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Nonlinear Analysis and Design of Phase-Locked Loops

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1. Introduction

Phase-locked loops (PLLs) are widely used in telecommunication and computer architectures. They were invented in the 1930s-1940s (De Bellescize, 1932; Wendt & Fredentall, 1943) and then intensive studies of the theory and practice of PLLs were carried out (Viterbi, 1966; Lindsey, 1972; Gardner, 1979; Lindsey and Chie, 1981; Leonov et al., 1992; Encinas, 1993; Nash, 1994; Brennan, 1996; Stensby, 1997).

One of the first applications of phase-locked loop (PLL) is related to the problems of data transfer by radio signal. In radio engineering PLL is applied to a carrier synchronization, carrier recovery, demodulation, and frequency synthesis (see, e.g., (Stephens, 2002; Ugrumov, 2000)).

After the appearance of an architecture with chips, operating on different frequencies, the phase-locked loops are used to generate internal frequencies of chips and synchronization of operation of different devices and data buses (Young et al., 1992; Egan, 2000; Kroupa, 2003; Razavi, 2003; Shu & Sanchez-Sinencio, 2005; Manassewitsch, 2005). For example, the modern computer motherboards contain different devices and data buses operating on different frequencies, which are often in the need for synchronization (Wainner & Richmond, 2003; Buchanan & Wilson, 2001).

Another actual application of PLL is the problem of saving energy. One of the solutions of this problem for processors is a decreasing of kernel frequency with processor load. The independent phase-locked loops permit one to distribute more uniformly a kernel load to save the energy and to diminish a heat generation on account of that each kernel operates on its own frequency. Now the phase-locked loops are widely used for the solution of the problems of clock skew and synchronization for the sets of chips of computer architectures and chip microarchitecture. For example, a clock skew is very important characteristic of processors (see, e.g., (Xanthopoulos, 2001; Bindal, 2003)).

Various methods for analysis of phase-locked loops are well developed by engineers and are considered in many publications (see, e.g., (Banerjee, 2006; Best, 2003; Kroupa, 2003; Bianchi, 2005; Egan, 2007)), but the problems of construction of adequate nonlinear models and nonlinear analysis of such models are still far from being resolved and require using special methods of qualitative theory of differential, difference, integral, and integro-differential equations (Gelig et al., 1978; Leonov et al., 1996a; Leonov et al., 1996b; Leonov & Smirnova, 2000; Abramovitch, 2002; Suarez & Quere, 2003; Margaris, 2004; Kudrewicz & Wasowicz, 2007; Kuznetsov, 2008; Leonov, 2006). We could not list here all references in the area of design and analysis of PLL, so readers should see mentioned papers and books and the references cited therein.

2. Mathematical model of PLL

In this work three levels of PLL description are suggested:
1) the level of electronic realizations,
2) the level of phase and frequency relations between inputs and outputs in block diagrams,
3) the level of difference, differential and integro-differential equations.
The second level, involving the asymptotical analysis of high-frequency oscillations, is necessary for the well-formed derivation of equations and for the passage to the third level of description.
Consider a PLL on the first level (Fig. 1)

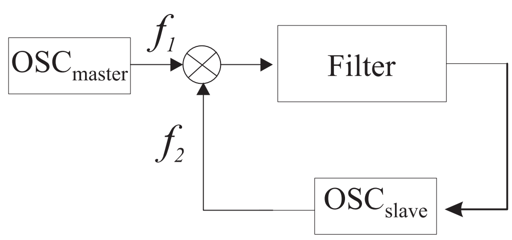

Fig. 1. Block diagram of PLL on the level of electronic realizations.

Here OSC_{master} is a master oscillator, OSC_{slave} is a slave (tunable) oscillator, which generate high-frequency "almost harmonic oscillations"

$$f_j(t) = A_j \sin(\omega_j(t)t + \psi_j) \quad j = 1, 2, \tag{1}$$

where A_j and ψ_j are some numbers, $\omega_j(t)$ are differentiable functions. Block $\otimes$ is a multiplier of oscillations of $f_1(t)$ and $f_2(t)$ and the signal $f_1(t)f_2(t)$ is its output. The relations between the input $\xi(t)$ and the output $\sigma(t)$ of linear filter have the form

$$\sigma(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)\,d\tau. \tag{2}$$

Here $\gamma(t)$ is an impulse transient function of filter, $\alpha_0(t)$ is an exponentially damped function, depending on the initial data of filter at the moment $t = 0$. The electronic realizations of generators, multipliers, and filters can be found in (Wolaver, 1991; Best, 2003; Chen, 2003; Giannini & Leuzzi, 2004; Goldman, 2007; Razavi, 2001; Aleksenko, 2004). In the simplest case it is assumed that the filter removes from the input the upper sideband with frequency $\omega_1(t) + \omega_2(t)$ but leaves the lower sideband $\omega_1(t) - \omega_2(t)$ without change.
Now we reformulate the high-frequency property of oscillations $f_j(t)$ and essential assumption that $\gamma(t)$ and $\omega_j(t)$ are functions of "finite growth". For this purpose we consider the great fixed time interval $[0, T]$, which can be partitioned into small intervals of the form $[\tau, \tau + \delta], (\tau \in [0, T])$ such that the following relations

$$\begin{aligned} &|\gamma(t) - \gamma(\tau)| \le C\delta, \;\; |\omega_j(t) - \omega_j(\tau)| \le C\delta, \\ &\forall\, t \in [\tau, \tau + \delta], \;\; \forall\, \tau \in [0, T], \end{aligned} \tag{3}$$

$$|\omega_1(\tau) - \omega_2(\tau)| \le C_1, \ \ \forall\, \tau \in [0, T], \tag{4}$$

$$\omega_j(t) \ge R, \ \ \forall\, t \in [0, T] \tag{5}$$

are satisfied. Here we assume that the quantity δ is sufficiently small with respect to the fixed numbers T, C, C_1, the number R is sufficiently great with respect to the number δ. The latter means that on the small intervals $[\tau, \tau + \delta]$ the functions $\gamma(t)$ and $\omega_j(t)$ are "almost constants" and the functions $f_j(t)$ rapidly oscillate as harmonic functions.
Consider two block diagrams shown in Fig. 2 and Fig. 3.

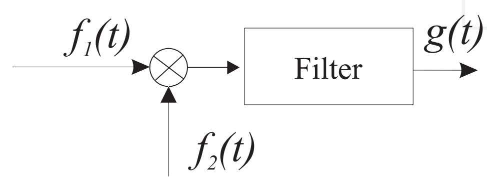

Fig. 2. Multiplier and filter.

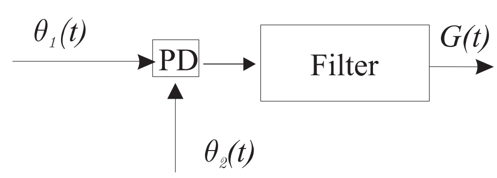

Fig. 3. Phase detector and filter.

Here $\theta_j(t) = \omega_j(t)t + \psi_j$ are phases of the oscillations $f_j(t)$, PD is a nonlinear block with the characteristic $\varphi(\theta)$ (being called a phase detector or discriminator). The phases $\theta_j(t)$ are the inputs of PD block and the output is the function $\varphi(\theta_1(t) - \theta_2(t))$. The shape of the phase detector characteristic is based on the shape of input signals.
The signals $f_1(t)f_2(t)$ and $\varphi(\theta_1(t) - \theta_2(t))$ are inputs of the same filters with the same impulse transient function $\gamma(t)$. The filter outputs are the functions $g(t)$ and $G(t)$, respectively.
A classical PLL synthesis is based on the following result:

Theorem 1. *(Viterbi, 1966) If conditions (3)–(5) are satisfied and we have*

$$\varphi(\theta) = \frac{1}{2} A_1 A_2 \cos\theta,$$

then for the same initial data of filter, the following relation

$$|G(t) - g(t)| \le C_2\delta, \ \ \forall\, t \in [0, T]$$

is satisfied. Here C_2 is a certain number being independent of δ.

