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Random Variational Inequalities with Applications to Equilibrium Problems under Uncertainty

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Abstract. In this contribution we introduce to the topic of Random Variational Inequalities (RVI) and present some of our recent results in this field. We show how the theory of monotone RVI, where random variables occur both in the operator and the constraints set, can be applied to model nonlinear equilibrium problems under uncertainty arising from economics and operations research, including migration and transportation science. In particular we treat Wardrop equilibria in traffic networks. We describe an approximation procedure for the statistical quantities connected to the equilibrium solution and illustrate this procedure by means of some small sized numerical examples.
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1. Introduction

Although relatively recent, the Variational Inequality (V.I.) approach to a variety of equilibrium problems arising in various fields of applied sciences, such as economics, game theory and transportation science, has developped very rapidly (see e.g. (10), (23), (8), (19)). Since the data of most of the above mentioned problems are often affected by uncertainty, the question arises of how to introduce this uncertainty, or randomness, in their V.I. formulation. In fact, while the topic of stochastic programming is already a well established field of optimization theory (see e.g.(25), (6)), the theory of random (or stochastic) variational inequalities is much less developped.

In (14) the author studied a class of V.I. with a linear random operator, presented an existence and discretization theory and applied this theory to a unilateral boundary value problem stemming from mechanics, where the coefficients of the elliptic differential operator are admitted to be random to model uncertainty in material parameters. The functional setting introduced therein, and extended in (15) in order to include randomness also in the constraints set, can also be utilized to model many finite dimensional random equilibrium problems, which only in special cases admit an optimization formulation (see e.g.(16)). Furthermore,

recently in (17), the authors have extended the theory in (15) to the monotone nonlinear case, while formulating their results in an abstract Hilbert space setting. However, apart from the generalization of the previous theory, this extension is motivated by the need to cope with the nonlinearity in many equilibrium problems arising in operations research, such as the random traffic equilibrium problem which is studied in detail in this article.
For a comparison between our approach and other ways to treat randomness in variational inequalities we refer the interested reader to (16). Here we just quote (11), (12), for the solution of stochastic variational inequalities with a (nonlinear) Fréchet differentiable mapping on a polyhedral subset in finite dimension via the sample-path method, (21) presenting a regularization method for stochastic programs with complementarity constraints, and (28), (29) for a systematic study of stochastic programs under equilibrium constraints.
The paper is structured in 6 sections. In the following Sect. 2 we specialize the abstract formulation of (17) to the case in which the deterministic variables belong to a finite dimensional space, so as to make our theory readily applicable to economics and operations research problems; in Sect. 3 we consider the special case where the deterministic and the random variables are separated; in Sect. 4 we recall and refine the approximation procedure given in (17). Then in Sect. 5, we show how the theory of monotone RVI, where random variables occur both in the operator and the constraints set, can be applied to model various nonlinear equilibrium problems under uncertainty arising from economics and operations research, including migration. In the last Sect. 6 we focus on the modelling of the nonlinear random traffic equilibrium problem and, in order to explain the role of monotonicity, we also discuss the fact that this problem (as every network equilibrium problem) can be formulated by using two different sets of variables, connected by a linear transformation. Finally we illustrate our approximation procedure by two small sized numerical examples of traffic equilibrium problems.

2. The Pointwise and the Integral Formulation

Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space. For all $\omega \in \Omega$, let $\mathcal{K}(\omega)$ be a closed, convex and nonempty subset of $\mathbb{R}^k$. Consider a random vector λ and a Carathéodory function $F : \Omega \times \mathbb{R}^k \mapsto \mathbb{R}^k$, i.e. for each fixed $x \in \mathbb{R}^k$, $F(\cdot, x)$ is measurable with respect to $\mathcal{A}$, and for every $\omega \in \Omega$, $F(\omega, \cdot)$ is continuous. Moreover, for each $\omega \in \Omega$, $F(\omega, \cdot)$ a monotone operator on $\mathbb{R}^k$, i.e. $\langle F(\omega, x) - F(\omega, y), x - y \rangle \geq 0$, $\forall x, y \in \mathbb{R}^k$.
With these data we consider the following

Problem 1. For each $\omega \in \Omega$, find $\hat{X}(\omega) \in \mathcal{K}(\omega)$ such that

$$\langle F(\omega, \hat{X}(\omega)), x - \hat{X}(\omega) \rangle \geq \langle \lambda(\omega), x - \hat{X}(\omega) \rangle , \qquad \forall x \in \mathcal{K}(\omega) . \tag{1}$$

Now we consider the set-valued map $\Sigma : \Omega \hookrightarrow \mathbb{R}^k$ which, to each $\omega \in \Omega$, associates the solution set of (1). The measurability of Σ (with respect to the algebra $\mathcal{B}(\mathbb{R}^k)$ of the Borel sets on $\mathbb{R}^k$ and to the σ-algebra $\mathcal{A}$ on Ω) has been proved in (15) for the case of a bilinear form on a general separable Hilbert space. However, the proof given therein can be adapted straightforwardly to nonlinear operators.
To progress in our analysis we shall confine ourselves to the case of strongly monotone operators, since it is known that the strong monotonicity assumption guarantees the existence of a unique solution to (1) (see (18)). Moreover we shall need the following sharpening of monotonicity.

Definition 2.1. *We call F uniformly strongly monotone, if there is some constant $c_0 > 0$ such that*

$$\langle F(\omega,x) - F(\omega,y), x - y\rangle \geq c_0\|x-y\|^2 \quad \forall x,y \in \mathbb{R}^k, \forall \omega \in \Omega.$$

In many applications, such as the traffic equilibrium problem, the modelling is often done with polynomial cost functions. We are then led to require the growth condition

$$\|F(\omega,z)\| \leq \alpha(\omega) + \beta(\omega)\|z\|^{p-1} \quad \forall z \in \mathbb{R}^k, \tag{2}$$

for some $p \geq 2$.
Since our final aim is to calculate statistical quantities such as the mean value or the variance of the solution of (1), we shall use the following result which has been proved in (17):

Theorem 2.1. *Let $(\Omega, \mathcal{A}, \mu)$ a complete σ-finite measure space, and $F(\omega,\cdot)$ a strongly monotone operator on $\mathbb{R}^k$ for all $\omega \in \Omega$. Then the variational inequality (1) admits a unique solution $\hat{X} : \omega \in \Omega \mapsto \hat{X}(\omega) \in \mathcal{K}(\omega)$. Moreover, suppose that, F is uniformly strongly monotone, that the random vector λ belongs to $L^p(\Omega, \mu, \mathbb{R}^k)$, that the growth condition (2) is satisfied and that there exists $z_0 \in L^{(p-1)p}(\Omega, \mu, \mathbb{R}^k) \bigcap L^p(\Omega, \mu, \mathbb{R}^k)$ such that $z_0(\omega)$ belongs to $\mathcal{K}(\omega)$. Then $\hat{X} \in L^p(\Omega, \mu, \mathbb{R}^k)$.*

Let us now introduce a probability space $(\Omega, \mathcal{A}, P)$ and for fixed $p \geq 2$, the reflexive Banach space $L^p(\Omega, P, \mathbb{R}^k)$ of random vectors V from Ω to $\mathbb{R}^k$ such that the expectation

$$E^P\|V\|^p = \int_\Omega \|V(\omega)\|^p dP(\omega) < \infty. \tag{3}$$

Furthermore we define the convex and closed set

$$K := \{V \in L^p(\Omega, P, \mathbb{R}^k) : V(\omega) \in \mathcal{K}(\omega), \text{P-almost sure}\}.$$

