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A survey for descriptor weakly nonlinear dynamic systems with applications

Athanasios D. Karageorgos¹, Athanasios A. Pantelous^{1,2,3}
and Grigoris I. Kalogeropoulos¹

¹Department of Mathematics, University of Athens, Greece

²School of Engineering and Mathematical Sciences, City University, London, UK

³Department of Mathematical Sciences, University of Liverpool, UK

1. Introduction

In many and very significant applications, for instance in mechanical, electrical and chemical engineering, in economy (the famous input-output Leontief model and its several important extensions, see Leontief (1986), Luenberger (1977), Campbell (1980) et al.), in actuarial science, Pantelous et al. (2008), in ecology and growth population (the Leslie growth population model and backward population projections, see Leslie (1945)), the descriptor dynamic systems' framework is required for the modelling procedure.

Example 1.1 For instance, practically speaking, we can consider a simple *circuit* network, as shown in **Figure 1**, Dai (1989) p. 10, where the voltage source $V_s(t)$ is the control input. R , L and C stand for the resistor, the inductor, and the capacity, respectively. Moreover, their voltages are denoted by $V_R(t)$, $V_L(t)$ and $V_C(t)$, respectively.

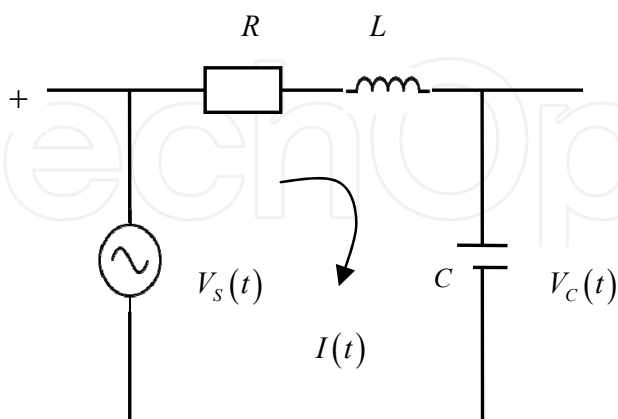


Fig. 1. A simple circuit network

Then, following Kirchoff's laws, we have the following systems of circuit equations

$$V_L(t) + V_C(t) + V_R(t) = V_S(t) \quad (1.1)$$

$$V_L(t) = L \cdot i'(t) \quad (1.2)$$

$$V_R(t) = R \cdot i(t) \quad (1.3)$$

$$V_C(t) = \frac{1}{C} \int i(\tau) d\tau. \quad (1.4)$$

Moreover, if we assume that

$$\underline{x}(t) = \begin{bmatrix} i(t) \\ V_L(t) \\ V_C(t) \\ V_R(t) \end{bmatrix},$$

as the state variable, we obtain

$$\begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i'(t) \\ V_L'(t) \\ V_C'(t) \\ V_R'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{C} & 0 & 0 & 0 \\ -R & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} i(t) \\ V_L(t) \\ V_C(t) \\ V_R(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} V_S(t).$$

Then, the following dynamic system is derived

$$F \underline{x}'(t) = G \underline{x}(t) + B \underline{u}(t), \quad (1.5a)$$

where

$$F = \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{C} & 0 & 0 & 0 \\ -R & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Example 1.2 (Analog-computer simulation) (Grispos, 1991)

$$\begin{cases} F \underline{x}'(t) = G \underline{x}(t) + B \underline{u}(t) \\ \underline{y}(t) = C \underline{x}(t) + D \underline{u}(t)' \end{cases} \quad (1.5b)$$

where $\det F = 0$ or $\det F \neq 0$.

In **figure 2**, an analog-computer simulation of a relevant descriptor non-autonomous dynamic system is sketched, which is accelerated by the use of *integrator* (\int). Moreover, in the above figure, an *adder* (\oplus) is included, where the output vector is equal to the sum of input

vectors and the *amplifier* (or *attenuator*) \boxed{A} . This simulation of system (1.5b) is true if we assume that the vector $F\underline{x}'(t)$ *does* exist.

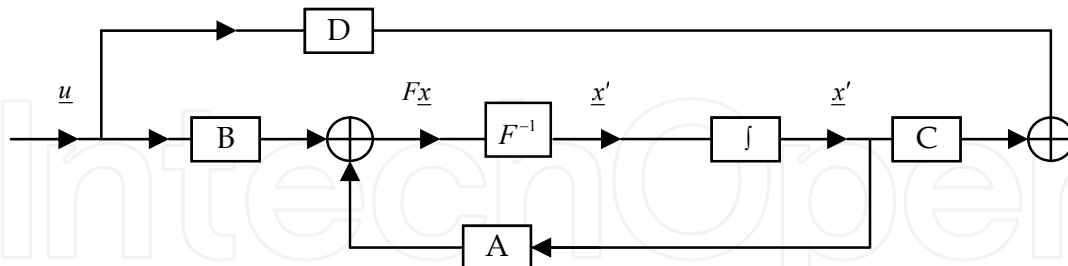


Fig. 2. Analog-computer simulation

As we can see, the systems (1.5a) and (1.5b) are linear descriptor differential systems. However, several of those electrical applications can be represented more effectively by combining linear and nonlinear parts of differential equations known, in the literature, as *weakly nonlinear differential systems*.

Weakly nonlinear (or semi-linear) regular differential systems of type (1.6) are investigated in this book chapter, see also Kalogeropoulos et al. (2008 a,b), Karageorgos et al. (2009), i.e.

$$F\underline{x}'(t) = G\underline{x}(t) + \underline{f}(t, \underline{x}(t)), \quad (1.6)$$

where $F, G \in \mathbb{C}^{n \times n}$ are time-invariant matrices, with $\det F = 0$, and $\underline{f}(t, \underline{x}(t))$ is a sufficiently differentiable n -vector function of t ; $\underline{x}(t)$, for $t \geq 0$ and $\|\underline{x}\| < \infty$.

This mixture of linear and nonlinear parts is a very productive in many applications of engineering including electrical circuits and networks; power system; aerospace engineering; nonlinear mechanical phenomena; cheap control, etc. Since 1970s, this kind of systems has attracted the attention of many researchers. However, more theoretical analysis of (1.6) is needed. This chapter is a step in that direction.

In the classical literature of generalized linear regular differential systems, see Campbell (1980, 1983), Karcanias (1981), Kalogeropoulos (1985) et al., one of the important features is that not every initial condition $\underline{x}(t_0)$ admits a functional solution.

This chapter is organized as follows: Section 2 provides some preliminary concepts and definitions from Matrix Pencil theory. In Section 3, the solution of weakly nonlinear regular differential systems is derived. The solution is provided for (non-) consistent initial conditions. The asymptotic stability is investigated for the solution of weakly nonlinear regular differential systems. Some important conditions are available in Section 4. Section 5 provides a standard linearization technique and the solution of the regular linearized system is provided. Finally, section 6 concludes the paper. Several extensions and interesting fields of research are also provided.

2. Mathematical Background – Elements of Matrix Pencil Theory

This preliminary section introduces some basic concepts and definitions from Matrix Pencil theory those are used throughout the paper.

Definition 2.1 Given $F, G \in \mathbb{C}^{m \times n}$ and an indeterminate s , the matrix pencil $sF - G$ is called regular when $m = n$ and $\det(sF - G) \neq 0$, where 0 is the zero polynomial. In any other case, the pencil will be called singular.

In the case where $sF - G$ is a regular pencil, the elementary divisors of the following type is obtained:

- e.d. of the type s^π are called zero finite elementary divisors (**z.e.d.**)
- e.d. of the type $(s - \lambda_j)^p$, $\lambda_j \neq 0$ are called nonzero finite elementary divisors (**nz. f.e.d.**)
- e.d. of the type \hat{s}^q are called infinite elementary divisors (**i.e.d.**)

Theorem 2.1 (Gantmacher, 1959) (**complex Weierstrass canonical form**)

For a regular matrix pencil $sF - G$, there exist non singular $\mathbb{C}^{n \times n}$ matrices P and Q such that.

$$PFQ = F_w = \begin{bmatrix} I_p & \mathbb{O}_{p,q} \\ \mathbb{O}_{q,p} & H_q \end{bmatrix} \quad (2.1)$$

$$PGQ = G_w = \begin{bmatrix} J_p & \mathbb{O}_{p,q} \\ \mathbb{O}_{q,p} & I_q \end{bmatrix} \quad (2.2)$$

Then, the complex Weierstrass canonical form $sF_w - G_w$ of the regular matrix pencil $sF - G$, is given by

$$sF_w - G_w \triangleq \text{block diag} \{ sI_p - J_p, sH_q - I_q \},$$

where the first normal Jordan type block $sI_p - J_p$ is uniquely defined by the set of f.e.d.

$$(s - \lambda_1)^{p_1}, \dots, (s - \lambda_v)^{p_v}, \sum_{j=1}^v p_j = p$$

of $sF - G$ and has the form $sI_p - J_p \triangleq \text{block diag} \{ sI_{p_1} - J_{p_1}(\lambda_1), \dots, sI_{p_v} - J_{p_v}(\lambda_v) \}$.

And the q blocks of the second uniquely defined block $sH_q - I_q$ correspond to the i.e.d.

$$(\hat{s})^{q_1}, \dots, (\hat{s})^{q_\mu}, \sum_{j=1}^\mu q_j = q$$

of $sF - G$ and has the form $sH_q - I_q \triangleq \text{block diag} \{ sH_{q_1} - I_{q_1}, \dots, sH_{q_\mu} - I_{q_\mu} \}$,

where $I_{p_j}, J_{p_j}(\lambda_j), H_{q_j}$, are given by (2.3) on the field of \mathbb{C} .

$$I_{p_j} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{p_j \times p_j}, \quad J_{p_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & 0 & \lambda_i \end{bmatrix}_{p_i \times p_i} \quad \text{and} \quad H_{q_j} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{q_j \times q_j} \quad (2.3)$$

the H_{q_j} is a nilpotent matrix of index $q^* = \max \{ q_j : j = 1, 2, \dots, \sigma \}$,

where

$$H_{q_j}^{\dot{}} = \mathbb{O}. \quad (2.4)$$

3. Solving weakly nonlinear regular differential systems

In this section, we are interested in solving weakly nonlinear regular differential systems by using the matrix pencil approach. However, before we go further, the following definition is necessary to be stated.

Definition 3.1 We shall call \underline{x}_o a consistent initial condition for (1.6) at t_o , if there is a differentiable solution to (1.6) defined on some interval $[t_o, t_o + \gamma]$, $\gamma > 0$ such that $\underline{x}(t_o) = \underline{x}_o$, see Campbell (1983).

Consider, now, an electrical circuit which is in use at time $t < t_o$. Moreover, at the exact time t_o , the system has initial condition

$$\underline{x}(t_o^-) = \lim_{t \rightarrow t_o^-} \underline{x}(t) \neq \lim_{t \rightarrow t_o^+} \underline{x}(t) = \underline{x}(t_o^+),$$

which is profoundly non-consistent with the (new) system. This result is due to the impulse behaviour of the system (1.6) at time t_o , which is translated to an effort to change (almost) instantly, i.e. in zero time, the state of the system in a new initial condition. From mathematical point of view, this approach can be modelled efficiently by using δ -function of Dirac and its derivatives. In this chapter, both cases are considered and discussed. The following lemma divides our initial system (1.6) into two equivalent, lower order differential systems.

Lemma 3.1 System (1.6) is divided into two subsystems

$$\underline{y}'_p(t) = J_p \underline{y}_p(t) + P_{p,n} \underline{f}(t, Q\underline{y}(t)) \quad (3.1)$$

with initial conditions $\underline{y}_p(t_o)$, and

$$H_q \underline{y}'_q(t) = \underline{y}_q(t) + P_{q,n} \underline{f}(t, Q\underline{y}(t)) \quad (3.2)$$

with initial conditions $\underline{y}_q(t_o)$.

