Delay-dependent exponential stability and filtering for time-delay stochastic systems with nonlinearities

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1. Introduction

It is well known that the time-delays are frequently encountered in a variety of dynamic systems such as engineering, biological, and chemical systems, etc., which are very often the main sources of instability and poor performance of systems. Also, in practice, uncertainties are unavoidable since it is very difficult to obtain an exact mathematical model of an object or process due to environmental noise, or slowly varying parameters, etc. Consequently, the problems of robust stability for time-delay systems have been of great importance and have received considerable attention for decades. The developed stability criteria are often classified into two categories according to their dependence on the size of the delays, namely, delay-independent criteria (Park, 2001) and delay-dependent criteria (Wang et al, 1992; Li et al, 1997; Kim, 2001; Moon et al, 2001; Jing et al, 2004; Kwon & Park, 2004; Wu et al, 2004). In general, the latter are less conservative than the former when the size of the time-delay is small. On the other hand, stochastic systems have received much attention since stochastic modelling has come to play an important role in many branches of science and industry. In the past decades, increasing attention has been devoted to the problems of stability of stochastic time-delay systems by a considerable number of researchers (Mao, 1996; Xie & Xie, 2000; Blythe et al, 2001; Xu & Chen, 2002; Lu et al, 2003). Very recently, the problem of exponential stability for delayed stochastic systems with nonlinearities has been extensively investigated by many researchers (Mao, 2002; Yue & Won, 2001; Chen et al, 2005). Motivated by the method for deterministic delayed systems introduced in (Wu et al, 2004), we extend it to uncertain stochastic time-varying delay systems with nonlinearities.
The filter design problem has long been one of the key problems in the areas of control and signal processing. Compared with the Kalman filter, the advantage of H_∞ filtering is that the noise sources are arbitrary signals with bounded energy or average power instead of being Gaussian, and no exact statistics are required to be known (Nagpal & Khargonekar, 1991). When parameter uncertainty appears in a system model, the robustness of H_∞ filters has to be taken into account. A great number of results on robust H_∞ filtering problem have been reported in the literature (Li & Fu, 1997; De Souza et al, 1993), and much attention has been

focused on the robust H∞ filtering problem for time-delay systems (Pila et al, 1999; Wang & Yang, 2002; Xu & Chen, 2004; Gao & Wang, 2003; Fridman et al, 2003; Xu & Van Dooren, 2002; Xu et al, 2003; Zhang et al, 2005; Wang et al, 2006; Wang et al, 2004; Wang et al, 2008; Liu et al, 2008; Zhang & Han, 2008). Depending on whether the existence conditions of filter include the information of delay or not, the existing results on H∞ filtering for time-delay systems can be classified into two types: delay-independent ones (Pila et al, 1999; Wang & Yang, 2002; Xu & Chen, 2004) and delay-dependent ones (Gao & Wang, 2003; Fridman et al, 2003; Xu & Van Dooren, 2002; Xu et al, 2003; Zhang et al, 2005; Wang et al, 2006; Wang et al, 2004; Wang et al, 2008; Liu et al, 2008; Zhang & Han, 2008). On the other hand, since the stochastic systems have gained growing interests recently, H∞ filtering for the time-delay stochastic systems have drawn a lot of attentions from researchers working in related areas (Zhang et al, 2005; Wang et al, 2006; Wang et al, 2008; Liu et al, 2008). It is also known that Markovian jump systems (MJSs) are a set of systems with transitions among the models governed by a Markov chain taking values in a finite set. These systems have the advantages of modeling the dynamic systems subject to abrupt variation in their structures. Therefore, filtering and control for MJSs have drawn much attention recently, see (Xu et al, 2003; Wang et al, 2004). Note that nonlinearities are often introduced in the form of nonlinear disturbances, and exogenous nonlinear disturbances may result from the linearization process of an originally highly nonlinear plant or may be an external nonlinear input, and thus exist in many real-world systems. Therefore, H∞ filtering for nonlinear systems has also been an attractive topic for many years both in the deterministic case (De Souza et al, 1993; Gao & Wang, 2003; Xu & Van Dooren, 2002)) and the stochastic case (Zhang et al, 2005; Wang et al, 2004; Wang et al, 2008; Liu et al, 2008).

Exponential stability is highly desired for filtering processes so that fast convergence and acceptable accuracy in terms of reasonable error covariance can be ensured. A filter is said to be exponential if the dynamics of the estimation error is stochastically exponentially stable. The design of exponential fast filters for linear and nonlinear stochastic systems is also an active research topic; see, e.g. (Wang et al, 2006; Wang et al, 2004). To the best of the authors' knowledge, however, up to now, the problem of delay-range-dependent robust exponential H∞ filtering problem for uncertain $It\hat{o}$ -type stochastic systems in the simultaneous presence of parameter uncertainties, Markovian switching, nonlinearities, and mode-dependent time-varying delays in a range has not been fully investigated, which still remains open and challenging. This motivates us to investigate the present study.

This chapter is organized as follows. In section 2, the main results are given. Firstly, delay-dependent exponentially mean-square stability for uncertain time-delay stochastic systems with nonlinearities is studied. Secondly, the robust H∞ exponential filtering problem for uncertain stochastic time-delay systems with Markovian switching and nonlinear disturbances is investigated. In section 3, numerical examples and simulations are presented to illustrate the benefits and effectiveness of our proposed theoretical results. Finally, the conclusions are given in section 4.

2. Main results

2.1 Exponential stability of uncertain time-delay nonlinear stochastic systems

Consider the following uncertain stochastic system with time-varying delay and nonlinear stochastic perturbations:

$$\begin{cases} dx(t) = \left[\left(A + \Delta A(t) \right) x(t) + \left(B + \Delta B(t) \right) x(t - \tau(t)) + f(t, x(t), x(t - \tau(t))) \right] dt + g(t, x(t), x(t - \tau(t))) d\omega(t), \\ x(t) = \phi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, A, B, C, D are known real constant matrices with appropriate dimensions, $\omega(t)$ is a scalar Brownian motion defined on a complete probability space (Ω, F, P) with a nature filtration $\{F_t\}_{t \geq 0}$. $\phi(t)$ is any given initial data in $L^2_{F_0}([-\tau, 0]; \mathbb{R}^n)$. $\tau(t)$ denotes the time-varying delay and is assumed to satisfy either (2a) or (2b):

$$0 \leq \tau(t) \leq \tau, \ \dot{\tau}(t) \leq d < 1, \tag{2a}$$

$$0 \leq \tau(t) \leq \tau, \tag{2b}$$

where τ and d are constants and the upper bound of $\tau(t)$ and $\dot{\tau}(t)$, respectively. $\Delta A(t)$, $\Delta B(t)$ are all unknown time-varying matrices with appropriate dimensions which represent the system uncertainty and stochastic perturbation uncertainty, respectively. We assume that the uncertainties are norm-bounded and can be described as follows:

$$\left[\Delta A(t) \quad \Delta B(t) \right] = EF(t) \left[G_1 \quad G_2 \right], \tag{3}$$

where E, G_1, G_2 are known real constant matrices with appropriate dimensions, $F(t)$ are unknown real matrices with Lebesgue measurable elements bounded by:

$$F^{\mathrm{T}}(t) F(t) \leq I. \tag{4}$$

$f(\cdot,\cdot,\cdot): R_+ \times R^n \times R^n \rightarrow R^n$ and $g(\cdot,\cdot,\cdot): R_+ \times R^n \times R^n \rightarrow R^{n \times m}$ denote the nonlinear uncertainties which is locally Lipschitz continuous and satisfies the following linear growth conditions

$$\| f(t, x(t), x(t - \tau(t))) \| \leq \| F_1 x(t) \| + \| F_2 x(t - \tau(t)) \|, \tag{5}$$

and

$$Trace\left[g^{\mathrm{T}}(t, x(t), x(t - \tau(t))) g(t, x(t), x(t - \tau(t))) \right] \leq \| H_1 x(t) \|^2 + \| H_2 x(t - \tau(t)) \|^2, \tag{6}$$

Throughout this paper, we shall use the following definition for the system (1).

Definition 1 (Chen et al, 2005**).** The uncertain nonlinear stochastic time-delay system (1) is said to be exponentially stable in the mean square sense if there exists a positive scalar $\alpha > 0$ such that for all admissible uncertainties

$$\lim_{t \to \infty} \sup \frac{1}{t} \log \mathrm{E} \| x(t) \|^2 \leq -\alpha. \tag{7}$$

Lemma 1 (Wang et al, 1992). For any vectors $x, y \in \mathbb{R}^n$, matrices $A, P \in \mathbb{R}^{n\times n}, D \in \mathbb{R}^{n\times n_f}$, $E \in \mathbb{R}^{n_f\times n}$, and $F \in \mathbb{R}^{n_f\times n_f}$ with $P>0, F^{\mathrm{T}}F \le \mathrm{I}$, and scalar $\varepsilon>0$, the following inequalities hold:

(i) $2x^{\mathrm{T}}y \le x^{\mathrm{T}}P^{-1}x + y^{\mathrm{T}}Py$,

(ii) $DFE + E^{\mathrm{T}}F^{\mathrm{T}}D^{\mathrm{T}} \le \varepsilon^{-1}DD^{\mathrm{T}} + \varepsilon E^{\mathrm{T}}E$,

(iii) If $P - \varepsilon DD^{\mathrm{T}} > 0$, then $(A+DFE)^{\mathrm{T}}P^{-1}(A+DFE) \le A^{\mathrm{T}}(P-\varepsilon DD^{\mathrm{T}})^{-1}A + \varepsilon E^{\mathrm{T}}E.$

For convenience, we let

$$y(t) = \left(A + \Delta A(t)\right)x(t) + \left(B + \Delta B(t)\right)x(t-\tau(t)) + f(t, x(t), x(t-\tau(t))), \tag{8}$$

and set

$$f(t) = f(t, x(t), x(t-\tau(t))), \quad g(t) = g(t, x(t), x(t-\tau(t))), \tag{9}$$

then system (1) becomes

$$dx(t) = y(t)dt + g(t)d\omega(t). \tag{10}$$

Then, for any appropriately dimensioned matrices $N_i, M_i, i = 1, 2, 3$, the following equations hold:

$$\Sigma_1 = 2\left[x^{\mathrm{T}}(t)N_1 + x^{\mathrm{T}}(t-\tau(t))N_2 + y^{\mathrm{T}}(t)N_3\right]\times\left[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} y(s)ds - \int_{t-\tau(t)}^{t} g(s)dw(s)\right] = 0, \tag{11}$$

and

$$\Sigma_2 = 2\left[x^{\mathrm{T}}(t)M_1 + x^{\mathrm{T}}(t-\tau(t))M_2 + y^{\mathrm{T}}(t)M_3\right]\times\left[\left(A + \Delta A(t)\right)x(t) + \left(B + \Delta B(t)\right)x(t-\tau(t)) + f(t) - y(t)\right] = 0, \tag{12}$$

where the free weighting matrices $N_i, M_i, i = 1, 2, 3$ can easily be determined by solving the corresponding LMIs.

On the other hand, for any semi-positive-definite matrix $X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ * & X_{22} & X_{23} \\ * & * & X_{33} \end{bmatrix} \ge 0$, the following holds:

$$\Sigma_3 = \tau\xi^{\mathrm{T}}(t)X\xi(t) - \int_{t-\tau(t)}^{t} \xi^{\mathrm{T}}(t)X\xi(t)ds \ge 0, \tag{13}$$

where $\xi^{\mathrm{T}}(t) = \left[x^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-\tau(t)) \quad y^{\mathrm{T}}(t)\right]$.

