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From Conformal Geometric Algebra to Spherical Harmonics for a Correlation with Lines

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1. Introduction

In this chapter we will apply the classic theory of Harmonic Analysis and the Conformal Geometric Algebra to evaluate the Radon transform on the unit sphere S^2 and on the rotation group $SO(3)$ to recover the 3D camera rotation. Since the images taken by omnidirectional sensors can be mapped to the sphere, the problem of attitude estimation of a 3D camera rotation can be treated as a problem of estimating rotations between spherical images.

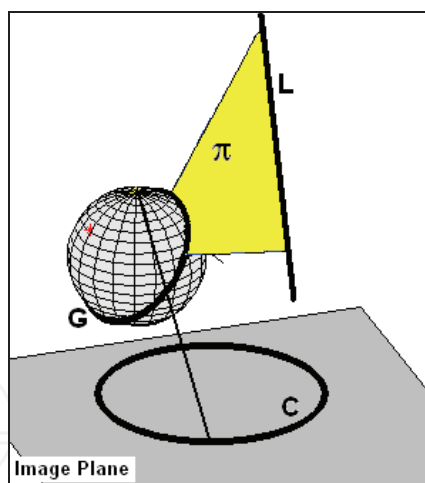


Fig. 1. A 3D line L as a circle C in the image plane.

From [Geyer & Daniilidis, 2000], we know that the parabolic, hyperbolic and elliptic mirrors are equivalent to the equivalent sphere, that is, a 3D-point P is projected to the sphere in a point η which is the intersection of the sphere and the line from the origin to P . Then, if the mirror is parabolic, η is stereographically projected from the north pole to the image plane $z=0$. See Figure 1. This is the reason to study harmonic analysis on S^2 and $SO(3)$. In recent years harmonic analysis has been used in computer vision to obtain 3D rotations with the Radon and Hough transforms. In [Geyer et al., 2004] harmonic analysis is used to obtain

the *essential matrix* of two omnidirectional images. In [Makadia et al., 2005] and [Makadia & Daniilidis, 2003] the Euler angles are obtained with the Radon transform as a correlation of points on S^2 and $SO(3)$. In [Falcon-Morales & Bayro-Corrochano, 2007] the Radon transform is defined as a correspondence of lines to obtain a 3D rotation. In reference to the notation and theory about spherical harmonics we are following [Arfken & Weber, 1966] and [Chirikjian & Kyatkin, 2001]. The objective is to combine the almost forgotten mathematical framework conformal geometric algebra with the classical analysis theory, to obtain a different approximation, focus on geometric entities, to a well know omnidirectional vision problem.

2. Geometric Algebra

The algebras of Clifford and Grassmann are well known to pure mathematicians, but since the beginning were abandoned by physicists in favor of the vector algebra of Gibbs, the commonly algebra used today in most areas of physics. The approach to Clifford algebra that we adopt here has been developed since the 1960's by David Hestenes. See [Hestenes & Sobczyk, 1984] and [Li et al., 2001].

2.1 Basic definitions

Let V_n be a vector space of dimension n . We are going to define and generate an algebra G_n , called *geometric algebra*. Let $\{e_1, e_2, \dots, e_n\}$ be a set of basis vectors of V_n . The *scalar multiplication* and *sum* in G_n are defined in the usual way of a vector space. The *product* or *geometric product* of elements of the basis of G_n will be simply denoted by juxtaposition. In this way, from any two basis vectors e_j and e_k , a new element of the algebra is obtained and denoted as $e_j e_k \equiv e_{jk}$. The product of basis vectors is anticommutative, that is,

$$e_j e_k \equiv -e_k e_j, \quad \forall j \neq k. \quad (1)$$

The basis vectors must *square* in $+1, -1$ or 0 , this means that there are no-negative integers p , q and r such that $n = p + q + r$ and

$$e_i e_i = e_i^2 = \begin{cases} +1 & \text{for } i = 1, \dots, p \\ -1 & \text{for } i = p+1, \dots, p+q. \\ 0 & \text{for } i = p+q+1, \dots, n \end{cases} \quad (2)$$

This *product* will be called the *geometric product* of G_n . With these operations G_n is an associative linear algebra with identity and it is called the *geometric algebra* or *Clifford algebra* of dimension $n = p + q + r$, generated by the vector space V_n . It is usual to write $G_{p,q,r}$ instead of G_n . The elements of this geometric algebra are called *multivectors*, because they are entities generated by the sum of elements of *mixed grade* of the basis set of G_n , such as

$$A = \langle A \rangle_0 + \langle A \rangle_1 + \dots + \langle A \rangle_n \quad (3)$$

where the multivector $A \in G_n$ is expressed by the addition of its 0-vector part (or scalar part) $\langle A \rangle_0$, its 1-vector part (or vector part) $\langle A \rangle_1$, its 2-vector part (or bivector part) $\langle A \rangle_2$, its 3-vector part (or trivector part) $\langle A \rangle_3$, and in general its n -vector part $\langle A \rangle_n$. A multivector $A \in G_n$ is called *homogeneous* of grade r if $A = \langle A \rangle_r$.