Proof. We make the transformation

$$\underline{x}(t) = Q\underline{y}(t). \quad (3.3)$$

Then, the system (1.6) is transformed to

$$FQ\underline{y}'(t) = GQ\underline{y}(t) + \underline{f}(t, Q\underline{y}(t)).$$

Multiplied by left by the nonsingular matrix P , we obtain

$$PFQ\underline{y}'(t) = PGQ\underline{y}(t) + P\underline{f}(t, Q\underline{y}(t)) \Leftrightarrow F_w \underline{y}'(t) = G_w \underline{y}(t) + P\underline{f}(t, Q\underline{y}(t)) \Leftrightarrow$$

$$\begin{bmatrix} I_p & \mathbb{O}_{p,q} \\ \mathbb{O}_{q,p} & H_q \end{bmatrix} \begin{bmatrix} \underline{y}'_p(t) \\ \underline{y}'_q(t) \end{bmatrix} = \begin{bmatrix} J_p & \mathbb{O}_{p,q} \\ \mathbb{O}_{q,p} & I_q \end{bmatrix} \cdot \begin{bmatrix} \underline{y}_p(t) \\ \underline{y}_q(t) \end{bmatrix} + \begin{bmatrix} P_{p,n} \\ P_{q,n} \end{bmatrix} \underline{f}(t, Q\underline{y}(t)).$$

Then, eqs. (3.1) and (3.2) derive.

Now, the initial conditions is obtained

$$\underline{x}(t_o) = Q\underline{y}(t_o) \Leftrightarrow \underline{y}(t_o) = Q^{-1}\underline{x}(t_o).$$

The $\underline{y}(t_0) = \begin{bmatrix} \underline{y}_p(t_0) \\ \underline{y}_q(t_0) \end{bmatrix}$. Thus, the initial conditions for system (3.1) is given by $\underline{y}_p(t_0)$ and for system (3.2) is provided by $\underline{y}_q(t_0)$.

Following the results of Lemma 3.1, the system (1.6) has been divided into two equivalent subsystems (3.1) and (3.2). Contrary to the matrix coefficients of the initial system (1.6), those systems have specified matrix coefficients, i.e. nilpotent, Jordan and identical matrices. This equivalence is very significant, since it provides the appropriate framework for a deeper system's analysis. The following Remarks and Lemmas provide us with the solution of system (1.6) with consistent and non-consistent initial conditions.

Remark 3.1 System (3.1) with initial conditions $\underline{y}_p(t_0)$ can be solved using some classical methods, and

$$\underline{y}_p(t) = e^{J_p(t-t_0)} \underline{y}_p(t_0) + \int_{t_0}^t e^{J_p(t-s)} P_{p,n} \underline{f}(s, Q\underline{y}(s)) ds. \quad (3.4)$$

In this part of the sections, we will solve the subsystem (3.3).

Lemma 3.2 Considering that

$$\underline{y}_q(t_0) = - \sum_{j=0}^{q^*-1} H_q^j P_{q,n} \frac{d^j}{dt^j} \underline{f}(t, Q\underline{y}(t)) \Big|_{t=t_0}, \quad (3.5)$$

the solution of subsystem (3.3) is given by

$$\underline{y}_q(t) = - \sum_{j=0}^{q^*-1} H_q^j P_{q,n} \frac{d^j}{dt^j} \underline{f}(t, Q\underline{y}(t)). \quad (3.6)$$

Proof. We start by observing that -as is well known- there exists a $q^* \in \mathbb{N}$ such that $H_q^{q^*} = \mathbb{O}$, i.e. the q^* is the annihilation index of H_q . We obtain

$$H_q \underline{y}'_q(t) = \underline{y}_q(t) + P_{q,n} \underline{f}(t, Q\underline{y}(t)) \quad (3.7)$$

whereby differentiating, and multiplying by H_q , we get

$$H_q^2 \underline{y}''_q(t) = H_q \underline{y}'_q(t) + H_q P_{q,n} \frac{d}{dt} \underline{f}(t, Q\underline{y}(t)) \quad (3.8)$$

and substituting (3.7) into (3.8), we have

$$H_q^2 \underline{y}''_q(t) = \underline{y}_q(t) + P_{q,n} \underline{f}(t, Q\underline{y}(t)) + H_q P_{q,n} \frac{d}{dt} \underline{f}(t, Q\underline{y}(t)). \quad (3.9)$$

By differentiating and multiplying by H_q again expression (3.9), we obtain

$$H_q^3 \underline{y}'''_q(t) = H_q \underline{y}_q(t) + H_q P_{q,n} \frac{d}{dt} \underline{f}(t, Q\underline{y}(t)) + H_q^2 P_{q,n} \frac{d^2}{dt^2} \underline{f}(t, Q\underline{y}(t)).$$

Repeating this argument a sufficient number of times we end up with

$$H_q^{q^*} \underline{y}^{(q^*)}_q(t) = \underline{y}_q(t) + \sum_{k=0}^{q^*-1} H_q^k P_{q,n} \frac{d^k}{dt^k} \underline{f}(t, Q\underline{y}(t)). \quad (3.10)$$

Taking into consideration that $H_q^{q^*} = \mathbb{O}$ and all the other similar relations up to and including (3.10), we arrive at (3.6), where

$$H_q^0 = I_q \text{ and } \frac{d^0}{dt^0} (f(t, Qy(t))) = f(t, Qy(t)).$$

The solution of system (3.2) has consistent initial conditions, since at $t = t_0$, we obtain (3.5).

The expression (3.5) is very crucial, since the consistency of the initial conditions is finally satisfied. Thus, a very smooth functional solution is provided.

Remark 3.2 Stepping on the previous results and considering eq. (3.3), the solution of system (1.6) with consistent initial conditions is given by

$$\begin{bmatrix} \underline{y}_p(t) \\ \underline{y}_q(t) \end{bmatrix} = \begin{bmatrix} e^{J_p(t-t_0)} \underline{y}_p(t_0) + \int_{t_0}^t e^{J_p(t-s)} P_{p,n} f(s, Qy(s)) ds \\ - \sum_{j=0}^{q^*-1} H_q^j P_{q,n} \frac{d^j}{dt^j} f(t, Qy(t)) \end{bmatrix}. \tag{3.11}$$

Denote that the matrix $Q = [Q_{n,p} \quad Q_{n,q}]$; $p + q = n$, then

$$\underline{x}(t) = Qy(t) = [Q_{n,p} \quad Q_{n,q}] \begin{bmatrix} \underline{y}_p(t) \\ \underline{y}_q(t) \end{bmatrix} = Q_{n,p} \underline{y}_p(t) + Q_{n,q} \underline{y}_q(t) \Leftrightarrow$$

$$\underline{x}(t) = Q_{n,p} e^{J_p(t-t_0)} \underline{y}_p(t_0) + Q_{n,p} \int_{t_0}^t e^{J_p(t-s)} P_{p,n} f(s, Qy(s)) ds - Q_{n,q} \sum_{j=0}^{q^*-1} H_q^j P_{q,n} \frac{d^j}{dt^j} f(t, Qy(t)) \tag{3.12}$$

with initial conditions at time $t = t_0$

$$\underline{x}_0 = Q_{n,p} \underline{y}_p(t_0) - Q_{n,q} \sum_{j=0}^{q^*-1} H_q^j P_{q,n} \frac{d^j}{dt^j} f(t, Qy(t))|_{t=t_0} \Leftrightarrow \underline{x}_0 = Q \begin{bmatrix} \underline{y}_p(t_0) \\ \underline{y}_q(t_0) \end{bmatrix}.$$

Now, we assume that vector function $f(t, \underline{x}(t))$ is $C^{q^*-1}[0, \infty)$, where q^* is the nilpotent matrix of H_q , see section 2, with the restricted condition that

$$\left. \frac{d^k}{dt^k} f(t, \underline{x}(t)) \right|_{t=t_0} = 0, \text{ and } \underline{x}(t_0) = \underline{0}$$

for all $k = 0, 1, \dots, q^* - 1$. The function $f(t, \underline{x}(t))$ can be a polynomial of type

$$f(t, \underline{x}(t)) = \sum_{k=2}^n a_k(t) (\underline{x}(t) - \underline{x}(t_0))^k, \text{ where } n \geq q^* - 1$$

and $a_k(t)$, $k = 2, \dots, q^* - 1$, are sufficiently differentiable functions of t . Denote also that

$$(\underline{x}(t) - \underline{x}(t_0))^k \triangleq \left[(x_1(t) - x_1(t_0))^k \quad \dots \quad (x_n(t) - x_n(t_0))^k \right]^T.$$

Example 3.1 (Kalogeropoulos et al. 2008a) Consider the dynamic differential system,

$$\begin{aligned} \dot{x}_2 - \dot{x}_3 + \dot{x}_5 + \dot{x}_6 - x_1 - x_2 + x_3 + x_4 + x_6 + x_7 - x_1^2 &= 0 \\ -\dot{x}_1 - 2\dot{x}_2 + \dot{x}_3 - \dot{x}_5 - \dot{x}_6 + 3x_1 + 4x_2 + 3x_3 - x_4 - 2x_5 - 4x_6 - 5x_7 - x_2^2 - x_4^2 &= 0 \\ \dot{x}_1 - \dot{x}_2 + \dot{x}_3 - \dot{x}_4 - \dot{x}_6 - \dot{x}_7 + x_1 - 2x_2 - x_4 + 2x_5 + x_6 - x_7 - x_4^2 &= 0 \\ -\dot{x}_2 + \dot{x}_3 - \dot{x}_5 - \dot{x}_6 + x_1 + x_2 + x_3 - x_4 - x_6 - 2x_7 - x_6^2 &= 0 \\ -x_3 + x_5 + x_6 - x_5^2 &= 0 \end{aligned}$$

$$\begin{aligned} \dot{x}_3 - \dot{x}_5 - \dot{x}_6 + x_1 - x_2 - x_3 - x_4 - x_6 - x_7 - x_6^2 &= 0 \\ -\dot{x}_1 - \dot{x}_2 + \dot{x}_6 + 3x_1 + 2x_2 + 3x_3 - x_4 - 2x_5 - 3x_6 - 4x_7 - x_7^2 &= 0 \end{aligned}$$

or in a matrix form, where $\underline{f}(t, \underline{x}(t)) = E\underline{x}^2(t)$, and

$$(S) \quad F\underline{x}'(t) = G\underline{x}(t) + E\underline{x}^2(t).$$

Note that $\underline{x}^2(t) \triangleq [x_1^2(t) \ x_2^2(t) \ x_3^2(t) \ x_4^2(t) \ x_5^2(t) \ x_6^2(t) \ x_7^2(t)]^T$,

with initial condition $\underline{x}(0) = \underline{x}_0 = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$,

where

$$F = \begin{bmatrix} 0 & 1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -2 & 1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & -1 & -1 \\ -3 & -4 & -3 & 1 & 2 & 4 & 5 \\ -1 & 2 & 0 & 1 & -2 & -1 & 1 \\ -1 & -1 & -1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & 1 \\ -3 & -2 & -3 & 1 & 2 & 3 & 4 \end{bmatrix}$$

and

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the regularity of $sF - G$, there exist non singular $\mathbb{C}^{n \times n}$ matrices

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

such that

$$PFG = F_w = \begin{bmatrix} I_2 & \mathbb{O}_{5,2} \\ \mathbb{O}_{5,2} & H_5 \end{bmatrix}, \quad \text{and} \quad PGQ = G_w = \begin{bmatrix} J_2(2) & \mathbb{O}_{5,2} \\ \mathbb{O}_{5,2} & I_5 \end{bmatrix},$$

where $I_2, J_2(2), H_5$ are given by (2.3).