Under the growth condition (2) with $\alpha \in L^p(\Omega, P), \beta \in L^\infty(\Omega, P)$, and assuming that $\lambda \in L^p(\Omega, P, \mathbb{R}^k)$, the integrals

$$\int_\Omega \langle F(\omega, U(\omega)), V(\omega) - U(\omega)\rangle\, dP(\omega), \int_\Omega \langle \lambda(\omega), V(\omega) - U(\omega)\rangle\, dP(\omega)$$

are well defined for all $U, V \in L^p(\Omega, P, \mathbb{R}^k)$. Therefore, we can consider the following

Problem 2. Find $U \in K$ such that, $\forall V \in K$,

$$\int_\Omega \langle F(\omega, U(\omega)),\, V(\omega) - U(\omega))\rangle\, dP(\omega) \geq \int_\Omega \langle \lambda(\omega), V(\omega) - U(\omega)\rangle\, dP(\omega). \tag{4}$$

Under our assumptions, (4) has a unique solution $U \in L^p(\Omega, P, \mathbb{R}^k)$. Thus, by uniqueness, Problem 1 and Problem 2 are equivalent in the sense that from the integral formulation in Problem 2 we obtain a pointwise solution that is only defined P-a.e. on Ω and that coincides there with the pointwise solution of Problem 1.

3. The Separable Case

Here and in the sequel we shall posit further assumptions on the structure of the random set and on the operator. More precisely, with a matrix $A \in \mathbb{R}^{m\times k}$ and a random m - vector D being given, we consider the random set

$$M(\omega) := \{x \in \mathbb{R}^k : Ax \leq D(\omega)\},\ \omega \in \Omega.$$

Moreover, let $G, H : \mathbb{R}^k \to \mathbb{R}^k$ be two (nonlinear) maps; $b, c \in \mathbb{R}^k$ fixed vectors and R and S two real valued random variables on Ω. Thus, we simplify Problem 1 to that of finding $\hat{X} : \Omega \to \mathbb{R}^k$, such that $\hat{X}(\omega) \in M(\omega)$ (P-a.s.) and the following inequality holds for P-almost every elementary event $\omega \in \Omega$ and $\forall x \in M(\omega)$

$$\langle S(\omega)\, G(\hat{X}(\omega)) + H(\hat{X}(\omega)), x - \hat{X}(\omega)\rangle \geq \langle b + R(\omega)\, c, x - \hat{X}(\omega)\rangle\,. \tag{5}$$

We assume that $S \in L^\infty(\Omega)$ and $R \in L^p(\Omega)$, while the operator F defined by

$$F(\omega, x) := S(\omega)\, G(x) + H(x)$$

is uniformly strongly monotone. The uniform strong monotonicity of F is ensured by the strong monotonicity of $\underline{s}\ G$ and H , where $\underline{s}$ is a positive constant such that there holds $S \geq \underline{s}$ P - a.s. (almost sure). We also require that F satisfies the growth condition (2).
Moreover, we assume that $D \in L^p_m(\Omega) := L^p(\Omega, P, \mathbb{R}^m)$. Hence we can introduce the following closed convex nonvoid subset of $L^p_k(\Omega)$:

$$M^P := \{V \in L^p_k(\Omega) : A\, V(\omega) \leq D(\omega),\ P - a.s.\}$$

and consider the following problem:
Find $\hat{U} \in M^P$ such that, $\forall V \in M^P$,

$$\int_\Omega \langle S(\omega)\, G(\hat{U}(\omega)) + H(\hat{U}(\omega)), V(\omega) - \hat{U}(\omega)\rangle\, dP(\omega) \geq \int_\Omega \langle b + R(\omega)\, c, V(\omega) - \hat{U}(\omega)\rangle\, dP(\omega)\,. \tag{6}$$

The r.h.s. of (6) defines a continuous linear form on $L^p_k(\Omega)$, while the l.h.s. defines a continuous form on $L^p_k(\Omega)$ which inherits strong monotonicity from the strong monotonicity of $\underline{s}\ G + H$. Therefore, (see e.g. (18)), there exists a unique solution in M^P to problem (6). By uniqueness, problems (5) and (6) are equivalent.
In order to get rid of the abstract sample space Ω, we consider the joint distribution $\mathbb{P}$ of the random vector (R, S, D) and work with the special probability space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{P})$, where the dimension $d := 2 + m$. To simplify our analysis we shall suppose that R, S and D are independent random vectors. Let $r = R(\omega)$, $s = S(\omega)$, $t = D(\omega)$, $y = (r, s, t)$. For each $y \in \mathbb{R}^d$, consider the set

$$M(y) := \{x \in \mathbb{R}^k : Ax \leq t\}$$

Then the pointwise version of our problem now reads:
Find $\hat{x}$ such that $\hat{x}(y) \in M(y)$, $\mathbb{P}$ - a.s., and the following inequality holds for $\mathbb{P}$ - almost every $y \in \mathbb{R}^d$ and $\forall x \in M(y)$,

$$\langle s\, G(\hat{x}(y)) + H(\hat{x}(y)), x - \hat{x}(y)\rangle \geq \langle b + r\, c, x - \hat{x}(y)\rangle\,. \tag{7}$$

In order to obtain the integral formulation of (7), consider the space $L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k)$ and introduce the closed convex nonvoid set

$$M_{\mathbb{P}} := \{v \in L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k) : Av(r,s,t) \le t,\ \mathbb{P} - a.s.\}.$$

This leads to the problem:
Find $\hat{u} \in M_{\mathbb{P}}$ such that, $\forall v \in M_{\mathbb{P}}$,

$$\int_{\mathbb{R}^d} \langle s\,G(\hat{u}(y)) + H(\hat{u}(y)), v(y) - \hat{u}(y)\rangle\, d\mathbb{P}(y) \ge \int_{\mathbb{R}^d} \langle b + r\,c, v(y) - \hat{u}(y)\rangle\, d\mathbb{P}(y). \quad (8)$$

By using the same arguments as in the ω-formulation in section 2, problems (7) and (8) are equivalent.

Remark 3.1. *Our approach and analysis here and in the next section readily applies also to more general finite Karhunen-Loève expansions*

$$\lambda(\omega) \quad = \quad b + \sum_{l=1}^{L} R_l(\omega)\, c_l\,,\ \ F(\omega,x) = H(x) + \sum_{l=1}^{L_F} S_l(\omega)\, G_l(x)\,.$$

However, such an extension does not only need a more lengthy notation, but - more importantly - leads to more computational work; see (15) for a more thorough discussion of those computational aspects.