It will be convenient to define other products between the elements of this algebra which will allow us to set up several geometric relations (unions, intersections, projections, etc.) between different geometric entities (points, lines, planes, spheres, etc.) in a very simple way.

Firstly, we define the *inner product* $a \cdot b$, and the *exterior or wedge product* $a \wedge b$, of any two 1-vectors a and b , as the *symmetric* and *antisymmetric* parts of the geometric product ab , respectively. That is, using the expression

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba) \quad (4)$$

we can define the inner product

$$a \cdot b = \frac{1}{2}(ab + ba) \quad (5)$$

and the outer or wedge product

$$a \wedge b = \frac{1}{2}(ab - ba). \quad (6)$$

Now, from (4), (5) and (6) we can express the geometric product of two vectors as

$$ab = a \cdot b + a \wedge b. \quad (7)$$

From (5) and (6), $a \cdot b = b \cdot a$, and $a \wedge b = -b \wedge a$.

Now we can define the inner and outer products for more general elements. For any two homogeneous multivectors A_r and B_s of grades r and s , we define the *inner product*

$$A_r \cdot B_s = \begin{cases} \langle A_r B_s \rangle_{|r-s|} & \text{if } r > 0 \text{ and } s > 0 \\ 0 & \text{if } r = 0 \text{ or } s = 0 \end{cases} \quad (8)$$

and the *outer or wedge product*

$$A_r \wedge B_s = \langle A_r B_s \rangle_{r+s}. \quad (9)$$

By definition, for a scalar α and a homogeneous multivector A , $\alpha \cdot A = 0$ and $\alpha \wedge A = \alpha A$. The dual, A^* , of the multivector A is defined as $A^* = AI_n^{-1}$ where $I_n \equiv e_{12\dots n}$ is the unit pseudoscalar of G_n . And the inverse of a multivector A , if it exists, is defined by the equation $A^{-1}A = 1$.

We say that an homogeneous vector A_r is an r -blade or a blade of grade r if $A_r = a_1 \wedge a_2 \wedge \dots \wedge a_r$, for 1-vectors a_1, a_2, \dots, a_r and $A_r \neq 0$.

From (8) it can be said that the inner product $A_r \cdot B_s$ lowers the grade of A_r by s units when $r \geq s > 0$, and from equation (9) that the outer product $A_r \wedge B_s$ raises the grade of A_r by s units for every $r, s \geq 0$.

The manipulation of multivectors is easier with the use of the next recursively equality of two blades $A_r = a_1 \wedge a_2 \wedge \dots \wedge a_r$, and $B_s = b_1 \wedge b_2 \wedge \dots \wedge b_s$,

$$A_r \cdot B_s = \begin{cases} ((a_1 \wedge a_2 \wedge \dots \wedge a_r) \cdot b_1) \cdot (b_2 \wedge b_3 \wedge \dots \wedge b_s) & \text{if } r \geq s \\ (a_1 \wedge a_2 \wedge \dots \wedge a_{r-1}) \cdot (a_r \cdot (b_1 \wedge b_2 \wedge \dots \wedge b_s)) & \text{if } r < s \end{cases} \quad (10)$$

where

$$(a_1 \wedge a_2 \wedge \dots \wedge a_r) \cdot b_1 = \sum_{i=1}^r (-1)^{r-i} a_1 \wedge \dots \wedge a_{i-1} \wedge (a_i \cdot b_1) \wedge a_{i+1} \wedge \dots \wedge a_r, \quad (11)$$

and

$$a_r \cdot (b_1 \wedge b_2 \wedge \dots \wedge b_s) = \sum_{i=1}^s (-1)^{i-1} b_1 \wedge \dots \wedge b_{i-1} \wedge (a_r \cdot b_i) \wedge b_{i+1} \wedge \dots \wedge b_s. \quad (12)$$

2.2 Conformal Geometric Algebra

The geometric algebra of a 3D Euclidean space $G_{3,0,0}$ has a point basis and the motor algebra $G_{3,0,1}$ a line basis. In the latter the lines expressed in terms of Plücker coordinates can be used to represent points and planes as well, [Bayro-Corrochano et al., 2000]. In the conformal geometric algebra the unit element is the sphere, which will allow us to represent other entities. We begin giving an introduction in conformal geometric algebra following the same formulation presented in [Li et al., 2001] and [Bayro-Corrochano, 2001] and showing how the Euclidean vector space R^n is represented in $R^{n+1,1}$.

