Some sufficient conditions for graphs to be (g, f, n)-critical graphs

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Abstract. Let G be a graph of order p, and let a and b and n be nonnegative integers with $1< a < b$, and let g and f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for all $x \in V(G)$. A (g, f)-factor of graph G is defined as a spanning sub graph F of G such that $g(x) \le d_F(x) \le f(x)$ for each $x \in V(G)$. Then a graph G is called a (g, f, n)-critical graph if after deleting any n vertices of G the remaining graph of G has a (g, f)-factor. In this paper, we prove that every graph G is a (g, f, n)-critical graph if its minimum degree is greater than $p + a + b - 2\sqrt{(a+1)p-bn+1}$. Furthermore, it is showed that the result in this paper is best possible in some sense.
Keywords: graph, minimum degree, $(g,$ f)-factor, (g, f, n)-critical graph.
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1. INTRODUCTION

In this paper, we consider a finite graph G with vertex set $V(G)$ and edge set $E(G)$, which has neither loops nor multiple edges. For any vertex x of G, we denote by $d_G(x)$ the degree of x in G. We denote by $\delta(G)$ the minimum vertex degree of G. For any $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$ and $G - S = G[V(G) \setminus S]$.

Let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for each $x \in V(G)$. A (g,f)-factor of graph G is defined as a spanning subgraph F of G such that $g(x) \le d_F(x) \le f(x)$ for each $x \in V(G)$ (Where of course d_F-denotes the degree in F). If $g(x)=f(x)$ for each $x \in V(G)$, then a (g,f)-factor is called an f-factor. If $g(x)=a$ and $f(x)=b$ for all $x \in V(G)$, then a (g,f)-factor is called an $[a,b]$-factor. If $g(x)=f(x)=k$ for all $x \in V(G)$, then a (g,f)-factor is called a k-factor. A graph G is called a (g,f,n)-critical graph if after deleting any n vertices of G the remaining graph of G has a (g,f)-factor. If G is a (g,f,n)-critical graph, then we also say that G is (g,f,n)-critical. If $g(x)=f(x)$ for each $x \in V(G)$, then a (g,f,n)-critical graph is an (f,n)-critical graph. If $g(x)=a$ and $f(x)=b$ for all $x \in V(G)$, then a (g,f,n)-critical graph is an (a,b,n)-critical graph. If $a=b=k$, then an (a,b,n)-critical graph is simply called a (k,n)-critical graph. In particular, a $(1,n)$-critical graph is simply called an n-critical graph. The other notations and definitions not given in this paper can be found in [1].

Q. Yu [2] gave the characterization of n-critical graphs. O. Favaron [3] studied the properties of n-critical graphs. G. Liu and Q. Yu [4] studied the characterization of (k,n)-critical graphs. The characterization of (a,b,n)-critical graphs with $a<b$ was given by G. Liu and J. Wang[5]. S. Zhou [6,7,8,13] gave some sufficient conditions for graphs to be (a,b,n)-critical graphs. J. Li [9] showed two sufficient conditions for graphs to be (a,b,n)-critical graphs. S. Zhou [10] obtained a sufficient condition for graphs to be (g,f,n)-critical graphs. The characterization of (g,f,n)-critical graphs was given by J. Li and H. Matsuda [11]. In this paper, we obtain two new sufficient conditions for graphs to be (g,f,n)-critical graphs. The main results will be given in the following section. The following results on k-factors and (a,b,k)-critical graphs and (g,f,n)-critical graphs are known.

In [12], Y. Egawa and H. Enomoto proved the following result for the existence of k-factors.

Theorem 1.[12] Let $k\geq 2$ be an integer, and let G be a graph of order n, kn is even. If

$$\delta(G) > n+2k-2\sqrt{kn+1},$$

then G has a k-factor.

In [8], S. Zhou and M. Zong gave a sufficient condition for graphs to be (a,b,k)-critical graphs.