4. An Approximation Procedure by Discretization of Distributions

Without loss of generality, we can suppose that $R \in L^p(\Omega,P)$ and $D \in L^p_m(\Omega,P)$ are nonnegative (otherwise we can use the standard decomposition in the positive part and the negative part). Moreover, we assume that the support (the set of possible outcomes) of $S \in L^\infty(\Omega,P)$ is the interval $[\underline{s},\bar{s}) \subset (0,\infty)$. Furthermore we assume that the distributions P_R, P_S, P_D are continuous with respect to the Lebesgue measure, so that according to the theorem of Radon-Nikodym, they have the probability densities $\varphi_R, \varphi_S, \varphi_{D_i}\, i = 1,\ldots,m$, respectively. Hence, $\mathbb{P} = P_R \otimes P_S \otimes P_D$, $dP_R(r) = \varphi_R(r)\,dr$, $dP_S(s) = \varphi_S(s)\,ds$ and $dP_{D_i}(t_i) = \varphi_{D_i}(t_i)\,dt_i$ for $i = 1,\ldots,m$. Let us note that $v \in L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k)$ means that $(r,s,t) \mapsto \varphi_R(r)\varphi_S(s)\varphi_D(t)v(r,s,t)$ belongs to the standard Lebesgue space $L^p(\mathbb{R}^d,\mathbb{R}^k)$ with respect to the Lebesgue measure, where shortly $\varphi_D(t) := \prod_i \varphi_{D_i}(t_i)$. Thus we arrive at the probabilistic integral formulation of our problem:
Find $\hat{u} \in M_{\mathbb{P}}$ such that, $\forall v \in M_{\mathbb{P}}$,

$$\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} \langle s\,G(\hat{u}) + H(\hat{u}), v - \hat{u}\rangle\, \varphi_R(r)\varphi_S(s)\varphi_D(t)\,dy \ge$$
$$\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} \langle b + r\,c, v - \hat{u}\rangle\, \varphi_R(r)\varphi_S(s)\varphi_D(t)\,dy.$$

In order to give an approximation procedure for the solution $\hat{u}$, let us start with a discretization of the space $X := L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k)$ and introduce a sequence $\{\pi_n\}_n$ of partitions of the support $\Upsilon := [0,\infty) \times [\underline{s},\bar{s}) \times \mathbb{R}^m_+$ of the probability measure $\mathbb{P}$ induced by the random elements R, S, D. To be precise, let $\pi_n = (\pi_n^R, \pi_n^S, \pi_n^D)$, where

$$\pi_n^R := (r_n^0,\ldots,r_n^{N_n^R}),\ \pi_n^S := (s_n^0,\ldots,s_n^{N_n^S}),\ \pi_n^{D_i} := (t_{n,i}^0,\ldots,t_{n,i}^{N_n^{D_i}})$$

$$0 = r_n^0 < r_n^1 < \dots r_n^{N_n^R} = n$$
$$\underline{s} = s_n^0 < s_n^1 < \dots s_n^{N_n^S} = \overline{s}$$
$$0 = t_{n,i}^0 < t_{n,i}^1 < \dots t_{n,i}^{N_n^{D_i}} = n \ (i = 1, \dots, m)$$
$$|\pi_n^R| := \max\{r_n^j - r_n^{j-1} : j = 1, \dots, N_n^R\} \to 0 \qquad (n \to \infty)$$
$$|\pi_n^S| := \max\{s_n^k - s_n^{k-1} : k = 1, \dots, N_n^S\} \to 0 \qquad (n \to \infty)$$
$$|\pi_n^{D_i}| := \max\{t_{n,i}^{h_i} - t_{n,i}^{h_i - 1} : h_i = 1, \dots, N_n^{D_i}\} \to 0 \qquad (i = 1, \dots, m; n \to \infty).$$

These partitions give rise to the exhausting sequence $\{Y_n\}$ of subsets of Y, where each Y_n is given by the finite disjoint union of the intervals:

$$I_{jkh}^n := [r_n^{j-1}, r_n^j) \times [s_n^{k-1}, s_n^k) \times I_h^n,$$

where we use the multiindex $h = (h_1, \cdots, h_m)$ and

$$I_h^n := \Pi_{i=1}^m [t_{n,i}^{h_i - 1}, t_{n,i}^{h_i}).$$

For each $n \in \mathbb{N}$ let us consider the space of the $\mathbb{R}^l$-valued simple functions ($l \in \mathbb{N}$) on Y_n, extended by 0 outside of Y_n:

$$X_n^l := \{v_n : v_n(r,s,t) = \sum_j \sum_k \sum_h v_{jkh}^n 1_{I_{jkh}^n}(r,s,t), v_{jkh}^n \in \mathbb{R}^l\}$$

where 1_I denotes the $\{0,1\}$-valued characteristic function of a subset I.
To approximate an arbitrary function $w \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R})$ we employ the mean value truncation operator μ_0^n associated to the partition π_n given by

$$\mu_0^n w := \sum_{j=1}^{N_n^R} \sum_{k=1}^{N_n^S} \sum_h (\mu_{jkh}^n w) 1_{I_{jkh}^n}, \tag{9}$$

where

$$\mu_{jkh}^n w := \begin{cases} \dfrac{1}{\mathbb{P}(I_{jkh})} \displaystyle\int_{I_{jkh}^n} w(y)\, d\mathbb{P}(y) & \text{if } \mathbb{P}(I_{jkh}^n) > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Likewise for a L^p vector function $v = (v_1, \dots, v_l)$, we define $\mu_0^n v := (\mu_0^n v_1, \dots, \mu_0^n v_l)$. From Lemma 2.5 in ((14)) (and the remarks therein) we obtain the following result.

Lemma 4.1. *For any fixed $l \in \mathbb{N}$, the linear operator $\mu_0^n : L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l) \to L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l)$ is bounded with $\|\mu_0^n\| = 1$ and for $n \to \infty$, μ_0^n converges pointwise in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l)$ to the identity.*

This lemma reflects the well-known density of the class of the simple functions in a L^p space. It shows that the mean value truncation operator μ_0^n, which acts as a projector on $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l)$, can be understood as a conditional expectation operator introduced by Kolmogorov in 1933, see also (7), and thus our approximation method is a projection method according to the terminology of (20).
In order to construct approximations for

$$M_{\mathbb{P}} = \{v \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) : Av(r,s,t) \leq t\,,\ \mathbb{P} - a.s.\}$$

we introduce the orthogonal projector $q : (r,s,t) \in \mathbb{R}^d \mapsto t \in \mathbb{R}^m$ and let, for each elementary quadrangle I_{jkh}^n,

$$\bar{q}_{jkh}^n = (\mu_{jkh}^n q) \in \mathbb{R}^m, \qquad (\mu_0^n q) = \sum_{jkh} \bar{q}_{jkh}^n 1_{I_{jkh}^n} \in X_n^m\,.$$

Thus we arrive at the following sequence of convex, closed sets

$$M_{\mathbb{P}}^n := \{v \in X_n^k : Av_{jkh}^n \leq \bar{q}_{jkh}^n\,,\ \forall j,k,h\}\,.$$

Note that the sets $M_{\mathbb{P}}^n$ are of polyhedral type. In (17) it has been proved that the sequence $\{M_{\mathbb{P}}^n\}$ approximate the set $M_{\mathbb{P}}$ in the sense of Mosco ((1), (22)), i.e.

$$\text{weak-limsup}_{n\to\infty} M_{\mathbb{P}}^n \subset M_{\mathbb{P}} \subset \text{strong-liminf}_{n\to\infty} M_{\mathbb{P}}^n\,.$$

Moreover we want to approximate the random variables R and S and introduce

$$\rho_n = \sum_{j=1}^{N_n^R} r_n^{j-1} 1_{[r_n^{j-1}, r_n^j)} \in X_n, \qquad \sigma_n = \sum_{k=1}^{N_n^S} s_n^{k-1} 1_{[s_n^{k-1}, s_n^k)} \in X_n\,.$$