Let $R^{n+1,1}$ be the vector space with an orthonormal vector basis given by $\{e_1, \dots, e_n, e_+, e_-\}$, with the property (1) expressed as:

$$e_i^2 = 1, e_+^2 = 1, e_-^2 = -1, \quad (13)$$

$$e_i \cdot e_+ = e_i \cdot e_- = e_+ \cdot e_- = 0 \quad (14)$$

for $i = 1, \dots, n$. Now, we define the *null basis* $\{e_0, e_\infty\}$ as

$$e_0 = \frac{1}{2}(e_- - e_+), \tag{15}$$

$$e_\infty = e_- + e_+ \tag{16}$$

where from (4) and (5) we have the properties

$$e_0^2 = e_\infty^2 = 0, e_\infty \cdot e_0 = -1. \tag{17}$$

A unit pseudoscalar $E \in \mathbb{R}^{1,1}$, representing the Minkowski plane, is defined by

$$E = e_\infty \wedge e_0, \tag{18}$$

and from (10), (11), (12), (17) and (18) we have that

$$E^2 = (e_\infty \wedge e_0) \cdot (e_\infty \wedge e_0) = ((e_\infty \wedge e_0) \cdot e_\infty) \cdot e_0 = (e_\infty \wedge (e_0 \cdot e_\infty) - (e_\infty \cdot e_\infty) \wedge e_0) \cdot e_0, \tag{19}$$

$$= (e_\infty \wedge (-1) - (0) \wedge e_0) \cdot e_0 = -e_\infty \cdot e_0 = 1, \tag{20}$$

that is, for the Minkowski plane E , $E^2 = 1$.

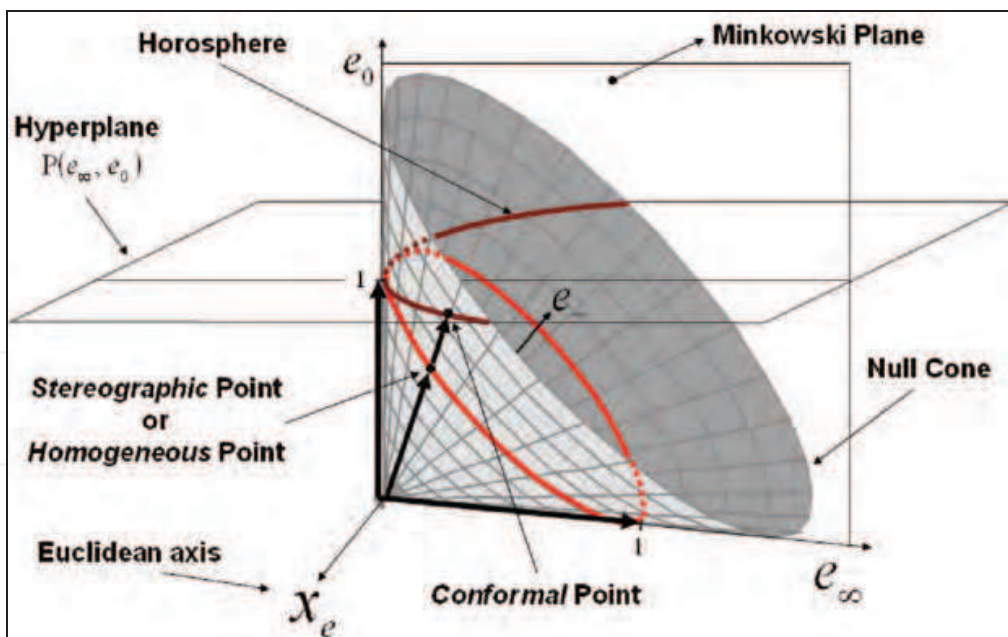


Fig. 2. The one dimensional null cone.

One of the results of the non-Euclidean geometry demonstrated by Nikolai Lobachevsky in the XIX century is that in spaces with hyperbolic structure we can find subsets which are isomorphic to a Euclidean space. In order to do this, Lobachevsky introduced two

constraints to the now so-called *conformal point* $x_c \in \mathbb{R}^{n+1,1}$. See Figure 2. The first constraint is the *homogeneous* representation of the conformal point x_c , which is obtained by the *normalization*

$$x_c \cdot e_\infty = -1, \quad (21)$$

and the second constraint is to make the conformal point a *null vector*, that is,

$$x_c^2 = 0. \quad (22)$$

Thus, conformal points are required to lie in the intersection space, denoted N_e^n , between the *null cone* N^{n+1} and the *hyperplane* $P(e_\infty, e_0)$, that is

$$N_e^n = N^{n+1} \cap P(e_\infty, e_0) = \{x_c \in \mathbb{R}^{n+1,1} \mid x_c^2 = 0, x_c \cdot e_\infty = -1\}. \quad (23)$$

The constraints (21) and (22) define an isomorphic mapping between the Euclidean and the conformal space. Thus, for each conformal point $x_c \in \mathbb{R}^{n+1,1}$ there is a unique Euclidean point $x_e \in \mathbb{R}^n$ and unique scalars α, β such that the mapping $x_e \mapsto x_c = x_e + \alpha e_\infty + \beta e_0$ is bijective. From (21) and (22) we can now obtain the values of the scalars, $\alpha = \frac{1}{2}x_e^2$ and $\beta = 1$. Then, the *standard form* of a conformal point x_c is

$$x_c = x_e + \frac{1}{2}x_e^2 e_\infty + e_0. \quad (24)$$

3. Orthogonal Expansion in Spherical Coordinates

We use the spherical coordinates as a parameterization of S^2 . Let θ be the meridian angle measure from the north pole which is called colatitude or polar angle. Let ϕ the angle measure on the equator in a counter-clockwise direction, and where $\phi = 0$ correspond to the x-axis. ϕ is called azimuth or longitude. By definition $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. See