Theorem 2.[8] Let a,b and k be nonnegative integers such that $1\leq a<b$ and G be a graph with order $n \geq \frac{(a-1)(a+1)(a+b)(a+b-1)}{a(b-1)} - \frac{(a+b)(ab+b-1)}{ab(b-1)} + k$. If $\delta(G) \geq a+k$, and

$$\max\{d_G(x), d_G(y)\} \geq \frac{an+bk}{a+b}$$

for any vertices x and y of $V(G)$ with $d(x,y)=2$. Then G is an (a,b,k)-critical graph.

In [10], S. Zhou showed a sufficient condition for graphs to be (g,f,n)-critical graphs.

Theorem 3.[10] Let G be a graph, and let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x)<f(x)$ for each $x \in V(G)$. If $g(x) \le d_G(x)$ and $f(x)(d_G(y)-n) \ge d_G(x)g(y)$ for each $x, y \in V(G)$, then G is a $(g,f,$ n)-critical graph. Here n is a nonnegative integer.

2. THE PROOF OF MAIN THEOREMS

In this paper, we obtain a new sufficient condition for graphs to be (g,f,n)-critical graphs. Our result is the extension of Theorem 1.

Theorem 4. Let G be a graph of order p, and let a, b and n be nonnegative integers such that $1 \le a < b$, and let g and f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$ for each $x \in V(G)$. If

$$\delta(G) > p + a + b - 2\sqrt{(a+1)p - bn + 1}\,, \tag{1}$$

then G is a (g,f,n)-critical graph.

In Theorem 4, if $n=0$, then we obtain the following corollary.

Corollary 1. Let G be a graph of order p, and let a and b be integers such that $1 \le a < b$, and let g and f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$ for each $x \in V(G)$. If

$$\delta(G) > p + a + b - 2\sqrt{(a+1)p + 1}\,,$$

then G has a (g,f)-factor.

According to Corollary 1 and the definition of $(g,f,$ n)-critical graph, we obtain easily the following result.

Theorem 5. Let G be a graph of order p, and let a, b and n be nonnegative integers such that $1 \le a < b$, and let g and f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$ for each $x \in V(G)$. If

$$\delta(G) > p + a + b - 2\sqrt{(a+1)p + 1} + n\,,$$

then G is a (g,f,n)-critical graph.

Let S and T be disjoint subsets of $V(G)$. We write $e_G(S,T)=|\{xy \in E(G) : x \in S, y \in T\}|$, $f(S)=\Sigma_{x\in S} f(x)$, $d_{G-S}(T)=\Sigma_{x\in T} d_{G-S}(x)$, and $g(T)=\Sigma_{x\in T} g(x)$. Our proof of Theorem 4 relies heavily on the following theorem.

Theorem 6.[11] Let G be a graph, $n \ge 0$ an integer, and let g and f be two integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. Then G is a (g,f,n)-

critical graph if and only if

$$\delta_G(S,T) = f(S)+d_{G-S}(T)-g(T) \geq \max\{f(N) : N \subseteq S, |N| = n\}$$

for all disjoint subsets S and T of $V(G)$ with $|S| \geq n$.

Proof of Theorem 4. Suppose a graph G satisfies the condition of the theorem, but it is not a (g, f, n)-critical graph. Then, by Theorem 6, there exist disjoint subsets S and T of $V(G)$ with $|S| \geq n$ such that

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \leq \max\{f(N) : N \subseteq S, |N| = n\} - 1. \quad (2)$$

We choose subsets S and T such that $|T|$ is minimum and S and T satisfy (2).

Claim 1. $d_{G-S}(x) \leq g(x) - 1 \leq b-2$ for each $x \in T$.
Proof. Suppose that there exists a vertex $x \in T$ such that $d_{G-S}(x) \geq g(x)$. Then the subsets S and $T - \{x\}$ satisfy (2), which contradicts the choice of T.
Completing the proof of Claim 1.