Moreover, taking into consideration the transformation (3.3) for

$$f(t, \underline{x}(t)) = E \underline{x}^2(t) = E(Q \underline{y}(t))^2 = \begin{bmatrix} (y_1(t) + y_4(t) - y_7(t))^2 \\ y_2^2(t) \\ (y_2(t) + y_5(t) + y_6(t) + y_7(t))^2 \\ (y_1(t) + y_3(t) + y_5(t) + y_6(t))^2 \\ (y_2(t) + y_3(t) + y_5(t))^2 \\ (-y_3(t) + y_6(t))^2 \\ y_4^2(t) \end{bmatrix}$$

and multiplying by P , then

$$B(t) \triangleq \begin{bmatrix} B_2(t) \\ B_5(t) \end{bmatrix} = \begin{bmatrix} (-y_3(t) + y_6(t))^2 - y_4^2(t) \\ (y_1(t) + y_4(t) - y_7(t))^2 + (-y_3(t) + y_6(t))^2 \\ -y_2^2(t) - (y_1(t) + y_3(t) + y_5(t) + y_6(t))^2 + (-y_3(t) + y_6(t))^2 + y_4^2(t) \\ (y_1(t) + y_4(t) - y_7(t))^2 + (-y_3(t) + y_6(t))^2 \\ -(y_1(t) + y_3(t) + y_5(t) + y_6(t))^2 + (y_2(t) + y_3(t) + y_5(t))^2 \\ (-y_3(t) + y_6(t))^2 \\ (y_2(t) + y_3(t) + y_5(t))^2 \end{bmatrix}.$$

The above system can be divided into the following two subsystems

$$(S1) \quad \underline{y}'_2(t) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \underline{y}_2(t) + B_2(t) \quad \text{and} \quad (S2) \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ & & & 0 & 1 \\ & & & & & 0 & 0 \end{bmatrix} \underline{y}'_5(t) = \underline{y}_5(t) + B_5(t).$$

Now, we consider the initial conditions

$$\underline{y}(0) = Q^{-1} \underline{x}_0 = \begin{bmatrix} \underline{y}_2(0) \\ \underline{y}_5(0) \end{bmatrix} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T.$$

Note also that

$$Q = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \triangleq [Q_{7,2} \quad \vdots \quad Q_{7,5}].$$

Finally, it is not difficult to verify that $H_5^3 = \mathbb{O}_5$, i.e. $q^* = 3$.

According to Remark 3.2, and the relevant expressions, the solution of the system (S) is given by

$$\underline{x}(t) = Q_{7,2} \left\{ e^{J_2(2)(t-t_0)} \underline{y}_2(0) + \int_0^t e^{J_2(2)(t-s)} B_2(s) ds \right\} - Q_{7,5} \left\{ B_5(t) + H_5 \frac{d}{dt} B_5(t) + H_5^2 \frac{d^2}{dt^2} B_5(t) \right\}$$

or equivalently, if we consider the initial condition then

$$\underline{x}(t) = Q_{7,2} \int_{t_0}^t e^{J_2(2)(t-s)} B_2(s) ds - Q_{7,5} \left\{ B_5(t) + H_5 \frac{d}{dt} B_5(t) + H_5^2 \frac{d^2}{dt^2} B_5(t) \right\}.$$

Up to now, we have considered that system (1.6) has consistent initial conditions. If this assumption eliminated then subsystem (3.2) had non-consistent initial conditions, i.e.

$$\underline{y}_{-q}(t_0) \neq - \sum_{j=0}^{q^*-1} H_q^j P_{q,n} \frac{d^j}{dt^j} f(t, Q\underline{y}(t)) \Big|_{t=t_0}, \quad (3.13)$$

then the solution of (3.2) can be found as follows.

Lemma 3.3 (Kalogeropoulos et al., 2008a) Assume that the condition (3.13) is true. The solution of the system (3.2) is given by

$$\underline{y}_q(t) = - \sum_{j=1}^{q^*-1} \delta^{(j-1)}(t) H_q^j \underline{y}_q(t_0) - \sum_{j=0}^{q^*-1} H_q^j P_{q,n} (\underline{f} * \delta^{(j)})(t), \quad (3.14)$$

where $\underline{f} * \delta^{(j)}$ is the convolution between two functionals

$$(\underline{f} * \delta^{(j)})(t) = \begin{bmatrix} (f_1 * \delta^{(j)})(t) \\ (f_2 * \delta^{(j)})(t) \\ \vdots \\ (f_n * \delta^{(j)})(t) \end{bmatrix} \quad (3.15)$$

and

$$(f_i * \delta^{(j)})(t) = \int_0^t f_i(s, Q\underline{y}(s)) \delta^{(j)}(t-s) ds, \quad (3.16)$$

where $\delta(t)$ is the Dirac function for every $j = 1, 2, \dots, q^* - 1$.

Proof. Let us start by observing that the q^* is the annihilation index of H_q . Whereby taking the Laplace transformation of (3.2), the following expression derives

$$H_q \mathfrak{L}\{\underline{y}'_q(t)\} = \mathfrak{L}\{\underline{y}_q(t)\} + P_{q,n} \mathfrak{L}\{f(t, Q\underline{y}(t))\}$$

and by defining $\mathfrak{L}\{\underline{y}_q(t)\} = X_q(s)$, $\mathfrak{L}\{\underline{y}'_q(t)\} = sX_q(s) - \underline{y}_q(t_0)$,

we obtain

$$H_q (sX_q(s) - \underline{y}_q(t_0)) = X_q(s) + P_{q,n} \mathfrak{L}\{f(t, Q\underline{y}(t))\}$$

or equivalently

$$(sH_q - I_q) X_q(s) = H_q \underline{y}_q(t_0) + P_{q,n} \mathfrak{L}\{f(t, Q\underline{y}(t))\}. \quad (3.17)$$

Since q^* is the annihilation index of H_q , it is known that $(sH_q - I_q)^{-1} = - \sum_{j=0}^{q^*-1} (sH_q)^j$, where

$H_q^0 = I_q$, see for instance Kalogeropoulos (1985) and Meyer (2001). Thus, substituting the above expression into the (3.17), the following equation is taken

$$\begin{aligned}
 X_q(s) &= -\sum_{j=0}^{q^*-1} (sH_q)^j H_q \underline{y}_q(t_o) - \sum_{j=0}^{q^*-1} (sH_q)^j P_{q,n} \mathfrak{F}\{f(t, Q\underline{y}(t))\} = \\
 &= -\sum_{j=0}^{q^*-1} s^j H_q^{j+1} \underline{y}_q(t_o) - \sum_{j=0}^{q^*-1} s^j H_q^j P_{q,n} \mathfrak{F}\{f(t, Q\underline{y}(t))\} = \\
 &= -(H_q + sH_q^2 + \dots + s^{q^*-1} H_q^{q^*}) \underline{y}_q(t_o) - \sum_{j=0}^{q^*-1} s^j H_q^j P_{q,n} \mathfrak{F}\{f(t, Q\underline{y}(t))\} \quad (\text{note } H_q^{q^*} = \mathbb{O}) = \\
 &= -(H_q + sH_q^2 + \dots + s^{q^*-2} H_q^{q^*-1}) \underline{y}_q(t_o) - \sum_{j=0}^{q^*-1} s^j H_q^j P_{q,n} \mathfrak{F}\{f(t, Q\underline{y}(t))\} \Leftrightarrow \\
 X_q(s) &= -\sum_{j=1}^{q^*-1} s^{j-1} H_q^j \underline{y}_q(t_o) - \sum_{j=0}^{q^*-1} (sH_q)^j P_{q,n} \mathfrak{F}\{f(t, Q\underline{y}(t))\}. \tag{3.18}
 \end{aligned}$$

Since $\mathfrak{F}\{\delta^{(j)}(t)\} = s^j$, the expression (3.18) is transformed into (3.19)

$$X_q(s) = -\sum_{j=1}^{q^*-1} \mathfrak{F}\{\delta^{(j-1)}(t)\} H_q^j \underline{y}_q(t_o) - \sum_{j=0}^{q^*-1} H_q^j P_{q,n} \mathfrak{F}\{\delta^{(j)}(t)\} \mathfrak{F}\{f(t, Q\underline{y}(t))\} \tag{3.19}$$

Now, by applying the inverse Laplace transformation into (3.19), we obtain

$$\begin{aligned}
 \mathfrak{F}^{-1}\left(\mathfrak{F}\{\delta^{(j)}(t)\} \mathfrak{F}\{f(t, Q\underline{y}(t))\}\right) &= \mathfrak{F}^{-1}\left(\mathfrak{F}\{\delta^{(j)}(t)\} \begin{bmatrix} \mathfrak{F}\{f_1(t, Q\underline{y}(t))\} \\ \mathfrak{F}\{f_2(t, Q\underline{y}(t))\} \\ \vdots \\ \mathfrak{F}\{f_n(t, Q\underline{y}(t))\} \end{bmatrix}\right) = \\
 &= \mathfrak{F}^{-1}\left(\begin{bmatrix} \mathfrak{F}\{\delta^{(j)}(t)\} \mathfrak{F}\{f_1(t, Q\underline{y}(t))\} \\ \mathfrak{F}\{\delta^{(j)}(t)\} \mathfrak{F}\{f_2(t, Q\underline{y}(t))\} \\ \vdots \\ \mathfrak{F}\{\delta^{(j)}(t)\} \mathfrak{F}\{f_n(t, Q\underline{y}(t))\} \end{bmatrix}\right) = (f * \delta^{(j)})(t).
 \end{aligned}$$

Because

$$\mathfrak{F}^{-1}\left(\mathfrak{F}\{\delta^{(j)}(t)\} \cdot \mathfrak{F}\{f_i(t, Q\underline{y}(t))\}\right) = (f_i * \delta^{(j)})(t) \quad \text{for every } i = 1, 2, \dots, n.$$

Moreover,

$$\underline{y}_q(t) = -\sum_{j=1}^{q^*-1} \delta^{(j-1)}(t) H_q^j \underline{y}_q(t_o) - \sum_{j=0}^{q^*-1} H_q^j P_{q,n} (f * \delta^{(j)})(t),$$

where the convolution of the two time function is

$$(\underline{f} * \delta^{(j)})(t) \triangleq \begin{bmatrix} (f_1 * \delta^{(j)})(t) \\ (f_2 * \delta^{(j)})(t) \\ \vdots \\ (f_n * \delta^{(j)})(t) \end{bmatrix},$$

and $(f_i * \delta^{(j)})(t) = \int_0^t f_i(\tau, Q\underline{y}(\tau)) \delta^{(j)}(t-\tau) d\tau$, for $i = 1, 2, \dots, n$.

Remark 3.3 The solution of system (1.6) is given by (3.21) when the assumption (3.13) is considered.

$$\underline{x}(t) = Q\underline{y}(t) = \begin{bmatrix} Q_{n,p} & Q_{n,q} \end{bmatrix} \begin{bmatrix} \underline{y}_p(t) \\ \underline{y}_q(t) \end{bmatrix} \Leftrightarrow \underline{x}(t) = Q_{n,p} e^{J_p(t-t_0)} \underline{y}_p(t_0) +$$

$$\int_{t_0}^t Q_{n,p} e^{J_p(t-s)} P_{p,n} \underline{f}(s, Q\underline{y}(s)) ds - Q_{n,q} \sum_{j=1}^{q^*-1} \delta^{(j-1)}(t) H_q^j \underline{y}_q(t_0) - Q_{n,q} \sum_{j=0}^{q^*-1} H_q^j P_{q,n} (\underline{f} * \delta^{(j)})(t) \quad (3.21)$$

In the next section, the asymptotic stability of systems (1.6) with consistent initial condition is considered.

4. The asymptotic stability of weakly nonlinear regular differential systems

In the beginning of this section, some fundamental elements of asymptotic stability are introduced and discussed.