Figure 3. Thus, any point $u = u(\theta, \phi) \in S^2$ has a unique representation on the unit sphere as $u = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$. The unit sphere in the Euclidean space \mathbb{R}^3 is a two dimensional surface denoted as S^2 defined by the constraint $x_1^2 + x_2^2 + x_3^2 = 1$. If f is a real-valued function on S^2 , its integral is performed as

$$\int_{S^2} f ds = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} f(\theta, \phi) \sin(\theta) d\theta d\phi. \quad (25)$$

Since sine function is defined only for μ , the volume element in (25) can be viewed as the product of the volume elements $[0, \pi]$ and S^2 , with the weighting factor $\sin(\theta)$.

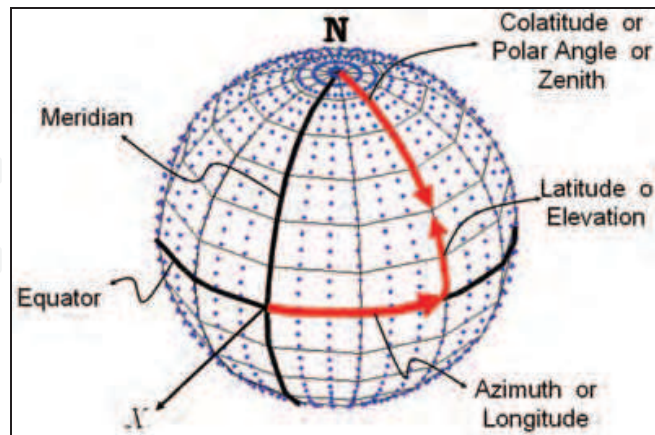


Fig. 3. Parametrization of S^2 with band-limit bw .

This allows us to use the Sturm-Liouville theory to generate orthogonal functions in these two domains separately. From classical theory an orthogonal basis for $L^2(S^1)$ is the set $\{e^{im\phi}\}_{m \in \mathbb{Z}}$ where S^1 is the unit circle and L^2 the Hilbert space of square integrable functions. Likewise, an orthogonal basis for $L^2([-1, 1], dx)$ is given by the Legendre polynomials $P_l(x)$. Using the change of variable $x = \cos(\theta)$, the functions

$$\sqrt{\frac{2l+1}{2}} P_l(\theta) \quad (26)$$

are an orthogonal basis of the space $L^2([0, \pi], \sin(\theta)d\theta)$, and the set of functions

$$\sqrt{\frac{2l+1}{4\pi}} P_l(\theta) e^{im\phi} \quad (27)$$

where $l = 0, 1, 2, \dots$, and $m \in \mathbb{Z}$, form a complete orthonormal set of functions on the sphere S^2 . It is much more common to choose the associated Legendre functions $\{P_l^m(\theta)\}$, where each integer $m \in \mathbb{Z}$ satisfy $|m| \leq l$. Thus, the elements of the orthogonal basis that we need to expand functions on the sphere are of the form

$$Y_l^m(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos \theta) e^{im\phi}, \quad (28)$$

and they are called *spherical harmonics*. Then, given any function $f \in L^2(S^2)$, its spherical Fourier series is given as

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \hat{f}(l, m) Y_l^m(\theta, \phi), \quad (29)$$

where

$$\hat{f}(l, m) = \int_{S^2} f \overline{Y_l^m} ds. \quad (30)$$

Each coefficient $\hat{f}(l, m)$ will be called the *spherical Fourier transform* of $f(\theta, \phi)$, and the set $\{\hat{f}(l, m)\}$ is called the *spectrum on the sphere* of $f(\theta, \phi)$. The spherical harmonics $Y_l^m(\theta, \phi)$ are the usual common functions used to expand functions on the sphere because they are eigenfunctions of the Laplacian operator. Indeed, for a constant radius $r = 1$ the Laplacian of a smooth function f in spherical coordinates is given as

$$\nabla^2(f) = \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2 f}{\partial \phi^2}. \quad (31)$$

In analogy with the Sturm-Liouville theory, an eigenfunction of the Laplacian operator is defined as

$$\nabla^2(f) = \lambda f \quad (32)$$

for some eigenvalues λ . Moreover, in spherical coordinates the boundary conditions are periodic in the variable ϕ , that is, $f(\theta, \phi + 2\pi) = f(\theta, \phi)$, and a solution set of eigenfunctions of the Laplacian (32) is the set of spherical harmonics (28) with eigenvalue $\lambda = -l(l+1)$.

4. Representation Theory on SO(3)

From a classical result of linear algebra a matrix R is a *rotation matrix* or a *special orthogonal matrix* if and only if $RR^T = I$, where I is the identity matrix, and the determinant satisfies $\det(R) = +1$. Let $SO(3)$ be the *special orthogonal group* or *rotation group* for three dimensional space.

