha_1^{\theta_1+1}\lambda\mathcal{B}(\beta_1, \lambda, \lambda_1) + \alpha_1^{\theta_1}\lambda\mathcal{B}(\beta_1, \lambda, \lambda_1) \\ &- \alpha_1^{\theta_2+1}\lambda\mathcal{B}(\beta_1, \lambda, \lambda_1) + \alpha_1^{\theta_2}\lambda\mathcal{B}(\beta_1, \lambda, \lambda_1) + \lambda^2\mathcal{B}(\beta_1, \lambda, \lambda_1) \\ &= 0, \end{aligned} \quad (71)$$

which reduces to

$$\mathcal{A}(\alpha_1, \lambda, \lambda_1)\mathcal{B}(\beta_1, \lambda, \lambda_1) = 0, \quad (72)$$

with  $\mathcal{A}(\alpha_1, \lambda, \lambda_1)$  and  $\mathcal{B}(\beta_1, \lambda, \lambda_1)$  as stated in (64).  $\square$

Finally, on putting all the results so far together gives.

**Theorem 8.** Consider the system

$$\begin{bmatrix} w(i+1, j) \\ z(i+1, j+1) \\ v(i, j+1) \end{bmatrix} = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_{12} & 0 \\ 0 & 0 & J_2 \end{bmatrix} \begin{bmatrix} w(i, j) \\ z(i, j) \\ v(i, j) \end{bmatrix} + \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{bmatrix} \begin{bmatrix} w(i - \theta_w, j) \\ u(i - \theta_u, j - \phi_u) \\ v(i, j - \phi_v) \end{bmatrix}, \quad (73)$$

or

$$\begin{bmatrix} w(i+1, j) \\ z(i+1, j+1) \\ v(i, j+1) \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{bmatrix} \begin{bmatrix} w(i, j) \\ z(i, j) \\ v(i, j) \end{bmatrix} + \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_{12} & 0 \\ 0 & 0 & J_2 \end{bmatrix} \begin{bmatrix} w(i - \theta_w, j) \\ u(i - \theta_u, j - \phi_u) \\ v(i, j - \phi_v) \end{bmatrix}, \quad (74)$$

where  $w(i, j), z(i, j), v(i, j)$  are subsystems as in lemma 3 and 4;  $J_1, J_{12}$  and  $J_2$  are Jordan matrices with eigenvalues  $\lambda_1, \lambda_{12}$  and  $\lambda_2$  respectively, and  $\Lambda$  is a diagonal matrix. Then the system is asymptotically stable if and only if there exist non-null  $\lambda_*$  ( $\forall \star$ ),  $\lambda$  ( $|\lambda| < 1$ ), and  $\alpha$ 's ( $|\alpha| < 1$ ) such that the solutions to (64) and (65) are Lagrange solutions vanishing as the indices increase.

## 5. Illustrative Example

In this section, a simple example is presented to show how the procedure described so far works. For this purpose, consider the system model described by the following system of difference equations

$$\begin{bmatrix} x_1(i+1, j) \\ x_2(i+1, j) \\ x_3(i, j+1) \end{bmatrix} = \begin{bmatrix} 0.825 & 0.222 & 0.623 \\ -1.850 & -0.207 & -1.455 \\ 0.050 & -0.102 & 0.082 \end{bmatrix} \begin{bmatrix} x_1(i, j) \\ x_2(i, j) \\ x_3(i, j) \end{bmatrix} + \begin{bmatrix} 0.181 & -0.014 & -0.041 \\ -0.489 & 0.147 & 0.118 \\ 0.170 & 0.049 & 0.273 \end{bmatrix} \begin{bmatrix} x_1(i-1, j) \\ x_2(i-1, j) \\ x_3(i, j-1) \end{bmatrix}, \quad (75)$$

which is assumed, in order to focus only on the essence of the work, to be the feedback control system originated by the means of pole assignment method.

Thus, hereafter the aim is to check whether the feedback control system is asymptotically stable.

For this, note that that a matrix composed by the eigenvectors of the second matrix on the right hand side of (75) is given by

$$T = \begin{bmatrix} 1.000 & 0.100 & 0.400 \\ 1.000 & 0.200 & 0.100 \\ 0.200 & 0.300 & 1.000 \end{bmatrix}. \quad (76)$$

Thus, the similarity transformation of (75) by means of (76) leads to

$$\begin{bmatrix} y_1(i+1, j) \\ y_2(i+1, j) \\ y_3(i, j+1) \end{bmatrix} = \begin{bmatrix} 0.500 & 0.100 & 0.300 \\ 0.000 & 0.400 & 0.300 \\ 0.000 & -0.300 & -0.200 \end{bmatrix} \begin{bmatrix} y_1(i, j) \\ y_2(i, j) \\ y_3(i, j) \end{bmatrix} + \begin{bmatrix} 0.200 & 0.000 & 0.000 \\ 0.000 & 0.100 & 0.000 \\ 0.000 & 0.000 & 0.300 \end{bmatrix} \begin{bmatrix} y_1(i-1, j) \\ y_2(i-1, j) \\ y_3(i, j-1) \end{bmatrix}. \quad (77)$$