If $T = \emptyset$, then by (2), $f(S) - 1 \geq \max\{f(N) : N \subseteq S, |N| = n\} - 1 \geq \delta_G(S, T) = f(S)$, a contradiction. Hence, $T \neq \emptyset$. Let

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

According to Claim 1, we have

$$0 \leq h \leq b-2,$$

and

$$\delta(G) \leq h + |S|. \quad (3)$$

According to (2) and $|S| + |T| \leq p$, we get that

$$\begin{aligned} bn - 1 &\geq \max\{f(N) : N \subseteq S, |N| = n\} - 1 \\ &\geq \delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) \\ &\geq (a+1)|S| + d_{G-S}(T) - (b-1)|T| \\ &\geq (a+1)|S| + h|T| - (b-1)|T| \\ &= (a+1)|S| - (b-h-1)|T| \\ &\geq (a+1)|S| - (b-h-1)(p-|S|) \\ &= (a+b-h)|S| - (b-h-1)p. \end{aligned}$$

Thus, we obtain

$$|S| \leq \frac{(b-h-1)p + bn - 1}{a+b-h}. \quad (4)$$

In view of (3) and (4), we have

$$\delta(G) \le h + |S| \le h + \frac{(b-h-1)p + bn - 1}{a+b-h}. \quad (5)$$

Let $f(h) = h + \frac{(b-h-1)p+bn-1}{a+b-h}$. In the range of $h \le a+b$, the function $f(\mathrm{h})$ attains its maximum value at $h = a+b-\sqrt{(a+1)p-bn+1}$. Since $0 \le h \le b-2$, then we have

$$f(h) \le f(a+b-\sqrt{(a+1)p-bn+1})$$
$$= p+a+b-2\sqrt{(a+1)p-bn+1},$$

that is,

$$\delta(G) \le p+a+b-2\sqrt{(a+1)p-bn+1},$$

this contradicts (1).

From the argument above, we deduce the contradiction. Hence, G is a (g, f, n)-critical graph.
Completing the proof of Theorem 4.

The binding number $bind(G)$ of G is the minimum value of $\frac{|N_G(X)|}{|X|}$ taken over all non-empty subsets X of $V(G)$ such that $N_G(X) \ne V(G)$. The binding number condition on (a, b, k)-critical graphs was given by Sizhong Zhou and Jiashang Jiang [7].

Theorem 7.[7] Let G be a graph of order n, and let a, b and k be nonnegative integers such that $1 \le a < b$. If the binding number $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}$ and $n \ge \frac{(a+b-1)(a+b-2)}{b} + \frac{bk}{b-1}$, then G is an (a, b, k)-critical graph.
In Theorem 7, if $k=0$, then we get the following corollary.
Corollary 2. Let G be a graph of order n, and let a and b be two integers such that $1 \le a < b$. If the binding number $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)+2}$ and $n \ge \frac{(a+b-1)(a+b-2)}{b}$, then G has an $[a, b]$-factor.

Let a, b and k be nonnegative integers such that $1 \le a < b$. The proof of Theorem 7 relies heavily on the following theorem.

Theorem 8.[5] Let G be a graph of order $n \geq a + k + 1$. Then G is (a, b, k)-critical if and only if for any $S \subseteq V(G)$ and $|S| \geq k$

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \leq b|S| - bk \text{ , or}$$

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \geq bk \text{ ,}$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \leq a-1\}$.

Proof of Theorem 7. Suppose a graph G satisfies the condition of the theorem, but it is not an (a, b, k)-critical graph. Then, by Theorem 8, there exists a subset S of $V(G)$ with $|S| \geq k$ such that

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \leq bk - 1, \tag{6}$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \leq a-1\}$. We choose subsets S and T such that $|T|$ is minimum and S and T satisfy (6).

If $T = \emptyset$, then by (6), $bk - 1 \geq \delta_G(S,T) = b|S| \geq bk$, a contradiction. Hence, $T \neq \emptyset$.

Let

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

According to the definition of T, we have

$$0 \leq h \leq a - 1.$$

We shall consider various cases according to the value of h and derive contradictions.

Case 1. $h = 0$.

At first, we prove the following claim.

Claim 2. $\dfrac{bn-(a+b)-bk+2}{n-1} > 1$.