In the same way that with the square integrable functions on the unit circle or on the line, we can say that a function is square integrable on the space $L^2(SO(3))$ if

$$\int_{SO(3)} |f(R)|^2 d(R) < \infty. \quad (33)$$

$L^2(SO(3))$ is a vector space with the usual addition and scalar multiplication, that is, $(f + g)(R) = f(R) + g(R)$ and $(\alpha f)(R) = \alpha f(R)$.

We can now define the convolution of two functions f and $g \in L^2(SO(3))$ as

$$(f * g)(R) = \int_{SO(3)} f(Q)g(Q^{-1}R)d(R). \quad (34)$$

See [Chirikjian & Kyatkin, 2001]. On the contrary to the classical case, the convolution on $L^2(SO(3))$ is not a commutative product.

With the Euler parameterization ZYZ now we can state one of the most important properties of the spherical harmonics

$$Y_l^n(R^T \eta) = \sum_{|k| \leq l} U_{kn}^l(R) Y_l^k(\eta) \quad (35)$$

where $\eta \in S^2$, $R \in SO(3)$ and the $(2l+1) \times (2l+1)$ unitary matrices U^l are the irreducible representation of $SO(3)$, where their components are given as

$$U_{mn}^l(R(\alpha, \beta, \gamma)) = i^{m-n} e^{-im\alpha} P_{mn}^l(\cos \beta) e^{-in\gamma}, \quad (36)$$

and P_{mn}^l are the generalized associated Legendre functions. In fact, the unitary matrices U^l are the representation group of $SO(3)$ and they constitute a basis which can be used to obtain a Fourier transform on the rotation group, that is

$$f(R) = \sum_l \sum_{|m| \leq l} \sum_{|k| \leq l} \hat{f}_{mk}^l U_{mk}^l(R) \quad (37)$$

where the Fourier coefficients of f on the rotation group $SO(3)$ are

$$\hat{f}_{mk}^l = \int_{R \in SO(3)} f(R) \overline{U_{mk}^l(R)} dR. \quad (38)$$

Now, from (35) we can obtain a *shift theorem*, relating the coefficients of the rotation functions

$$h(\eta) = f(R^{-1}\eta) \Leftrightarrow \hat{h}_m^l = \sum_{|k| \leq l} \hat{f}_k^l U_{mk}^l(R). \quad (39)$$

This *shift theorem* is telling us that the effect caused by a rotation R on a spherical function f , is equivalent in the Fourier space to the effect of the Fourier coefficients \hat{f}_k^l by the unitary matrices U_{mk}^l of the irreducible representation of $SO(3)$. More explicitly, while the spherical functions are rotated by orthogonal matrices, the Fourier coefficients of longitude $(2l+1)$ are affected by the unitary matrices U^l .

As expected, this theory can be extended to the direct product group $SO(3) \times SO(3)$ acting on the homogenous space $S^2 \times S^2$. Thus, the expansion of functions on $S^2 \times S^2$ is given as

$$f(\omega_1, \omega_2) = \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} \sum_{n \in \mathbb{N}} \sum_{|p| \leq n} \hat{f}_{np}^{ln} Y_l^m(\omega_1) Y_n^p(\omega_2) \quad (40)$$

where the Fourier transform on $S^2 \times S^2$ is

$$\hat{f}_{mp}^{ln} = \int_{\omega_1 \in S^2} \int_{\omega_2 \in S^2} f(\omega_1, \omega_2) \overline{Y_l^m(\omega_1) Y_n^p(\omega_2)} d\omega_1 d\omega_2. \quad (41)$$

Also, a *shift theorem* exists for functions on $S^2 \times S^2$ given as

$$h(\omega_1, \omega_2) = f(R_1^T \omega_1, R_2^T \omega_2) \Leftrightarrow \hat{h}_{mp}^{ln} = \sum_{|r| \leq l} \sum_{|q| \leq n} U_{rm}^l(R_1) U_{qp}^n(R_2) \hat{f}_{rq}^{ln}. \quad (42)$$

These expressions have been used in [Makadia et al., 2005] and [Makadia & Daniilidis, 2003] to obtain the Euler angles of a 3D rotation, using the *point correlation* of two given images without correspondences.

5. Radon Transform with Lines

In this section we will extend the way of obtain the Euler angles of a 3D rotation as presented in [Makadia et al., 2005]. As these authors used correlation between *points* of two images, we will use *lines* instead of points. Note that lines are less noise sensitive. From [Makadia et al., 2005] the Radon transform on points is defined as

$$G(R, t) = \int_{p \in S^2} \int_{q \in S^2} g(p, q) \Delta(Rp, q, t) dp dq \quad (43)$$

where the *similarity* function g is based on the SIFT points. See [Lowe, 2004] for details of the SIFT algorithm. For sake of simplicity from now on we will call *sift points* to the sift descriptors returned by the SIFT algorithm. The Δ function is the Kronecker delta function relating the points of two images with the epipolar constraint of a stereo camera system. Now, to extend the Radon transform to a correlation of lines to estimate pure rotations, we

need to define analogue *similarity* and *delta* functions for lines, instead of points, as well as a *constraint* for lines instead of the epipolar constraint for points.