Proof of Theorem 1 (Leonov, 2006)
For $t \in [0, T]$ we obviously have

$$
\begin{aligned}
& g(t) - G(t) = \\
& = \int_0^t \gamma(t-s)\Big[A_1 A_2\Big(\sin\left(\omega_1(s)s + \psi_1\right)\sin\left(\omega_2(s)s + \psi_2\right)\Big) - \\
& - \varphi\Big(\omega_1(s)s - \omega_2(s)s + \psi_1 - \psi_2\Big)\Big]ds = \\
& = -\frac{A_1 A_2}{2}\int_0^t \gamma(t-s)\left[\cos\Big(\left(\omega_1(s) + \omega_2(s)\right)s + \psi_1 + \psi_2\Big)\right]ds.
\end{aligned}
$$

Consider the intervals $[k\delta, (k+1)\delta]$, where $k = 0, \ldots, m$ and the number m is such that $t \in [m\delta, (m+1)\delta]$. From conditions (3)–(5) it follows that for any $s \in [k\delta, (k+1)\delta]$ the relations

$$\gamma(t-s) = \gamma(t-k\delta) + O(\delta) \tag{6}$$

$$\omega_1(s) + \omega_2(s) = \omega_1(k\delta) + \omega_2(k\delta) + O(\delta) \tag{7}$$

are valid on each interval $[k\delta, (k+1)\delta]$. Then by (7) for any $s \in [k\delta, (k+1)\delta]$ the estimate

$$\cos\Big(\left(\omega_1(s) + \omega_2(s)\right)s + \psi_1 + \psi_2\Big) = \cos\Big(\left(\omega_1(k\delta) + \omega_2(k\delta)\right)s + \psi_1 + \psi_2\Big) + O(\delta) \tag{8}$$

is valid. Relations (6) and (8) imply that

$$
\begin{aligned}
& \int_0^t \gamma(t-s)\left[\cos\Big(\left(\omega_1(s) + \omega_2(s)\right)s + \psi_1 + \psi_2\Big)\right]ds = \\
& = \sum_{k=0}^{m} \gamma(t-k\delta) \int_{k\delta}^{(k+1)\delta} \left[\cos\Big(\left(\omega_1(k\delta) + \omega_2(k\delta)\right)s + \psi_1 + \psi_2\Big)\right]ds + O(\delta).
\end{aligned}
\tag{9}
$$

From (5) we have the estimate

$$\int_{k\delta}^{(k+1)\delta} \left[\cos\Big(\left(\omega_1(k\delta) + \omega_2(k\delta)\right)s + \psi_1 + \psi_2\Big)\right]ds = O(\delta^2)$$

and the fact that R is sufficiently great as compared with δ. Then

$$\int_0^t \gamma(t-s)\left[\cos\Big(\left(\omega_1(s) + \omega_2(s)\right)s + \psi_1 + \psi_2\Big)\right]ds = O(\delta).$$

Theorem 1 is completely proved. ■

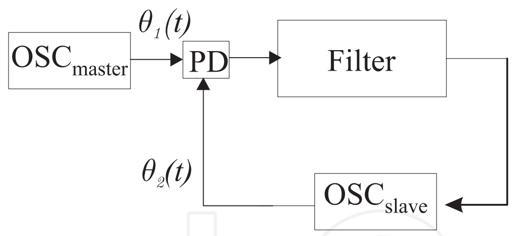

Fig. 4. Block diagram of PLL on the level of phase relations

Thus, the outputs $g(t)$ and $G(t)$ of two block diagrams in Fig. 2 and Fig. 3, respectively, differ little from each other and we can pass (from a standpoint of the asymptotic with respect to δ) to the following description level, namely to the second level of phase relations.
In this case a block diagram in Fig. 1 becomes the following block diagram (Fig. 4).
Consider now the high-frequency impulse oscillators, connected as in diagram in Fig. 1. Here

$$f_j(t) = A_j \text{sign}\left(\sin(\omega_j(t)t + \psi_j)\right). \tag{10}$$

We assume, as before, that conditions (3)– (5) are satisfied.
Consider 2π-periodic function $\varphi(\theta)$ of the form

$$\varphi(\theta) = \begin{cases} A_1 A_2 (1 + 2\theta/\pi) \text{ for } \theta \in [-\pi, 0], \\ A_1 A_2 (1 - 2\theta/\pi) \text{ for } \theta \in [0, \pi]. \end{cases} \tag{11}$$

and block diagrams in Fig. 2 and Fig. 3.

Theorem 2. *(Leonov, 2006)*
If conditions (3)–(5) are satisfied and the characteristic of phase detector $\varphi(\theta)$ has the form (11), then for the same initial data of filter the following relation

$$|G(t) - g(t)| \le C_3 \delta, \ \forall\, t \in [0, T]$$

is satisfied. Here C_3 is a certain number being independent of δ.

Proof of Theorem 2
In this case we have

$$g(t) - G(t) =$$
$$= \int_0^t \gamma(t-s) \Big[A_1 A_2 \text{sign}\Big(\sin\left(\omega_1(s)s + \psi_1\right) \sin\left(\omega_2(s)s + \psi_2\right) \Big) -$$
$$- \varphi\left(\omega_1(s)s - \omega_2(s)s + \psi_1 - \psi_2\right) \Big]\, ds.$$

Partitioning the interval $[0, t]$ into the intervals $[k\delta, (k+1)\delta]$ and making use of assumptions (5) and (10), we replace the above integral with the following sum

$$\sum_{k=0}^{m} \gamma(t - k\delta) \Big[\int_{k\delta}^{(k+1)\delta} A_1 A_2 \text{sign} \Big[\cos\Big((\omega_1(k\delta)\omega_2(k\delta))k\delta + \psi_1 - \psi_2 \Big) - \\ - \cos\Big((\omega_1(k\delta) + \omega_2(k\delta))s + \psi_1 + \psi_2 \Big) \Big] ds - \\ -\varphi\Big((\omega_1(k\delta) - \omega_2(k\delta))k\delta + \psi_1 - \psi_2 \Big)\delta \Big].$$

The number m is chosen in such a way that $t \in [m\delta, (m+1)\delta]$. Since $(\omega_1(k\delta) + \omega_2(k\delta))\delta \gg 1$, the relation

$$\int_{k\delta}^{(k+1)\delta} A_1 A_2 \text{sign} \Big[\cos\Big((\omega_1(k\delta) - \omega_2(k\delta))k\delta + \psi_1 - \psi_2 \Big) - \\ - \cos\Big((\omega_1(k\delta) + \omega_2(k\delta))s + \psi_1 + \psi_2 \Big) \Big] ds \approx \\ \approx \varphi\Big((\omega_1(k\delta) - \omega_2(k\delta))k\delta + \psi_1 - \psi_2 \Big)\delta, \tag{12}$$

is satisfied. Here we use the relation

$$A_1 A_2 \int_{k\delta}^{(k+1)\delta} \text{sign}\big[\cos\alpha - \cos(\omega s + \psi_0) \big] ds \approx \varphi(\alpha)\delta$$

for $\omega\delta \gg 1$, $\alpha \in [-\pi, \pi]$, $\psi_0 \in R^1$.
Thus, Theorem 2 is completely proved. ■
Theorem 2 is a base for the synthesis of PLL with impulse oscillators. For the impulse clock oscillators it permits one to consider two block diagrams simultaneously: on the level of electronic realization (Fig. 1) and on the level of phase relations (Fig. 4), where general principles of the theory of phase synchronization can be used (Leonov & Seledzhi, 2005b; Kuznetsov et al., 2006; Kuznetsov et al., 2007; Kuznetsov et al., 2008; Leonov, 2008).