We observe that $\sigma_n(r,s,t) \to \sigma(r,s,t) = s$ in $L^\infty(\mathbb{R}^d, \mathbb{P})$ while, as a consequence of the Chebyshev inequality (see e.g. (3)), $\rho_n(r,s,t) \to \rho(r,s,t) = r$ in $L^p(\mathbb{R}^d, \mathbb{P})$.
Thus we are led to consider, $\forall n \in \mathbb{N}$, the following substitute problem:
Find $\hat{u}_n \in M_{\mathbb{P}}^n$ such that, $\forall v_n \in M_{\mathbb{P}}^n$,

$$\int_{\mathbb{R}^d} \langle \sigma_n(y)\, G(\hat{u}_n(y)) + H(\hat{u}_n(y)), v_n(y) - \hat{u}_n(y)\rangle \, d\mathbb{P}(y) \geq \int_{\mathbb{R}^d} \langle b + \rho_n(y)\, c, v_n(y) - \hat{u}_n(y)\rangle \, d\mathbb{P}(y)\,. \tag{10}$$

We observe that (10) splits in a finite number of finite dimensional strongly monotone variational inequalities:
For $\forall n \in \mathbb{N}$, $\forall j,k,h$ find $\hat{u}_{jkh}^n \in M_{jkh}^n$ such that, $\forall v_{jkh}^n \in M_{jkh}^n$,

$$\langle \tilde{F}_k^n(\hat{u}_{jkh}^n), v_{jkh}^n - \hat{u}_{jkh}^n\rangle \geq \langle \tilde{c}_j^n, v_{jkh}^n - \hat{u}_{jkh}^n\rangle\,, \tag{11}$$

where

$$\begin{aligned} M_{jkh}^n &:= \{v_{jkh}^n \in \mathbb{R}^k : Av_{jkh}^n \leq \bar{q}_{jkh}^n\}\,, \\ \tilde{F}_k^n &:= s_n^{k-1}\, G + H\,,\ \ \tilde{c}_j^n := b + r_n^{j-1}\, c\,. \end{aligned}$$

Clearly, this gives

$$\hat{u}_n = \sum_j \sum_k \sum_h \hat{u}_{jkh}^n 1_{I_{jkh}^n} \in X_n^k\,.$$

Now, we can state the following convergence result (whose proof can be found in (17)).

Theorem 4.1. *The sequence $\hat{u}_n$ generated by the substitute problems in (10) converges strongly in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ for $n \to \infty$ to the unique solution $\hat{u}$ of (8).*

This theorem can be refined under the additional assumption of Lipschitz continuity, because in this case (and in virtue of uniform strong monotonicity), it is enough to solve the finite dimensional substitute problem (10) only inaccurately.

Theorem 4.2. *Suppose, both maps G and H are uniformly strongly monotone and Lipschitz continuous. Let $\varepsilon_n > 0$. Introduce the monotone operator T_n by*

$$T_n(u)(y) := \sigma_n(y)\, G(u)(y) + H(u)(y) - b - \rho_n(y)\, c$$

and the associated natural map

$$F_n^{nat}(u) = u - Proj_{M_{\mathbb{P}}^n}[u - T_n(u)],$$

both acting in $X_n^l(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ (where Proj is the minimum norm projection). Let $\tilde{u}_n \in M_{\mathbb{P}}^n$ satisfy

$$\|F_n^{nat}(\tilde{u}_n)\| \leq \varepsilon_n. \tag{12}$$

Suppose that in (12), $\varepsilon_n \to 0$ for $n \to \infty$. Then the sequence $\tilde{u}_n$ converges strongly in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ to the unique solution $\hat{u}$ of (8).

Proof. It will be enough to show that $\lim_n \|\tilde{u}_n - \hat{u}_n\| = 0$.
Let us observe that obviously a zero $\hat{u}_n$ of F_n^{nat} is an exact solution of (10). Instead we solve (10) only inaccurately. In fact, we can estimate (see (8) Volume I, Theorem 2.3.3)

$$\|\tilde{u}_n - \hat{u}_n\| \leq \frac{L_n + 1}{c_n} \|F_n^{\text{nat}}(\tilde{u}_n)\|,$$

where L_n, respectively c_n is the Lipschitz constant, respectively the uniform monotonicity constant of T_n. Since the support of the random variable $S \in L^\infty(\Omega, P)$ is the interval $[\underline{s}, \bar{s}) \subset (0, \infty)$ and $\underline{s}G + H$ is uniformly strongly monotone with some constant $c_0 > 0$, respectively $\bar{s}G + H$ is Lipschitz continuous with some constant L_0, we have $0 < c_0 \leq c_n, L_n \leq L_0 < \infty$. Therefore by construction, $\lim_n \|\tilde{u}_n - \hat{u}_n\| = 0$ follows.

□

5. Some Random Nonlinear Equilibrium Problems

In this section we describe some simple equilibria problems from economics and migration theory, while equilibrium problems using a more involved network structure are deferred to the next section. Here we discuss where uncertainty can enter in the data of the problems and show how our theory of RVI, where we can admit that random variables occur both in the operator and the constraints set, can be applied to model those nonlinear equilibrium problems under uncertainty.

5.1 A random Cassel-Wald economic equilibrium model

We follow (19) and describe a Cassel-Wald type economic equilibrium model. This model deals with n commodities and m pure factors of production. Let c_k be the price of the k-th commodity, let b_i be the total inventory of the i-th factor, and let a_{ij} be the consumption rate of the i-th factor which is required for producing one unit of the j-th commodity, so that we set $c = (c_1, \dots, c_n)^T, b = (b_1, \dots, b_m)^T, A = (a_{ij})_{m\times n}$. Next let x_j denote the output of the j-th commodity and p_i denote the shadow price of the i-th factor, so that $x = (x_1, \dots, x_n)^T$ and $p = (p_1, \dots, p_m)^T$. In this model it is assumed that the prices are dependent on the outputs, so that $c : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is a given mapping. Now in contrast to (19) we do not consider b as a fixed vector, but we admit that the total inventory vector may be uncertain and model it as a random vector $b = B(\omega)$. Thus we arrive at the following
Problem CW-1. For each $\omega \in \Omega$, find $\hat{X}(\omega) \in \mathbb{R}^n_+, \hat{P}(\omega) \in \mathbb{R}^n_+$ such that

$$\langle c(\hat{X}(\omega)), \hat{X}(\omega) - x\rangle + \langle \hat{P}(\omega), Ax - A\hat{X}(\omega)\rangle \geq 0, \quad \forall x \in \mathbb{R}^n_+;$$
$$\langle p - \hat{P}(\omega), B(\omega) - A\hat{X}(\omega)\rangle \geq 0, \quad \forall p \in \mathbb{R}^n_+.$$

This is nothing but the optimality condition for the variational inequality problem:
Problem CW-2. For each $\omega \in \Omega$, find $\hat{X}(\omega) \in \mathcal{K}(\omega)$ such that

$$\langle c(\hat{X}(\omega)), \hat{X}(\omega) - x\rangle \geq 0, \quad \forall x \in \mathcal{K}(\omega),$$

where here

$$\mathcal{K}(\omega) = \{x \in \mathbb{R}^n \mid x \geq 0, Ax \leq B(\omega)\}.$$

Both problems CW-1 and CW-2 are special instances of the general Problem 1, where randomness in CW-1 only occurs via the generally nonlinear mapping c, while randomness in CW-2 also affects the comstraints set.