Theorem 1. When (2a) holds, then for any scalars $\tau > 0, d < 1$, the system (1) is exponentially stable in mean square for all time-varying delays and for all admissible uncertainties, if there exist $P>0, Q>0, R>0, S>0$, scalars $\rho > 0, \mu > 0$, $\varepsilon_j > 0, j = 0,1,\ldots,7$, a symmetric semi-positive-definite matrix $X \geq 0$ and any appropriately dimensioned matrices $M_i, N_i, i = 1, 2, 3$, such that the following LMIs hold

$$\Pi = \begin{bmatrix} X_{11} & X_{12} & X_{13} & N_1 \\ * & X_{22} & X_{23} & N_2 \\ * & * & X_{33} & N_3 \\ * & * & * & (1-d)Q \end{bmatrix} \geq 0, \tag{14}$$

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & N_1 & M_1 & M_1 & M_1E & M_1E & 0 & 0 & 0 & 0 \\ * & \Theta_{22} & \Theta_{23} & N_2 & M_2 & M_2 & 0 & 0 & M_2E & M_2E & 0 & 0 \\ * & * & \Theta_{33} & N_3 & M_3 & M_3 & 0 & 0 & 0 & 0 & M_3E & M_3E \\ * & * & * & -S & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_0 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_3 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon_4 I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\varepsilon_5 I & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\varepsilon_6 I & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -\varepsilon_7 I \end{bmatrix} < 0, \tag{15}$$

$$P \leq \rho I, \tag{16}$$

$$S \leq \mu I, \tag{17}$$

where

$\Theta_{11} = R + N_1 + N_1^{\mathrm{T}} + M_1 A + A^{\mathrm{T}} M_1^{\mathrm{T}} + \tau X_{11} + (\varepsilon_2 + \varepsilon_4 + \varepsilon_6) G_1^{\mathrm{T}} G_1 + \varepsilon_0 F_1^{\mathrm{T}} F_1 + (\rho + \frac{\tau\mu}{1-d}) H_1^{\mathrm{T}} H_1$,

$\Theta_{12} = -N_1 + N_2^{\mathrm{T}} + M_1 B + A^{\mathrm{T}} M_2^{\mathrm{T}} + \tau X_{12}$, $\quad \Theta_{13} = P + N_3^{\mathrm{T}} - M_1 + A^{\mathrm{T}} M_3^{\mathrm{T}} + \tau X_{13}$,

$\Theta_{22} = -(1-d)R - N_2 - N_2^{\mathrm{T}} + M_2 B + B^{\mathrm{T}} M_2^{\mathrm{T}} + \tau X_{22} + (\varepsilon_3 + \varepsilon_5 + \varepsilon_7) G_2^{\mathrm{T}} G_2 + \varepsilon_1 F_2^{\mathrm{T}} F_2 + (\rho + \frac{\tau\mu}{1-d}) H_2^{\mathrm{T}} H_2$,

$\Theta_{23} = -N_3^{\mathrm{T}} - M_2 + B^{\mathrm{T}} M_3^{\mathrm{T}} + \tau X_{23}$, $\quad \Theta_{33} = \tau Q - M_3 - M_3^{\mathrm{T}} + \tau X_{33}$.

Proof. Construct the Lyapunov-Krasovskii functional candidate for system (1) as follows:

$$V(t) = \sum_{i=1}^{4} V_i(t),$$

where

$$V_1(t) = x^{\mathrm{T}}(t) P x(t), \quad V_2(t) = \int_{-\tau(t)}^{0} \int_{t+\theta}^{t} y^{\mathrm{T}}(s) Q y(s) ds d\theta, \quad V_3(t) = \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s) R x(s) ds,$$
$$V_4(t) = \frac{1}{1-d} \int_{-\tau(t)}^{0} \int_{t+\beta}^{t} trace\left[g^{\mathrm{T}}(s) S g(s) \right] ds d\beta. \tag{18}$$

Defining x_t by $x_t(s) = x(t+s), -2\tau \le s \le 0$, the weak infinitesimal operator L of the stochastic process $\{x_t, t \ge 0\}$ along the evolution of $V_1(t)$ is given by (Blythe et al, 2001):

$$LV_1(t) = 2x^{\mathrm{T}}(t)Py(t) + trace\left[g^{\mathrm{T}}(t)Pg(t)\right]. \tag{19}$$

The weak infinitesimal operator L for the evolution of $V_2(t), V_3(t), V_4(t)$ can be computed directly as follows

$$LV_2(t) = \tau(t)y^{\mathrm{T}}(t)Qy(t) - (1-\dot{\tau}(t))\int_{t-\tau(t)}^{t} y^{\mathrm{T}}(s)Qy(s)ds, \tag{20}$$

$$LV_3(t) = x^{\mathrm{T}}(t)Rx(t) - (1-\dot{\tau}(t))x^{\mathrm{T}}(t-\tau(t))Rx(t-\tau(t)), \tag{21}$$

$$LV_4(t) = \frac{1}{1-d}\tau(t)trace\left[g^{\mathrm{T}}(t)Sg(t)\right] - \frac{1}{1-d}(1-\dot{\tau}(t))\int_{t-\tau(t)}^{t} trace\left[g^{\mathrm{T}}(s)Sg(s)\right]ds. \tag{22}$$

Therefore, using (2a) and adding Eqs. (11)-(13) to Eqs. (19)-(22), then the weak infinitesimal operator of $V(t)$ along the trajectory of system (1) yields

$$\begin{aligned} LV(t) \le\ & 2x^{\mathrm{T}}(t)Py(t) + trace\left[g^{\mathrm{T}}(t)Pg(t)\right] + \tau y^{\mathrm{T}}(t)Qy(t) - (1-d)\int_{t-\tau(t)}^{t} y^{\mathrm{T}}(s)Qy(s)ds + x^{\mathrm{T}}(t)Rx(t) \\ & -(1-d)x^{\mathrm{T}}(t-\tau(t))Rx(t-\tau(t)) + \frac{\tau}{1-d}trace\left[g^{\mathrm{T}}(t)Sg(t)\right] - \int_{t-\tau(t)}^{t} trace\left[g^{\mathrm{T}}(s)Sg(s)\right]ds + \Sigma_1 + \Sigma_2 + \Sigma_3 \end{aligned} \tag{23}$$

It follows from (i) of Lemma 1 that

$$\begin{aligned} & -2\left[x^{\mathrm{T}}(t)N_1 + x^{\mathrm{T}}(t-\tau(t))N_2 + y^{\mathrm{T}}(t)N_3\right]\int_{t-\tau(t)}^{t} g(s)d\omega(s) \\ & \le \xi^{\mathrm{T}}(t)NS^{-1}N^{\mathrm{T}}\xi(t) + \left(\int_{t-\tau(t)}^{t} g(s)d\omega(s)\right)^{\mathrm{T}} S\left(\int_{t-\tau(t)}^{t} g(s)d\omega(s)\right), \end{aligned} \tag{24}$$

where $N^{\mathrm{T}} = \left[N_1^{\mathrm{T}} \quad N_2^{\mathrm{T}} \quad N_3^{\mathrm{T}}\right]$.

Moreover, from Lemma 1 and (5)

$$\begin{aligned} & 2\left[x^{\mathrm{T}}(t)M_1 + x^{\mathrm{T}}(t-\tau(t))M_2 + y^{\mathrm{T}}(t)M_3\right]f(t) \\ & \le \xi^{\mathrm{T}}(t)(\varepsilon_0^{-1} + \varepsilon_1^{-1})MM^{\mathrm{T}}\xi(t) + x^{\mathrm{T}}(t)\varepsilon_0 F_1 F_1^{\mathrm{T}}x(t) + x^{\mathrm{T}}(t-\tau(t))\varepsilon_1 F_2 F_2^{\mathrm{T}}x(t-\tau(t)), \end{aligned} \tag{25}$$

where $M^{\mathrm{T}} = \left[M_1^{\mathrm{T}} \quad M_2^{\mathrm{T}} \quad M_3^{\mathrm{T}}\right]$.

Taking note of (6) together with (16) and (17) imply

$$trace[g^{\mathrm{T}}(t)Pg(t)]+\frac{\tau}{1-d}trace[g^{\mathrm{T}}(t)Sg(t)]\le(\rho+\frac{\tau\mu}{1-d})[x^{\mathrm{T}}(t)H_1H_1^{\mathrm{T}}x(t)+x^{\mathrm{T}}(t-\tau(t))H_2H_2^{\mathrm{T}}x(t-\tau(t))]. \tag{26}$$

Noting that

$$\mathrm{E}\left(\int_{t-\tau(t)}^{t}g(s)d\omega(s)\right)^{\mathrm{T}}S\left(\int_{t-\tau(t)}^{t}g(s)d\omega(s)\right)=\mathrm{E}\int_{t-\tau(t)}^{t}trace\left[g^{\mathrm{T}}(s)Sg(s)\right]ds. \tag{27}$$

For the positive scalars $\varepsilon_k>0, k=2,3,\ldots,7$, it follows from (3), (4) and Lemma 1 that

$$2x^{\mathrm{T}}(t)M_1\Delta A(t)x(t)\le\varepsilon_2^{-1}x^{\mathrm{T}}(t)M_1EE^{\mathrm{T}}M_1^{\mathrm{T}}x(t)+\varepsilon_2x^{\mathrm{T}}(t)G_1^{\mathrm{T}}G_1x(t), \tag{28}$$

$$2x^{\mathrm{T}}(t)M_1\Delta B(t)x(t-\tau(t))\le\varepsilon_3^{-1}x^{\mathrm{T}}(t)M_1EE^{\mathrm{T}}M_1^{\mathrm{T}}x(t)+\varepsilon_3x^{\mathrm{T}}(t-\tau(t))G_2^{\mathrm{T}}G_2x(t-\tau(t)), \tag{29}$$

$$2x^{\mathrm{T}}(t-\tau(t))M_2\Delta A(t)x(t)\le\varepsilon_4^{-1}x^{\mathrm{T}}(t-\tau(t))M_2EE^{\mathrm{T}}M_2^{\mathrm{T}}x(t-\tau(t))+\varepsilon_4x^{\mathrm{T}}(t)G_1^{\mathrm{T}}G_1x(t), \tag{30}$$

$$2x^{\mathrm{T}}(t-\tau(t))M_2\Delta B(t)x(t-\tau(t))\le\varepsilon_5^{-1}x^{\mathrm{T}}(t-\tau(t))M_2EE^{\mathrm{T}}M_2^{\mathrm{T}}x(t-\tau(t))+\varepsilon_5x^{\mathrm{T}}(t-\tau(t))G_2^{\mathrm{T}}G_2x(t-\tau(t)), \tag{31}$$

$$2y^{\mathrm{T}}(t)M_3\Delta A(t)x(t)\le\varepsilon_6^{-1}y^{\mathrm{T}}(t)M_3EE^{\mathrm{T}}M_3^{\mathrm{T}}y(t)+\varepsilon_6x^{\mathrm{T}}(t)G_1^{\mathrm{T}}G_1x(t), \tag{32}$$

$$2y^{\mathrm{T}}(t)M_3\Delta B(t)x(t-\tau(t))\le\varepsilon_7^{-1}y^{\mathrm{T}}(t)M_3EE^{\mathrm{T}}M_3^{\mathrm{T}}y(t)+\varepsilon_7x^{\mathrm{T}}(t-\tau(t))G_2^{\mathrm{T}}G_2x(t-\tau(t)). \tag{33}$$