3. Differential equations of PLL

Let us make a remark necessary for derivation of differential equations of PLL.
Consider a quantity

$$\dot{\theta}_j(t) = \omega_j(t) + \dot{\omega}_j(t)t.$$

For the well-synthesized PLL such that it possesses the property of global stability, we have exponential damping of the quantity $\dot{\omega}_j(t)$:

$$|\dot{\omega}_j(t)| \leq Ce^{-\alpha t}.$$

Here C and α are certain positive numbers being independent of t. Therefore, the quantity $\dot{\omega}_j(t)t$ is, as a rule, sufficiently small with respect to the number R (see conditions (3)– (5)). From the above we can conclude that the following approximate relation $\dot{\theta}_j(t) \approx \omega_j(t)$ is valid. In deriving the differential equations of this PLL, we make use of a block diagram in Fig. 4 and exact equality

$$\dot{\theta}_j(t) = \omega_j(t). \tag{13}$$

Note that, by assumption, the control law of tunable oscillators is linear:

$$\omega_2(t) = \omega_2(0) + LG(t). \tag{14}$$

Here $\omega_2(0)$ is the initial frequency of tunable oscillator, L is a certain number, and $G(t)$ is a control signal, which is a filter output (Fig. 4). Thus, the equation of PLL is as follows

$$\dot{\theta}_2(t) = \omega_2(0) + L\Big(\alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi\big(\theta_1(\tau) - \theta_2(\tau)\big)d\tau\Big).$$

Assuming that the master oscillator is such that $\omega_1(t) \equiv \omega_1(0)$, we obtain the following relations for PLL

$$\big(\theta_1(t) - \theta_2(t)\big)' + L\Big(\alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi\big(\theta_1(\tau) - \theta_2(\tau)\big)d\tau\Big) = \omega_1(0) - \omega_2(0). \tag{15}$$

This is an equation of standard PLL. Note, that if the filter (2) is integrated with the transfer function $W(p) = (p+\alpha)^{-1}$

$$\dot{\sigma} + \alpha\sigma = \varphi(\theta)$$

then for $\phi(\theta) = \cos(\theta)$ instead of equation (15) from (13) and (14) we have

$$\ddot{\tilde{\theta}} + \alpha\dot{\tilde{\theta}} + L\sin\tilde{\theta} = \alpha\big(\omega_1(0) - \omega_2(0)\big) \tag{16}$$

with $\tilde{\theta} = \theta_1 - \theta_2 + \dfrac{\pi}{2}$. So, if here phases of the input and output signals mutually shifted by $\pi/2$ then the control signal $G(t)$ equals zero.

Arguing as above, we can conclude that in PLL it can be used the filters with transfer functions of more general form

$$K(p) = a + W(p),$$

where a is a certain number, $W(p)$ is a proper fractional rational function. In this case in place of equation (15) we have

$$\begin{aligned}\big(\theta_1(t) - \theta_2(t)\big)' + L\Big[a\varphi\big(\theta_1(t) - \theta_2(t)\big) + \alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi\big(\theta_1(\tau) - \theta_2(\tau)\big)d\tau\Big] = \\ = \omega_1(0) - \omega_2(0).\end{aligned} \tag{17}$$

In the case when the transfer function of the filter $a + W(p)$ is non-degenerate, i.e. its numerator and denominator do not have common roots, equation (17) is equivalent to the following system of differential equations

$$\begin{aligned}\dot{z} &= Az + b\psi(\sigma) \\ \dot{\sigma} &= c^*z + \rho\psi(\sigma).\end{aligned} \tag{18}$$

Here $\sigma = \theta_1 - \theta_2$, A is a constant $(n \times n)$-matrix, b and c are constant (n)-vectors, ρ is a number, and $\psi(\sigma)$ is 2π-periodic function, satisfying the relations:

$$\rho = -aL,$$
$$W(p) = L^{-1}c^*(A - pI)^{-1}b,$$
$$\psi(\sigma) = \varphi(\sigma) - \frac{\omega_1(0) - \omega_2(0)}{L(a + W(0))}.$$

The discrete phase-locked loops obey similar equations

$$\begin{aligned} z(t+1) &= Az(t) + b\psi(\sigma(t)) \\ \sigma(t+1) &= \sigma(t) + c^*z(t) + \rho\psi(\sigma(t)), \end{aligned} \tag{19}$$

where $t \in Z$, Z is a set of integers. Equations (18) and (19) describe the so-called standard PLLs (Shakhgil'dyan & Lyakhovkin, 1972; Leonov, 2001). Note that there exist many other modifications of PLLs and some of them are considered below.

4. Mathematical analysis methods of PLL

The theory of phase synchronization was developed in the second half of the last century on the basis of three applied theories: theory of synchronous and induction electrical motors, theory of auto-synchronization of the unbalanced rotors, theory of phase-locked loops. Its main principle is in consideration of the problem of phase synchronization at three levels: (i) at the level of mechanical, electromechanical, or electronic models, (ii) at the level of phase relations, and (iii) at the level of differential, difference, integral, and integro-differential equations. In this case the difference of oscillation phases is transformed into the control action, realizing synchronization. These general principles gave impetus to creation of universal methods for studying the phase synchronization systems. Modification of the direct Lyapunov method with the construction of periodic Lyapunov-like functions, the method of positively invariant cone grids, and the method of nonlocal reduction turned out to be most effective. The last method, which combines the elements of the direct Lyapunov method and the bifurcation theory, allows one to extend the classical results of F. Tricomi and his progenies to the multidimensional dynamical systems.

4.1 Method of periodic Lyapunov functions

Here we formulate the extension of the Barbashin–Krasovskii theorem to dynamical systems with a cylindrical phase space (Barbashin & Krasovskii, 1952). Consider a differential inclusion

$$\dot{x} \in f(x), \;\; x \in R^n, \;\; t \in R^1, \tag{20}$$

where $f(x)$ is a semicontinuous vector function whose values are the bounded closed convex set $f(x) \subset R^n$. Here R^n is an n-dimensional Euclidean space. Recall the basic definitions of the theory of differential inclusions.

Definition 1. *We say that $U_\varepsilon(\Omega)$ is an ε-neighbourhood of the set Ω if*

$$U_\varepsilon(\Omega) = \{x \mid \inf_{y\in\Omega} |x - y| < \varepsilon\},$$

where $|\cdot|$ is an Euclidean norm in R^n.

Definition 2. *A function $f(x)$ is called semicontinuous at a point x if for any $\varepsilon > 0$ there exists a number $\delta(x, \varepsilon) > 0$ such that the following containment holds:*

$$f(y) \in U_\varepsilon(f(x)), \;\; \forall\, y \in U_\delta(x).$$

Definition 3. *A vector function $x(t)$ is called a solution of differential inclusion if it is absolutely continuous and for the values of t, at which the derivative $\dot{x}(t)$ exists, the inclusion*

$$\dot{x}(t) \in f(x(t))$$

holds.

Under the above assumptions on the function $f(x)$, the theorem on the existence and continuability of solution of differential inclusion (20) is valid (Yakubovich et al., 2004). Now we assume that the linearly independent vectors $d_1, \ldots, d_m$ satisfy the following relations:

$$f(x + d_j) = f(x), \;\; \forall\, x \in R^n. \tag{21}$$

Usually, $d_j^* x$ is called the phase or angular coordinate of system (20). Since property (21) allows us to introduce a cylindrical phase space (Yakubovich et al., 2004), system (20) with property (21) is often called a system with cylindrical phase space.
The following theorem is an extension of the well–known Barbashin–Krasovskii theorem to differential inclusions with a cylindrical phase space.

Theorem 3. *Suppose that there exists a continuous function $V(x) : R^n \to R^1$ such that the following conditions hold:*
1) $V(x + d_j) = V(x), \;\; \forall x \in R^n, \;\; \forall j = 1, \ldots, m$;
2) $V(x) + \sum\limits_{j=1}^{m} (d_j^* x)^2 \to \infty$ as $|x| \to \infty$;
3) for any solution $x(t)$ of inclusion (20) the function $V(x(t))$ is nonincreasing;
4) if $V(x(t)) \equiv V(x(0))$, then $x(t)$ is an equilibrium state.
Then any solution of inclusion (20) tends to stationary set as $t \to +\infty$.

Recall that the tendency of solution to the stationary set Λ as t means that

$$\lim_{t \to +\infty} \inf_{z \in \Lambda} |z - x(t)| = 0.$$

A proof of Theorem 3 can be found in (Yakubovich et al., 2004).

4.2 Method of positively invariant cone grids. An analog of circular criterion

This method was proposed independently in the works (Leonov, 1974; Noldus, 1977). It is sufficiently universal and "fine" in the sense that here only two properties of system are used such as the availability of positively invariant one-dimensional quadratic cone and the invariance of field of system (20) under shifts by the vector d_j (see (21)).
Here we consider this method for more general nonautonomous case

$$\dot{x} = F(t, x), \quad x \in \mathbb{R}^n, \; t \in \mathbb{R}^1,$$

where the identities $F(t, x + d_j) = F(t, x)$ are valid $\forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}^1$ for the linearly independent vectors $d_j \in \mathbb{R}^n$ $(j = 1, ..., m)$. Let $x(t) = x(t, t_0, x_0)$ is a solution of the system such that $x(t_0, t_0, x_0) = x_0$.