Let $R \in SO(3)$ be a 3D rotation relating two 3D lines, l and l' which were projected to two omnidirectional images $Im1$ and $Im2$. Then we can write $l' = Rl$ or $l = R^T l'$. The three dimensional line l is associated to a great circle C on the sphere S^2 . Let $\eta \in S^2$ be an orthogonal vector to the plane containing the great circle C , then η and $R^T l'$ are orthogonal, that is, $\eta^T R^T l' = 0$. We will use this constraint as the delta function to define our desired integral, that is, as the constraint for lines. Thus, the integral of the *Radon transform on lines* would be

$$G(R) = \int_{l' \in S^2} \int_{\eta \in S^2} g(\eta, l') \Delta(\eta^T R^T l') d\eta dl' \quad (44)$$

where g is a *similarity function* between the lines of both images and Δ the delta Kronecker function over the constraint with lines. So, (44) can be used as a correlation function between g and Δ , where $g, \Delta: S^2 \times S^2 \rightarrow \{0, 1\}$.

Although we know how to calculate the analytical expressions of the continuous Fourier and Radon transforms in spherical coordinates, it is necessary a discretization process for their applications with real omnidirectional images. Thus, given a function on the space $L^2(S^2)$ with band-limit bw , its spherical Fourier transform SFT can be obtained with the FFT algorithm of order $O((bw)^2 \log^2(bw))$ on S^2 , see [Driscoll & Healy, 1994] for details. Similarly, we can use the FFT algorithm of order $O((bw)^3 \log^2(bw))$ in the case of the rotation space $SO(3)$, see [Kostelec & Rockmore, 2003] for details.

Then, applying the spherical Fourier expansion (40) to the similarity function g of (44), we have that

$$g(\omega_1, \omega_2) = \sum_{l_1=0}^{bw-1} \sum_{|m_1| \leq l_1} \sum_{l_2=0}^{bw-1} \sum_{|m_2| \leq l_2} g_{m_1 m_2}^{l_1 l_2} Y_{l_1}^{m_1}(\omega_1) Y_{l_2}^{m_2}(\omega_2), \quad (45)$$

where we used $g(\omega_1, \omega_2)$ instead of $g(\eta, l')$ to simplify notation. Likewise, we get $\Delta(\omega_1, R\omega_2)$, the expansion of the Δ function for each $(\omega_1, \omega_2) \in S^2 \times S^2$ and $R \in SO(3)$ as

$$\Delta(\omega_1, R\omega_2) = \sum_{p_1=0}^{bw-1} \sum_{|k_1| \leq p_1} \sum_{p_2=0}^{bw-1} \sum_{|k_2| \leq p_2} \hat{\Delta}_{k_1 k_2}^{p_1 p_2}(R) Y_{p_1}^{k_1}(\omega_1) Y_{p_2}^{k_2}(\omega_2). \quad (46)$$

Now, because $0 = \omega_1^T R \omega_2 = (\omega_1^T R \omega_2)^T = \omega_2^T R^T \omega_1$, we can write $\hat{\Delta}_{k_1 k_2}^{p_1 p_2}(\omega_1, R^T \omega_2)$ instead of $\hat{\Delta}_{k_1 k_2}^{p_1 p_2}(R)$ in (46), and by the shift theorem

$$\Delta_{k_1 k_2}^{p_1 p_2}(\omega_1, R^T \omega_2) = \sum_{|b| \leq p_2} U_{bk_2}^{p_2}(R) \hat{\Delta}_{k_1 b}^{p_1 p_2}, \quad (47)$$

where $\hat{\Delta}_{k_1 b}^{p_1 p_2} \equiv \hat{\Delta}_{k_1 b}^{p_1 p_2}(\omega_1, \omega_2)$.

Substituting (45), (46) and (47) in the Radon transform on lines (44) we get

$$G(R) = \int_{\omega_1 \in S^2} \int_{\omega_2 \in S^2} (B_1)(B_2) d\omega_1 d\omega_2 \quad (48)$$

where

$$B_1 = \sum_{l_1} \sum_{|m_1| \leq l_1} \sum_{l_2} \sum_{|m_2| \leq l_2} \hat{g}_{m_1 m_2}^{l_1 l_2} Y_{l_1}^{m_1}(\omega_1) Y_{l_2}^{m_2}(\omega_2), \quad (49)$$

$$B_2 = \sum_{p_1} \sum_{|k_1| \leq p_1} \sum_{p_2} \sum_{|k_2| \leq p_2} (B_3) \overline{Y_{p_1}^{k_1}(\omega_1) Y_{p_2}^{k_2}(\omega_2)} \quad (50)$$

and

$$B_3 = \sum_{|b| \leq p_2} \overline{U_{bk_2}^{p_2}(R) \hat{\Delta}_{k_1 b}^{p_1 p_2}}. \quad (51)$$