Proof. Since $n \geq \dfrac{(a+b-1)(a+b-2)}{b} + \dfrac{bk}{b-1}$, then we have

$$\begin{aligned} bn-(a+b)-bk+2-(n-1) &= (b-1)n-(a+b)-bk+3 \\ &\geq (b-1)\left(\frac{(a+b-1)(a+b-2)}{b}+\frac{bk}{b-1}\right)-(a+b)-bk+3 \\ &= \frac{(b-1)(a+b-1)(a+b-2)}{b}-(a+b)+3 \\ &\geq (a+b-2)-(a+b)+3>0 \end{aligned}$$

Thus, we have

$$\frac{bn-(a+b)-bk+2}{n-1}>1.$$

Completing the proof of Claim 2.

Let $m=|\{x : x\in T,\ d_{G-S}(x)=0\}|$, and let $Y=V(G)\backslash S$. Then $N_G(Y)\neq V(G)$ since h=0. In view of the definition of the binding number $bind(G)$, we get that

$$|N_G(Y)|\geq bind(G)\,|Y|.$$

Thus, we obtain

$$n-m\geq |N_G(Y)|\geq bind(G)|Y|=bind(G)(n-|S|),$$

that is,

$$|S|\geq n-\frac{n-m}{bind(G)}. \tag{7}$$

Using $|S|+|T|\leq n$ and (6) and (7) and Claim 2, we have

$$\begin{aligned} bk-1\geq \delta_G(S,T) &= b|S|+d_{G-S}(T)-a|T| \\ &\geq b|S|-(a-1)|T|-m \\ &\geq b|S|-(a-1)(n-|S|)-m \\ &= (a+b-1)|S|-(a-1)n-m \\ &\geq (a+b-1)\left(n-\frac{n-m}{bind(G)}\right)-(a-1)n-m \\ &= bn-(a+b-1)\frac{n-m}{bind(G)}-m \\ &> bn-(a+b-1)\frac{(n-m)(bn-(a+b)-bk+2)}{(a+b-1)(n-1)}-m \\ &= bn-\frac{(n-m)(bn-(a+b)-bk+2)}{n-1}-m \end{aligned}$$

$$\begin{aligned} &\geq bn - \frac{(n-1)(bn-(a+b)-bk+2)}{n-1} - 1 \\ &= bk + (a+b) - 3 \\ &\geq bk, \end{aligned}$$

which is a contradiction.

Case 2. $1 \leq h \leq a-1$.

Let x_1 be a vertex in T such that $d_{G-S}(x_1)=h$, and let $Y = (V(G) \setminus S) \setminus N_{G-S}(x_1)$. Then $x_1 \in Y \setminus N_G(Y)$, so $Y \neq \emptyset$ and $N_G(Y) \neq V(G)$. In view of the definition of the binding number $bind(G)$, we obtain

$$\frac{|N_G(Y)|}{|Y|} \geq bind(G).$$

Thus, we get that

$$n-1 \geq |N_G(Y)| \geq bind(G)\,|Y| = bind(G)(n-h-|S|),$$

that is,

$$|S| \geq n-h-\frac{n-1}{bind(G)}. \tag{8}$$

By $|S|+|T| \leq n$ and (6) and (8), we obtain

$$\begin{aligned} bk-1 \geq \delta_G(S,T) &= b|S| + d_{G-S}(T) - a|T| \\ &\geq b|S| - (a-h)|T| \\ &\geq b|S| - (a-h)(n-|S|) \\ &= (a+b-h)|S| - (a-h)n \\ &\geq (a+b-h)(n-h-\frac{n-1}{bind(G)}) - (a-h)n \\ &> (a+b-h)(n-h-\frac{bn-(a+b)-bk+2}{a+b-1}) - (a-h)n, \end{aligned}$$

that is,

$$bk-1 > (a+b-h)(n-h-\frac{bn-(a+b)-bk+2}{a+b-1}) - (a-h)n. \tag{9}$$

Let $f(h) = (a+b-h)(n-h-\frac{bn-(a+b)-bk+2}{a+b-1}) - (a-h)n$. In fact, the function $f(h)$ attains its minimum value at $h=1$ since $1 \leq h \leq a-1$ is an integer. Then, we have

$$f(h) \geq f(1).$$

Combining this with (9), we obtain

$$\begin{aligned} bk-1 > f(1) &= (a+b-1)(n-1-\frac{bn-(a+b)-bk+2}{a+b-1})-(a-1)n \\ &= (a+b-1)(n-1)-(bn-(a+b)-bk+2)-(a-1)n \\ &= bk-1, \end{aligned}$$

that is a contradiction.