Then, taking the mathematical expectation of both sides of (23) and combining (24)-(27) with (28)-(33), it can be concluded that

$$E\{LV(t)\}\le E\left\{\xi^{\mathrm{T}}(t)\Xi\xi(t)\right\}-\int_{t-\tau(t)}^{t}E\left\{\xi^{\mathrm{T}}(t,s)\Pi\xi(t,s)\right\}ds, \tag{34}$$

where

$$\xi^{\mathrm{T}}(t,s)=\begin{bmatrix}x^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-\tau(t)) & y^{\mathrm{T}}(t) & y^{\mathrm{T}}(s)\end{bmatrix},\quad \Xi=\begin{bmatrix}\Xi_{11} & \Xi_{12} & \Xi_{13}\\ * & \Xi_{22} & \Xi_{23}\\ * & * & \Xi_{33}\end{bmatrix},$$

$$\Xi_{11}=\Theta_{11}+N_1S^{-1}N_1^{\mathrm{T}}+(\varepsilon_0^{-1}+\varepsilon_1^{-1})M_1M_1^{\mathrm{T}}+(\varepsilon_2^{-1}+\varepsilon_3^{-1})M_1EE^{\mathrm{T}}M_1^{\mathrm{T}},$$

$$\Xi_{12}=\Theta_{12}+N_1S^{-1}N_2^{\mathrm{T}}+(\varepsilon_0^{-1}+\varepsilon_1^{-1})M_1M_2^{\mathrm{T}},\quad \Xi_{13}=\Theta_{13}+N_1S^{-1}N_3^{\mathrm{T}}+(\varepsilon_0^{-1}+\varepsilon_1^{-1})M_1M_3^{\mathrm{T}},$$

$$\Xi_{22}=\Theta_{22}+N_2S^{-1}N_2^{\mathrm{T}}+(\varepsilon_0^{-1}+\varepsilon_1^{-1})M_2M_2^{\mathrm{T}}+(\varepsilon_4^{-1}+\varepsilon_5^{-1})M_2EE^{\mathrm{T}}M_2^{\mathrm{T}},$$

$$\Xi_{23}=\Theta_{23}+N_2S^{-1}N_3^{\mathrm{T}}+(\varepsilon_0^{-1}+\varepsilon_1^{-1})M_2M_3^{\mathrm{T}},$$

$$\Xi_{33}=\Theta_{33}+N_3S^{-1}N_3^{\mathrm{T}}+(\varepsilon_0^{-1}+\varepsilon_1^{-1})M_3M_3^{\mathrm{T}}+(\varepsilon_6^{-1}+\varepsilon_7^{-1})M_3EE^{\mathrm{T}}M_3^{\mathrm{T}}.$$

By applying the Schur complement techniques, $\Xi < 0$ is equivalent to LMI (15). Therefore, if LMIs (14) and (15) are satisfied, one can show that (34) implies

$$E\{LV(t)\} \le E\left\{\xi^{\mathrm{T}}(t)\Xi\xi(t)\right\}. \tag{35}$$

Now we proceed to prove system (1) is exponential stable in mean square, using the similar method of (Chen et al, 2005). Set $\lambda_0 = \lambda_{\min}(-\Xi), \lambda_1 = \lambda_{\min}(P)$, by (35),

$$E\{LV(t)\} \le -\lambda_0 E\left\{\xi^{\mathrm{T}}(t)\xi(t)\right\} \le -\lambda_0 E\left\{x^{\mathrm{T}}(t)x(t)\right\}. \tag{36}$$

From the definitions of $V(t)$ and $y(t)$, there exist positive scalars β_1, β_2 such that

$$\lambda_1 \|x(t)\|^2 \le V(t) \le \beta_1 \|x(t)\|^2 + \beta_2 \int_{t-2\tau}^{t} \|x(s)\|^2 ds. \tag{37}$$

Defining a new function as $W(t) = e^{\beta_0 t} V(t)$, its weak infinitesimal operator is given by

$$L\{W(t)\} = \beta_0 e^{\beta_0 t} V(t) + e^{\beta_0 t} L\{V(t)\}, \tag{38}$$

Then, from (36)-(38), by using the generalized $\mathrm{I}t\hat{o}$ formula, we can obtain that

$$E\{W(t)\} - E\{W(t_0)\} \le E\int_{t_0}^{t} e^{\beta_0 s}\left[\beta_0\left(\beta_1\|x(s)\|^2 + \beta_2\int_{s-2\tau}^{s}\|x(\theta)\|^2 d\theta\right) - \lambda_0\|x(s)\|^2\right] ds. \tag{39}$$

Since the following inequality holds (Chen et al, 2005)

$$\int_{t_0}^{t} e^{\beta_0 s} ds \int_{s-2\tau}^{s} \|x(\theta)\|^2 d\theta \le 2\tau e^{2\beta_0 \tau} \int_{t_0-2\tau}^{t} \|x(s)\|^2 e^{\beta_0 s} ds. \tag{40}$$

Therefore, it follows that from (39) and (40),

$$E\{W(t)\} - E\{W(t_0)\} \le E\int_{t_0}^{t} e^{\beta_0 s}\left[\beta_0(\beta_1 + 2\tau\beta_2 e^{2\beta_0\tau}) - \lambda_0\right]\|x(s)\|^2 ds + C_0(t_0), \tag{41}$$

where $C_0(t_0) = 2\tau\beta_0\beta_2 e^{2\beta_0\tau} \int_{t_0-2\tau}^{t_0} \mathrm{E}\,\|x(s)\|^2 e^{\beta_0 s} ds.$

Choose a positive scalar $\beta_0 > 0$ such that (Chen et al, 2005)

$$\beta_0(\beta_1 + 2\tau\beta_2 e^{2\beta_0\tau}) \le \lambda_0. \tag{42}$$

Then, by (41) and (42), it is easily obtain

$$\lim_{t\to\infty}\sup\frac{1}{t}\log \mathrm{E}\|x(t)\|^2 \le -\beta_0,$$

which implies that system (1) is exponentially stable in mean square by Definition 1. This completes the proof. □

In the case of the conditon (2b) for system (1), which is derivative-independent, or in the case of $\tau(t)$ is not differentiable. According to the proof of Theorem 1, the following theorem is followed:

Theorem 2. When (2b) holds, then for any scalars $\tau > 0$, the stochastic system (1) is exponentially mean-square stable for all admissible uncertainties, if there exist $P>0, Q>0, S>0$, scalars $\rho>0, \mu>0, \varepsilon_j>0, j=0,1,\ldots,7$, matrix $X\ge 0$ and any appropriately dimensioned matrices $M_i, N_i, i=1,2,3$, such that (16),(17) and the following LMI holds

$$\tilde{\Pi} = \begin{bmatrix} X_{11} & X_{12} & X_{13} & N_1 \\ * & X_{22} & X_{23} & N_2 \\ * & * & X_{33} & N_3 \\ * & * & * & Q \end{bmatrix} \ge 0, \tag{43}$$

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_{11} & \Theta_{12} & \Theta_{13} & N_1 & M_1 & M_1 & M_1E & M_1E & 0 & 0 & 0 & 0 \\ * & \tilde{\Theta}_{22} & \Theta_{23} & N_2 & M_2 & M_2 & 0 & 0 & M_2E & M_2E & 0 & 0 \\ * & * & \Theta_{33} & N_3 & M_3 & M_3 & 0 & 0 & 0 & 0 & M_3E & M_3E \\ * & * & * & -S & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_0 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_3 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon_4 I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\varepsilon_5 I & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\varepsilon_6 I & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -\varepsilon_7 I \end{bmatrix} < 0, \tag{44}$$

Where

$$\tilde{\Theta}_{11} = N_1 + N_1^{\mathrm{T}} + M_1A + A^{\mathrm{T}}M_1^{\mathrm{T}} + \tau X_{11} + (\varepsilon_2+\varepsilon_4+\varepsilon_6)G_1^{\mathrm{T}}G_1 + \varepsilon_0 F_1^{\mathrm{T}}F_1 + (\rho+\tau\mu)H_1^{\mathrm{T}}H_1,$$
$$\tilde{\Theta}_{22} = -N_2 - N_2^{\mathrm{T}} + M_2B + B^{\mathrm{T}}M_2^{\mathrm{T}} + \tau X_{22} + (\varepsilon_3+\varepsilon_5+\varepsilon_7)G_2^{\mathrm{T}}G_2 + \varepsilon_1 F_2^{\mathrm{T}}F_2 + (\rho+\tau\mu)H_2^{\mathrm{T}}H_2.$$

Remark 1. Theorem 1 and 2 provides delay-dependent exponentially stable criteria in mean square for stochastic system (1) in terms of the solvability of LMIs. By using them, one can obtain the MADB τ by solving the following optimization problems:

$$\begin{cases}\max \ \tau \\ s.t. \quad X \geq 0, P>0, Q>0, R>0, Z>0, \rho>0, \mu>0, \varepsilon_j>0, M_i, N_i, (14)-(17), \ i=1,2,3; \ j=0,1,...,7,\end{cases} \tag{45}$$

or

$$\begin{cases}\max \ \tau \\ s.t. \quad X \geq 0, P>0, Q>0, Z>0, \rho>0, \mu>0, \varepsilon_j>0, \ M_i, N_i, (16),(17),(43),(44), \ i=1,2,3; \ j=0,1,...,7.\end{cases} \tag{46}$$

2.2 H∞ exponential filtering for uncertain Markovian switching time-delay stochastic systems with nonlinearities

We consider the following uncertain nonlinear stochastic systems with Markovian jump parameters and mode-dependent time delays

$$\begin{aligned}(\Sigma): dx(t) = &[A(t,r_t)x(t) + A_d(t,r_t)x(t-\tau_{r_t}(t)) + D_1(r_t)f(x(t), x(t-\tau_{r_t}(t)), r_t) + B_1(t,r_t)v(t)]dt \\ &+[E(t,r_t)x(t) + E_d(t,r_t)x(t-\tau_{r_t}(t)) + G(t,r_t)v(t)]d\omega(t),\end{aligned} \tag{47}$$

$$y(t) = C(t,r_t)x(t) + C_d(t,r_t)x(t-\tau_{r_t}(t)) + D_2(r_t)g(x(t), x(t-\tau_{r_t}(t)), r_t) + B_2(t,r_t)v(t), \tag{48}$$

$$z(t) = L(r_t)x(t), \tag{49}$$

$$x(t) = \phi(t), \qquad r(t) = r(0), \ \ \forall t \in [-\tau_2, 0], \tag{50}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $v(t) \in \mathbb{R}^p$ is the exogenous disturbance input which belongs to $L_2[0,\infty)$; $y(t) \in \mathbb{R}^q$ is the measurement; $z(t) \in \mathbb{R}^m$ is the signal to be estimated; $\omega(t)$ is a zero-mean one-dimensional Wiener process (Brownian Motion) satisfying $\mathrm{E}[\omega(t)] = 0$ and $\mathrm{E}[\omega^2(t)] = t$; $\{r_t, t \geq 0\}$ is a continuous-time Markovian process with right continuous trajectories and taking values in a finite set $S = \{1, 2, \ldots, N\}$ with transition probability matrix $\Pi \triangleq \{\pi_{ij}\}$ given by

$$\Pr\{r_{t+\Delta} = j \mid r_t = i\} = \begin{cases}\pi_{ij}\Delta + o(\Delta), & if \ i \neq j, \\ 1 + \pi_{ii}\Delta + o(\Delta), & if \ i = j,\end{cases} \tag{51}$$

where $\Delta > 0$, $\lim_{\Delta \to 0}(o(\Delta)/\Delta) = 0$; $\pi_{ij} \geq 0$ for $i \neq j$, is the transition rate from mode i at time t to mode j at time $t+\Delta$ and

$$\pi_{ii} = -\sum_{j=1, j\neq i}^{N} \pi_{ij}. \tag{52}$$

In system (Σ), $\tau_{r_t}(t)$ denotes the time-varying delay when the mode is in r_t and satisfies