Definition 4.1 Considering a system of ordinary differential equations, then the solution $\underline{x}(t)$ is asymptotically stable (see figure 3) if for every other solution $\tilde{\underline{x}}(t)$, the following condition is satisfied

$$\lim_{t \rightarrow +\infty} \|\underline{x}(t) - \tilde{\underline{x}}(t)\| = 0.$$

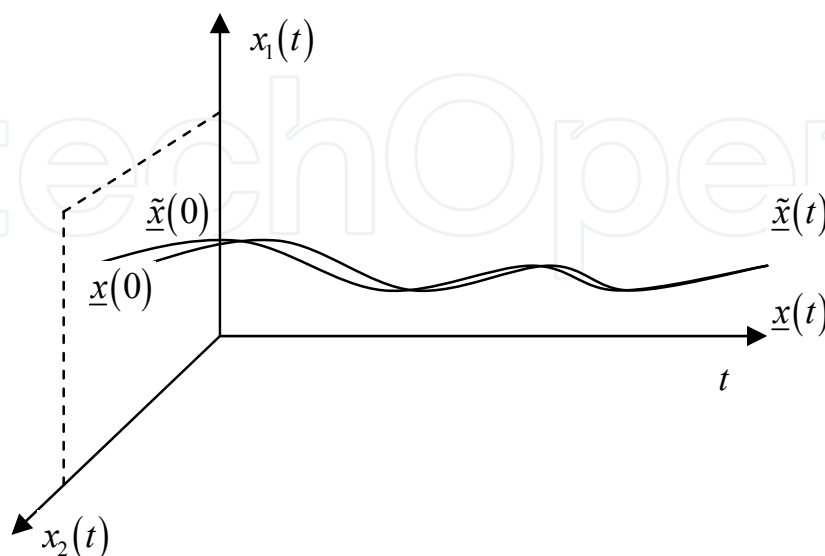


Fig. 3. The asymptotic stability of two-dimension vector $\underline{x}(t)$

Consequently, some elements of norms for rectangular matrices are needed.

Definition 4.2 Consider the function $v: \mathbb{C}^{m \times n} \rightarrow \mathbb{R}$ which is called norm in the space $\mathbb{C}^{m \times n}$ or matrix norm, if the following conditions are satisfied simultaneously.

1. $A \neq \mathbb{O} \Rightarrow v(A) > 0$
2. $v(\alpha A) = |\alpha|v(A)$
3. $v(A + B) \leq v(A) + v(B)$

Definition 4.3 The matrix $A \in \mathbb{C}^{m \times n}$. Denote the **Frobenius** norm of matrix A ,

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |\alpha_{ij}|^2} = \text{trace}(A^H A)^{1/2} \quad (4.1)$$

where $A^H = (\bar{A})^t$ (the conjugate transpose of matrix A).

Remark 4.1 The Frobenius norm has a significant property, which does not appear in other norms for rectangular matrices. Analytically, consider the matrices $A \in \mathbb{C}^{m \times p}$ and $B \in \mathbb{C}^{p \times n}$ then

$$\|A \cdot B\|_F \leq \|A\|_F \cdot \|B\|_F \quad (4.2)$$

In the next lines of this chapter, we are using the Frobenius norm. If contrary, it will be mentioned. First, we consider the asymptotic stability of system (1.6) with consistent initial conditions. Thus, we assume that $\tilde{x}(t)$ is a different solution of system (1.6) with the initial conditions $\tilde{x}(t_0) = \tilde{x}_0$, and

$$\tilde{x}(t) = Q\tilde{y}(t) = Q \begin{bmatrix} \tilde{y}_p(t) \\ \tilde{y}_q(t) \end{bmatrix}.$$

Following the results of section 3, the solution of (1.6) is given by

$$\tilde{x}(t) = Q_{n,p} e^{J_p(t-t_0)} \tilde{y}_p(t_0) + Q_{n,p} \int_{t_0}^t e^{J_p(t-s)} P_{p,n} \underline{f}(s, Q\tilde{y}(s)) ds - Q_{n,q} \sum_{j=0}^{q^*-1} H_q^j P_{q,n} \frac{d^j}{dt^j} (f(t, Q\tilde{y}(t))).$$

Moreover, we obtain

$$\begin{aligned} \|\underline{x}(t) - \tilde{x}(t)\| &= \left\| Q_{n,p} e^{J_p(t-t_0)} (\underline{y}_p(t_0) - \tilde{y}_p(t_0)) + Q_{n,p} \int_{t_0}^t e^{J_p(t-s)} P_{p,n} (f(s, Q\underline{y}(s)) - f(s, Q\tilde{y}(s))) ds \right. \\ &\quad \left. - Q_{n,q} \sum_{j=0}^{q^*-1} H_q^j P_{q,n} \left(\frac{d^j}{dt^j} (f(t, Q\underline{y}(t)) - f(t, Q\tilde{y}(t))) \right) \right\|. \end{aligned}$$

Afterwards, using some properties for norms and the eq. (4.2), we obtain

$$\begin{aligned} \|\underline{x}(t) - \tilde{x}(t)\| &\leq \|Q_{n,p}\| \cdot \|e^{J_p(t-t_0)}\| \cdot \|\underline{y}_p(t_0) - \tilde{y}_p(t_0)\| \\ &\quad + \|Q_{n,p}\| \cdot \int_{t_0}^t \|e^{J_p(t-s)}\| \cdot \|P_{p,n}\| \cdot \|f(s, Q\underline{y}(s)) - f(s, Q\tilde{y}(s))\| ds \\ &\quad + \|Q_{n,q}\| \cdot \sum_{j=0}^{q^*-1} \|H_q^j\| \cdot \|P_{q,n}\| \cdot \left\| \frac{d^j}{dt^j} (f(t, Q\underline{y}(t)) - f(t, Q\tilde{y}(t))) \right\|. \end{aligned} \quad (4.3)$$

Continuing the section, some significant Lemmas are proven. These results are consequence of important properties of norm.

Lemma 4.1 The following inequality holds

$$\|e^{J_p(t-t_0)}\| \leq \sum_{i=1}^{\nu} \left(e^{\lambda_i(t-t_0)} \left(\sum_{j=0}^{p_i-1} (p_i-j) \frac{|(t-t_0)^{2j}|}{|(j!)^2|} \right)^{1/2} \right). \quad (4.4)$$

Proof. Since, we have used the Frobenius norm, we have

$$e^{J_p(t-t_0)} = \text{block diag} \left\{ e^{J_{p_1}(\alpha_1)(t-t_0)}, \dots, e^{J_{p_\nu}(\alpha_\nu)(t-t_0)} \right\}$$

where

$$e^{J_{p_i}(\alpha_i)(t-t_0)} = \begin{bmatrix} e^{\alpha_i(t-t_0)} & e^{\alpha_i(t-t_0)}(t-t_0) & e^{\alpha_i(t-t_0)} \frac{(t-t_0)^2}{2!} & \dots & e^{\alpha_i(t-t_0)} \frac{(t-t_0)^{p_i}}{p_i!} \\ & e^{\alpha_i(t-t_0)} & & & e^{\alpha_i(t-t_0)} \frac{(t-t_0)^{p_i-1}}{(p_i-1)!} \\ & & & \dots & \\ \textcircled{0} & & & & e^{\alpha_i(t-t_0)} \end{bmatrix}_{p_i \times p_i}$$

for every $i = 1, \dots, \nu$. Thus

$$\begin{aligned} \|e^{J_p(t-t_0)}\| &= \left(\sum_{i=1}^{\nu} \sum_{j=0}^{p_i-1} (p_i-j) \left| e^{\alpha_i(t-t_0)} \frac{(t-t_0)^j}{j!} \right|^2 \right)^{1/2} = \left(\sum_{i=1}^{\nu} \left(|e^{2\alpha_i(t-t_0)}| \sum_{j=0}^{p_i-1} (p_i-j) \frac{|(t-t_0)^{2j}|}{|(j!)^2|} \right) \right)^{1/2} = \\ &= \left(|e^{2\alpha_1(t-t_0)}| \sum_{j=0}^{p_1-1} (p_1-j) \frac{|(t-t_0)^{2j}|}{|(j!)^2|} + \dots + |e^{2\alpha_\nu(t-t_0)}| \sum_{j=0}^{p_\nu-1} (p_\nu-j) \frac{|(t-t_0)^{2j}|}{|(j!)^2|} \right)^{1/2} \leq \\ &\leq |e^{2\alpha_1(t-t_0)}|^{1/2} \left(\sum_{j=0}^{p_1-1} (p_1-j) \frac{|(t-t_0)^{2j}|}{|(j!)^2|} \right)^{1/2} + \dots + |e^{2\alpha_\nu(t-t_0)}|^{1/2} \left(\sum_{j=0}^{p_\nu-1} (p_\nu-j) \frac{|(t-t_0)^{2j}|}{|(j!)^2|} \right)^{1/2} = \end{aligned}$$

(The inequality is due to the expression $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$)

$$= |e^{\alpha_1(t-t_0)}| \left(\sum_{j=0}^{p_1-1} (p_1-j) \frac{|(t-t_0)^{2j}|}{|(j!)^2|} \right)^{1/2} + \dots + |e^{\alpha_\nu(t-t_0)}| \left(\sum_{j=0}^{p_\nu-1} (p_\nu-j) \frac{|(t-t_0)^{2j}|}{|(j!)^2|} \right)^{1/2}.$$

The factor $e^{\alpha_i(t-t_0)} = e^{(\lambda_i + \theta m_i)(t-t_0)}$, where $a_i = \lambda_i + \theta m_i$, $\theta^2 = -1$ (the imagine unit) and $\lambda_i, m_i \in \mathbb{R}$ for every $i = 1, 2, \dots, \nu$. Then,

$$e^{\alpha_i(t-t_0)} = e^{\lambda_i(t-t_0)} e^{\theta m_i(t-t_0)} = e^{\lambda_i(t-t_0)} (\cos(m_i(t-t_0)) + \theta \sin(m_i(t-t_0))),$$

for every $i = 1, 2, \dots, \nu$. Thus

$$|e^{\alpha_i(t-t_0)}| = |e^{\lambda_i(t-t_0)}| |\cos(m_i(t-t_0)) + \theta \sin(m_i(t-t_0))| = e^{\lambda_i(t-t_0)} \cdot 1 = e^{\lambda_i(t-t_0)}$$

for every $i = 1, 2, \dots, \nu$. Consequently,

$$\|e^{J_p(t-t_0)}\| \leq e^{\lambda_1(t-t_0)} \left(\sum_{j=0}^{p_1-1} (p_1-j) \frac{|(t-t_0)^{2j}|}{|(j!)^2|} \right)^{1/2} + \dots + e^{\lambda_\nu(t-t_0)} \left(\sum_{j=0}^{p_\nu-1} (p_\nu-j) \frac{|(t-t_0)^{2j}|}{|(j!)^2|} \right)^{1/2} \Leftrightarrow$$

$$\Leftrightarrow \|e^{J_p(t-t_0)}\| \leq \sum_{i=1}^v \left(e^{\lambda_i(t-t_0)} \left(\sum_{j=0}^{p_i-1} (p_i - j) \frac{|(t-t_0)^{2j}|}{|(j!)^2|} \right)^{1/2} \right).$$

Remark 4.2 Considering the results of Lemma 4.1, we obtain

$$\|e^{J_p(t-s)}\| \leq \sum_{i=1}^v \left(e^{\lambda_i(t-s)} \left(\sum_{j=0}^{p_i-1} (p_i - j) \frac{|(t-s)^{2j}|}{|(j!)^2|} \right)^{1/2} \right) \tag{4.5}$$

with $t_0 \leq s \leq t$.

Remark 4.3 Assuming, now, that $t \geq t_0$ and $s \in [t_0, t]$, then the (4.4) and (4.5) are transposed to the following expressions, respectively

$$\|e^{J_p(t-t_0)}\| \leq \sum_{i=1}^v \left(e^{\lambda_i(t-t_0)} \left(\sum_{j=0}^{p_i-1} (p_i - j) \frac{(t-t_0)^{2j}}{(j!)^2} \right)^{1/2} \right), \tag{4.6}$$

and

$$\|e^{J_p(t-s)}\| \leq \sum_{i=1}^v \left(e^{\lambda_i(t-s)} \left(\sum_{j=0}^{p_i-1} (p_i - j) \frac{(t-s)^{2j}}{(j!)^2} \right)^{1/2} \right). \tag{4.7}$$

Furthermore, using the above two expressions, the eq. (4.3) is benefited as following

$$\begin{aligned} \|\underline{x}(t) - \tilde{\underline{x}}(t)\| &\leq \|Q_{n,p}\| \cdot \sum_{i=1}^v \left(e^{\lambda_i(t-t_0)} \left(\sum_{j=0}^{p_i-1} (p_i - j) \frac{(t-t_0)^{2j}}{(j!)^2} \right)^{1/2} \right) \cdot \|\underline{y}_p(t_0) - \tilde{\underline{y}}_p(t_0)\| + \\ &+ \|Q_{n,p}\| \cdot \int_{t_0}^t \sum_{i=1}^v \left(e^{\lambda_i(t-s)} \left(\sum_{j=0}^{p_i-1} (p_i - j) \frac{(t-s)^{2j}}{(j!)^2} \right)^{1/2} \right) \cdot \|P_{p,n}\| \cdot \|\underline{f}(s, Q\underline{y}(s)) - \underline{f}(s, Q\tilde{\underline{y}}(s))\| ds + \\ &+ \|Q_{n,q}\| \cdot \sum_{j=0}^{q-1} \|H_q^j\| \cdot \|P_{q,n}\| \cdot \left\| \frac{d^j}{dt^j} (\underline{f}(t, Q\underline{y}(t)) - \underline{f}(t, Q\tilde{\underline{y}}(t))) \right\|. \end{aligned} \tag{4.8}$$

Remark 4.3 Looking beyond the lines, the eq. (4.8) says clearly that if $\underline{f}(t, \underline{x}(t))$ is an arbitrary vector function then the asymptotic stability is not obtained. Thus, our main target is to determine the necessary conditions such as the asymptotic stability derives for system (1.6). Consequently, the following properties are taken. Note that these properties are quite general.