From the argument above, we deduce the contradictions, so the hypothesis cannot hold. Hence, G is $(a, b,$ k)-critical.
Completing the proof of Theorem 7.

Remark 1. Let us show that the condition $\delta(G) > p+a+b-2\sqrt{(a+1)p-bn+1}$ in Theorem 4 cannot be replaced by $\delta(G) \geq p+a+b-2\sqrt{(a+1)p-bn+1}$. Let $a=2,$ $b=3$, and $n\geq 0$ an integer. Let $H = K_{n+1} \vee (K_a \cup K_a)$. Then $p=2a+n+1$ and $\delta(H)=a+n$. Thus, we obtain easily $p+a+b-2\sqrt{(a+1)p-bn+1} = a+n$, that is, $\delta(H) = p+a+b-2\sqrt{(a+1)p-bn+1}$. Let $S=V(K_{n+1})\subseteq V(H)$, $T=V(Ka \cup K_a)\subseteq V(H)$. Since $a\leq g(x) < f(x) \leq b$ and $b=a+1$, then we have $g(x)=a$ and $f(x)=b=a+1$. Thus, we get

$$\begin{aligned} \delta_H(S,T) &= f(S)+d_{H-S}(T)-g(T) \\ &= b|S|+(a-1)|T|-a|T| \\ &= b|S|-|T| \\ &= b(n+1)-2a \\ &= bn+b-2a \\ &= bn-1 \qquad (\textit{Since } a=2 \textit{ and } b=3) \\ &< bn = \max\{f(N): N\subseteq S, |N|=n\}. \end{aligned}$$

By Theorem 6, H is not a (g, f, n)-critical graph. In the above sense, the result of Theorem 4 is best possible.

Remark 2. We may adopt the similar way to argue the condition $\delta(G) > p+a+b-2\sqrt{(a+1)p+1}+n$ in Theorem 5, and the condition $\delta(G) > p+a+b-2\sqrt{(a+1)p+1}+n$ in Theorem 5 is the best possible in some sense.

Remark 3. Let us show that the condition $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}$ in Theorem 7 cannot be replaced by $bind(G) \geq \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}$. Let $b > a \geq 2, k \geq 0$ be three integers such that $a+b+k$

is odd, and let $n=\frac{(a+b-1)(a+b-2)+(a+b-2)+(a+2b-1)k}{b}$ is an integer, and let $l=\frac{a+b+k-1}{2}$ and $m=n-2l=n-(a+b+k-1)=\frac{(a+b-1)(a-2)+(a+b-2)+(a+b-1)k}{b}$. Clearly, m is an integer. Let $H=K_m \vee lK_2$. Let $X=V(lK_2)$, for any $x \in X$, then $|N_H(X \setminus x)| = n-1$. By the definition of $bind(H)$, $bind(H) = \frac{|N_H(X\setminus x|}{|X\setminus x|}=\frac{n-1}{2l-1}=\frac{n-1}{a+b+k-2}=\frac{(a+b-1)(n-1}{bn-(a+b)-bk+2}$. Let $S=V(Km) \subseteq V(H)$, $T=V(lK_2)\subseteq V(H)$, then $|S|=m\geq k$, $|T|=2l$.
Thus, we get

$$\begin{aligned}\delta_H(S,T) &= b|S|-a|T|+d_{H-S}(T)\\ &= b|S|-a|T|+|T|= b|S|-(a-1)|T|\\ &= b\frac{(a+b-1)(a-2)+(a+b-2)+(a+b-1)k}{b}-(a-1)(a+b+k-1)\\ &= bk-1<bk.\end{aligned}$$

By Theorem 8, H is not an (a,b,k)-critical graph. In the above sense, the result in Theorem 7 is best possible.

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