5.2 A random distributed market equilibrium model

We follow (13) and consider a single commodity that is produced at n supply markets and consumed at m demand markets. There is a total supply g_i in each supply market i, where $i = 1, \dots, n$. Likewise there is a total demand f_j in each demand market j, where $j = 1, \dots, m$. Since the markets are spatially separated, x_{ij} units of the commodity are transported from i to j. Introducing the excess supply s_i and the excess demand t_j we must have

$$g_i = \sum_{j=1}^{m} x_{ij} + s_i, \qquad i = 1, \dots, n; \tag{13}$$

$$f_j = \sum_{i=1}^{n} x_{ij} + t_j, \qquad j = 1, \dots, m; \tag{14}$$

Moreover the transportation from i to j gives rise to unit costs π_{ij}. Further we associate with each supply market i a supply price p_i and with each demand market j a demand price q_j. We assume there is given a fixed minimum supply price $\underline{p}_i \geq 0$ ('price floor') for each supply market i and also a fixed maximum demand price $\bar{q}_j > 0$ ('price ceiling') for each demand market j. These bounds can be absent and the standard spatial price equilibrium model due to Dafermos ((5), see also (19)) results, where the markets are required to be cleared, i.e.

$$s_i = 0 \quad \text{for } i = 1, \dots, n; \qquad t_j = 0 \quad \text{for } j = 1, \dots, m$$

are required to hold. Since $s_i \geq 0$ and $t_j \geq 0$ are admitted, the model is also termed a disequilibrium model. As is common in operations research models, we also include upper bounds $\bar{x}_{ij} > 0$ for the transportation fluxes x_{ij} on our bipartite graph of distributed markets.
Let us group the introduced quantities in vectors omitting the indices i and j: We have the total supply vector $g \in \mathbb{R}^n$, the supply price vector $p \in \mathbb{R}^n$, the total demand vector $f \in \mathbb{R}^m$, the demand price vector $q \in \mathbb{R}^m$, the flux vector $x \in \mathbb{R}^{nm}$, and the unit cost vector $\pi \in \mathbb{R}^{nm}$. Thus in our constrained distibuted market model the feasible set for the unknown vector $u = [p,q,x]$ is given by the product set

$$M := \prod_{i=1}^{n} [\underline{p}_i, \infty) \times \prod_{j=1}^{m} [0, \bar{q}_j] \times \prod_{i=1}^{n} \prod_{j=1}^{m} [0, \bar{x}_{ij}] \, .$$

As soon as the given bounds are uncertain and we model these bounds as random variables, we obtain the random constraints set

$$\mathcal{M}(\omega) := \prod_{i=1}^{n} [\underline{p}_i(\omega), \infty) \times \prod_{j=1}^{m} [0, \bar{q}_j(\omega)] \times \prod_{i=1}^{n} \prod_{j=1}^{m} [0, \bar{x}_{ij}(\omega)] \, .$$

Assuming perfect equilibrium the economic market conditions take the following form

$$s_i > 0 \quad \Rightarrow \quad p_i = \underline{p}_i \, , \quad p_i > \underline{p}_i \Rightarrow s_i = 0 \qquad i = 1, \ldots, n; \tag{15}$$

$$t_j > 0 \quad \Rightarrow \quad q_j = \bar{q}_j \, , \quad q_j < \bar{q}_j \Rightarrow t_j = 0 \qquad j = 1, \ldots, m; \tag{16}$$

$$p_i + \pi_{ij} \begin{cases} \geq q_j & \text{if } x_{ij} = 0 \\ = q_j & \text{if } 0 < x_{ij} < \bar{x}_{ij} \\ \leq q_j & \text{if } x_{ij} = \bar{x}_{ij} \end{cases} \qquad i = 1, \ldots, n \, ; j = 1, \ldots, m \, . \tag{17}$$

The last condition (17) extends the classic equilibrium conditions in that $p_i + \pi_{ij} < q_j$ can occur because of the flux constraint $x_{ij} \leq \bar{x}_{ij}$. As in unconstrained market equilibria ((5)) we assume that we are given the functions

$$g = \breve{g}(p) \, , f = \breve{f}(q) \, , \pi = \breve{\pi}(x) \, .$$

Then under the natural assumptions that for each $i = 1, \ldots, n; j = 1, \ldots, m$ there holds

$$q_j = 0 \Rightarrow \breve{f}_j(q) \geq 0 \, ; \quad x_{ij} > 0 \Rightarrow \breve{\pi}_{ij}(x) > 0 \, .$$

it can be shown (see (13)) that a market equilibrium $u = (p,q,x)$ introduced above by the conditions (13) – (17) can be characterized as a solution to the following Variational Inequality: Find $u = (p,q,x) \in M$ such that

$$\sum_{i=1}^{n} (\breve{g}_i(p) - \sum_{j=1}^{m} x_{ij})(\tilde{p}_i - p_i) \; - \; \sum_{j=1}^{m} (\breve{f}_j(q) - \sum_{i=1}^{n} x_{ij})(\tilde{q}_j - q_j)$$
$$+ \; \sum_{i=1}^{n} \sum_{j=1}^{m} (p_i + \breve{\pi}_{ij}(x) - q_j)(\tilde{x}_{ij} - x_{ij}) \; \geq \; 0 \, , \qquad \forall \tilde{u} = (\tilde{p}, \tilde{q}, \tilde{x}) \in M \, .$$

Now also the functions $\breve{g}, \breve{f}, \breve{\pi}$ may be not precisely known, but are affected by uncertainness, so may be modelled as random. Thus we obtain the random distributed market problem:
Problem DM. For each $\omega \in \Omega$, find $(\hat{P}, \hat{Q}, \hat{X})(\omega) \in \mathcal{M}(\omega)$ such that

$$\sum_{i=1}^{n}(\breve{g}_i(\omega,\hat{P}(\omega)) - \sum_{j=1}^{m}\hat{X}_{ij}(\omega))(\tilde{p}_i - \hat{P}_i(\omega))$$

$$- \sum_{j=1}^{m}(\breve{f}_j(\omega,\hat{Q}(\omega)) - \sum_{i=1}^{n}\hat{X}_{ij})(\omega))(\tilde{q}_j - \hat{Q}_j(\omega))$$

$$+ \sum_{i=1}^{n}\sum_{j=1}^{m}(\hat{P}_i(\omega) + \breve{\pi}_{ij}(\omega,\hat{X}) - \hat{Q}_j(\omega))(\tilde{x}_{ij} - \hat{X}_{ij}(\omega)) \geq 0, \forall \tilde{u} = (\tilde{p},\tilde{q},\tilde{x}) \in \mathcal{M}(\omega).$$

Obviously Problem DM is a special instance of Problem 1 with randomness both in the operator and in the constraints set.