(i) $\lambda_i < 0$ for every $i = 1, 2, \dots, v$. (4.9)

(ii) $\|\underline{f}(t, \underline{x}_1(t)) - \underline{f}(t, \underline{x}_2(t))\| \leq \Pi(t)$, (4.10)

where $\Pi(t) = \sum_{\kappa=1}^{\sigma} \beta_{\kappa} t^{\gamma_{\kappa}}$ with $\beta_{\kappa} \in \mathbb{R}$ (or \mathbb{C}) and $\gamma_{\kappa} \in \mathbb{R}$, $\gamma_{\kappa} = \text{finite}$ for every $\kappa = 1, \dots, \sigma$ and $\sigma = \text{finite}$ for every $\underline{x}_1(t), \underline{x}_2(t) \in \mathbb{R}^n$ (or \mathbb{C}^n) and every $t \geq t_0$.

(iii) $\lim_{t \rightarrow \infty} \left\| \frac{d^j}{dt^j} (\underline{f}(t, \underline{x}_1(t)) - \underline{f}(t, \underline{x}_2(t))) \right\| = 0$, (4.11)

for every $\underline{x}_1(t), \underline{x}_2(t) \in \mathbb{R}^n$ (or \mathbb{C}^n).

Lemma 4.2

$$\begin{aligned} & \int_{t_0}^t \sum_{i=1}^{\nu} \left(e^{\lambda_i(t-s)} \left(\sum_{j=0}^{p_i-1} (p_i-j) \frac{(t-s)^{2j}}{(j!)^2} \right)^{1/2} \right) \cdot \|P_{q,n}\| \cdot \|f(s, Q\underline{y}(s)) - f(s, Q\tilde{y}(s))\| ds \leq \\ & \leq \|P_{q,n}\| \cdot \sum_{i=1}^{\nu} e^{\lambda_i(t-t_0)} \left(\sum_{j=0}^{p_i-1} (p_i-j) \frac{(t-t_0)^{2j}}{(j!)^2} \right)^{1/2} \cdot \int_{t_0}^t \Pi(s) ds. \end{aligned} \quad (4.12)$$

Proof. It is known that both $e^{\lambda_i(t-s)}$ when $\lambda_i < 0$ and $\sum_{j=0}^{p_i-1} (p_i-j) \frac{(t-s)^{2j}}{(j!)^2}$ when $s \in [t_0, t]$ are decreasing functions. Thus,

$$e^{\lambda_i(t-s)} \leq e^{\lambda_i(t-t_0)} \quad \text{and} \quad \sum_{j=0}^{p_i-1} (p_i-j) \frac{(t-s)^{2j}}{(j!)^2} \leq \sum_{j=0}^{p_i-1} (p_i-j) \frac{(t-t_0)^{2j}}{(j!)^2}.$$

Moreover, we have that if the inequality $f(x) \leq g(x)$ exists and $\Pi(x)$ is a positive function, then

$$f(x)\Pi(x) \leq g(x)\Pi(x) \Rightarrow \int_{\alpha}^{\beta} f(x)\Pi(x)dx \leq \int_{\alpha}^{\beta} g(x)\Pi(x)dx.$$

Consequently,

$$\begin{aligned} & \int_{t_0}^t \sum_{i=1}^{\nu} \left(e^{\lambda_i(t-s)} \left(\sum_{j=0}^{p_i-1} (p_i-j) \frac{(t-s)^{2j}}{(j!)^2} \right)^{1/2} \right) \cdot \|P_{q,n}\| \cdot \|f(s, Q\underline{y}(s)) - f(s, Q\tilde{y}(s))\| ds \leq \\ & \leq \int_{t_0}^t \sum_{i=1}^{\nu} \left(e^{\lambda_i(t-s)} \left(\sum_{j=0}^{p_i-1} (p_i-j) \frac{(t-s)^{2j}}{(j!)^2} \right)^{1/2} \right) \cdot \|P_{q,n}\| \cdot \Pi(s) ds \leq \\ & \leq \|P_{q,n}\| \cdot \int_{t_0}^t \sum_{i=1}^{\nu} \left(e^{\lambda_i(t-t_0)} \left(\sum_{j=0}^{p_i-1} (p_i-j) \frac{(t-t_0)^{2j}}{(j!)^2} \right)^{1/2} \right) \Pi(s) ds = \\ & = \|P_{q,n}\| \cdot \sum_{i=1}^{\nu} \left(e^{\lambda_i(t-t_0)} \left(\sum_{j=0}^{p_i-1} (p_i-j) \frac{(t-t_0)^{2j}}{(j!)^2} \right)^{1/2} \right) \cdot \int_{t_0}^t \Pi(s) ds. \end{aligned}$$

Using the results of Lemma 4.2, the eqs (4.9) and (4.10), the condition (4.8) is transformed to the following.

$$\begin{aligned} & \|\underline{x}(t) - \tilde{\underline{x}}(t)\| \leq \|Q_{n,p}\| \cdot \sum_{i=1}^{\nu} \left(e^{\lambda_i(t-t_0)} \left(\sum_{j=0}^{p_i-1} (p_i-j) \frac{(t-t_0)^{2j}}{(j!)^2} \right)^{1/2} \right) \cdot \|\underline{y}_p(t_0) - \tilde{\underline{y}}_p(t_0)\| + \\ & + \|Q_{n,p}\| \cdot \|P_{q,n}\| \cdot \sum_{i=1}^{\nu} \left(e^{\lambda_i(t-t_0)} \left(\sum_{j=0}^{p_i-1} (p_i-j) \frac{(t-t_0)^{2j}}{(j!)^2} \right)^{1/2} \right) \cdot \int_{t_0}^t \Pi(s) ds + \\ & + \|Q_{n,q}\| \cdot \sum_{j=0}^{q^*-1} \|H_q^j\| \cdot \|P_{q,n}\| \cdot \left\| \frac{d^j}{dt^j} (f(t, Q\underline{y}(t)) - f(t, Q\tilde{y}(t))) \right\| = \\ & = \|Q_{n,p}\| \cdot \sum_{i=1}^{\nu} \left(e^{\lambda_i(t-t_0)} \left(\sum_{j=0}^{p_i-1} (p_i-j) \frac{(t-t_0)^{2j}}{(j!)^2} \right)^{1/2} \right) \left[\|\underline{y}_p(t_0) - \tilde{\underline{y}}_p(t_0)\| + \|P_{q,n}\| \cdot \int_{t_0}^t \Pi(s) ds \right] + \\ & + \|Q_{n,q}\| \cdot \sum_{j=0}^{q^*-1} \left(\|H_q^j\| \cdot \|P_{q,n}\| \cdot \left\| \frac{d^j}{dt^j} (f(t, Q\underline{y}(t)) - f(t, Q\tilde{y}(t))) \right\| \right). \end{aligned} \quad (4.13)$$

Lemma 4.3 Consider that $\lambda < 0$ and $\theta(t) = \sum_{\kappa=1}^{\mu} d_{\kappa} t^{c_{\kappa}}$ with $c_{\kappa} \in \mathbb{R}$ (finite real powers) for every $\kappa = 1, \dots, \mu$ and $d_{\kappa} \in \mathbb{R}$, with finite μ , then $\lim_{t \rightarrow +\infty} (e^{\lambda t} \theta(t)) = 0$.

Proof. Calculating the $\lim_{t \rightarrow +\infty} (e^{\lambda t} t^c)$ with $\lambda < 0$, and c finite, then using c -times De L' Hospital law, we take

$$\lim_{t \rightarrow +\infty} (e^{\lambda t} t^c) = 0.$$

A straightforward generalization of the above result is given below

$$\lim_{t \rightarrow +\infty} (e^{\lambda t} \theta(t)) = 0.$$

Remark 4.4 The factor, which is included in eq. (4.13),

$$\left(\sum_{j=0}^{p_i-1} (p_i - j) \frac{(t - t_0)^{2j}}{(j!)^2} \right)^{1/2} \left[\|\underline{y}_p(t_0) - \tilde{\underline{y}}_p(t_0)\| + \|P_{q,n}\| \cdot \int_{t_0}^t \Pi(s) ds \right] \triangleq \Delta(t)$$

where $\Delta(t)$ is constructed equivalently to $\Pi(t)$, has a greatest power index

$$p_i - 1 + \gamma_{\kappa} + 1 = p_i + \gamma_{\kappa}.$$

Following the results of Lemma 4.3

$$\lim_{t \rightarrow +\infty} (e^{\lambda_i(t-t_0)} \Delta(t)) = 0.$$

Proposition 4.1 The solution of system (1.6) with *consistent* initial conditions is asymptotically stable when expressions (4.9), (4.10) and (4.11) exist simultaneously.

Proof. Considering the eq. (4.13), the Remark 4.4 and the condition (4.11), we have

$$\begin{aligned} \lim_{t \rightarrow +\infty} \|\underline{x}(t) - \tilde{\underline{x}}(t)\| &\leq \lim_{t \rightarrow +\infty} \left[\|Q_{n,p}\| \cdot \sum_{i=1}^v e^{\lambda_i(t-t_0)} \Delta(t) + \right. \\ &\left. + \|Q_{n,q}\| \cdot \sum_{j=0}^{q^*-1} \|H_q^j\| \cdot \|P_{q,n}\| \cdot \left\| \frac{d^j}{dt^j} (f(t, Q\underline{y}(t)) - f(t, Q\tilde{\underline{y}}(t))) \right\| \right] = \\ &= \|Q_{n,p}\| \cdot \sum_{i=1}^v \lim_{t \rightarrow +\infty} (e^{\lambda_i(t-t_0)} \Delta(t)) + \|Q_{n,q}\| \cdot \|P_{q,n}\| \cdot \sum_{j=0}^{q^*-1} \left(\|H_q^j\| \cdot \left(\lim_{t \rightarrow +\infty} \left\| \frac{d^j}{dt^j} (f(t, Q\underline{y}(t)) - f(t, Q\tilde{\underline{y}}(t))) \right\| \right) \right) \\ &= \|Q_{n,p}\| \cdot 0 + \|Q_{n,q}\| \cdot \|P_{q,n}\| \cdot 0 = 0. \end{aligned}$$

Thus

$$\lim_{t \rightarrow +\infty} \|\underline{x}(t) - \tilde{\underline{x}}(t)\| = 0.$$

Afterwards, in this part of the section, the asymptotic stability when the system (1.6) has non-consistent initial conditions is studied. The solution of the system (1.6), as we have already seen in Remark 3.3, is given by

$$\begin{aligned} \underline{x}(t) &= Q_{n,p} e^{J_p(t-t_0)} \underline{y}_p(t_0) + \int_{t_0}^t Q_{n,p} e^{J_p(t-s)} P_{p,n} \underline{f}(s, Q\underline{y}(s)) ds - \\ &- Q_{n,q} \sum_{j=1}^{q^*-1} \delta^{(j-1)}(t) H_q^j \underline{y}_q(t_0) - Q_{n,q} \sum_{j=0}^{q^*-1} H_q^j P_{q,n} (f * \delta^{(j)})(t). \end{aligned}$$