Interchanging integrals and summation in (48) we can write it now as

$$G(R) = \sum_{l_1} \sum_{|m_1| \leq l_1} \sum_{l_2} \sum_{|m_2| \leq l_2} \sum_{p_1} \sum_{|k_1| \leq p_1} \sum_{p_2} \sum_{|k_2| \leq p_2} (D_1)(D_2)(D_3), \quad (52)$$

where

$$D_1 = \sum_{|b| \leq p_2} \hat{g}_{m_1 m_2}^{l_1 l_2} \overline{U_{bk_2}^{p_2}(R) \hat{\Delta}_{k_1 b}^{p_1 p_2}}, \quad (53)$$

$$D_2 = \int_{\omega_1 \in S^2} Y_{l_1}^{m_1}(\omega_1) \overline{Y_{p_1}^{k_1}(\omega_1)} d\omega_1 \quad (54)$$

and

$$D_3 = \int_{\omega_2 \in S^2} Y_{l_2}^{m_2}(\omega_2) \overline{Y_{p_2}^{k_2}(\omega_2)} d\omega_2. \quad (55)$$

Now, applying the orthonormal property

$$\int_{\omega \in S^2} Y_l^m(\omega) \overline{Y_{l'}^{m'}(\omega)} d\omega = \delta_{ll'} \delta_{mm'}, \quad (56)$$

to (54) and (55), the expression (48) for the Radon transform $G(R)$ is reduced now to

$$G(R) = \sum_{l_1} \sum_{|m_1| \leq l_1} \sum_{l_2} \sum_{|m_2| \leq l_2} \sum_{|b| \leq p_2} \hat{g}_{m_1 m_2}^{l_1 l_2} \overline{\hat{\Delta}_{m_1 b}^{l_1 l_2}} U_{bm_2}^{l_2}(R). \quad (57)$$

We know that in $SO(3)$ the unitary elements U 's are an orthonormal set too, that is,

$$\int_{R \in SO(3)} U_{mk}^l(R) \overline{U_{m'k'}^{l'}(R)} dR = \delta_{ll'} \delta_{mm'} \delta_{kk'}. \quad (58)$$

Then, for each $l_2 = 0, 1, \dots, (bw-1)$ and $b', m_2' \in \{-l_2, \dots, -1, 0, 1, \dots, l_2\}$ we can multiply (57) by $U_{b'm_2'}^{l_2'}(R)$, and take the integration on $SO(3)$ to obtain

$$\int_{R \in SO(3)} G(R) U_{b'm_2'}^{l_2'} dR = \sum_{l_1} \sum_{|m_1| \leq l_1} \sum_{l_2} \sum_{|m_2| \leq l_2} \sum_{|b| \leq p_2} \hat{g}_{m_1 m_2}^{l_1 l_2} \overline{\hat{\Delta}_{m_1 b}^{l_1 l_2}} \int_{R \in SO(3)} U_{b'm_2'}^{l_2'}(R) \overline{U_{bm_2}^{l_2}(R)} dR \quad (59)$$

and from (58)

$$\int_{R \in SO(3)} G(R) U_{b'm_2'}^{l_2'} dR = \sum_{l_1} \sum_{|m_1| \leq l_1} \hat{g}_{m_1 m_2'}^{l_1 l_2'} \overline{\hat{\Delta}_{m_1 b'}^{l_1 l_2'}}. \quad (60)$$

From (60) we can have now the 3D Fourier transform \hat{G} of G on $SO(3)$. Indeed, rewriting indices without primes in (60), the Fourier transform \hat{G} of G on the rotation group $SO(3)$ is given as

$$\hat{G} = \left(\hat{G}_{bm_2}^{l_2} \right) \quad (61)$$

where $l_2 = 0, 1, \dots, (bw-1)$ and $b, m_2 \in \{-l_2, \dots, -1, 0, 1, \dots, l_2\}$, and

$$\hat{G}_{bm_2}^{l_2} = \sum_{l_1} \sum_{|m_1| \leq l_1} \hat{g}_{m_1 m_2}^{l_1 l_2} \overline{\hat{\Delta}_{m_1 b}^{l_1 l_2}}. \quad (62)$$

The expressions (61) and (62) obtained for lines and a pure rotation is consistent with the formula for points obtained in [Makadia et al., 2005].

6. Radon Transform with Lines using Conformal Geometric Algebra

The conformal geometric algebra is a mathematical framework that helps to unify matrices, vectors, transformations, complex numbers in one unique theory using the geometric product, with its inner and wedge products, to generate the former mathematical concepts.