Now, since a matrix composed by the eigenvectors of the first matrix on the right hand side of (77) is given by

$$\bar{T} = \begin{bmatrix} 1.000 & 0.500 & 0.833 \\ 0.000 & 1.000 & 3.333 \\ 0.000 & -1.000 & 0.000 \end{bmatrix}, \quad (78)$$

apply the similarity transformation on (77), but considering the entries of the second matrix set all to the maximum singular values. Thus, the system turns into

$$\begin{bmatrix} z_1(i+1, j) \\ z_2(i+1, j) \\ z_3(i, j+1) \end{bmatrix} = \begin{bmatrix} 0.500 & 0.000 & 0.000 \\ 0.000 & 0.100 & 1.000 \\ 0.000 & 0.000 & 0.100 \end{bmatrix} \begin{bmatrix} z_1(i, j) \\ z_2(i, j) \\ z_3(i, j) \end{bmatrix} + \begin{bmatrix} 0.300 & 0.000 & 0.000 \\ 0.000 & 0.300 & 0.000 \\ 0.000 & 0.000 & 0.300 \end{bmatrix} \begin{bmatrix} z_1(i-1, j) \\ z_2(i-1, j) \\ z_3(i, j-1) \end{bmatrix}. \quad (79)$$

Thus, the first difference equation

$$z_1(i+1, j) - 0.5z_1(i, j) - 0.3z_1(i-1, j) = 0, \quad (80)$$

gives

$$z_1(i, j) \in \{(0.852)^i, (-0.352)^i\}. \quad (81)$$

On the other hand, the second and third vector terms in (79)

$$\begin{aligned} z_3(i, j) &= z_2(i+1, j) - 0.1z_1(i, j) - 0.3z_1(i-1, j), \\ z_3(i, j+1) - 0.1z_3(i, j) - 0.3z_3(i, j-1) &= 0 \end{aligned} \quad (82)$$

yield

$$\begin{aligned} z_2(i, j) &\in \{(-0.500)^i(-0.500)^j, (-0.500)^i(0.600)^j, \\ &(0.600)^i(-0.500)^j, (0.600)^i(0.600)^j\}, \end{aligned} \quad (83)$$

from which the solutions  $z_3(i, j)$  can be easily computed by using the first equation in (82). Finally, to complete the stability analysis, one should repeat the computations so far for system (79) with the first diagonal matrix replaced by a matrix with minimum value. However, due to the fact that the all diagonal entries are less than unit, let us to conclude that the system (75) is asymptotically stable.

## 6. Final Remarks

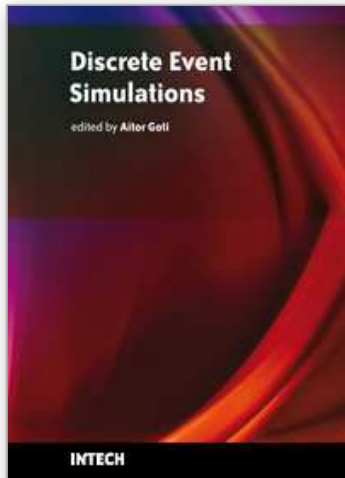
This work investigated indirectly the conditions for 2-d discrete control systems with delays to be asymptotically stable when interconnected by feedback control laws. The point key point is the stability analysis is accomplish on the basis of the doubly similarity approach. Moreover, unlike the related investigations so far, the analysis procedure is not split into delay dependent and independent cases, because the delay elements appear naturally as the degrees of the polynomials that one has to solve in order to obtain the solutions to the doubly transformed systems. Finally, an example was presented to show the procedures obtained.

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## **Discrete Event Simulations**

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Considered by many authors as a technique for modelling stochastic, dynamic and discretely evolving systems, this technique has gained widespread acceptance among the practitioners who want to represent and improve complex systems. Since DES is a technique applied in incredibly different areas, this book reflects many different points of view about DES, thus, all authors describe how it is understood and applied within their context of work, providing an extensive understanding of what DES is. It can be said that the name of the book itself reflects the plurality that these points of view represent. The book embraces a number of topics covering theory, methods and applications to a wide range of sectors and problem areas that have been categorised into five groups. As well as the previously explained variety of points of view concerning DES, there is one additional thing to remark about this book: its richness when talking about actual data or actual data based analysis. When most academic areas are lacking application cases, roughly the half part of the chapters included in this book deal with actual problems or at least are based on actual data. Thus, the editor firmly believes that this book will be interesting for both beginners and practitioners in the area of DES.

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