We take a different second solution $\tilde{\underline{x}}(t)$ of the system (1.6). Then

$$\tilde{\underline{x}}(t) = Q_{n,p} e^{J_p(t-t_0)} \tilde{\underline{y}}_p(t_0) + \int_{t_0}^t Q_{n,p} e^{J_p(t-s)} P_{p,n} \underline{f}(s, Q\tilde{\underline{y}}(s)) ds -$$

$$- Q_{n,q} \sum_{j=1}^{q^*-1} \delta^{(j-1)}(t) H_q^j \tilde{y}_q(t_0) - Q_{n,q} \sum_{j=0}^{q^*-1} H_q^j P_{q,n} (\tilde{f} * \delta^{(j)})(t). \quad (4.14)$$

Remark 4.5 The

$$(\tilde{f} * \delta^{(j)})(t) = \begin{bmatrix} (\tilde{f}_1 * \delta^{(j)})(t) \\ (\tilde{f}_2 * \delta^{(j)})(t) \\ \vdots \\ (\tilde{f}_n * \delta^{(j)})(t) \end{bmatrix}$$

and

$$(\tilde{f}_i * \delta^{(j)})(t) = \int_0^t f_i(s, Q\tilde{y}(s)) \cdot \delta^{(j)}(t-s) ds. \quad (4.15)$$

The factor \tilde{f} is indifferent to f , since \tilde{f} depends on $\tilde{y}(s)$ while f on $y(s)$. Moreover, we are interested on the difference

$$\begin{aligned} & \|x(t) - \tilde{x}(t)\| = \\ & = \|Q_{n,p} e^{J_p(t-t_0)} (y_p(t_0) - \tilde{y}_p(t_0)) + \int_{t_0}^t Q_{n,p} e^{J_p(t-s)} P_{p,n} [f(s, Qy(s)) - f(s, Q\tilde{y}(s))] ds - \\ & - Q_{n,q} \sum_{j=1}^{q^*-1} \delta^{(j-1)}(t) H_q^j [y_q(t_0) - \tilde{y}_q(t_0)] - Q_{n,q} \sum_{j=0}^{q^*-1} H_q^j P_{q,n} ((f * \delta^{(j)})(t) - (\tilde{f} * \delta^{(j)})(t))\|. \end{aligned} \quad (4.16)$$

Considering some basic properties of Frobenius norm, we obtain

$$\begin{aligned} & \|x(t) - \tilde{x}(t)\| \leq \|Q_{n,p}\| \cdot \|e^{J_p(t-t_0)}\| \cdot \|y_p(t_0) - \tilde{y}_p(t_0)\| + \\ & + \int_{t_0}^t \|Q_{n,p}\| \cdot \|e^{J_p(t-s)}\| \cdot \|P_{p,n}\| \cdot \|f(s, Qy(s)) - f(s, Q\tilde{y}(s))\| ds + \\ & + \|Q_{n,q}\| \cdot \sum_{j=1}^{q^*-1} |\delta^{(j-1)}(t)| \cdot \|H_q^j\| \cdot \|y_q(t_0) - \tilde{y}_q(t_0)\| + \|Q_{n,q}\| \cdot \sum_{j=0}^{q^*-1} \|H_q^j\| \cdot \|P_{q,n}\| \cdot \|(f * \delta^{(j)})(t) - (\tilde{f} * \delta^{(j)})(t)\|. \end{aligned}$$

In order to find the $\lim_{t \rightarrow \infty} \|x(t) - \tilde{x}(t)\|$, the following Lemmas are significant.

Lemma 4.4

$$\|(f * \delta^{(j)})(t) - (\tilde{f} * \delta^{(j)})(t)\| = \left(\sum_{i=1}^n \left| \int_0^t [f_i(s, Qy(s)) - f_i(s, Q\tilde{y}(s))] \delta^{(j)}(t-s) ds \right|^2 \right)^{1/2}. \quad (4.17)$$

Proof. It is easy to verify that

$$\begin{aligned} \|(f * \delta^{(j)})(t) - (\tilde{f} * \delta^{(j)})(t)\| &= \left\| \begin{bmatrix} \int_0^t [f_1(s, Qy(s)) - f_1(s, Q\tilde{y}(s))] \delta^{(j)}(t-s) ds \\ \vdots \\ \int_0^t [f_n(s, Qy(s)) - f_n(s, Q\tilde{y}(s))] \delta^{(j)}(t-s) ds \end{bmatrix} \right\| = \\ &= \left(\sum_{i=1}^n \left| \int_0^t [f_i(s, Qy(s)) - f_i(s, Q\tilde{y}(s))] \delta^{(j)}(t-s) ds \right|^2 \right)^{1/2}. \end{aligned}$$

Denote that

$$f_i(s, Qy(s)) - f_i(s, Q\tilde{y}(s)) = R_i(s) \quad (4.18)$$

where $R_i(s)$ is a function which is n -times differentiable in $[0, +\infty)$.

Lemma 4.5
$$\int_0^t R_i(s)\delta^{(j)}(t-s)ds = -\left(R_i^{(j)}(t) + \sum_{\kappa=1}^j R_i^{(\kappa-1)}(t)\delta^{(j-\kappa)}(0) \right). \tag{4.19}$$

Proof. We have

$$\begin{aligned} \int_0^t R_i(s)\delta^{(j)}(t-s)ds &= \left[R_i(s)(-1)\delta^{(j-1)}(t-s) \right]_0^t - \int_0^t R_i'(s)(-1)\delta^{(j-1)}(t-s)ds = \\ &= -R_i(t)\delta^{(j-1)}(0) + R_i(0)\delta^{(j-1)}(t) + \int_0^t R_i'(s)\delta^{(j-1)}(t-s)ds. \end{aligned}$$

However, for $t \neq 0$, $\delta(t) = 0$, and $\delta^{(j-1)}(t) = 0$. Consequently, we obtain

$$\int_0^t R_i(s)\delta^{(j)}(t-s)ds = -R_i(t)\delta^{(j-1)}(0) + \int_0^t R_i'(s)\delta^{(j-1)}(t-s)ds.$$

Continuing as above, we have

$$\begin{aligned} \int_0^t R_i(s)\delta^{(j)}(t-s)ds &= -R_i(t)\delta^{(j-1)}(0) - R_i^{(1)}(t)\delta^{(j-2)}(0) - R_i^{(2)}(t)\delta^{(j-3)}(0) - \dots - \\ &\quad - R_i^{(j-1)}(t)\delta(0) + \int_0^t R_i^{(j)}(s)\delta(t-s)ds. \end{aligned}$$

Since $\int_0^t R_i^{(j)}(s)\delta(t-s)ds = -R_i^{(j)}(t)$. We finally take

$$\int_0^t R_i(s)\delta^{(j)}(t-s)ds = -\sum_{\kappa=1}^j R_i^{(\kappa-1)}(t)\delta^{(j-\kappa)}(0) - R_i^{(j)}(t).$$

Remark 4.6 Since the eq. (4.19) includes the elements $\delta^{(j-\kappa)}(0)$ for every $\kappa = 1, 2, \dots, j$, then

the $\lim_{t \rightarrow +\infty} \int_0^t R_i(s)\delta^{(j)}(t-s)ds$ is not zero. Thus, we have

$$\lim_{t \rightarrow +\infty} \|\underline{x}(t) - \tilde{\underline{x}}(t)\| \neq 0.$$

Therefore, the solution of system (1.6) with non-consistent initial conditions is not asymptotically stable.

However, under the following assumptions, we can have asymptotic stability.

1. The real parts of finite elementary divisors of matrix pencil $sF - G$ are negative numbers, i.e. the condition (4.9) is true.
2. The vector function \underline{f} is only t -time variable, i.e. $\underline{f}(t)$.

Analytically, we consider the above assumption, the eq. (4.17) is

$$\|(\underline{f} * \delta^{(j)})(t) - (\tilde{\underline{f}} * \delta^{(j)})(t)\| = 0$$

because

$$f_i(s, Q\underline{y}(s)) - f_i(s, Q\tilde{\underline{y}}(s)) = 0 \text{ for every } i = 1, 2, \dots, n.$$

Moreover, the eq. (4.16) is given by

$$\|\underline{x}(t) - \tilde{\underline{x}}(t)\| = \left\| Q_{n,p} e^{J_p(t-t_0)} (\underline{y}_p(t_0) - \tilde{\underline{y}}_p(t_0)) - Q_{n,q} \sum_{j=1}^{q^*-1} \delta^{(j-1)}(t) H_q^j [\underline{y}_q(t_0) - \tilde{\underline{y}}_q(t_0)] \right\|.$$

Then

$$\|\underline{x}(t) - \tilde{\underline{x}}(t)\| \leq \|Q_{n,p}\| \cdot \|e^{J_p(t-t_0)}\| \cdot \|\underline{y}_p(t_0) - \tilde{\underline{y}}_p(t_0)\| + \|Q_{n,q}\| \cdot \sum_{j=1}^{q^*-1} |\delta^{(j-1)}(t)| \cdot \|H_q^j\| \cdot \|\underline{y}_q(t_0) - \tilde{\underline{y}}_q(t_0)\|. \tag{4.21}$$

Lemma 4.6 Assuming that the condition (4.9) is true then $\lim_{t \rightarrow +\infty} \|e^{J_p(t-t_0)}\| = 0$.

Proof. We have proved in Lemma 4.1 that

$$\|e^{J_p(t-t_0)}\| \leq \sum_{i=1}^{\nu} \left(e^{\lambda_i(t-t_0)} \left(\sum_{j=0}^{p_i-1} (p_i-j) \frac{|(t-t_0)^{2j}|}{|(j!)^2|} \right)^{1/2} \right).$$

Since $\lambda_i < 0$. Then,

$$\lim_{t \rightarrow +\infty} \sum_{i=1}^{\nu} \left(e^{\lambda_i(t-t_0)} \left(\sum_{j=0}^{p_i-1} (p_i-j) \frac{|(t-t_0)^{2j}|}{|(j!)^2|} \right)^{1/2} \right) = 0.$$

Finally, $\lim_{t \rightarrow +\infty} \|e^{J_p(t-t_0)}\| = 0$.

Lemma 4.7 The $\lim_{t \rightarrow +\infty} \|Q_{n,q}\| \cdot \sum_{j=1}^{q^*-1} |\delta^{(j-1)}(t)| \cdot \|H_q^j\| \cdot \|y_q(t_0) - \tilde{y}_q(t_0)\| = 0$.

Proof. This result is a straightforward application of the definition of Dirac δ -function, i.e.

$$\lim_{t \rightarrow +\infty} |\delta^{(m)}(t)| = 0 \text{ for every } m = 0, 1, 2, \dots$$

We conclude the section by considering the following Proposition.

Proposition 4.2 The solution of the system (1.6) with *non-consistent* initial conditions is asymptotically stable when the results (1) (or (4.9)) and (2) of Remark 4.6 hold.

Proof. Considering the Remark 4.6 and Lemmas 4.6 and 4.7, the asymptotic stability of the solution of the system (1.6) with non-consistent initial conditions is derived.

5. Studying the linearized solution of weakly nonlinear regular differential systems

In order to solve nonlinear systems, it is very common to apply a linearization technique. In this section, a standard approach for the study of nonlinear dynamical systems based on classical linearization's technique (Taylor expansion for time t) is provided, see Kalogeropoulos et al. (2008b). Analytically

$$\underline{f}(t, \underline{x}(t)) = (\underline{\omega} \circ \underline{x})(t), \quad (5.1)$$

where $\underline{\omega}: \mathbb{R}^n \rightarrow \mathbb{R}^n$, such as $\underline{\omega} = [\underline{\omega}_1^T \ \dots \ \underline{\omega}_n^T]^T \in \mathbb{R}^{n^2}$ and $\omega_i: \mathbb{R}^n \rightarrow \mathbb{R}$ for $i = 1, 2, \dots, n$.