We assume that such a cone of the form $\Omega = \{x^* Hx \leq 0\}$, where H is a symmetrical matrix such that one eigenvalue is negative and all the rest are positive, is positively invariant. The latter means that on the boundary of cone $\partial\Omega = \{xHx = 0\}$ the relation

$$\dot{V}(x(t)) < 0$$

is satisfied for all $x(t)$ such that $\{x(t) \neq 0, x(t) \in \partial\Omega\}$ (Fig. 5).

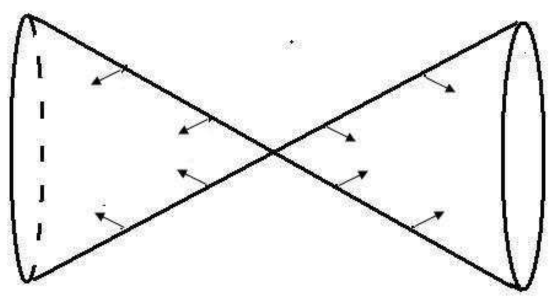

Fig. 5. Positively invariant cone.

By the second property, namely the invariance of vector field under shift by the vectors kd_j, $k \in Z$, we multiply the cone in the following way

$$\Omega_k = \{(x - kd_j)H(x - kd_j) \leq 0\}.$$

Since it is evident that for the cones Ω_k the property of positive invariance holds true, we obtain a positively invariant cone grid shown in Fig. 6. As can be seen from this figure, all the

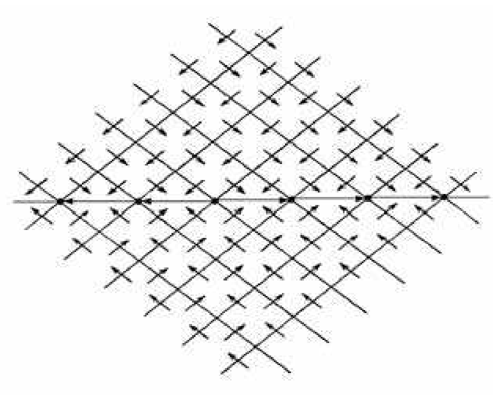

Fig. 6. Positively invariant cone grid.

solutions $x(t, t_0, x_0)$ of system, having these two properties, are bounded on $[t_0, +\infty)$.
If the cone Ω has only one point of intersection with the hyperplane $\{d_j^* x = 0\}$ and all solutions $x(t)$, for which at the time t the inequality

$$x(t)^* Hx(t) \geq 0$$

is satisfied, have property $\dot{V}(x(t)) \leq -\varepsilon |x(t)|^2$ (here ε is a positive number), then from Fig. 6 it is clear that the system is Lagrange stable (all solutions are bounded on the interval $[0, +\infty)$).

Thus, the proposed method is simple and universal. By the Yakubovich–Kalman frequency theorem it becomes practically efficient (Gelig et al., 1978; Yakubovich et al., 2004).
Consider, for example, the system

$$\dot{x} = Px + q\varphi(t,\sigma), \ \ \sigma = r^* x, \tag{22}$$

where P is a constant singular $n \times n$-matrix, q and r are constant n-dimensional vectors, and $\varphi(t,\sigma)$ is a continuous 2π-periodic in σ function $R^1 \times R^1 \to R^1$, satisfying the relations

$$\mu_1 \leq \frac{\varphi(t,\sigma)}{\sigma} \leq \mu_2, \ \forall\, t \in R^1, \ \forall \sigma \neq 0, \quad \varphi(t,0) = 0.$$

Here μ_1 and μ_2 are some numbers, which by virtue of periodicity of $\varphi(t,\sigma)$ in σ, without loss of generality, can be assumed to be negative, $\mu_1 < 0$, and positive, $\mu_2 > 0$, respectively.
We introduce the transfer function of system (22)

$$\chi(p) = r^*(P - pI)^{-1}q,$$

which is assumed to be nondegenerate. Consider now quadratic forms $V(x) = x^* Hx$ and

$$G(x,\xi) = 2x^* H[(P + \lambda I)x + q\xi] + (\mu_2^{-1}\xi - r^* x)(\mu_1^{-1}\xi - r^* x),$$

where λ is a positive number.
By the Yakubovich–Kalman theorem, for the existence of the symmetrical matrix H with one negative and $n-1$ positive eigenvalues and such that the inequality $G(x,\xi) < 0, \forall x \in R^n, \xi \in R^1, x \neq 0$ is satisfied, it is sufficient that
(C1) the matrix $(P + \lambda I)$ has $(n-1)$ eigenvalues with negative real part and
(C2) the frequency inequality

$$\mu_1^{-1}\mu_2^{-1} + (\mu_1^{-1} + \mu_2^{-1})\mathrm{Re}\chi(i\omega - \lambda) + |\chi(i\omega - \lambda)|^2 < 0, \ \forall \omega \in R^1$$

is satisfied.
It is easy to see that the condition $G(x,\xi) < 0, \forall x \neq 0, \forall \xi$ implies the relation

$$\dot{V}\big(x(t)\big) + 2\lambda V\big(x(t)\big) < 0, \ \forall x(t) \neq 0.$$

This inequality assures the positive invariance of the considered cone Ω.
Thus, we obtain the following analog of the well-known circular criterion.

Theorem 4. *(Leonov, 1974; Gelig et al., 1978; Yakubovich et al., 2004)*
If there exists a positive number λ such that the above conditions (C1) and (C2) are satisfied, then any solution $x(t,t_0,x_0)$ of system (22) is bounded on the interval $(t_0,+\infty)$.

A more detailed proof of this fact can be found in (Leonov & Smirnova 2000; Gelig et al., 1978; Yakubovich et al., 2004). We note that this theorem is also true under the condition of nonstrict inequality in (C2) and in the cases when $\mu_1 = -\infty$ or $\mu_2 = +\infty$ (Leonov & Smirnova 2000; Gelig et al., 1978; Yakubovich et al., 2004).
We apply now an analog of the circular criterion, formulated with provision for the above remark, to the simplest case of the second-order equation

$$\ddot{\theta} + \alpha\dot{\theta} + \varphi(t,\theta) = 0, \tag{23}$$

where α is a positive parameter (equation (16) can be transformed into (23) by $\tilde{\theta} = \theta + \arcsin\left[\alpha\big(\omega_1(0) - \omega_2(0)\big)/L\right]$). This equation can be represented as system (22) with $n = 2$ and the transfer function

$$\chi(p) = \frac{1}{p(p+\alpha)}.$$

Obviously, condition (C1) of theorem takes the form $\lambda \in (0, \alpha)$ and for $\mu_1 = -\infty$ and $\mu_2 = \alpha^2/4$ condition (C2) is equivalent to the inequality

$$-\omega^2 + \lambda^2 - \alpha\lambda + \alpha^2/4 \leq 0, \ \forall \omega \in R^1.$$

This inequality is satisfied for $\lambda = \alpha/2$. Thus, if in equation (23) the function $\varphi(t, \theta)$ is periodic with respect to θ and satisfies the inequality

$$\frac{\varphi(t,\theta)}{\theta} \leq \frac{\alpha^2}{4}, \tag{24}$$

then any its solution $\theta(t)$ is bounded on $(t_0, +\infty)$.

It is easily seen that for $\varphi(t, \theta) \equiv \varphi(\theta)$ (i.e. $\varphi(t, \theta)$ is independent of t) equation (23) is dichotomic. It follows that in the autonomous case if relation (24) is satisfied, then any solution of (23) tends to certain equilibrium state as $t \to +\infty$.

Here we have interesting analog of notion of absolute stability for phase synchronization systems. If system (22) is absolutely stable under the condition that for any nonlinearity φ from the sector $[\mu_1, \mu_2]$ any its solution tends to certain equilibrium state, then for equation (23) with $\varphi(t, \theta) \equiv \varphi(\theta)$ this sector is $(-\infty, \alpha^2/4]$.