5.3 A random migration equilibrium model

We follow (19) in simplifying a more involved migration model of (23). This model involves a set of nodes (locations) $\mathbf{N}$. For each $i \in \mathbf{N}$ let b_i denote the the initial fixed population in location i; let h_{ij} denote the value of the migration flow from i to j, and let x_i denote the current population in location i. Set $x = \{x_i | i \in \mathbf{N}\}$ and $h = \{h_{ij} | i,j \in \mathbf{N}, i \neq j\}$. Because of nonnegativity of the migration flow and due to the conservation of flows while preventing any chain migration we have the feasible set

$$\begin{aligned} M \quad := \quad & \{(x,h) \,|\, h \geq 0, \sum_{j \neq i} h_{ij} \leq b_i, \\ & x_i = b_i + \sum_{j \neq i} h_{ji} - \sum_{j \neq i} h_{ij}, \ \ \forall i \in \mathbf{N}\}. \end{aligned}$$

With each location i there is associated the utility u_i that is assumed to be dependent on the population, i.e. $u_i = \breve{u}_i(x)$. Also with each pair of loactions $i, j; i \neq j$ there is associated the migration cost c_{ij} that is assumed to be dependent on the migration flow, i.e. $c_{ij} = \breve{c}_{ij}(h)$. Now a pair $(x,h) \in M$ is considered to be in equilibrium, if (note the similarities to the equlibrium conditions (15) and (17)!)

$$\breve{u}_i(x) - \breve{u}_j(x) + \breve{c}_{ij}(h) + \mu_i \begin{cases} \geq 0 & \text{if } h_{ij} = 0, \\ = 0 & \text{if } h_{ij} > 0; \end{cases} \qquad \forall i,j \in \mathbf{N}, i \neq j;$$

$$\mu_i \begin{cases} \geq 0 & \text{if } \sum_{l \neq i} h_{il} = b_i, \\ = 0 & \text{if } \sum_{l \neq i} h_{il} < b_i; \end{cases} \qquad \forall i \in \mathbf{N}.$$

These equilibrium conditions can be equivalently expressed in the form of the variational inequality:
Find a pair $(x,h) \in M$ such that

$$\sum_{i \in \mathbf{N}} \breve{u}_i(x)(x_i - \tilde{x}_i) + \sum_{i,j \in \mathbf{N}, i \neq j} \breve{c}_{ij}(h)(\tilde{h}_{ij} - h_{ij}) \geq 0, \qquad \forall (\tilde{x},\tilde{h}) \in M.$$

Now the functions $\breve{u}_i, \breve{c}_{ij}$ may be not precisely known, but are affected by uncertainness, so may be modelled as random. Thus we obtain the random migration problem:

Problem M. For each $\omega \in \Omega$, find $(\hat{X},\hat{H})(\omega) \in M$ such that

$$\sum_{i\in\mathbf{N}} \breve{u}_i(\omega,\hat{X}(\omega))(\hat{X}_i(\omega) - \tilde{x}_i) + \sum_{i,j\in\mathbf{N},i\neq j} \breve{c}_{ij}(\omega,\hat{H}(\omega))(\tilde{h}_{ij} - \hat{H}_{ij}(\omega)) \geq 0\,, \qquad \forall(\tilde{x},\tilde{h}) \in M\,.$$

Obviously Problem DM is a special instance of Problem 1 now with randomness only in the operator.

6. A Random Traffic Equilibrium Problem

In this Section 6 we apply our results to network equilibrium problems. A common characteristic of these problems is that they admit two different formulations based either on link variables or on path variables. These are actually related to each other through a linear transformation; we stress that in general, in the path approach, the strong monotonicity assumption is not reasonable. However, we are able to fill this gap by proving a Mosco convergence result for the transformed sequence of sets and working in the "right" group of variables. To be more precise we need first some notation commonly used to state the standard traffic equilibrium problem from the user's point of view in the stationary case (see for instance (27), (4), (24)).

A traffic network consists of a triple (N,A,W) where $N = \{N_1,\ldots,N_p\}$, $p \in \mathbb{N}$, is the set of nodes, $A = (A_1,\ldots,A_n)$, $n \in \mathbb{N}$, represents the set of the directed arcs connecting couples of nodes and $W = \{W_1,\ldots,W_m\} \subset N \times N$, $m \in \mathbb{N}$ is the set of the origin– destination (O,D) pairs. The flow on the arc A_i is denoted by f_i, this gives the arc flow vector $f = (f_1,\ldots,f_n)$; for the sake of simplicity we shall consider arcs with infinite capacity. We call a set of consecutive arcs a path, and assume that each (O_j,D_j) pair W_j is connected by r_j, $r_j \in \mathbb{N}$, paths whose set is denoted by P_j, $j = 1,\ldots,m$. All the paths in the network are grouped in a vector $(R_1,\ldots,R_k)$, $k \in \mathbb{N}$, We can describe the arc structure of the paths by using the arc–path incidence matrix

$$\Delta = (\delta_{ir})_{\substack{i=1,\ldots,n\\ r=1,\ldots,k}} \text{ with the entries } \delta_{ir} = \begin{cases} 1 & \text{if } A_i \in R_r \\ 0 & \text{if } A_i \notin R_r \end{cases}. \tag{18}$$

To each path R_r there corresponds a flow F_r. The path flows are grouped in a vector $(F_1,\ldots,F_k)$ which is called the path (network) flow. The flow f_i on the arc A_i is equal to the sum of the flows on the paths which contain A_i, so that $f = \Delta F$. Let us now introduce the unit cost of transversing A_i as a given real-valued function $c_i(f) \geq 0$ of the flows on the network, so that $c(f) = (c_1(f),\ldots,c_n(f))$ denotes the arc cost vector on the network. The meaning of the cost is usually that of the travel time. Analogously, one can define a cost on the paths as $C(F) = (C_1(F),\ldots,C_k(F))$. Usually $C_r(F)$ is just the sum of the costs on the arcs which build that path, hence $C_r(F) = \sum_{i=1}^n \delta_{ir} c_i(f)$ or in compact form,

$$C(F) = \Delta^T c(\Delta F)\,. \tag{19}$$

For each (O,D) pair W_j there is a given traffic demand $D_j \geq 0$, so that $(D_1,\ldots,D_m)$ is the demand vector. Feasible flows are nonnegative flows which satisfy the demands, i.e. belong to the set

$$K = \left\{F \in \mathbb{R}^k : \; F_r \geq 0 \text{ for any } r = 1,\ldots,k \text{ and } \Phi F = D\right\},$$

where Φ is the pair–path incidence matrix whose elements, say φ_{jr}, $j = 1, \dots, m$, $r = 1, \dots, k$, are

$$\varphi_{jr} = \begin{cases} 1 & \text{if the path } R_r \text{ connects the pair } W_j \\ 0 & \text{elsewhere} \end{cases}.$$

A path flow H is called an equilibrium flow or *Wardrop Equilibrium*, if and only if $H \in K$ and for any $W_j \in W$ and any $R_q, R_s \in P_j$ there holds

$$C_q(H) < C_s(H) \implies H_s = 0. \tag{20}$$

This statement is equivalent (see for instance (4) and (27)) to

$$H \in K \quad \text{and} \quad \langle C(H), F - H \rangle \geq 0, \quad \forall F \in K. \tag{21}$$

Roughly speaking, the meaning of Wardrop Equilibrium is that the road users choose minimum cost paths. Let us note that condition (20) implies that all the used paths of a given (O, D) pair have the same cost.
Although the Wardrop equilibrium principle is expressed in the path variables, it is clear that the "physical" (and measured) quantities are expressed in the arc (link) variables; moreover, the strong monotonicity hypothesis on $c(f)$ is quite common, but as noticed for instance in (2) this does not imply the strong monotonicity of $C(F)$ in (19), unless the matrix $\Delta^T \Delta$ is nonsingular. Although one can give a procedure for buildings networks preserving the strong monotonicity property (see for instance (9)), the condition fails for a generic network, even for a very simple one as we shall illustrate in the sequel. Thus, it is useful to consider the following variational inequality problem:

$$h \in \Delta K \text{ and } \langle c(h), f - h \rangle \geq 0 \quad \forall f \in \Delta K. \tag{22}$$