The linearized approach of $\underline{f}(t, \underline{x}(t))$ is given by the Taylor expansion method using the following expression

$$\underline{f}(t, \underline{x}(t)) \cong \underline{f}(t_0, \underline{x}(t_0)) + \begin{bmatrix} \frac{\theta \omega_1}{\theta x_1}(\underline{x}(t_0)), & \frac{\theta \omega_1}{\theta x_2}(\underline{x}(t_0)), & \dots & \frac{\theta \omega_1}{\theta x_n}(\underline{x}(t_0)) \\ \vdots & & & \\ \vdots & & & \\ \frac{\theta \omega_n}{\theta x_1}(\underline{x}(t_0)), & \dots & \dots & \frac{\theta \omega_n}{\theta x_n}(\underline{x}(t_0)) \end{bmatrix} \begin{bmatrix} x'_1(t_0) \\ x'_2(t_0) \\ \vdots \\ x'_n(t_0) \end{bmatrix} (t-t_0). \quad (5.2)$$

Define

$$J(t_0) = \begin{bmatrix} \frac{\theta\omega_1}{\theta x_1}(\underline{x}(t_0)), & \dots & \frac{\theta\omega_1}{\theta x_n}(\underline{x}(t_0)) \\ \vdots & & \\ \frac{\theta\omega_n}{\theta x_1}(\underline{x}(t_0)), & \dots & \frac{\theta\omega_n}{\theta x_n}(\underline{x}(t_0)) \end{bmatrix} \quad \text{and} \quad \underline{x}'(t_0) = \begin{bmatrix} x'_1(t_0) \\ x'_2(t_0) \\ \vdots \\ x'_n(t_0) \end{bmatrix}$$

Thus, we obtain

$$\underline{f}(t, \underline{x}(t)) \cong \underline{f}(t_0, \underline{x}(t_0)) + J(t_0)\underline{x}'(t_0)(t - t_0). \quad (5.3)$$

Consequently, taking into account (5.3) the system (1.6) is transformed to a linear as follows

$$F\underline{x}'(t) = G\underline{x}(t) + J(t_0)\underline{x}'(t_0)(t - t_0) + \underline{f}(t_0, \underline{x}(t_0)) \quad (5.4)$$

with initial conditions $\underline{x}(t_0) = \underline{x}_0$.

As we have already discussed in section 2, there exist non-singular matrices P, Q , such as

$$PFQ = F_w = \begin{bmatrix} I_p & \mathbb{O}_{p,q} \\ \mathbb{O}_{q,p} & H_q \end{bmatrix} \quad \text{and} \quad PGQ = G_w = \begin{bmatrix} J_p & \mathbb{O}_{p,q} \\ \mathbb{O}_{q,p} & I_q \end{bmatrix}.$$

Then, equivalently as in section 3, the following lemmas derive.

Lemma 5.1 The system (5.4) is divided into two subsystems as follows

$$\underline{y}'_p(t) = J_p \underline{y}_p(t) + P_{p,n} \left(J(t_0) Q \underline{y}'(t_0) (t - t_0) + \underline{f}(t_0, Q \underline{y}(t_0)) \right) \quad (5.5)$$

with initial conditions $\underline{y}_p(t_0)$,

$$H_q \underline{y}'_q(t) = \underline{y}_q(t) + P_{q,n} \left[J(t_0) Q \underline{y}'(t_0) (t - t_0) + \underline{f}(t_0, Q \underline{y}(t_0)) \right] \quad (5.6)$$

with initial conditions $\underline{y}_q(t_0)$.

Proof. We apply the same transformation (3.3) in (5.4)

$$\underline{x}(t) = Q \underline{y}(t).$$

Then, the system (5.4) is transposed to

$$F Q \underline{y}'(t) = G Q \underline{y}(t) + J(t_0) Q \underline{y}'(t_0) (t - t_0) + \underline{f}(t_0, Q \underline{y}(t_0)).$$

Multiplying by left by the non-singular matrix P , we obtain

$$\begin{aligned} PFQ \underline{y}'(t) &= PGQ \underline{y}(t) + P \left[J(t_0) Q \underline{y}'(t_0) (t - t_0) + \underline{f}(t_0, Q \underline{y}(t_0)) \right] \Leftrightarrow \\ &\Leftrightarrow F_w \underline{y}'(t) = G_w \underline{y}(t) + P \left[J(t_0) Q \underline{y}'(t_0) (t - t_0) + \underline{f}(t_0, Q \underline{y}(t_0)) \right] \Leftrightarrow \end{aligned}$$

$$\begin{bmatrix} I_p & \mathbb{O}_{p,q} \\ \mathbb{O}_{q,p} & H_q \end{bmatrix} \cdot \begin{bmatrix} \underline{y}'_p(t) \\ \underline{y}'_q(t) \end{bmatrix} = \begin{bmatrix} J_p & \mathbb{O}_{p,q} \\ \mathbb{O}_{q,p} & I_q \end{bmatrix} \cdot \begin{bmatrix} \underline{y}_p(t) \\ \underline{y}_q(t) \end{bmatrix} + \begin{bmatrix} P_{p,n} \\ P_{q,n} \end{bmatrix} \left(J(t_0) Q \underline{y}'(t_0) (t - t_0) + \underline{f}(t_0, Q \underline{y}(t_0)) \right).$$

Then, eqs. (5.5) and (5.6) derive. Now, the initial conditions is obtained

$$\underline{x}(t_0) = Q \underline{y}(t_0) \Leftrightarrow \underline{y}(t_0) = Q^{-1} \underline{x}(t_0).$$

The $\underline{y}(t_0) = \begin{bmatrix} \underline{y}_p(t_0) \\ \underline{y}_q(t_0) \end{bmatrix}$. Thus, the initial conditions for system (5.5) is given by $\underline{y}_p(t_0)$ and for

system (5.6) is provided by $\underline{y}_q(t_0)$.

Remark 5.1 System (5.5) with initial conditions $\underline{y}_p(t_0)$ can be solved using the classical methods, and

$$\underline{y}_p(t) = e^{J_p(t-t_0)} \underline{y}_p(t_0) + \int_{t_0}^t e^{J_p(t-s)} P_{p,n} \left[J(t_0) Q \underline{y}'(t_0) (s-t_0) + \underline{f}(t_0, Q \underline{y}(t_0)) \right] ds. \quad (5.7)$$

In this part of the sections, we will solve the subsystem (5.6).

Lemma 5.2 The solution of subsystem (5.6) is given by

$$\underline{y}_q(t) = -\left((t-t_0)I_q + H_q\right) P_{q,n} J(t_0) Q \underline{y}'(t_0) - P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)), \quad (5.8)$$

when the following condition is satisfied

$$\underline{y}_q(t_0) = -H_q P_{q,n} J(t_0) Q \underline{y}'(t_0) - P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)) \quad (5.9)$$

i.e. we have consistent initial conditions.

Proof. Equivalently as in section 3, we differentiate the expression (5.6)

$$H_q \underline{y}_q''(t) = \underline{y}_q'(t) + P_{q,n} J(t_0) Q \underline{y}'(t_0).$$

Multiply by left with the nilpotent matrix H_q , we obtain

$$\begin{aligned} H_q^2 \underline{y}_q''(t) &= H_q \underline{y}_q'(t) + H_q P_{q,n} J(t_0) Q \underline{y}'(t_0) \Leftrightarrow \\ \Leftrightarrow H_q^2 \underline{y}_q''(t) &= \underline{y}_q(t) + P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)) + P_{q,n} J(t_0) Q \underline{y}'(t_0) (t-t_0) + H_q P_{q,n} J(t_0) Q \underline{y}'(t_0). \end{aligned}$$

Continuing the above procedure we obtain the following recursive expression,

$$H_q^{q^*} \underline{y}_q^{(q^*)}(t) = \underline{y}_q(t) + P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)) + P_{q,n} J(t_0) Q \underline{y}'(t_0) (t-t_0) + H_q P_{q,n} J(t_0) Q \underline{y}'(t_0).$$

However, since q^* is the annihilation index of matrix H_q , i.e. $H_q^{q^*} = \mathbb{O}$. Then

$$\underline{y}_q(t) = -P_{q,n} J(t_0) Q \underline{y}'(t_0) (t-t_0) - H_q P_{q,n} J(t_0) Q \underline{y}'(t_0) - P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)),$$

for $t = t_0$, we obtain

$$\underline{y}_q(t_0) = -H_q P_{q,n} J(t_0) Q \underline{y}'(t_0) - P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0))$$

and finally expression (5.8) is derived.

Lemma 5.3 The solution of subsystem (5.7) for non-consistent initial conditions, i.e.

$$\underline{y}_q(t_0) \neq -H_q P_{q,n} J(t_0) Q \underline{y}'(t_0) - P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0))$$

is given by

$$\begin{aligned} \underline{y}_q(t) &= -\sum_{\kappa=0}^{q^*-3} \delta^{(\kappa)}(t) H_q^{\kappa+1} \left\{ \underline{y}_q(t_0) + (-t_0 I_q + H_q) P_{q,n} J(t_0) Q \underline{y}'(t_0) \right\} - \\ &- \delta^{(q^*-2)}(t) H_q^{q^*-1} \left(\underline{y}_q(t_0) - t_0 P_{q,n} J(t_0) Q \underline{y}'(t_0) \right) + (t_0 I_q - H_q) P_{q,n} J(t_0) Q \underline{y}'(t_0) - t P_{q,n} J(t_0) Q \underline{y}'(t_0) - \\ &- \sum_{\kappa=1}^{q^*-1} \delta^{(\kappa-1)}(t) P_{q,n} H_q^\kappa \underline{f}(t_0, Q \underline{y}(t_0)) - P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)). \end{aligned} \quad (5.10)$$

Proof. First, we apply in eq. (5.7) the Laplace transformation. Thus, we obtain

$$H_q \mathfrak{L}\{\underline{y}_q'(t)\} = \mathfrak{L}\{\underline{y}_q(t)\} + P_{q,n} J(t_0) Q \underline{y}'(t_0) \mathfrak{L}\{t-t_0\} + P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)) \mathfrak{L}\{1\}.$$

Since

$$\mathfrak{Z}\{\underline{y}(t)\} = X_q(s), \quad \mathfrak{Z}\{t - t_0\} = \frac{1 - t_0s}{s^2}, \quad \mathfrak{Z}\{y'_q(t)\} = sX_q(s) - \underline{y}_q(t_0) \quad \text{and} \quad \mathfrak{Z}\{1\} = \frac{1}{s}.$$

we obtain

$$\begin{aligned} H_q \left(sX_q(s) - \underline{y}_q(t_0) \right) &= X_q(s) + P_{q,n}J(t_0)Q\underline{y}'(t_0) \frac{1 - t_0s}{s^2} + P_{q,n}\underline{f}(t_0, Q\underline{y}(t_0)) \frac{1}{s} \Leftrightarrow \\ \Leftrightarrow (sH_q - I_q)X_q(s) &= H_q\underline{y}_q(t_0) + P_{q,n}J(t_0)Q\underline{y}'(t_0) \frac{1 - t_0s}{s^2} + P_{q,n}\underline{f}(t_0, Q\underline{y}(t_0)) \frac{1}{s}. \end{aligned}$$

Moreover, we have $H_q^{q^*} = \mathbb{O}$ and

$$(sH_q - I_q)^{-1} = -\sum_{\kappa=0}^{q^*-1} (sH_q)^\kappa, \quad \text{where } H_q^0 = I_q.$$

Thus, we obtain

$$\begin{aligned} X_q(s) &= -\left(\sum_{\kappa=0}^{q^*-1} s^\kappa H_q^\kappa \right) H_q\underline{y}_q(t_0) - \left(\sum_{\kappa=0}^{q^*-1} s^\kappa H_q^\kappa \right) P_{q,n}J(t_0)Q\underline{y}'(t_0) \frac{1 - t_0s}{s^2} - \left(\sum_{\kappa=0}^{q^*-1} s^\kappa H_q^\kappa \right) P_{q,n}\underline{f}(t_0, Q\underline{y}(t_0)) \frac{1}{s} = \\ &= -\sum_{\kappa=0}^{q^*-1} s^\kappa H_q^{\kappa+1} \underline{y}_q(t_0) - \sum_{\kappa=0}^{q^*-1} s^{\kappa-2} H_q^\kappa P_{q,n}J(t_0)Q\underline{y}'(t_0) + \\ &+ t_0 \sum_{\kappa=0}^{q^*-1} s^{\kappa-1} H_q^\kappa P_{q,n}J(t_0)Q\underline{y}'(t_0) - \sum_{\kappa=0}^{q^*-1} s^{\kappa-1} H_q^\kappa P_{q,n}\underline{f}(t_0, Q\underline{y}(t_0)). \end{aligned} \tag{5.11}$$