At the same time, in the classical theory of absolute stability (without the assumption that φ is periodic), for $\varphi(t, \theta) \equiv \varphi(\theta)$ we have two sectors: the sector of absolute stability $(0, +\infty)$ and the sector of absolute instability $(-\infty, 0)$.

Thus, the periodicity alone of φ allows one to cover a part of sector of absolute stability and a complete sector of absolute instability: $(-\infty, \alpha^2/4] \supset (-\infty, 0) \cup (0, \alpha^2/4]$ (see Fig. 7).

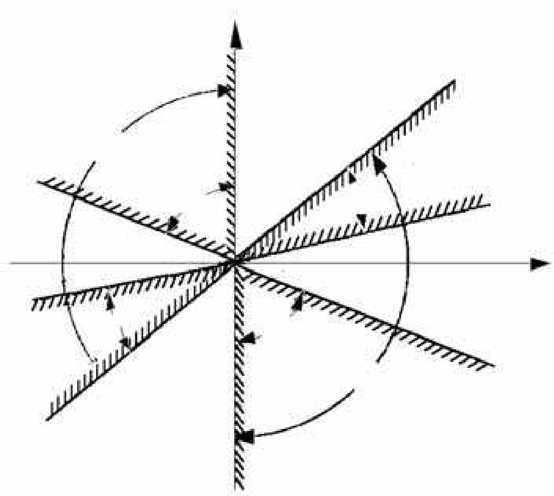

Fig. 7. Sectors of stability and instability.

More complex examples of using the analog of circular criterion can be found in (Leonov & Smirnova 2000; Gelig et al., 1978; Yakubovich et al., 2004).

4.3 Method of nonlocal reduction

We describe the main stages of extending the theorems of Tricomi and his progenies, obtained for the equation

$$\ddot{\theta} + \alpha\dot{\theta} + \psi(\theta) = 0, \tag{25}$$

to systems of higher dimensions.
Consider first the system

$$\begin{aligned} \dot{z} &= Az + b\psi(\sigma) \\ \dot{\sigma} &= c^* z + \rho\psi(\sigma), \end{aligned} \tag{26}$$

describing a standard PLL. We assume, as usual, that $\psi(\sigma)$ is 2π-periodic, A is a stable $n \times n$-matrix, b and c are constant n-vectors, and ρ is a number.
Consider the case when any solution of equation (25) or its equivalent system

$$\begin{aligned} \dot{\eta} &= -\alpha\eta - \psi(\theta) \\ \dot{\theta} &= \eta \end{aligned} \tag{27}$$

tends to the equilibrium state as $t \to +\infty$. In this case it is possible to demonstrate (Barbashin & Tabueva, 1969) that for the equation

$$\frac{d\eta}{d\theta} = \frac{-\alpha\eta - \psi(\theta)}{\eta} \tag{28}$$

equivalent to (27) there exists a solution $\eta(\theta)$ such that $\eta(\theta_0) = 0$, $\eta(\theta) \neq 0, \forall\theta \neq \theta_0$,

$$\lim_{\theta\to+\infty} \eta(\theta) = -\infty, \ \lim_{\theta\to-\infty} \eta(\theta) = +\infty. \tag{29}$$

Here θ_0 is a number such that $\psi(\theta_0) = 0$, $\psi'(\theta_0) < 0$.
We consider now the function

$$V(z,\sigma) = z^* Hz - \frac{1}{2}\eta(\sigma)^2,$$

which induces the cone $\Omega = \{V(z,\sigma) \leq 0\}$ in the phase space $\{z,\sigma\}$. This is a generalization of quadratic cone shown in Fig. 5. We prove that under certain conditions this cone is positively invariant. Consider the expression

$$\begin{aligned} \frac{dV}{dt} + 2\lambda V = 2z^* H\left[(A+\lambda I)z + b\psi(\sigma)\right] - \lambda\eta(\sigma)^2 - \eta(\sigma)\frac{d\eta(\sigma)}{d\sigma}(c^* z + \rho\psi(\sigma)) = \\ = 2z^* H\left[(A+\lambda I)z + b\psi(\sigma)\right] - \lambda\eta(\sigma)^2 + \psi(\sigma)(c^* z + \rho\psi(\sigma)) + \alpha\eta(\sigma)(c^* z + \rho\psi(\sigma)). \end{aligned}$$

Here we make use of the fact that $\eta(\sigma)$ satisfies equation (28).
We note that if the frequency inequalities

$$\begin{aligned} &\operatorname{Re} W(i\omega - \lambda) - \varepsilon|K(i\omega - \lambda)|^2 > 0, \\ &\lim_{\omega\to\infty} \omega^2(\operatorname{Re} K(i\omega - \lambda) - \varepsilon|K(i\omega - \lambda)|^2) > 0, \end{aligned} \tag{30}$$

where $K(p) = c^*(A - pI)^{-1}b - \rho$, are satisfied, then by the Yakubovich–Kalman frequency theorem there exists H such that for ξ and all $z \neq 0$ the following relation

$$2z^* H[(A+\lambda I)z + b\xi] + \xi(c^* z + \rho\xi) + \varepsilon|(c^* z + \rho\xi)|^2 < 0$$

is valid. Here ε is a positive number. If $A + \lambda I$ is a stable matrix, then $H > 0$.
Thus, if $(A + \lambda I)$ is stable, (30) and $\alpha^2 \leq 4\lambda\varepsilon$ are satisfied, then we have

$$\frac{dV}{dt} + 2\lambda V < 0, \ \forall z(t) \neq 0$$

and, therefore, Ω is a positively invariant cone.
We can make a breeding of the cones $\Omega_k = \{z^* Hz - \frac{1}{2}\eta_k(\sigma)^2 \leq 0\}$ in the same way as in the last section and construct a cone grid (Fig. 6), using these cones. Here $\eta_k(\sigma)$ is the solution $\eta(\sigma)$, shifted along the axis σ by the quantity $2k\pi$. The cone grid is a proof of boundedness of solutions of system (26) on the interval $(0, +\infty)$. Under these assumptions there occurs a dichotomy. This is easily proved by using the Lyapunov function

$$z^* Hz + \int_0^\sigma \psi(\sigma) d\sigma.$$

Thus we prove the following

Theorem 5. *If for certain $\lambda > 0$ and $\varepsilon > 0$ the matrix $A + \lambda I$ is stable, conditions (30) are satisfied, and system (27) with $\alpha = 2\sqrt{\lambda\varepsilon}$ is a globally stable system (all solutions tend to stationary set as $t \to +\infty$), then system (26) is also a a globally stable system.*

Various generalizations of this theorem and numerous examples of applying the method of nonlocal reduction, including the applying to synchronous machines, can be found in the works (Leonov, 1975; Leonov, 1976; Gelig et al., 1978; Leonov et al., 1992; Leonov et al., 1996a). Various criteria for the existence of circular solutions and second-kind cycles were also obtained within the framework of this method (Gelig et al., 1978; Leonov et al., 1992; Leonov et al., 1996a; Yakubovich et al., 2004).

5. Floating PLL

The main requirement to PLLs for digital signal processors is that they must be floating in phase. This means that the system must eliminate the clock skew completely. Let us clarify the problem of eliminating the clock skew in multiprocessor systems when parallel algorithms are applied. Consider a clock C transmitting clock pulses through a bus to processors P_k operating in parallel (Fig. 8).

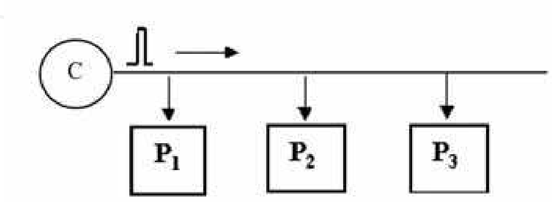

Fig. 8. Clock C that transmits clock pulses through a bus to processors P_k working in parallel.

In realizing parallel algorithms, the processors must perform a certain sequence of operations simultaneously. These operations are to be started at the moments of arrival of clock pulses to

processors. Since the paths along which the pulses run from the clock to every processor are of different length, a mistiming between processors arises. This phenomenon is called a clock skew.
The elimination of the clock skew is one of the most important problems in parallel computing and information processing (as well as in the design of array processors (Kung, 1988)).
Several approaches to the solution of the problem of eliminating the clock skew have been devised for the last thirty years.
In developing the design of multiprocessor systems, a way was suggested (Kung, 1988; Saint-Laurent et al., 2001) for joining the processors in the form of an H-tree, in which (Fig. 9) the lengths of the paths from the clock to every processor are the same.