$$0 \le \tau_{1i} \le \tau_i(t) \le \tau_{2i},\ \dot{\tau}_i(t) \le d_i < 1,\ \forall r_t = i, i \in S \tag{53}$$

where τ_{1i}, τ_{2i} and d_i are known real constants scalars for any $i \in S$. In (50), $\tau_2 = \max\{\tau_{2i}, i \in S\}$, and $\phi(t)$ is a vector-valued initial continuous function defined on $[-\tau_2, 0]$. $A(t,r_t), A_d(t,r_t), D_1(r_t), B_1(t,r_t), E(t,r_t), E_d(t,r_t), G(t,r_t), C(t,r_t), C_d(t,r_t), D_2(r_t), B_2(t,r_t)$ and $L(r_t)$ are matrix functions governed by Markov process r_t, and

$$\begin{aligned}
&A(t,r_t) = A(r_t) + \Delta A(t,r_t),\ \ A_d(t,r_t) = A_d(r_t) + \Delta A_d(t,r_t),\ \ B_1(t,r_t) = B_1(r_t) + \Delta B_1(t,r_t),\\
&E(t,r_t) = E(r_t) + \Delta E(t,r_t),\ \ E_d(t,r_t) = E_d(r_t) + \Delta E_d(t,r_t),\ \ G(t,r_t) = G(r_t) + \Delta G(t,r_t),\\
&C(t,r_t) = C(r_t) + \Delta C(t,r_t),\ \ C_d(t,r_t) = C_d(r_t) + \Delta C_d(t,r_t),\ \ B_2(t,r_t) = B_2(r_t) + \Delta B_2(t,r_t).
\end{aligned}$$

where $A(r_t), A_d(r_t), B_1(r_t), E(r_t), E_d(r_t), G(r_t), C(r_t), C_d(r_t), B_2(r_t)$ and $L(r_t)$ are known real matrices representing the nominal system for all $r_t \in S$, and $\Delta A(t,r_t), \Delta A_d(t,r_t)$, $\Delta E(t,r_t), \Delta E_d(t,r_t), \Delta G(t,r_t), \Delta C(t,r_t), \Delta C_d(t,r_t)$ and $\Delta B_2(t,r_t)$ are unknown matrices representing parameter uncertainties, which are assumed to be of the following form

$$\begin{bmatrix} \Delta A(t,r_t) & \Delta A_d(t,r_t) & \Delta B_1(t,r_t) \\ \Delta E(t,r_t) & \Delta E_d(t,r_t) & \Delta G(t,r_t) \\ \Delta C(t,r_t) & \Delta C_d(t,r_t) & \Delta B_2(t,r_t) \end{bmatrix} = \begin{bmatrix} M_1(r_t) \\ M_2(r_t) \\ M_3(r_t) \end{bmatrix} F(t,r_t)\begin{bmatrix} N_1(r_t) & N_2(r_t) & N_3(r_t) \end{bmatrix}, \forall r_t \in S, \tag{54}$$

where $M_1(r_t), M_2(r_t), N_1(r_t), N_2(r_t)$ and $N_3(r_t)$ are known real constant matrices for all $r_t \in S$, and $F(t,r_t)$ is time-varying matrices with Lebesgue measurable elements satisfying

$$F^{\mathrm{T}}(t,r_t)F(t,r_t) \le I, \quad \forall r_t \in S. \tag{55}$$

Assumption 1: For a fixed system mode $r_t \in S$, there exist known real constant mode-dependent matrices $F_1(r_t) \in \mathbb{R}^{n\times n}$, $F_2(r_t) \in \mathbb{R}^{n\times n}$, $H_1(r_t) \in \mathbb{R}^{n\times n}$ and $H_2(r_t) \in \mathbb{R}^{n\times n}$ such that the unknown nonlinear vector functions $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot,\cdot)$ satisfy the following boundedness conditions:

$$\left| f(x(t), x(t-\tau_{r_t}(t)), r_t) \right| \le \left| F_1(r_t)x(t) \right| + \left| F_2(r_t)x(t-\tau_{r_t}(t)) \right|, \tag{56}$$

$$\left| g(x(t), x(t-\tau_{r_t}(t)), r_t) \right| \le \left| H_1(r_t)x(t) \right| + \left| H_2(r_t)x(t-\tau_{r_t}(t)) \right|. \tag{57}$$

For the sake of notation simplification, in the sequel, for each possible $r_t = i, i \in S$, a matrix $M(t, r_t)$ will be denoted by $M_i(t)$; for example, $A(t, r_t)$ is denoted by $A_i(t)$, and $B(t, r_t)$ by B_i, and so on.

For each $i \in S$, we are interested in designing an exponential mean-square stable, Markovian jump, full-order linear filter described by

$$(\Sigma_f): \quad d\hat{x}(t) = A_{fi}\hat{x}(t)dt + B_{fi}y(t)dt, \tag{58}$$

$$\hat{z}(t) = L_{fi}\hat{x}(t), \tag{59}$$

where $\hat{x}(t) \in \mathbb{R}^n$ and $\hat{z}(t) \in \mathbb{R}^q$ for $i \in S$, and the constant matrices A_{fi}, B_{fi} and L_{fi} are filter parameters to be determined.

Denote

$$\tilde{x}(t) = x(t) - \hat{x}(t), \quad \tilde{z}(t) = z(t) - \hat{z}(t), \quad \xi(t) = [x(t) \quad \tilde{x}(t)]^{\mathrm{T}}, \tag{60}$$

Then, for each $r_t = i, i \in S$, the filtering error dynamics from the systems (Σ) and (Σ_f) can be described by

$$\begin{aligned}(\tilde{\Sigma}): d\xi(t) = {} & [\tilde{A}_i(t)\xi(t) + \tilde{A}_{di}(t)H\xi(t-\tau_i(t)) + \tilde{D}_{1i}f(H\xi(t), H\xi(t-\tau_i(t)), i) \\ & - \tilde{D}_{2i}g(H\xi(t), H\xi(t-\tau_i(t)), i) + \tilde{B}_{1i}(t)v(t)]dt \\ & + [\tilde{E}_i(t)H\xi(t) + \tilde{E}_{di}(t)H\xi(t-\tau_i(t)) + \tilde{G}_i(t)v(t)]d\omega(t),\end{aligned} \tag{61}$$

$$\tilde{z}(t) = \tilde{L}_i\xi(t), \tag{62}$$

where

$$\tilde{A}_i(t) = \tilde{A}_i + \Delta\tilde{A}_i(t), \qquad \tilde{A}_{di}(t) = \tilde{A}_{di} + \Delta\tilde{A}_{di}(t), \qquad \tilde{B}_i(t) = \tilde{B}_i + \Delta\tilde{B}_i(t),$$

$$\tilde{E}_i(t) = \tilde{E}_i + \Delta\tilde{E}_i(t), \qquad \tilde{E}_{di}(t) = \tilde{E}_{di} + \Delta\tilde{E}_{di}(t), \qquad \tilde{G}_i(t) = \tilde{G}_i + \Delta\tilde{G}_i(t),$$

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ A_i - A_{fi} - B_{fi}C_i & A_{fi} \end{bmatrix}, \quad \Delta\tilde{A}_i(t) = \begin{bmatrix} \Delta A_i(t) & 0 \\ \Delta A_i(t) - B_{fi}\Delta C_i(t) & 0 \end{bmatrix}, \quad \tilde{A}_{di} = \begin{bmatrix} A_{di} \\ A_{di} - B_{fi}C_{di} \end{bmatrix},$$

$$\Delta\tilde{A}_{di}(t) = \begin{bmatrix} \Delta A_{di}(t) \\ \Delta A_{di}(t) - B_{fi}\Delta C_{di}(t) \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} B_{1i} \\ B_{1i} - B_{fi}B_{2i} \end{bmatrix}, \quad \Delta\tilde{B}_i(t) = \begin{bmatrix} \Delta B_{1i}(t) \\ \Delta B_{1i}(t) - B_{fi}\Delta B_{2i}(t) \end{bmatrix},$$

$$\tilde{E}_i = \begin{bmatrix} E_i \\ E_i \end{bmatrix}, \quad \Delta\tilde{E}_i(t) = \begin{bmatrix} \Delta E_i(t) \\ \Delta E_i(t) \end{bmatrix}, \quad \tilde{E}_{di} = \begin{bmatrix} E_{di} \\ E_{di} \end{bmatrix}, \quad \Delta\tilde{E}_{di}(t) = \begin{bmatrix} \Delta E_{di}(t) \\ \Delta E_{di}(t) \end{bmatrix}, \quad \tilde{G}_i = \begin{bmatrix} G_i \\ G_i \end{bmatrix},$$

$$\Delta\tilde{G}_i(t) = \begin{bmatrix} \Delta G_i(t) \\ \Delta G_i(t) \end{bmatrix}, \quad \tilde{D}_{1i} = \begin{bmatrix} D_{1i} \\ D_{1i} \end{bmatrix}, \quad \tilde{D}_{2i} = \begin{bmatrix} 0 \\ B_{fi}D_{2i} \end{bmatrix}, \quad \tilde{L}_i = \begin{bmatrix} L_i - L_{fi} & L_{fi} \end{bmatrix}, \quad H = \begin{bmatrix} I & 0 \end{bmatrix}.$$

Observe the filtering error system (61)-(62) and let $\xi(t;\varsigma)$ denote the state trajectory from the initial data $\xi(\theta)=\varsigma(\theta)$ on $-\tau_2 \le \theta \le 0$ in $L^2_{F_0}\left([-\tau_2,0];\mathbb{R}^{2n}\right)$. Obviously, the system (61)-(62) admits a trivial solution $\xi(t;0)=0$ corresponding to the initial data $\varsigma=0$. Throughout this paper, we adopt the following definition.

Definition 2 (Wang et al, 2004**)**: For every $\varsigma \in L^2_{F_0}\left([-\tau_2,0];\mathbb{R}^{2n}\right)$, the filtering error system (61)-(62) is said to be robustly exponentially mean-square stable if, when $v(t)=0$, for every system mode, there exist constant scalars $\alpha>0$ and $\beta>0$, such that

$$\mathrm{E}\left|\xi(\mathrm{t};\varsigma)\right|^2 \le \alpha e^{-\beta t} \sup_{-\tau_2\le\theta\le 0} \mathrm{E}\left|\varsigma(\theta)\right|^2 . \tag{63}$$

We are now in a position to formulate the robust H∞ filter design problem to be addressed in this paper as follows: given the system (Σ) and a prescribed $\gamma>0$, determine an filter (Σ_f) such that, for all admissible uncertainties, nonlinearities as well as delays, the filtering error system $(\tilde{\Sigma})$ is robustly exponentially mean-square stable and

$$\left\|\tilde{z}(t)\right\|_{\mathrm{E}_2} \le \gamma \left\|v(t)\right\|_2 \tag{64}$$

under zero-initial conditions for any nonzero $v(t)\in L_2[0,\infty)$, where $\left\|\tilde{z}(t)\right\|_{\mathrm{E}_2} = \mathrm{E}\left\{\int_0^\infty \left|\tilde{z}(t)\right|^2 dt\right\}^{1/2}$.

The following lemmas will be employed in the proof of our main results.

Lemma 2 (Xie, L., 1996). Let $x\in\mathbb{R}^n, y\in\mathbb{R}^n$ and a scalar $\varepsilon>0$. Then we have $x^{\mathrm{T}}y + y^{\mathrm{T}}x \le \varepsilon x^{\mathrm{T}}x + \varepsilon^{-1}y^{\mathrm{T}}y$.