If c is strongly monotone, one can prove that for each solution H of (21), $C(H)$ =const., i.e. all possibly nonunique solutions of (21) share the same cost. From an algorithmic point of view it is worth noting that one advantage in working in the path variables is the simplicity of the corresponding convex set but the price to be paid is that the number of paths grows exponentially with the size of the network.
Let us now consider the random version of (21) and (22), which results from an uncertain demand and uncertain costs. In the path-flow variables the random Wardrop equilibrium problem reads:
For each $\omega \in \Omega$, find $H(\omega) \in K(\omega)$ such that

$$\langle C(\omega, H(\omega)), F - H(\omega) \rangle \geq 0, \quad \forall F \in K(\omega), \tag{23}$$

where, for any $\omega \in \Omega$,

$$K(\omega) = \left\{ F \in \mathbb{R}^k : F_r \geq 0 \text{ for any } r = 1, \dots, k \text{ and } \Phi F = D(\omega) \right\},$$

Analogously in the link-flow variables the random Wardrop equilibrium problem reads:
For each $\omega \in \Omega$, find $h(\omega) \in \Delta K(\omega)$ such that

$$\langle c(\omega, h(\omega)), f - h(\omega) \rangle \geq 0, \quad \forall f \in \Delta K(\omega). \tag{24}$$

Clearly (23) is equivalent to the random Wardrop principle: For any $\omega \in \Omega$, for any $H(\omega) \in K(\omega)$, and for any $W_j \in W$, $R_q, R_s \in P_j$, there holds

$$C_q(\omega, H(\omega)) < C_s(\omega, H(\omega)) \implies H_s(\omega) = 0.$$

Moreover, both problems are special instances of Problem 1 (pointwise formulation).
In order to use our approximation scheme we require the separability assumption . However this assumption is very natural in many applications where the random perturbation is treated as a *modulation* of a deterministic process. Under the above mentioned assumptions, (23) assumes the particular form:

$$S(\omega)\langle A(H(\omega)), F - H(\omega)\rangle \geq R(\omega)\langle b, F - H(\omega)\rangle, \; \forall F \in K(\omega) \tag{25}$$

In equation (25), both the l.h.s. and the r.h.s. can, be replaced with any (finite) linear combination of monotone and separable terms, where each term satisfies the hypothesis of the previous sections:

$$\sum_i S_i(\omega)\langle A_i(H(\omega)), F - H(\omega)\rangle \geq \sum_j R_j(\omega)\langle b_j, F - H(\omega), \; \forall F \in K(\omega). \tag{26}$$

In this way, in (25) $R(\omega), S(\omega)$ can be replaced by a random vector, respectively by a random matrix.
Hence, in the traffic network, we could consider the case where the random perturbation has a different weight for each path.

Remark 6.1. *When applying our theory to the random traffic equilibrium problem we shall consider the integrated form of* (25)*, which, after the transformation to the image space, is defined on the feasible set:*

$$K_{\mathbb{P}} = \{F \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) : \Phi F(r,s,t) = t, F(r,s,t) \geq 0 \quad \mathbb{P} - a.s.\}$$

Let $K^n_{\mathbb{P}}$ be the approximate sets constructed as described in section 4. It can be easily verified that the sets $K^n_{\mathbb{P}}$ are uniformly bounded. Moreover, the arc-path incidence matrix Δ induces a linear operator mapping $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ to $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^n)$. This operator, which by abuse of notation is still denoted by Δ, is weak-weak, as well as strong-strong continuous. Thus, from the Mosco convergence $K^n_{\mathbb{P}} \to K_{\mathbb{P}}$ it follows easily that also $\Delta K^n_{\mathbb{P}} \to \Delta K_{\mathbb{P}}$ in Mosco's sense.

In what follows we present two examples of small size. In the first example we build a small network and we study the random variational inequality in the path-flow variables. The network is built in such a way that if the cost operator is strongly monotone in the link-flow variables, the transformed operator, is still strongly monotone in the path-flow variables. Moreover, this small network can be considered as an elementary block of an arbitrarily large network with the same property of preserving strong monotonicity. On the other hand, the second example, which we solve exactly, shows that even very simple networks can fail to preserve the strong monotonicity of the operator when passing from the link to the path-flow variables. In this last case, two possible strategies can be followed. The first possibility is to work from the beginning in the link-variables and use the previous remark to apply our approximation procedure. The other option is to regularize (in the sense of Tichonov) the problem in the path-variables. We stress the fact that if one is interested in the cost shared by the network users, it does not matter which solution is obtained from the regularized problem, because, thanks to the particular structure of the operator, the cost is constant on the whole solution set.

Example 6.1. In the network under consideration, (see Fig.1), there are 7 links and one origin-destination pair, $1 - 6$, which can be connected by 3 paths, namely*:*
$R_1 = A_1 A_2 A_7$

N	$\langle F_1\rangle$	$\langle F_2\rangle$	$\langle F_3\rangle$	σ_1^2	σ_2^2	σ_3^2
10	4.5396	1.4756	4.4346	0.0153	0.0017	0.0148
100	4.5590	1.4821	4.4537	0.0154	0.0017	0.0149
1000	4.5610	1.4821	4.4556	0.0154	0.0017	0.0149
10000	4.5612	1.4828	4.4558	0.0154	0.0017	0.0149

Table 1. Mean values corresponding to various approximations for $d \in [10, 11]$ and $\rho = 0.1$

$R_2 = A_1 A_6 A_4$
$R_3 = A_5 A_3 A_4$
The traffic demand is represented by the non negative random variable d, so that $F_1 + F_2 + F_3 = d$, while link-cost functions are given by:

$t_1 = \rho f_1^2 + f_1;$ $\quad t_5 = \rho f_5^2 + f_5$
$t_2 = \rho f_2^2 + 2f_2;$ $\quad t_6 = \rho f_6^2 + 2f_6$
$t_3 = \rho f_3^2 + f_3;$ $\quad t_7 = \rho f_7^2 + f_7 + 0.5f_5$
$t_4 = \rho f_4^2 + 2f_4 + f_6.$

The linear part of the operator above is represented by a nonsymmetric positive definite matrix, while the nonnegative parameter ρ represents the weight of the non linear terms. Such a functional form is quite common in many network equilibrium problems ((23)). Since we want to solve the variational inequality associated to the Wardrop Equilibrium we have to perform the transformation to the path-flow variables, which yields for the cost functions the following expressions:

$C_1 = 3\rho F_1^2 + \rho F_2^2 + 2\rho F_1 F_2 + 4F_1 + F_2 + 0.5F_3;$
$C_2 = \rho F_1^2 + 3\rho F_2^2 + \rho F_3^2 + 2\rho F_1 F_2 + 2\rho F_2 F_3 + F_1 + 6F_2 + 2F_3;$
$C_3 = \rho F_2^2 + 3\rho F_3^2 + 2\rho F_2 F_3 + 4F_3 + 3F_2$

For the numerical solution of the discretized, finite dimensional variational inequalities, many algorithms are available. Due to the simple structure of our example we employ the extragradient algorithm, see e.g. (8).
In the tables 3 and 4 we show mean values and variances for various choices of the parameters in the case of uniform distribution. We observe that the variances in the second table are quite large. This is due to the fact that if $\rho = 1$, when the other parameter varies the equilibrium pattern changes qualitatively. In particular, near $d = 10$ there is a zero component (H_2) in the solution, while in most of the interval the equilibrium solution has nonzero components.