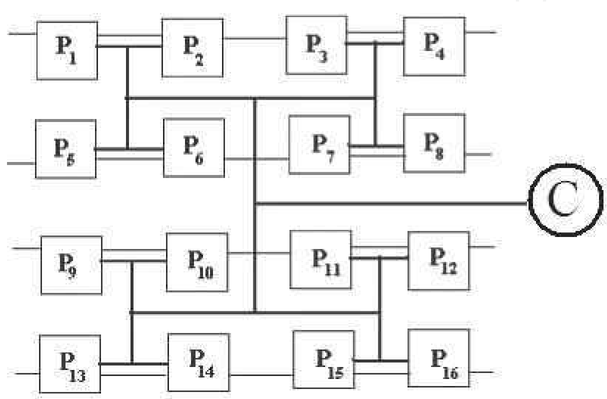

Fig. 9. Join of processors in the form of an H-tree.

However, in this case the clock skew is not eliminated completely because of heterogeneity of the wires (Kung, 1988). Moreover, for a great number of processors, the configuration of communication wires is very complicated. This leads to difficult technological problems.
The solution of the clock skew problem at a hard- and software levels has resulted in the invention of asynchronous communication protocols, which can correct the asynchronism of operations by waiting modes (Kung, 1988). In other words, the creation of these protocols permits one not to distort the final results by delaying information at some stages of the execution of a parallel algorithm. As an advantage of this approach, we may mention the fact that we need not develop a special complicated hardware support system. Among the disadvantages we note the deceleration of performance of parallel algorithms.
In addition to the problem of eliminating the clock skew, one more important problem arose. An increase in the number of processors in multiprocessor systems required an increase in the power of the clock. But powerful clocks lead to produce significant electromagnetic noise.
About ten years ago, a new method for eliminating the clock skew and reducing the generator's power was suggested. It consists of introducing a special distributed system of clocks controlled by PLLs. An advantage of this method, in comparison with asynchronous communication protocols, is the lack of special delays in the performance of parallel algorithms. This approach allows one to reduce significantly the power of clocks.
Consider the general scheme of a distributed system of oscillators (Fig. 10).
By Theorem 2, we can make the design of a block diagram of floating PLL, which plays a role of the function of frequency synthesizer and the function of correction of the clock-skew (see parameter τ in Fig. 11). Its block diagram differs from that shown in Fig. 4 with the phase

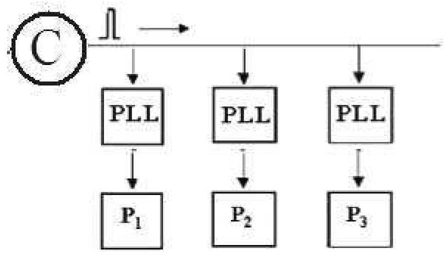

Fig. 10. General scheme of a distributed system of oscillators with PLL.

detector characteristic (11) only in that a relay element with characteristic sign G is inserted after the filter.
Such a block diagram is shown in Fig. 11.

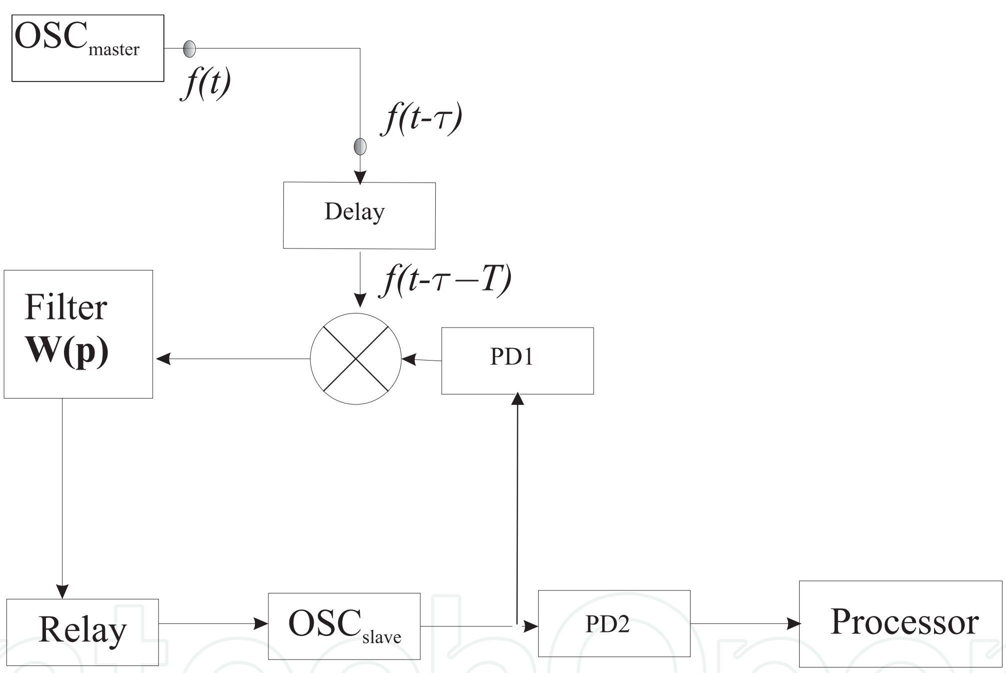

Fig. 11. Block diagram of floating PLL

Here OSC_{master} is a master oscillator, *Delay* is a time-delay element, *Filter* is a filter with transfer function

$$W(p) = \frac{\beta}{p + \alpha},$$

OSC_{slave} is a slave oscillator, PD1 and PD2 are programmable dividers of frequencies, and *Processor* is a processor.
The *Relay* element plays a role of a floating correcting block. The inclusion of it allows us to null a residual clock skew, which arises for the nonzero initial difference of frequencies of master and slave oscillators.

We recall that it is assumed here that the master oscillator $\dot{\theta}_1(t) \equiv \omega_1(t) \equiv \omega_1(0) = \omega_1$ is highly stable. The parameter of delay line T is chosen in such a way that $\omega_1(0)(T+\tau) = 2\pi k + 3\pi/2$. Here k is a certain natural number, $\omega_1(0)\tau$ is a clock skew.
By Theorem 2 and the choice of T the block diagram, shown in Fig. 11, can be changed by the close block diagram, shown in Fig. 12. Here $\varphi(\theta)$ is a 2π-periodic characteristic of phase detector. It has the form

$$\varphi(\theta) = \begin{cases} 2A_1A_2\theta/\pi & \text{for } \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 2A_1A_2(1-\theta/\pi) & \text{for } \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}], \end{cases} \tag{31}$$

$\theta_2(t) = \dfrac{\theta_3(t)}{M}$, $\theta_4(t) = \dfrac{\theta_3(t)}{N}$, where the natural numbers M and N are parameters of programmable divisions PD1 and PD2, respectively.

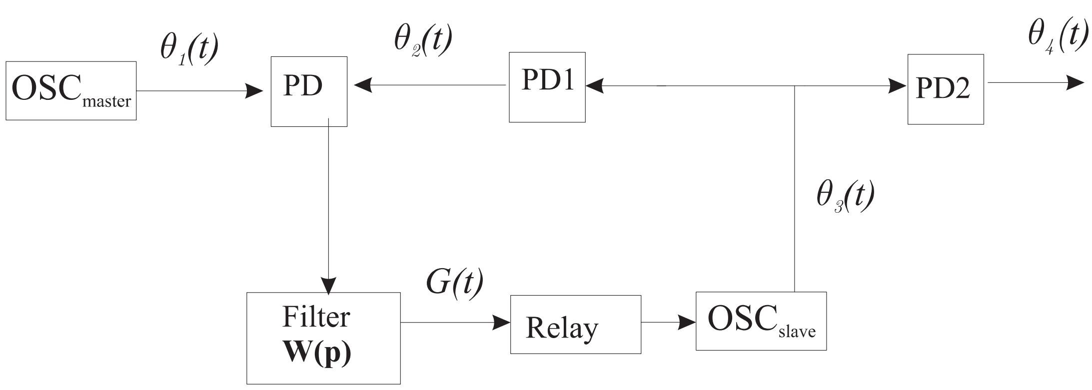

Fig. 12. Equivalent block diagram of PLL

For a transient process (a capture mode) the following conditions

$$\lim_{t\to+\infty}\left(\theta_4(t) - \frac{M}{N}\theta_1(t)\right) = \frac{2\pi k M}{N} \tag{32}$$

(a phase capture) and

$$\lim_{t\to+\infty}\left(\dot{\theta}_4(t) - \frac{M}{N}\dot{\theta}_1(t)\right) = 0 \tag{33}$$

(a frequency capture), must be satisfied.
Relations (32) and (33) are the main requirements to PLL for array processors. The time of transient processors depends on the initial data and is sufficiently large for multiprocessor system (Kung, 1988; Leonov & Seledzhi, 2002).
Assuming that the characteristic of relay is of the form $\Psi(G) = \text{sign}G$ and the actuating element of slave oscillator is linear, we have

$$\dot{\theta}_3(t) = R\,\text{sign}G(t) + \omega_3(0), \tag{34}$$

where R is a certain number, $\omega_3(0)$ is the initial frequency, and $\theta_3(t)$ is a phase of slave oscillator.