Lemma 3 (Xie, L., 1996). Given matrices $Q=Q^{\mathrm{T}}, H, E$ and $R=R^{\mathrm{T}}>0$ of appropriate dimensions, $Q+HFE+E^{\mathrm{T}}F^{\mathrm{T}}H^{\mathrm{T}}<0$ for all F satisfying $F^{\mathrm{T}}F\le R$, if and only if there exists some $\lambda>0$ such that $Q+\lambda HH^{\mathrm{T}}+\lambda^{-1}E^{\mathrm{T}}RE<0$.

To this end, we provide the following theorem to establish a delay-dependent criterion of robust exponential mean-square stability with H∞ performance of system $(\tilde{\Sigma})$, which will be fundamental in the design of the expected H∞ filter.

Theorem 3. Given scalars $\tau_{1i}, \tau_{2i}, d_i$ and $\gamma>0$, for any delays $\tau_i(t)$ satisfying (7), the filtering error system $(\tilde{\Sigma})$ is robustly exponentially mean-square stable and (64) is satisfied under zero-initial conditions for any nonzero $v(t)\in L_2[0,\infty)$ and all admissible uncertainties if there exist matrices $P_i>0, i=1,2,\ldots,N, Q>0$ and sclars $\varepsilon_{1i}>0, \varepsilon_{2i}>0$ such that the following LMI holds for each $i\in S$

$$\Phi_i = \begin{bmatrix} \Phi_{11} & P_i\tilde{A}_{di}(t) & P_i\tilde{B}_i(t) & H^{\mathrm{T}}\tilde{E}_i^{\mathrm{T}}(t)P_i & P_i\tilde{D}_{1i} & P_i\tilde{D}_{2i} \\ * & \Phi_{22} & 0 & \tilde{E}_{di}^{\mathrm{T}}(t)P_i & 0 & 0 \\ * & * & -\gamma^2 I & \tilde{G}_i^{\mathrm{T}}(t)P_i & 0 & 0 \\ * & * & * & -P_i & 0 & 0 \\ * & * & * & * & -\varepsilon_{1i}I & 0 \\ * & * & * & * & * & -\varepsilon_{2i}I \end{bmatrix} < 0, \tag{65}$$

where

$$\Phi_{11} = \sum_{j=1}^{N}\pi_{ij}P_j + P_i\tilde{A}_i(t) + \tilde{A}_i^{\mathrm{T}}(t)P_i + \mu H^{\mathrm{T}}QH + 2\varepsilon_{1i}H^{\mathrm{T}}F_{1i}^{\mathrm{T}}F_{1i}H + 2\varepsilon_{2i}H^{\mathrm{T}}H_{1i}^{\mathrm{T}}H_{1i}H + \tilde{L}_i^{\mathrm{T}}\tilde{L}_i,$$

$$\Phi_{22} = 2\varepsilon_{1i}F_{2i}^{\mathrm{T}}F_{2i} + 2\varepsilon_{2i}H_{2i}^{\mathrm{T}}H_{2i} - (1-d_i)Q,$$

$$\mu = 1+\rho(\tau_2-\tau_1), \quad \rho = \max\left\{\left|\pi_{ii}\right|, i\in S\right\}, \quad \tau_1 = \min\left\{\tau_{1i}, i\in S\right\}, \quad \tau_2 = \max\left\{\tau_{2i}, i\in S\right\}.$$

Proof. Define $x_t(s) = x(t+s), t-\tau_{r_t}(t) \le s \le t,$ then $\{(x_t, r_t), t\ge 0\}$ is a Markov process with initial state $(\phi(\cdot), r_0)$. Now, define a stochastic Lyapunov-Krasovskii functional as

$$V(\xi_t, r_t) = \xi^{\mathrm{T}}(t)P(r_t)\xi(t) + \int_{t-\tau_{r_t}(t)}^{t}\xi^{\mathrm{T}}(s)H^{\mathrm{T}}QH\xi(s)ds + \rho\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t}\xi^{\mathrm{T}}(s)H^{\mathrm{T}}QH\xi(s)dsd\theta, \tag{66}$$

Let L be the weak infinitesimal operator of the stochastic process $\{(x_t, r_t), t\ge 0\}$. By *Itô* differential formula, the stochastic differential of $V(\xi_t, r_t)$ along the trajectory of system $(\tilde{\Sigma})$ with $v(t)$=0 for $r_t = i, i\in S$ is given by

$$dV(\xi_t, i) = L[V(\xi_t, i)] + 2\xi^{\mathrm{T}}(t)P_i[\tilde{E}_i(t)H\xi(t) + \tilde{E}_{di}(t)H\xi(t-\tau_i(t))], \tag{67}$$

where

$$\begin{aligned} L[V(\xi_t,i)] = {} & \xi^{\mathrm{T}}(t)(\sum_{j=1}^{N}\pi_{ij}P_j)\xi(t) + 2\xi^{\mathrm{T}}(t)P_i[\tilde{A}_i(t)\xi(t) + \tilde{A}_{di}(t)H\xi(t-\tau_i(t)) \\ & + \tilde{D}_{1i}f(H\xi(t), H\xi(t-\tau_i(t)), i) - \tilde{D}_{2i}g(H\xi(t), H\xi(t-\tau_i(t)), i)] \\ & + [\tilde{E}_i(t)H\xi(t) + \tilde{E}_{di}(t)H\xi(t-\tau_i(t))]^{\mathrm{T}}P_i[\tilde{E}_i(t)H\xi(t) + \tilde{E}_{di}(t)H\xi(t-\tau_i(t))] \\ & + \sum_{j=1}^{N}\pi_{ij}\int_{t-\tau_j(t)}^{t}\xi^{\mathrm{T}}(s)H^{\mathrm{T}}QH\xi(s)ds + \xi^{\mathrm{T}}(t)H^{\mathrm{T}}QH\xi(t) - (1-\dot{\tau}_i(t))\xi^{\mathrm{T}}(t-\tau_i(t))H^{\mathrm{T}}QH\xi(t-\tau_i(t)) \\ & + \rho(\tau_2-\tau_1)\xi^{\mathrm{T}}(t)H^{\mathrm{T}}QH\xi(t) - \rho\int_{t-\tau_2}^{t-\tau_1}\xi^{\mathrm{T}}(s)H^{\mathrm{T}}QH\xi(s)ds \end{aligned} \tag{68}$$

Noting $\pi_{ij} \geq 0$ for $i \neq j$, and $\pi_{ii} \leq 0$, we have

$$\sum_{j=1}^{N} \pi_{ij} \int_{t-\tau_j(t)}^{t} \xi^{\mathrm{T}}(s) H^{\mathrm{T}} Q H \xi(s) ds \leq -\pi_{ii} \int_{t-\tau_2}^{t-\tau_1} \xi^{\mathrm{T}}(s) H^{\mathrm{T}} Q H \xi(s) ds \leq \rho \int_{t-\tau_2}^{t-\tau_1} \xi^{\mathrm{T}}(s) H^{\mathrm{T}} Q H \xi(s) ds. \quad (69)$$

Noting (56), (57) and using Lemma 2, we have

$$\begin{aligned} &2\xi^{\mathrm{T}}(t) P_i \tilde{D}_{1i} f(H\xi(t), H\xi(t-\tau_i(t)), i) \\ &\leq \varepsilon_{1i}^{-1} \xi^{\mathrm{T}}(t) P_i \tilde{D}_{1i} \tilde{D}_{1i}^{\mathrm{T}} P \xi(t) + 2\varepsilon_{1i} (\xi^{\mathrm{T}}(t) H^{\mathrm{T}} F_{1i}^{\mathrm{T}} F_{1i} H \xi(t) + \xi^{\mathrm{T}}(t-\tau_i(t)) H F_{2i}^{\mathrm{T}} F_{2i} H \xi(t-\tau_i(t))), \end{aligned} \quad (70)$$

and

$$\begin{aligned} &-2\xi^{\mathrm{T}}(t) P_i \tilde{D}_{2i} g(H\xi(t), H\xi(t-\tau_i(t)), i) \\ &\leq \varepsilon_{2i}^{-1} \xi^{\mathrm{T}}(t) P_i \tilde{D}_{2i} \tilde{D}_{2i}^{\mathrm{T}} P_i \xi(t) + 2\varepsilon_{2i} (\xi^{\mathrm{T}}(t) H^{\mathrm{T}} H_{1i}^{\mathrm{T}} H_{1i} H \xi(t) + \xi^{\mathrm{T}}(t-\tau_i(t)) H H_{2i}^{\mathrm{T}} H_{2i} H \xi(t-\tau_i(t))), \end{aligned} \quad (71)$$

Substituting (69)-(71) into (68), then, it follows from (68) that for each $r_t = i, i \in S$

$$L[V(\xi_t, i)] \leq \eta^{\mathrm{T}}(t) \Theta_i \eta(t), \quad (72)$$

where

$$\eta(t) = \begin{bmatrix} \xi^{\mathrm{T}}(t) & \xi^{\mathrm{T}}(t-\tau_i(t)) H^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \qquad \Theta_i = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ * & \Theta_{22} \end{bmatrix},$$

$$\begin{aligned} \Theta_{11} &= \sum_{j=1}^{N} \pi_{ij} P_j + P_i \tilde{A}_i(t) + \tilde{A}_i^{\mathrm{T}}(t) P_i + \varepsilon_{1i}^{-1} P_i \tilde{D}_{1i} \tilde{D}_{1i}^{\mathrm{T}} P + 2\varepsilon_{1i} H^{\mathrm{T}} F_{1i}^{\mathrm{T}} F_{1i} H + \varepsilon_{2i}^{-1} P_i \tilde{D}_{2i} \tilde{D}_{2i}^{\mathrm{T}} P_i \\ &\quad + 2\varepsilon_{2i} H^{\mathrm{T}} H_{1i}^{\mathrm{T}} H_{1i} H + H^{\mathrm{T}} \tilde{E}_i^{\mathrm{T}}(t) P_i \tilde{E}_i(t) H + \mu H^{\mathrm{T}} Q H, \\ \Theta_{12} &= P_i \tilde{A}_{di}(t) + H^{\mathrm{T}} \tilde{E}_i^{\mathrm{T}}(t) P_i \tilde{E}_{di}(t), \ \Theta_{22} = 2\varepsilon_{1i} F_{2i}^{\mathrm{T}} F_{2i} + 2\varepsilon_{2i} H_{2i}^{\mathrm{T}} H_{2i} + \tilde{E}_{di}^{\mathrm{T}}(t) P_i \tilde{E}_{di}(t) - (1-d_i) Q, \end{aligned}$$

By the Schur complement, it is ease to see that LMI in (65) implies that $\Theta_i < 0$. Therefore, from (72) we obtain

$$L[V(\xi_t, i)] \leq -\delta \eta^{\mathrm{T}}(t) \eta(t), \quad (73)$$

where $\delta = \min_{i \in S} \{\lambda_{\min}(-\Theta_i)\}$. By Dynkin's formula, we can obtain

$$E\{V(\xi_t, i)\} - E\{V(\xi_0, r_0)\} = E\left\{\int_0^t L[V(\xi_s, i)] ds\right\} \leq -\delta \int_0^t E\{\xi^{\mathrm{T}}(s) \xi(s)\} ds. \quad (74)$$

On the other hand, it is follows from (66) that

$$E\left\{V(\xi_t,i)\right\} \geq \lambda_p E\left\{\xi^{\mathrm{T}}(t)\xi(t)\right\}, \tag{75}$$

where $\lambda_p = \min_{i\in S}\left\{\lambda_{\min}(P_i)\right\} > 0.$ Therefore, by (74) and (75),

$$E\left\{\xi^{\mathrm{T}}(t)\xi(t)\right\} \leq \lambda_p^{-1}V(\xi_0,r_0) - \delta\lambda_p^{-1}\int_0^t E\left\{\xi^{\mathrm{T}}(s)\xi(s)\right\}ds. \tag{76}$$

Then, applying Gronwall-Bellman lemma to (76) yields

$$E\left\{\xi^{\mathrm{T}}(t)\xi(t)\right\} \leq \lambda_p^{-1}V(\xi_0,r_0)e^{-\delta\lambda_p^{-1}t}.$$

Noting that there exists a scalar $\alpha > 0$ such that $\lambda_p^{-1}V(\xi_0,r_0) \leq \alpha \sup_{-\tau_2\leq\theta\leq 0}\left|\varsigma(\theta)\right|^2$.