Example 6.2. We consider the simple network of Fig. 2 below which consists of four arcs and one origin–destination pair, which can be connected by four different paths. Let us assume that the traffic demand between O and D is given by a random variable $t \in \mathbb{R}$, and that the link cost functions are given by $c_1 = 2f_1, c_2 = 3f_2, c_3 = f_3, c_4 = f_4$. The link flows belong to the set

$$\{f \in \mathbb{R}^4 : \exists F \in K(t),\ f = \Delta F\},$$

N	$\langle F_1 \rangle$	$\langle F_2 \rangle$	$\langle F_3 \rangle$	σ_1^2	σ_2^2	σ_3^2
10	3.1602	2.6005	4.6891	4.2853	2.3442	0.1010
100	3.6964	2.1968	4.6017	3.0077	1.6456	0.0759
1000	3.6460	2.2390	4.6143	3.1668	1.7326	0.0791
10000	3.6505	2.2436	4.6157	3.1837	1.7418	0.0794

Table 2. Mean values corresponding to various approximations for $d \in [10, 11]$ and $\rho = 1$

where $K(t)$ is the feasible set in the path flow variables

$$K(t) = \{F_1, F_2, F_3, F_4 \geq 0 \text{ such that } F_1 + F_2 + F_3 + F_4 = t,\ t \in [0, T]\},$$

and Δ is the link-path matrix. Hence, if F is known, one can derive f from the equations

$$f_1 = F_1 + F_2,$$
$$f_2 = F_3 + F_4,$$
$$f_3 = F_1 + F_3,$$
$$f_4 = F_2 + F_4.$$

The path–cost functions are given by the relations

$$C_1 = c_1 + c_3 = 3F_1 + 2F_2 + F_3,$$
$$C_2 = c_1 + c_4 = 2F_1 + 3F_2 + F_4,$$
$$C_3 = c_2 + c_3 = F_1 + 4F_3 + 3F_4,$$
$$C_4 = c_2 + c_4 = F_2 + 3F_3 + 4F_4$$

The associated variational inequality can be solved exactly (see e.g. (9) for a non iterative algorithm) and the solution expressed in term of the second path variable is

$$\left(\frac{3t}{5} - G(t), G(t), G(t) - \frac{t}{10}, -G(t) + \frac{t}{2}\right)$$

where $G : [0, T] \mapsto \mathbb{R}$ is any functions which satisfies the constraint $G(t) \in \left[\frac{t}{10}, \frac{t}{2}\right]$. Let us observe that for each feasible choice of $G(t)$ the cost at the corresponding solution is always equal to $\frac{17}{10} t(1, 1, 1, 1)$. One can also solve the variational inequality in the link variables by using the relations

$$f_1 + f_2 = t,$$
$$f_3 + f_4 = t.$$

We are then left with the problem:

$$(c_2 - c_1)(f_2 - h_2(t)) + (c_4 - c_3)(f_4 - h_4(t)) \geq 0,$$

which yields

$$h(t) = t\left(\frac{3}{5}, \frac{2}{5}, \frac{1}{2}, \frac{1}{2}\right).$$

As an example we assume that our random parameter follows the *lognormal distribution*. This statistical distribution is used for numerous applications to model random phenomena described by nonnegative quantities. It is also known as the Galton Mc Alister distribution and, in economics, is sometimes called the Cobb-Douglas distribution, and has been used to model production data. Thus, let:

$$g_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

the *normal* distribution, then, the *lognormal distribution* is defined by:

$$\begin{cases} (1/x) g_{\mu,\sigma^2}(\log x), & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

The numerical evaluation of the mean values and variances, corresponding to $\mu = 0$ and $\sigma = 1$ yields:

$$(\langle h_1 \rangle, \langle h_2 \rangle, \langle h_3 \rangle, \langle h_4 \rangle) = 1.64(3/5, 2/5, 1/2, 1/2)$$

$$(\sigma^2(h_1), \sigma^2(h_2), \sigma^2(h_3), \sigma^2(h_4)) = 4.68(3/5, 2/5, 1/2, 1/2).$$

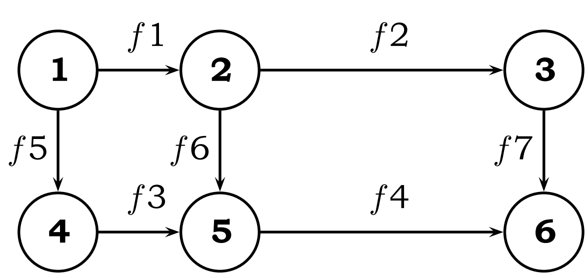

Fig. 1. Network which preserves strong monotonicity

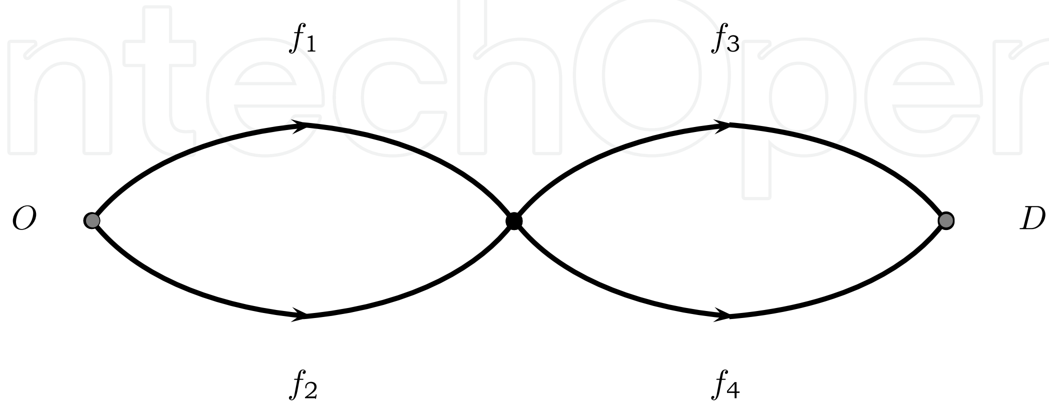

Fig. 2. Loss of strong monotonicity through a linear mapping

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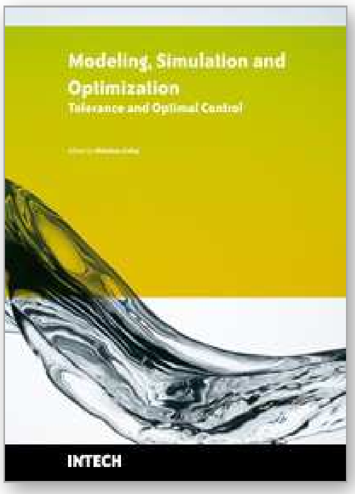

Modeling Simulation and Optimization - Tolerance and Optimal Control
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Parametric representation of shapes, mechanical components modeling with 3D visualization techniques using object oriented programming, the well known golden ratio application on vertical and horizontal displacement investigations of the ground surface, spatial modeling and simulating of dynamic continuous fluid flow process, simulation model for waste-water treatment, an interaction of tilt and illumination conditions at flight simulation and errors in taxiing performance, plant layout optimal plot plan, atmospheric modeling for weather prediction, a stochastic search method that explores the solutions for hill climbing process, cellular automata simulations, thyristor switching characteristics simulation, and simulation framework toward bandwidth quantization and measurement, are all topics with appropriate results from different research backgrounds focused on tolerance analysis and optimal control provided in this book.