Taking into account relations (34), (1), (31) and the block diagram in Fig. 12, we have the following differential equations of PLL

$$\begin{aligned} &\dot{G} + \alpha G = \beta\varphi(\theta), \\ &\dot{\theta} = -\frac{R}{M}\text{sign}G + \left(\omega_1 - \frac{\omega_3(0)}{M}\right). \end{aligned} \tag{35}$$

Here $\theta(t) = \theta_1(t) - \theta_2(t)$. In general case, we get the following PLL equations:

$$\begin{aligned} &\dot{z} = Az + b\varphi(\sigma) \\ &\dot{\sigma} = g(c^*z), \end{aligned} \tag{36}$$

where $\sigma = \theta_1 - \theta_2$, the matrix A and the vectors b and c are such that

$$W(p) = c^*(A - pI)^{-1}b,$$

$$g(G) = -L(\text{sign}\, G) + \big(\omega_1(0) - \omega_2(0)\big).$$

Rewrite system (35) as follows

$$\begin{aligned} &\dot{G} = -\alpha G + \beta\varphi(\theta), \\ &\dot{\theta} = -F(G), \end{aligned} \tag{37}$$

where

$$F(G) = \frac{R}{M}\text{sign}G - \left(\omega_1 - \frac{\omega_3(0)}{M}\right).$$

Theorem 6. *If the inequality*

$$|R| > |M\omega_1 - \omega_3(0)| \tag{38}$$

is valid, then any solution of system (37) tends to a certain equilibrium state as $t \to +\infty$. If the inequality

$$|R| < |M\omega_1 - \omega_3(0)| \tag{39}$$

is valid, then all the solutions of system (37) tend to infinity as $t \to +\infty$.

Consider equilibrium states for system (37). For any equilibrium state we have

$$\dot{\theta}(t) \equiv 0, \quad G(t) \equiv 0, \quad \theta(t) \equiv \pi k.$$

Theorem 7. *We assume that relation (38) is valid. In this case if $R > 0$, then the following equilibria*

$$G(t) \equiv 0, \quad \theta(t) \equiv 2k\pi \tag{40}$$

are locally asymptotically stable and the following equilibria

$$G(t) \equiv 0, \quad \theta(t) \equiv (2k+1)\pi \tag{41}$$

are locally unstable. If $R < 0$, then equilibria (41) are locally asymptotically stable and equilibria (40) are locally unstable.

Thus, for relations (32) and (33) to be satisfied it is necessary to choose the parameters of system in such a way that the inequality holds

$$R > |M\omega_1 - \omega_3(0)|. \tag{42}$$

Proofs of Theorems 6 and 7. Let $R > |M\omega_1 - \omega_3(0)|$. Consider the Lyapunov function

$$V(G,\theta) = \int_0^G \Phi(u)du + \beta \int_0^\theta \varphi(u)du,$$

where $\Phi(G)$ is a single-valued function coinciding with $F(G)$ for $G \neq 0$. For $G = 0$, the function $\Phi(G)$ can be defined arbitrary. At points t such that $G(t) \neq 0$, we have

$$\dot{V}\big(G(t),\theta(t)\big) = -\alpha G(t)F(G(t)). \tag{43}$$

Note that, for $G(t) = 0$, the first equation of system (35) yields

$$\dot{G}(t) \neq 0 \text{ for } \theta(t) \neq k\pi.$$

It follows that there are no sliding solutions of system (35). Then, relation (43) and the inequality $F(G)G > 0, \ \forall G \neq 0$ imply that conditions 3) and 4) of Theorem 3 are satisfied. Moreover, $V(G,\theta + 2\pi) \equiv V(G,\theta)$ and $V(G,\theta) \to +\infty$ as $G \to +\infty$. Therefore, conditions (1) and (2) of Theorem 3 are satisfied. Hence, any solution of system (35) tends to the stationary set as $t \to +\infty$. Since the stationary set of system (35) consists of isolated points, any solution to system (35) tends to equilibrium state as $t \to +\infty$.
If the inequality

$$-R > |M\omega_1 - \omega_3(0)|, \tag{44}$$

is valid, then, in place of the function $V(G,\theta)$, one should consider the Lyapunov function $W(G,\theta) = -V(G,\theta)$ and repeat the above considerations.
Under inequality (39), we have the relation $F(G) \neq 0, \forall G \in R^1$. Together with the second equation of system (37), this implies that

$$\lim_{t\to+\infty} \theta(t) = \infty.$$

Thus, Theorem 6 is completely proved.
To prove Theorem 7, we note that if condition (42) holds in a neighbourhood of points $G = 0$, $\theta = 2\pi k$, then the function $V(G,\theta)$ has the property

$$V(G,\theta) > 0 \quad \text{for} \quad |G| + |\theta - 2k\pi| \neq 0.$$

Together with equality (43), this implies the asymptotic stability of these equilibrium states.
In a neighbourhood of points $G = 0$, $\theta = (2k+1)\pi k$, the function $V(G,\theta)$ has the property $V(0,\theta) < 0 \quad \text{for} \quad \theta \neq (2k+1)\pi$. Together with equality (43), this implies the instability of these equilibrium states.
If inequality (44) holds, then, in place of the function $V(G,\theta)$, we can consider the function $W(G,\theta) = -V(G,\theta)$ and repeat the considerations. ■

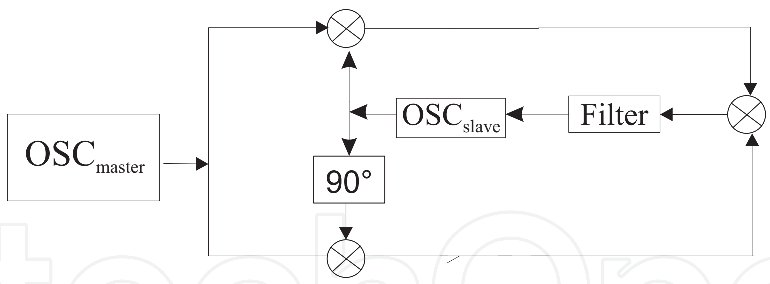

Fig. 13. Costas loop

6. Costas loop

Consider now a block diagram of the Costas loop (Fig. 13). Here all denotations are the same as in Fig. 1, 90^o is a quadrature component. As before, we consider here the case of the high-frequency harmonic and impulse signals $f_j(t)$.
However together with the assumption that conditions (3) and (5) are valid we assume also that (4) holds for the signal of the type (1) and the relation

$$|\omega_1(\tau) - 2\omega_2(\tau)| \le C_1, \quad \forall \tau \in [0, T], \tag{45}$$

is valid for the signal of the type (10). Applying the similar approach we can obtain differential equation for the Costas loop, where

$$\begin{aligned} \dot{z} &= Az + b\Psi(\sigma), \\ \dot{\sigma} &= c^* z + \rho\Psi(\sigma). \end{aligned} \tag{46}$$

Here A is a constant $n \times n$-matrix, b and c are constant n-vectors, ρ is a number, and $\Psi(\sigma)$ is a 2π- periodic function, satisfying the following relations

$$\begin{aligned} &\psi(\sigma) = \frac{1}{8} A_1^2 A_2^2 \sin\sigma - \frac{\omega_1(0) - \omega_2(0)}{L(a + W(0))}, \\ &\sigma = 2\theta_1 - 2\theta_2 \end{aligned}$$

in the case of harmonic oscillations (1) and

$$\begin{aligned} &\psi(\sigma) = P(\sigma) - \frac{\omega_1(0) - 2\omega_2(0)}{2L(a + W(0))}, \\ &P(\sigma) = \begin{cases} -2A_1^2 A_2^2 \left(1 + \dfrac{2\sigma}{\pi}\right), & \sigma \in [0, \pi] \\ -2A_1^2 A_2^2 \left(1 - \dfrac{2\sigma}{\pi}\right), & \sigma \in [-\pi, 0] \end{cases} \\ &\sigma = \theta_1 - 2\theta_2 \end{aligned}$$

in the case of impulse oscillations (10), where $\rho = -2aL, \quad W(p) = (2L)^{-1} c^* (A - pI)^{-1} b$. This implies that for deterministic (when the noise is lacking) description of the Costas loops the conventional introduction of additional filters turns out unnecessary. Here a central filter plays their role.