Defining $\beta = \delta\lambda_p^{-1} > 0$, then we have $\mathrm{E}\left|\xi(\mathrm{t})\right|^2 \leq \alpha e^{-\beta t} \sup_{-\tau_2\leq\theta\leq 0} \mathrm{E}\left|\varsigma(\theta)\right|^2$,

and, hence, the robust exponential mean-square stability of the filtering error system $(\tilde{\Sigma})$ with $v(t)$=0 is established.

Now, we shall establish the H∞ performance for the system $(\tilde{\Sigma})$, we introduce

$$J(t) = \mathrm{E}\int_0^t [\tilde{z}^{\mathrm{T}}(s)\tilde{z}(s) - \gamma^2 v^{\mathrm{T}}(s)v(s)]ds, \tag{77}$$

where $t > 0$. Noting under the zero initial condition and $\mathrm{E}V(\xi_t,i) \geq 0,$ by the Lyapunov-Krasovskii functional (66), it can be shown that for any nonzero $v(t) \in L_2\left[0,\infty\right)$

$$J(t) = \mathrm{E}\left\{\int_0^t [\tilde{z}^{\mathrm{T}}(s)\tilde{z}(s) - \gamma^2 v^{\mathrm{T}}(s)v(s) + LV(\xi_s,i)]ds\right\} - \mathrm{E}V(\xi_t,i) \leq \mathrm{E}\left\{\int_0^t \eta^{\mathrm{T}}(s)\Psi_i\eta(s)ds\right\}, \tag{78}$$

where

$$\eta(s) = \begin{bmatrix}\xi^{\mathrm{T}}(s) & v^{\mathrm{T}}(s)\end{bmatrix}^{\mathrm{T}},\ \Psi_i(t) = \begin{bmatrix}\Theta_{11} + \tilde{L}_i^{\mathrm{T}}\tilde{L}_i & \Theta_{12} & P_i\tilde{B}_i(t) + H^{\mathrm{T}}\tilde{E}_i^{\mathrm{T}}(t)P_i\tilde{G}_i(t), \\ * & \Theta_{22} & \tilde{E}_{di}^{\mathrm{T}}(t)P_i\tilde{G}_i(t), \\ * & * & \tilde{G}_i^{\mathrm{T}}(t)P_i\tilde{G}_i(t) - \gamma^2 I\end{bmatrix},$$

Now, applying Schur complement to (65), we have $\Psi_i(t) < 0$. This together with (78) implies that $J(t) < 0$ for any nonzero $v(t) \in L_2\left[0,\infty\right)$. Therefore, under zero conditions and for any nonzero $v(t) \in L_2\left[0,\infty\right)$, letting $t \to \infty$, we have $\left\|\tilde{z}(t)\right\|_{\mathrm{E}_2} \leq \gamma\left\|v(t)\right\|_{\mathrm{E2}}$ if (65) is satisfied. This completes the proof. □

Now, we are in a position to present a solution to the H∞ exponential filter design problem.
Theorem 4. Consider the uncertain Markovian jump stochastic system (Σ). Given scalars $\tau_{1i}, \tau_{2i}, d_i$ and $\gamma > 0$, for any delays $\tau_i(t)$ satisfying (7), the filtering error system $(\tilde{\Sigma})$ is robustly exponentially mean-square stable and (64) is satisfied under zero-initial conditions for any nonzero $v(t) \in L_2[0, \infty)$ and all admissible uncertainties, if for each $i \in S$ there exist matrices $P_{1i} > 0, P_{2i} > 0, Q > 0, W_i, Z_i$ and sclars $\varepsilon_{1i} > 0, \varepsilon_{2i} > 0, \varepsilon_{3i} > 0, \varepsilon_{4i} > 0$ such that the following LMI holds

$$\Xi_i = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & E_i^{\mathrm{T}} P_{1i} & E_i^{\mathrm{T}} P_{2i} & P_{1i} D_{1i} & 0 & P_{1i} M_{1i} & 0 & L_i^{\mathrm{T}} - L_{fi}^{\mathrm{T}} \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & 0 & 0 & P_{2i} D_{1i} & Z_i D_{2i} & P_{2i} M_{1i} - Z_i M_{3i} & 0 & L_{fi}^{\mathrm{T}} \\ * & * & \Xi_{33} & \Xi_{34} & E_{di}^{\mathrm{T}} P_{1i} & E_{di}^{\mathrm{T}} P_{2i} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{44} & G_i^{\mathrm{T}} P_{1i} & G_i^{\mathrm{T}} P_{2i} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -P_{1i} & 0 & 0 & 0 & 0 & P_{1i} M_{2i} & 0 \\ * & * & * & * & * & -P_{2i} & 0 & 0 & 0 & P_{2i} M_{2i} & 0 \\ * & * & * & * & * & * & -\varepsilon_{1i} I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_{2i} I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon_{3i} I & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\varepsilon_{4i} I & 0 \\ * & * & * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0, \quad (79)$$

Where

$$\Xi_{11} = \sum_{j=1}^{N} \pi_{ij} P_{1j} + P_{1i} A_i + A_i^{\mathrm{T}} P_{1i} + 2\varepsilon_{1i} F_{1i}^{\mathrm{T}} F_{1i} + 2\varepsilon_{2i} H_{1i}^{\mathrm{T}} H_{1i} + \mu Q + \varepsilon_i N_{1i}^{\mathrm{T}} N_{1i},$$

$$\Xi_{12} = A_i^{\mathrm{T}} P_{2i} - W_i^{\mathrm{T}} - C_i^{\mathrm{T}} Z_i^{\mathrm{T}}, \quad \Xi_{13} = P_{1i} A_{di} + \varepsilon_i N_{1i}^{\mathrm{T}} N_{2i}, \quad \Xi_{14} = P_{1i} B_{1i} + \varepsilon_i N_{1i}^{\mathrm{T}} N_{3i},$$

$$\Xi_{22} = \sum_{j=1}^{N} \pi_{ij} P_{2j} + W_i + W_i^{\mathrm{T}}, \quad \Xi_{23} = P_{2i} A_{di} - Z_i C_{di}, \quad \Xi_{24} = P_{2i} B_{1i} - Z_i B_{2i},$$

$$\Xi_{33} = 2\varepsilon_{1i} F_{2i}^{\mathrm{T}} F_{2i} + 2\varepsilon_{2i} H_{2i}^{\mathrm{T}} H_{2i} - (1 - d_i) Q + \varepsilon_i N_{2i}^{\mathrm{T}} N_{2i}, \quad \Xi_{34} = \varepsilon_i N_{2i}^{\mathrm{T}} N_{3i},$$

$$\Xi_{44} = -\gamma^2 I + \varepsilon_i N_{3i}^{\mathrm{T}} N_{3i}, \qquad \varepsilon_i = \varepsilon_{3i} + \varepsilon_{4i}, \qquad \mu = 1 + \rho(\tau_2 - \tau_1).$$

In this case, a desired robust Markovian jump exponential H∞ filter is given in the form of (58)-(59) with parameters as follows

$$A_{fi} = P_{2i}^{-1} W_i, \quad B_{fi} = P_{2i}^{-1} Z_i, \quad L_{fi}, \ i \in S. \quad (80)$$

Proof. Noting that for $r_t = i,\ i \in S$

$$\begin{bmatrix} \Delta\tilde{A}_i(t) & \Delta\tilde{A}_{di}(t) & \Delta\tilde{B}_i(t) \end{bmatrix} = \tilde{M}_{1i}F_i(t)\begin{bmatrix} \tilde{N}_{1i} & \tilde{N}_{2i} & \tilde{N}_{3i} \end{bmatrix}, \tag{81}$$

and

$$\begin{bmatrix} \Delta\tilde{E}_i(t) & \Delta\tilde{E}_{di}(t) & \Delta\tilde{G}_i(t) \end{bmatrix} = \tilde{M}_{2i}F_i(t)\begin{bmatrix} N_{1i} & N_{2i} & N_{3i} \end{bmatrix}, \tag{82}$$

where

$$\tilde{M}_{1i} = \begin{bmatrix} M_{1i} \\ M_{1i} - B_{fi}M_{3i} \end{bmatrix},\ \tilde{M}_{2i} = \begin{bmatrix} M_{2i} \\ M_{2i} \end{bmatrix},\ \tilde{N}_{1i} = \begin{bmatrix} N_{1i} & 0 \end{bmatrix},\ \tilde{N}_{2i} = N_{2i},\ \tilde{N}_{3i} = N_{3i}.$$

Then, it is readily to see that (65) can be written in the form as

$$\Phi_i = \Phi_{i0} + \Lambda_{i1}F_i(t)\Gamma_{i1} + \Gamma_{i1}^{\mathrm{T}}F_i^{\mathrm{T}}(t)\Lambda_{i1}^{\mathrm{T}} + \Lambda_{i2}F_i(t)\Gamma_{i2} + \Gamma_{i2}^{\mathrm{T}}F_i^{\mathrm{T}}(t)\Lambda_{i2}^{\mathrm{T}} < 0, \tag{83}$$

where

$$\Phi_{i0} = \begin{bmatrix} \Phi_{110} & P_i\tilde{A}_{di} & P_i\tilde{B}_i & H^{\mathrm{T}}\tilde{E}_i^{\mathrm{T}}P_i & P_i\tilde{D}_{1i} & P_i\tilde{D}_{2i} \\ * & \Phi_{22} & 0 & \tilde{E}_{di}^{\mathrm{T}}P_i & 0 & 0 \\ * & * & -\gamma^2 I & \tilde{G}_i^{\mathrm{T}}P_i & 0 & 0 \\ * & * & * & -P_i & 0 & 0 \\ * & * & * & * & -\varepsilon_{1i}I & 0 \\ * & * & * & * & * & -\varepsilon_{2i}I \end{bmatrix},$$

$$\Phi_{110} = \sum_{j=1}^{N}\pi_{ij}P_j + P_i\tilde{A}_i + \tilde{A}_i^{\mathrm{T}}P_i + 2\varepsilon_{1i}H^{\mathrm{T}}F_{1i}^{\mathrm{T}}F_{1i}H + 2\varepsilon_{2i}H^{\mathrm{T}}H_{1i}^{\mathrm{T}}H_{1i}H + \mu H^{\mathrm{T}}QH + \tilde{L}_i^{\mathrm{T}}\tilde{L}_i,$$

$$\Lambda_{i1} = \begin{bmatrix} \tilde{M}_{1i}^{\mathrm{T}}P_i & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}},\quad \Gamma_{i1} = \begin{bmatrix} \tilde{N}_{1i} & \tilde{N}_{2i} & \tilde{N}_{3i} & 0 & 0 & 0 \end{bmatrix},$$

$$\Lambda_{i2} = \begin{bmatrix} 0 & 0 & 0 & \tilde{M}_{2i}^{\mathrm{T}}P_i & 0 & 0 \end{bmatrix}^{\mathrm{T}},\quad \Gamma_{i2} = \begin{bmatrix} N_{1i}H & N_{2i} & N_{3i} & 0 & 0 & 0 \end{bmatrix},$$