Now, we are working with the following sum

$$\begin{aligned} -\sum_{\kappa=0}^{q^*-1} s^\kappa H_q^{\kappa+1} \underline{y}_q(t_0) &= -\sum_{\kappa=0}^{q^*-3} s^\kappa H_q^{\kappa+1} \underline{y}_q(t_0) - s^{q^*-2} H_q^{q^*-1} \underline{y}_q(t_0) - \\ -s^{q^*-1} H_q^{q^*} \underline{y}_q(t_0) &\stackrel{(H_q^{q^*}=\mathbb{O})}{=} \sum_{\kappa=0}^{q^*-3} s^\kappa H_q^{\kappa+1} \underline{y}_q(t_0) - s^{q^*-2} H_q^{q^*-1} \underline{y}_q(t_0). \end{aligned}$$

Then

$$\begin{aligned} \sum_{\kappa=0}^{q^*-1} s^{\kappa-2} H_q^\kappa &= \sum_{m=-2}^{q^*-3} s^m H_q^{m+2} = \sum_{m=0}^{q^*-3} s^m H_q^{m+2} + s^{-1} H_q + s^{-2} I_q, \\ \sum_{\kappa=0}^{q^*-1} s^{\kappa-1} H_q^\kappa &= \sum_{m=-1}^{q^*-2} s^m H_q^{m+1} = \sum_{m=0}^{q^*-3} s^m H_q^{m+1} + s^{q^*-2} H_q^{q^*-1} + s^{-1} I_q. \end{aligned}$$

Thus, considering the above results the eq. (5.11) is given by

$$\begin{aligned} X_q(s) &= -\sum_{\kappa=0}^{q^*-3} s^\kappa H_q^{\kappa+1} \underline{y}_q(t_0) - s^{q^*-2} H_q^{q^*-1} \underline{y}_q(t_0) - \\ &- \sum_{\kappa=0}^{q^*-3} s^\kappa H_q^{\kappa+2} P_{q,n}J(t_0)Q\underline{y}'(t_0) - s^{-1} H_q P_{q,n}J(t_0)Q\underline{y}'(t_0) - s^{-2} P_{q,n}J(t_0)Q\underline{y}'(t_0) + \\ &+ t_0 \sum_{\kappa=0}^{q^*-3} s^\kappa H_q^{\kappa+1} P_{q,n}J(t_0)Q\underline{y}'(t_0) + t_0 s^{q^*-2} H_q^{q^*-1} P_{q,n}J(t_0)Q\underline{y}'(t_0) + \\ &+ t_0 s^{-1} P_{q,n}J(t_0)Q\underline{y}'(t_0) - \sum_{\kappa=0}^{q^*-1} s^{\kappa-1} H_q^\kappa P_{q,n}\underline{f}(t_0, Q\underline{y}(t_0)) = \end{aligned}$$

$$\begin{aligned}
&= -\sum_{\kappa=0}^{q^*-3} s^\kappa H_q^{\kappa+1} \left(\underline{y}_{-q}(t_0) + H_q P_{q,n} J(t_0) Q \underline{y}'(t_0) - t_0 P_{q,n} J(t_0) Q \underline{y}'(t_0) \right) - \\
&\quad - s^{q^*-2} H_q^{q^*-1} \left(\underline{y}_{-q}(t_0) - t_0 P_{q,n} J(t_0) Q \underline{y}'(t_0) \right) \\
&\quad + s^{-1} (t_0 I_q - H_q) P_{q,n} J(t_0) Q \underline{y}'(t_0) - s^{-2} P_{q,n} J(t_0) Q \underline{y}'(t_0) - \\
&\quad - \sum_{\kappa=1}^{q^*-1} s^{\kappa-1} H_q^\kappa P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)) - \frac{1}{s} P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)).
\end{aligned}$$

Since, it is known that $\mathfrak{I}\{\delta^{(\kappa)}(t)\} = s^\kappa$, we take

$$\begin{aligned}
X_q(s) &= -\sum_{\kappa=0}^{q^*-3} \mathfrak{I}\{\delta^{(\kappa)}(t)\} H_q^{\kappa+1} \left(\underline{y}_{-q}(t_0) + H_q P_{q,n} J(t_0) Q \underline{y}'(t_0) - t_0 P_{q,n} J(t_0) Q \underline{y}'(t_0) \right) - \\
&\quad - \mathfrak{I}\{\delta^{(q^*-2)}(t)\} H_q^{q^*-1} \left(\underline{y}_{-q}(t_0) - t_0 P_{q,n} J(t_0) Q \underline{y}'(t_0) \right) + \\
&\quad + \mathfrak{I}\{1\} (t_0 I_q - H_q) P_{q,n} J(t_0) Q \underline{y}'(t_0) - \mathfrak{I}\{t\} P_{q,n} J(t_0) Q \underline{y}'(t_0) - \\
&\quad - \sum_{\kappa=1}^{q^*-1} \mathfrak{I}\{\delta^{(\kappa-1)}(t)\} H_q^\kappa P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)) - \mathfrak{I}\{1\} P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)).
\end{aligned}$$

Applying the inverse Laplace transformation, eq. (5.10) derives.

Remark 5.2 The solution of the linearized system (5.4) with consistent initial conditions is given by

$$\begin{aligned}
\underline{x}(t) &= Q_{n,p} \left[e^{J_p(t-t_0)} \underline{y}_p(t_0) + \int_{t_0}^t e^{J_p(t-s)} P_{p,n} \left[J(t_0) Q \underline{y}'(t_0) (s-t_0) + \underline{f}(t_0, Q \underline{y}(t_0)) \right] ds - \right. \\
&\quad \left. - Q_{n,q} \left[(t-t_0) I_q + H_q \right] P_{q,n} J(t_0) Q \underline{y}'(t_0) - Q_{n,q} P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)) \right]. \quad (5.12)
\end{aligned}$$

Proof. It is true that

$$\underline{x}(t) = Q \underline{y}(t) \Leftrightarrow \underline{x}(t) = \begin{bmatrix} Q_{n,p} & Q_{n,q} \end{bmatrix} \begin{bmatrix} \underline{y}_p(t) \\ \underline{y}_q(t) \end{bmatrix} \Leftrightarrow \underline{x}(t) = Q_{n,p} \underline{y}_p(t) + Q_{n,q} \underline{y}_q(t).$$

Using the eqs. (5.7) and (5.8), we obtain (5.12).

Remark 5.3 The solution of the linearized system (5.4) with non-consistent initial conditions is given by

$$\begin{aligned}
\underline{x}(t) &= Q_{n,p} \left[e^{J_p(t-t_0)} \underline{y}_p(t_0) + \int_{t_0}^t e^{J_p(t-s)} P_{p,n} \left[J(t_0) Q \underline{y}'(t_0) (s-t_0) + \underline{f}(t_0, Q \underline{y}(t_0)) \right] ds - \right. \\
&\quad - Q_{n,q} \sum_{\kappa=0}^{q^*-3} \delta^{(\kappa)}(t) H_q^{\kappa+1} \left\{ \underline{y}_{-q}(t_0) + (H_q - t_0 I_q) P_{q,n} J(t_0) Q \underline{y}'(t_0) \right\} - \\
&\quad - Q_{n,q} \delta^{(q^*-2)}(t) H_q^{q^*-1} \left(\underline{y}_{-q}(t_0) - t_0 P_{q,n} J(t_0) Q \underline{y}'(t_0) \right) + \\
&\quad + Q_{n,q} (t_0 I_q - H_q) P_{q,n} J(t_0) Q \underline{y}'(t_0) - t Q_{n,q} P_{q,n} J(t_0) Q \underline{y}'(t_0) - \\
&\quad \left. - Q_{n,q} \sum_{\kappa=1}^{q^*-1} \delta^{(\kappa-1)}(t) H_q^\kappa P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)) - Q_{n,q} P_{q,n} \underline{f}(t_0, Q \underline{y}(t_0)) \right]. \quad (5.13)
\end{aligned}$$

Proof. It is known that

$$\underline{x}(t) = Q\underline{y}(t) = \begin{bmatrix} Q_{n,p} & Q_{n,q} \end{bmatrix} \begin{bmatrix} \underline{y}_p(t) \\ \underline{y}_q(t) \end{bmatrix} = Q_{n,p}\underline{y}_p(t) + Q_{n,q}\underline{y}_q(t)$$

Using the eqs. (5.7) and (5.10), we obtain (5.13).

6. Conclusion

In this chapter, the class of weakly nonlinear (regular) differential systems is investigated. This kind of systems has a great importance in the modelling procedure of several applications in many scientific fields (engineering, population growth models, finance, actuarial science models etc). Thus, the analytic solution is provided, considering consistency and non-consistency in the initial conditions. Moreover, the asymptotic stability is discussed. Some necessary conditions are derived in order to obtain the asymptotic stabilization. In both cases, Propositions summarize the necessary conditions, which have not only to do with the real part of the eigenvalues, as it happens in non-descriptor systems.

Furthermore, using a classical linearization technique (Taylor expansion) and the complex Weierstrass canonical form, the linearized system is decomposed into two subsystems, whose solutions are also obtained.

Finally, the forms of solutions for the (non-) consistent initial conditions are provided. Practically speaking, the results of this chapter can be constructed and simulated. Although, there is in abstract mathematical format, the comparison of the solutions provides a first step for extending further our knowledge. Some numerical examples are also appeared. Moreover, the simulation of the linearized solution can be constructed using Matlab m-files (see for instance, Matlab DAE suite, Shampine et al. 1999).

Finally, it should be stressed some possible directions for further research. Firstly, we can transposed the system (1.6) into a control problem by introducing an input vector $\underline{u}(t)$. So,

$$F\underline{x}'(t) = G\underline{x}(t) + B\underline{u}(t) + \underline{f}(t, \underline{x}(t))$$

Thus, the controllability, the observability, the pole assignment, the elimination of impulse behaviour, and so on, can be examined. Moreover, since the system above can include a fixed, finite input $\underline{u}(t)$ (that is a step), then any resulting oscillations in the output will decay, and the output will tend asymptotically to a new final, steady-state value. If in the system, a Dirac delta impulse is given as an input, then the induced oscillations will die away and the system will return to its previous value. If the oscillations do not die away, or the system does not return to its original output when an impulse is applied, then does the system be marginally stable? Those questions are practically very important and very few things are really known.

Furthermore, a better approximation (instead of the Taylor expansion) of the nonlinear parameter can be considered. Under this approximation, a new linear system derives. Many questions about this approach can also be stated.

The last direction considers the introduction of a stochastic point of view for the system (1.6) by

$$F\underline{x}'(t) = G\underline{x}(t) + B\underline{u}(t) + \underline{f}(t, \underline{x}(t)) + B\underline{\xi}(t)$$

where the input $\underline{\xi}$ is a (fractional) white noise. This approach transforms the deterministic dynamic system into a stochastic one.

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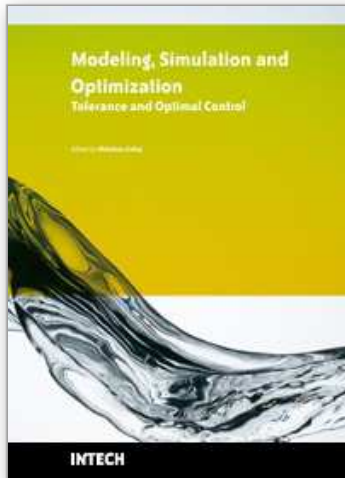
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51000 Rijeka, Croatia
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Phone: +86-21-62489820
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