Let $R \in SO(3)$ be a rotation relating two 3D lines, L_1 and L_2 , which were projected to two omnidirectional images $Im1$ and $Im2$ using the equivalent sphere as depicted in Figure 1. In

conformal geometric algebra the 3D rotation can be expressed as $R = e^{-\frac{\varphi}{2}\hat{n}}$, where \hat{n} is unit bivector which represents the dual of the rotation axis, and the φ angle, which represents the amount of the rotation. Then, these lines satisfy $L_2 = RL_1R^{-1}$, where the lines are expressed in conformal form too, that is, the dual form is $L_j^* = x_{j1} \wedge x_{j2} \wedge e_\infty$, $j \in \{1,2\}$ for two points $x_{j1}, x_{j2} \in L_j^*$. Using the origin e_0 of the conformal space, we can obtain the planes Π_j generated by the 3D line L_j and the origin e_0 , that is, the dual planes are

$$\Pi_j^* = L_j^* \wedge e_0 = x_{j1} \wedge x_{j2} \wedge e_\infty \wedge e_0 = x_{j1} \wedge x_{j2} \wedge E, \quad (63)$$

where E represents the Minkowski plane. The intersection these planes and the unit sphere S^2 are two great circles, expressed by the multivector $\eta_j^* = z_{j1} \wedge z_{j2} \wedge z_{j3}$, with dual orthogonal axis η_j , where z_j are conformal points on the great circle. Thus, if $L_2 = RL_1R^{-1}$, then the dual plane $\Pi_1^* = L_1^* \wedge e_0$ is orthogonal to the dual orthogonal axis η_2 of the dual plane $\Pi_2^* = L_2^* \wedge e_0$, and

$$\Pi_1^* \cdot \eta_2^* = 0, \quad (64)$$

that is

$$(x_{11} \wedge x_{12} \wedge E) \cdot (z_{21} \wedge z_{22} \wedge z_{23}) / I = 0, \quad (65)$$

where I is the unit pseudoscalar of the five dimensional conformal space. Thus, we have a correspondence between 3D points obtained from x_{11} and $x_{12} \in L_1$ and points η_2 on the unit sphere S^2 . Then the characteristic function Δ must obey the constraint for lines

$$\Delta(R\eta_1, \eta_2) = \delta\left(\left(R\eta_1R^{-1}\right) \cdot \eta_2\right) = \begin{cases} 1 & \text{if } \left|\left(R\eta_1R^{-1}\right) \cdot \eta_2\right| < \varepsilon, \\ 0 & \text{other case} \end{cases}, \quad (66)$$

for $\varepsilon > 0$ depending of the band-limit used in the discretization process. Thus, the characteristic function (66) is measuring how close the two lines L_1 and L_2 are.

Finally, following the work in [Makadia & Daniilidis, 2003] and [Makadia et al., 2005], we take the similarity function as

$$g(l_1, l_2) = e^{-\|l_1 - l_2\|}, \quad (67)$$

where $\|\cdot\|$ is the Euclidean norm and l_1 and l_2 are the 128-dimensional sift vectors of the sift algorithm applied to each omnidirectional image. See Figure 4.

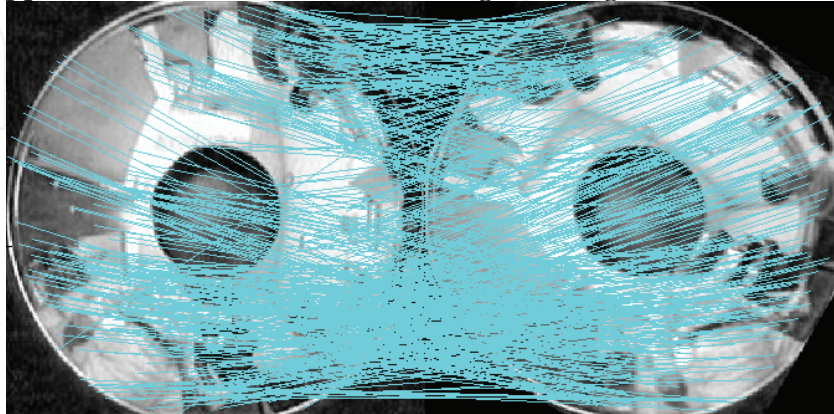


Fig. 4. SIFT descriptors between two omnidirectional images related by a rotation.

Notice that in the Radon transform with lines (44) the domain of the integral is now on the unit sphere but in conformal representation.

Future implementations with real and simulated images will be used to verify and compare the efficiency of the theory with respect to the works in [Makadia et al., 2005] and [Falcon & Bayro-Corrochano, 2007].

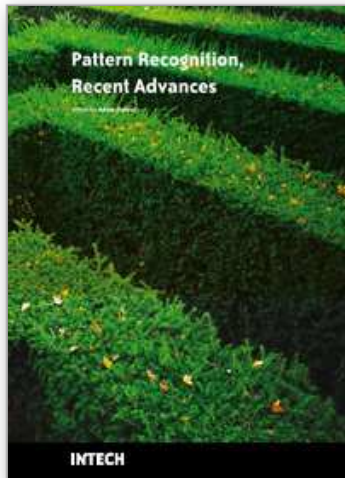
7. Conclusions

This chapter can give us an idea about how conformal geometric algebra can be used with traditional mathematical theory in order to expand the applications of this almost forgotten framework. The authors believe that this theoretical framework can be used to obtain different approximations to old and new computer vision problems. The author believe the use of harmonic analysis based on Radon transform using lines and conformal geometric algebra on incidence algebra is promising for omnidirectional image processing.

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