From (83) and by using Lemma 3, there exists positive scalars $\varepsilon_{3i} > 0, \varepsilon_{4i} > 0$ such that the following inequality holds

$$\Phi_{i0} + \varepsilon_{3i}^{-1}\Lambda_{i1}\Lambda_{i1}^{\mathrm{T}} + \varepsilon_{3i}\Gamma_{i1}^{\mathrm{T}}\Gamma_{i1} + \varepsilon_{4i}^{-1}\Lambda_{i2}\Lambda_{i2}^{\mathrm{T}} + \varepsilon_{4i}\Gamma_{i2}^{\mathrm{T}}\Gamma_{i2} < 0, \tag{84}$$

then, by applying the Schur complement to (84), we have

$$\tilde{\Phi}_i = \begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} & \tilde{\Phi}_{13} & H^{\mathrm{T}}\tilde{E}_i^{\mathrm{T}}P_i & P_i\tilde{D}_{1i} & P_i\tilde{D}_{2i} & P_i\tilde{M}_{1i} & 0 \\ * & \tilde{\Phi}_{22} & \tilde{\Phi}_{23} & \tilde{E}_{di}^{\mathrm{T}}P_i & 0 & 0 & 0 & 0 \\ * & * & \tilde{\Phi}_{33} & \tilde{G}_i^{\mathrm{T}}P_i & 0 & 0 & 0 & 0 \\ * & * & * & -P_i & 0 & 0 & 0 & P_i\tilde{M}_{2i} \\ * & * & * & * & -\varepsilon_{1i}I & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_{2i}I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_{3i}I & 0 \\ * & * & * & * & * & * & * & -\varepsilon_{4i}I \end{bmatrix}, \tag{85}$$

where

$$\begin{aligned}
\tilde{\Phi}_{11} &= \sum_{j=1}^{N}\pi_{ij}P_j + P_i\tilde{A}_i + \tilde{A}_i^{\mathrm{T}}P_i + 2\varepsilon_{1i}H^{\mathrm{T}}F_{1i}^{\mathrm{T}}F_{1i}H + 2\varepsilon_{2i}H^{\mathrm{T}}H_{1i}^{\mathrm{T}}H_{1i}H + \mu H^{\mathrm{T}}QH \\
&\quad + \tilde{L}_i^{\mathrm{T}}\tilde{L}_i + \varepsilon_{3i}\tilde{N}_{1i}^{\mathrm{T}}\tilde{N}_{1i} + \varepsilon_{4i}H^{\mathrm{T}}N_{1i}^{\mathrm{T}}N_{1i}H, \\
\tilde{\Phi}_{12} &= P_i\tilde{A}_{di} + \varepsilon_{3i}\tilde{N}_{1i}^{\mathrm{T}}\tilde{N}_{2i} + \varepsilon_{4i}H^{\mathrm{T}}N_{1i}^{\mathrm{T}}N_{2i}, \quad \tilde{\Phi}_{13} = P_i\tilde{B}_i + \varepsilon_{3i}\tilde{N}_{1i}^{\mathrm{T}}\tilde{N}_{3i} + \varepsilon_{4i}H^{\mathrm{T}}N_{1i}^{\mathrm{T}}N_{3i}, \\
\tilde{\Phi}_{22} &= 2\varepsilon_{1i}F_{2i}^{\mathrm{T}}F_{2i} + 2\varepsilon_{2i}H_{2i}^{\mathrm{T}}H_{2i} - (1-d_i)Q + \varepsilon_{3i}\tilde{N}_{2i}^{\mathrm{T}}\tilde{N}_{2i} + \varepsilon_{4i}N_{2i}^{\mathrm{T}}N_{2i}, \\
\tilde{\Phi}_{23} &= \varepsilon_{3i}\tilde{N}_{2i}^{\mathrm{T}}\tilde{N}_{3i} + \varepsilon_{4i}N_{2i}^{\mathrm{T}}N_{3i}, \qquad \tilde{\Phi}_{33} = -\gamma^2 I + \varepsilon_{3i}\tilde{N}_{3i}^{\mathrm{T}}\tilde{N}_{3i} + \varepsilon_{4i}N_{3i}^{\mathrm{T}}N_{3i}.
\end{aligned}$$

For each $r_t = i,\ i \in S$, we define the matrix $P_i > 0$ by

$$P_i = \begin{bmatrix} P_{1i} & 0 \\ 0 & P_{2i} \end{bmatrix}.$$

Then, substituting the matrix P_i, the matrices $\tilde{A}_i, \tilde{A}_{di}, \tilde{B}_i, \tilde{E}_i, \tilde{E}_{di}, \tilde{B}_i, \tilde{E}_i, \tilde{E}_{di}, \tilde{G}_i, \tilde{D}_{1i}, \tilde{D}_{2i}, \tilde{L}_i$, H defined in (61)-(62) into (85) and by introducing some matrices given by $W_i = P_{2i}A_{fi}, Z_i = P_{2i}B_{fi}$, then, we can obtain the results in Theorem 4. This completes the proof. □

3. Numerical Examples and Simulations

Example 1: Consider the uncertain stochastic time-delay system with nonlinearities

$$dx(t) = \left[\left(A+\Delta A(t)\right)x(t)+\left(B+\Delta B(t)\right)x(t-\tau(t))\right]dt + g(t, x(t), x(t-\tau(t)))d\omega(t), \tag{86}$$

where

$$A=\begin{bmatrix}-2 & 0\\ 1 & -1\end{bmatrix},\quad B=\begin{bmatrix}-1 & 0\\ -0.5 & -1\end{bmatrix},\quad \|\Delta A(t)\|\le 0.1,\quad \|\Delta B(t)\|\le 0.1,$$

$$trace\left[g^{\mathrm{T}}(t,x(t),x(t-\tau(t)))g(t,x(t),x(t-\tau(t)))\right]\le 0.1\|x(t)\|^2+0.1\|x(t-\tau(t))\|^2 .$$

$$E=\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix}, G_1=G_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, H_1=H_2=\begin{bmatrix}\sqrt{0.1} & 0\\ 0 & \sqrt{0.1}\end{bmatrix}.$$

For the time-invariant system, applying Theorem 1, it has been found that by using MATLAB LMI Toolbox that system (86) is exponentially stable in mean square for any delay $0\le\tau\le 1.0898$. It is note that the result of (Yue & Won, 2001) guarantees the exponential stability of (86) when $0\le\tau\le 0.8635$, whereas by the method of (Mao, 1996) the delay is only allowed 0.1750. According to Theorem 1, the MADB for different d is shown in Table 1. For a comparison with the results of other researchers, a summary is given in the following Table 1. It is obvious that the result in this paper is much less conservative and is an improvement of the results than that of (Mao, 1996) and (Yue & Won, 2001).
The stochastic perturbation of the system is Brownian motion and it can be depicted in Fig.1. The simulation of the state response for system (86) with $\tau=1.0898$ was depicted in Fig.2.

Methods	$d=0$	$d=0.5$	$d=0.9$
(Mao, 1996)	0.1750	-	-
(Yue & Won, 2001)	0.8635	-	-
Theorem 1	1.0898	0.5335	0.1459

Table1. Maximum allowable time delay to different d

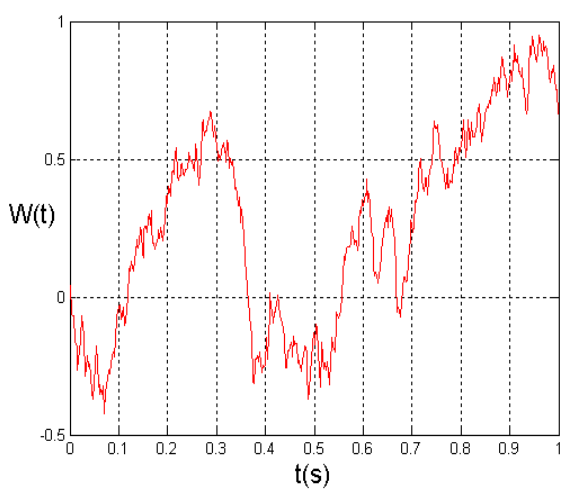

Fig. 1. The trajectory of Brownian motion

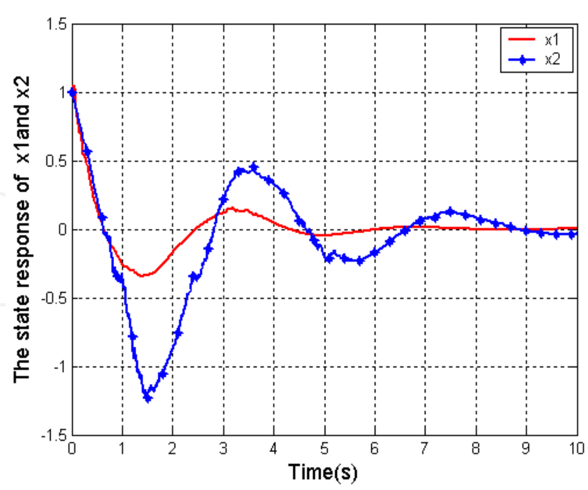

Fig. 2. The state response of system (47)

Example 2. Consider the uncertain Markovian jump stochastic systems in the form of (47)-(48) with two modes. For mode 1, the parameters as the following:

$$A_1 = \begin{bmatrix} -3 & 1 & 0 \\ 0.3 & -4.5 & 1 \\ -0.1 & 0.3 & -3.8 \end{bmatrix},\ A_{d1} = \begin{bmatrix} -0.2 & 0.1 & 0.6 \\ 0.5 & -1 & -0.8 \\ 0 & 1 & -2.5 \end{bmatrix},\ D_{11} = \begin{bmatrix} 0 & 0.1 & 0 \\ 0.1 & 0.1 & 0 \\ 0.1 & 0.2 & 0.2 \end{bmatrix},\ B_{11} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.1 & -0.1 & 0.2 \\ 0.3 & 0.3 & -0.4 \\ 0.1 & 0.1 & -0.3 \end{bmatrix}, E_{d1} = \begin{bmatrix} 0.1 & -0.1 & 0.2 \\ 0.3 & 0.3 & -0.4 \\ 0.1 & 0.1 & -0.3 \end{bmatrix},\ G_1 = \begin{bmatrix} 0.2 \\ 0 \\ 0.1 \end{bmatrix},\ C_1 = \begin{bmatrix} 0.8 & 0.3 & 0 \end{bmatrix},$$

$$C_{d1} = \begin{bmatrix} 0.2 & -0.3 & -0.6 \end{bmatrix},\ D_{21} = 0.1,\ B_{21} = 0.2, L_1 = \begin{bmatrix} 0.5 & -0.1 & 1 \end{bmatrix},$$

$$F_{11} = F_{21} = H_{11} = H_{21} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},\ M_{11} = \begin{bmatrix} 0.1 \\ 0 \\ 0.2 \end{bmatrix},\ M_{21} = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \end{bmatrix},\ M_{31} = 0.2,$$

$$N_{11} = \begin{bmatrix} 0.2 & 0 & 0.1 \end{bmatrix},\ N_{21} = \begin{bmatrix} 0.1 & 0.2 & 0 \end{bmatrix},\ N_{31} = 0.2.$$

and the time-varying delay $\tau(t)$ satisfies (53) with $\tau_{11} = 0.2, \tau_{21} = 1.3, d_1 = 0.2.$
For mode 2, the dynamics of the system are describe as