7. Bifurcations in digital PLL

For the study of stability of discrete systems, just as in the case of continuous ones, a discrete analog of direct Lyapunov method can be applied. By the frequency theorem similar to that of Yakubovich – Kalman for discrete systems (Leonov & Seledzhi, 2002), the results of applying the direct Lyapunov method can be formulated in the form of frequency inequalities. In the same way as in the continuous case, the phase systems have certain specific properties and for these systems it is necessary to use the direct Lyapunov method together with another research methods. For discrete systems there exist analogs of the method of cone grids and the reduction procedure of Bakaev–Guzh (Bakaev, 1959; Leonov & Smirnova, 2000).
Here we consider bifurcation effects, arising in discrete models of PLL (Osborne, 1980; Leonov & Seledzhi, 2002).
Discrete phase-locked loops (Simpson, 1994; Lapsley et al., 1997; Smith, 1999; Solonina et al., 2000; Solonina et al., 2001; Aleksenko, 2002; Aleksenko, 2004) with sinusoidal characteristic of phase discriminator are described in details in (Banerjee & Sarkar, 2008). Here a description of bifurcations of a filter-free PLL with a sine-shaped characteristic of phase detector (see (Belykh & Lebedeva, 1982; Leonov & Seledzhi, 2002; Lindsey & Chie, 1981; Osborne, 1980)) is considered. If the initial frequencies of the master and slave oscillators coincide, then the equation of the PLL is of the form

$$\sigma(x+1) = \sigma(x) - r\sin\big(\sigma(x)\big), \tag{47}$$

where r is a positive number. It is easy to see (Abramovich et al., 2005; Leonov & Seledzhi, 2002; Leonov & Seledzhi, 2005a) that this system is globally asymptotically stable for $r \in (0,2)$. Now we study equation (47) for $r > 2$. Let $r \in (2, r_1)$, where r_1 is a root of the equation $\sqrt{r^2-1} = \pi + arccos\frac{1}{r}$. Then (47) maps $[-\pi,\pi]$ into itself, that is $\sigma(t) \in [-\pi,\pi]$ for $\sigma(0) \in [-\pi,\pi]$ $t = 1, 2, \ldots$.
In the system (47) there is transition to chaos via the sequence of period doubling bifurcations (Fig. 14). Equation (47) is not unimodal, so we can not directly apply the usual Renorm-Group

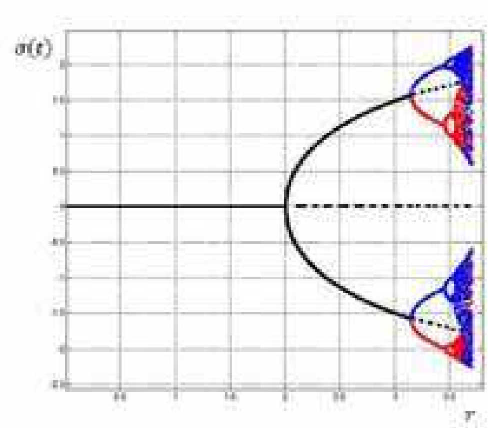

Fig. 14. Bifurcation tree.

method for its analytical investigation. Some first bifurcation parameters can be calculated analytically (Osborne, 1980), the others can be found only by means of numerical calculations (Abramovich et al., 2005; Leonov & Seledzhi, 2005a).

The first 13 calculated bifurcation parameters of period doubling bifurcations of (47) are the following

$$
\begin{array}{ll}
r_1 = 2 & r_2 = \pi \\
r_3 = 3.445229223301312 & r_4 = 3.512892457411257 \\
r_5 = 3.527525366711579 & r_6 = 3.530665376391086 \\
r_7 = 3.531338162105000 & r_8 = 3.531482265584890 \\
r_9 = 3.531513128976555 & r_{10} = 3.531519739097210 \\
r_{11} = 3.531521154835959 & r_{12} = 3.531521458080261 \\
r_{13} = 3.531521523045159 &
\end{array}
$$

Here r_2 is bifurcation of splitting global stable cycle of period 2 into two local stable cycles of period 2. The other r_j correspond to period doubling bifurcations.

We have here following Feigenbaum's effect of convergence for $\delta_{n+1} = \dfrac{r_{n+1} - r_n}{r_{n+2} - r_{n+1}}$:

$$
\begin{array}{ll}
\delta_2 = 3.759733732581654 & \delta_3 = 4.487467584214882 \\
\delta_4 = 4.624045206680584 & \delta_5 = 4.660147831971297 \\
\delta_6 = 4.667176508904449 & \delta_7 = 4.668767988303247 \\
\delta_8 = 4.669074658227896 & \delta_9 = 4.669111696537520 \\
\delta_{10} = 4.669025736544542 & \delta_{11} = 4.668640891299296 \\
\delta_{12} = 4.667817727564633. &
\end{array}
$$

8. Conclusion

The theory of phase synchronization was developed in the second half of the last century on the basis of three applied theories: theory of synchronous and induction electrical motors, theory of auto-synchronization of the unbalanced rotors, theory of phase-locked loops. Its main principle is consideration of the problems of phase synchronization at the three levels:
(i) at the level of mechanical, electromechanical, or electronic model,
(ii) at the level of phase relations,
(iii) at the level of difference, differential, and integro-differential equations.
In this case the difference of oscillation phases is transformed in the control action, realizing synchronization. These general principles gave impetus to creation of universal methods for studying the phase synchronization systems. Modification of the direct Lyapunov method with the construction of periodic Lyapunov-like functions, the method of positive invariant cone grids, and the method of nonlocal reduction turned out to be most effective. The last method, which combines the elements of the direct Lyapunov method and the bifurcation theory, allows one to extend the classical results of F. Tricomi and his progenies to the multi-dimensional dynamical systems.

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The present edited book is a collection of 18 chapters written by internationally recognized experts and well-known professionals of the field. Chapters contribute to diverse facets of automation and control. The volume is organized in four parts according to the main subjects, regarding the recent advances in this field of engineering. The first thematic part of the book is devoted to automation. This includes solving of assembly line balancing problem and design of software architecture for cognitive assembling in production systems. The second part of the book concerns different aspects of modelling and control. This includes a study on modelling pollutant emission of diesel engine, development of a PLC program obtained from DEVS model, control networks for digital home, automatic control of temperature and flow in heat exchanger, and non-linear analysis and design of phase locked loops. The third part addresses issues of parameter estimation and filter design, including methods for parameters estimation, control and design of the wave digital filters. The fourth part presents new results in the intelligent control. This includes building a neural PDF strategy for hydroelectric satation simulator, intelligent network system for process control, neural generalized predictive control for industrial processes, intelligent system for forecasting, diagnosis and decision making based on neural networks and self-organizing maps, development of a smart semantic middleware for the Internet , development of appropriate AI methods in fault-tollerant control, building expert system in rotary railcar dumpers, expert system for plant asset management, and building of a image retrieval system in heterogeneous database. The content of this thematic book admirably reflects the complementary aspects of theory and practice which have taken place in the last years. Certainly, the content of this book will serve as a valuable overview of theoretical and practical methods in control and automation to those who deal with engineering and research in this field of activities.

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