$$A_2 = \begin{bmatrix} -2.5 & 0.5 & -0.1 \\ 0.1 & -3.5 & 0.3 \\ -0.1 & 1 & -3.2 \end{bmatrix},\ A_{d2} = \begin{bmatrix} 0 & -0.3 & 0.6 \\ 0.1 & 0.5 & 0 \\ -0.6 & 1 & -0.8 \end{bmatrix},\ D_{12} = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0.2 & 0 \\ 0.2 & 0.1 & 0.1 \end{bmatrix},\ B_{12} = \begin{bmatrix} -0.6 \\ 0.5 \\ 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0.1 & -1 & 0.2 \\ 0.3 & 0.3 & -0.4 \\ 1 & 0.1 & 0.3 \end{bmatrix}, E_{d2} = \begin{bmatrix} 0.1 & -0.1 & 0.2 \\ 0.3 & 0.3 & -0.4 \\ 0.1 & 0.1 & 0.3 \end{bmatrix},\ G_2 = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \end{bmatrix},\ C_2 = \begin{bmatrix} -0.5 & 0.2 & 0.3 \end{bmatrix},$$

$$C_{d2} = \begin{bmatrix} 0 & -0.6 & 0.2 \end{bmatrix},\ D_{22} = 0.1,\ B_{22} = 0.5, L_2 = \begin{bmatrix} 0 & 1 & 0.6 \end{bmatrix},$$

$$F_{12} = F_{22} = H_{12} = H_{22} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},\ M_{12} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0 \end{bmatrix},\ M_{22} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0 \end{bmatrix},\ M_{32} = 0.1,$$

$$N_{12} = \begin{bmatrix} 0.1 & 0.1 & 0 \end{bmatrix},\ N_{22} = \begin{bmatrix} 0 & -0.1 & 0.2 \end{bmatrix},\ N_{32} = 0.1.$$

and the time-varying delay $\tau(t)$ satisfies (53) with $\tau_{12} = 0.1, \tau_{22} = 1.1, d_2 = 0.3.$

Suppose the transition probability matrix to be $\Pi = \begin{bmatrix} -0.5 & 0.5 \\ 0.3 & -0.3 \end{bmatrix}.$

The objective is to design a Markovian jump H∞ filter in the form of (58)-(59), such that for all admissible uncertainties, the filtering error system is exponentially mean-square stable and (64) holds. In this example, we assume the disturbance attenuation level $\gamma = 1.2$.

By using Matlab LMI Control Toolbox to solve the LMI in (77), we can obtain the solutions as follows:

$$P_{11} = \begin{bmatrix} 0.7952 & 0.0846 & 0.0051 \\ 0.0846 & 0.6355 & -0.1857 \\ 0.0051 & -0.1857 & 0.7103 \end{bmatrix}, P_{21} = \begin{bmatrix} 0.6847 & 0.0549 & -0.0341 \\ 0.0549 & 0.4614 & -0.0350 \\ -0.0341 & -0.0350 & 0.5624 \end{bmatrix}, P_{12} = \begin{bmatrix} 1.0836 & 0.1117 & -0.0355 \\ 0.1117 & 0.9508 & -0.1800 \\ -0.0355 & -0.1800 & 0.7222 \end{bmatrix},$$

$$P_{22} = \begin{bmatrix} 0.5974 & 0.0716 & -0.0536 \\ 0.0716 & 0.5827 & 0.0814 \\ -0.0536 & 0.0814 & 0.3835 \end{bmatrix}, Q = \begin{bmatrix} 1.4336 & -0.0838 & -0.0495 \\ -0.0838 & 2.1859 & -1.1472 \\ -0.0495 & -1.1472 & 1.9649 \end{bmatrix}, W_1 = \begin{bmatrix} -0.9731 & 0.3955 & 0.4457 \\ -0.3560 & -1.1939 & 0.5584 \\ -0.6309 & -0.5217 & -1.2038 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} -0.8741 & -0.0117 & 0.1432 \\ -0.0276 & -1.1101 & -0.0437 \\ -0.1726 & 0.1087 & -0.8501 \end{bmatrix}, Z_1 = \begin{bmatrix} -0.3844 \\ 0.1797 \\ 1.2608 \end{bmatrix}, Z_2 = \begin{bmatrix} 0.0072 \\ -0.0572 \\ -0.0995 \end{bmatrix},$$

$$e_{11} = 1.2704,\ e_{21} = 1.1626,\ e_{31} = 1.0887,\ e_{41} = 1.0670,\ e_{12} = 1.2945,\ e_{22} = 1.2173,\ e_{32} = 1.2434,\ e_{42} = 1.2629.$$

Then, by Theorem 4, the parameters of desired robust Markovian jump H∞ filter can be obtained as follows

$$A_{f1} = \begin{bmatrix} -1.4278 & 0.7459 & 0.4686 \\ -0.6967 & -2.7564 & 0.9989 \\ -1.2519 & -1.0541 & -2.0500 \end{bmatrix}, B_{f1} = \begin{bmatrix} -0.4989 \\ 0.6196 \\ 2.2503 \end{bmatrix}, L_{f1} = \begin{bmatrix} 0.3042 & 0.0467 & 0.7872 \end{bmatrix};$$

$$A_{f2} = \begin{bmatrix} -1.5571 & 0.2940 & 0.0074 \\ 0.2446 & -2.0474 & 0.2408 \\ -0.7197 & 0.7592 & -2.2669 \end{bmatrix}, B_{f2} = \begin{bmatrix} -0.0024 \\ -0.0635 \\ -0.2463 \end{bmatrix}, L_{f2} = \begin{bmatrix} 0.0037 & 0.5730 & 0.3981 \end{bmatrix}.$$

The simulation result of the state response of the real states $x(t)$ and their estimates $\hat{x}(t)$ are displayed in Fig. 3. Fig. 4 is the simulation result of the estimation error response of $\tilde{z}(t) = z(t) - \hat{z}(t)$. The simulation results demonstrate that the estimation error is robustly exponentially mean-square stable, and thus it can be seen that the designed filter satisfies the specified performance requirements and all the expected objectives are well achieved.

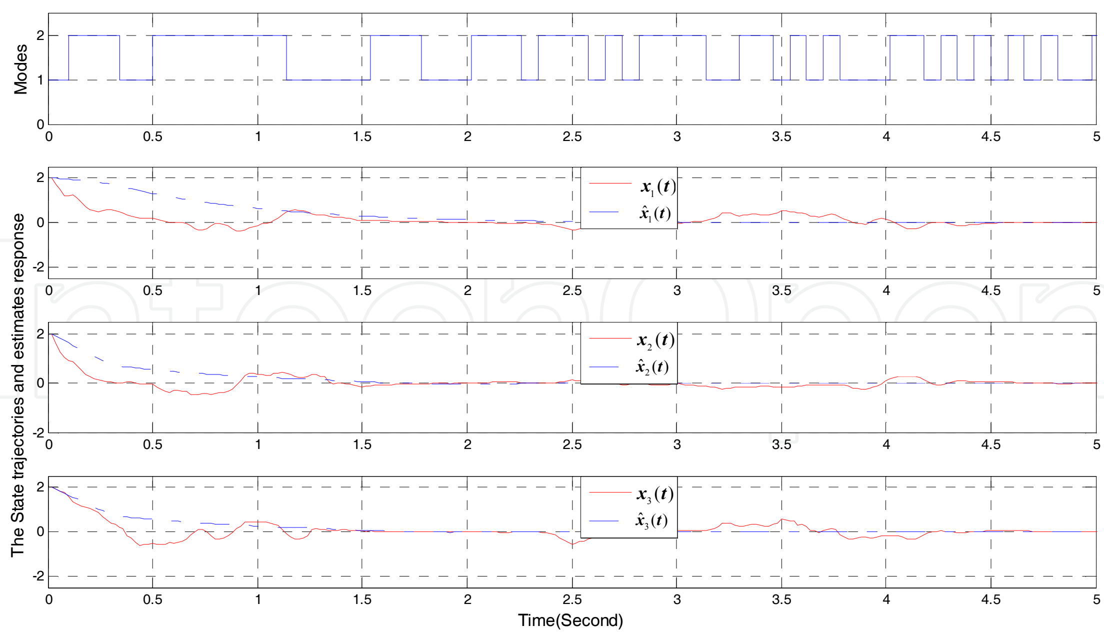

Fig. 3. The state trajectories and estimates response

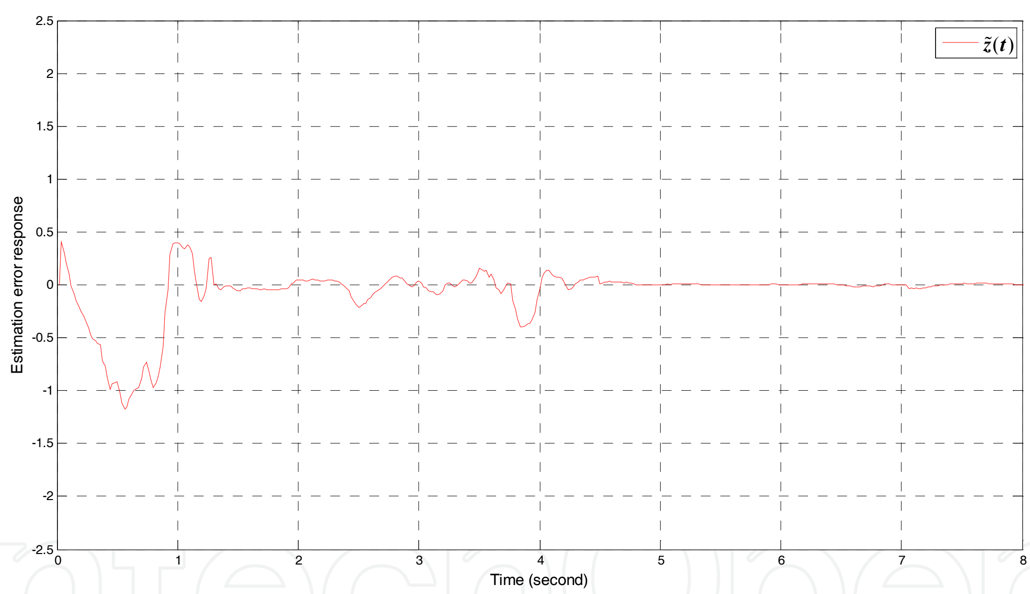

Fig. 4. The estimation error response

4. Conclusion

Both delay-dependent exponential mean-square stability and robust H∞ filtering for time-delay a class of *Itô* stochastic systems with time-varying delays and nonlinearities has addressed in this chapter. Novel stability criteria and H∞ exponential filter design methods are proposed in terms of LMIs. The new criteria are much less conservative than some existing results. The desired filter can be constructed through a convex optimization problem. Numerical examples and simulations have demonstrated the effectiveness and usefulness of the proposed methods.

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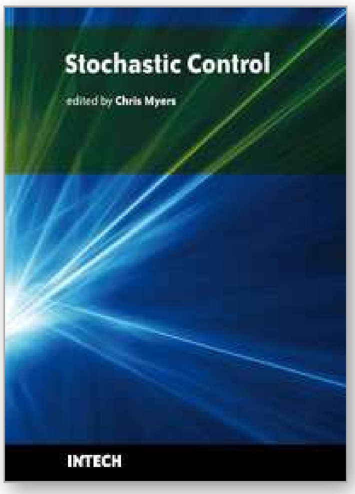

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Uncertainty presents significant challenges in the reasoning about and controlling of complex dynamical systems. To address this challenge, numerous researchers are developing improved methods for stochastic analysis. This book presents a diverse collection of some of the latest research in this important area. In particular, this book gives an overview of some of the theoretical methods and tools for stochastic analysis, and it presents the applications of these methods to problems in systems theory, science, and economics.