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Entropic Geometry of Crowd Dynamics

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1. Introduction

In this Chapter we propose a nonlinear entropic model of crowd generic psycho–physical[1] dynamics. For this we use Feynman's action–amplitude formalism, operating on microscopic, mesoscopic and macroscopic synergetic levels, which correspond to individual, group (aggregate) and full crowd behavior dynamics, respectively. In all three levels, goal–directed behavior operates under entropy conservation, $\partial_t S = 0$, while naturally chaotic behavior operates under (monotonically) increasing entropy, $\partial_t S > 0$. Between these two distinct behavioral phases lies a topological phase transition with a chaotic inter-phase. We formulate a geometrical representation of this behavioral transition in terms of the Perelman-Ricci flow on the crowd's Riemannian configuration manifold.

Recall that in psychology the term *cognition*[2] refers to an information processing view of an individual psychological functions (see [3; 4; 68; 81; 88]). More generally, cognitive processes can be natural and artificial, conscious and not conscious; therefore, they are analyzed from different perspectives and in different contexts, e.g., anesthesia, neurology, psychology, philosophy, logic (both Aristotelian and mathematical), systemics, computer science, artificial intelligence (AI) and computational intelligence (CI). Both in psychology and in AI/CI, cognition refers to the mental functions, mental processes and states of intelligent entities (humans, human organizations, highly autonomous robots), with a particular focus toward the study of comprehension, inferencing, decision–making, planning and learning (see, e.g. [11]). The recently developed Scholarpedia, the free peer reviewed web encyclopedia of computational neuroscience is largely based on cognitive neuroscience (see, e.g. [79]). The concept of cognition is closely related to such abstract concepts as mind, reasoning, perception, intelligence, learning, and many others that describe numerous capabilities of the human mind and expected properties of AI/CI (see [51; 57] and references therein).

Yet disembodied cognition is a myth, albeit one that has had profound influence in Western science since Rene Descartes and others gave it credence during the Scientific Revolution. In fact, the mind-body separation had much more to do with explanation of method than with explanation of the mind and cognition, yet it is with respect to the latter that its impact is most widely felt. We find it to be an unsustainable assumption in the realm of crowd behavior.

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[1] The new term "psychophysical" should not be confused with the reserved psychological term "psychophysics". By psycho-physical we mean cognitive–to–physical transition behavior: from mental idea to physical manifestation.

[2] Latin: "cognoscere = to know"

Mental intention is (almost immediately) followed by a physical action, that is, a human or animal movement [82]. In animals, this physical action would be jumping, running, flying, swimming, biting or grabbing. In humans, it can be talking, walking, driving, or shooting, etc. Mathematical description of human/animal movement in terms of the corresponding neuro-musculo-skeletal equations of motion, for the purpose of prediction and control, is formulated within the realm of biodynamics (see [43; 44; 45; 46; 47; 48; 49; 55]).

The crowd (or, collective) behavior is clearly formed by some kind of *superposition*, *contagion*, *emergence*, or *convergence* from the individual agents' behavior. Le Bon's 1895 contagion theory, presented in "The Crowd: A Study of the Popular Mind" influenced many 20th century figures. Sigmund Freud criticized Le Bon's concept of "collective soul," asserting that crowds do not have a soul of their own. The main idea of Freudian crowd behavior theory was that people who were in a crowd acted differently towards people than those who were thinking individually: the minds of the group would merge together to form a collective way of thinking. This idea was further developed in Jungian famous "collective unconscious" [63]. The term "collective behavior" [8] refers to social processes and events which do not reflect existing social structure (laws, conventions, and institutions), but which emerge in a "spontaneous" way. Collective behavior might also be defined as action which is neither conforming (in which actors follow prevailing norms) nor deviant (in which actors violate those norms). According to the emergence theory [86], crowds begin as collectivities composed of people with mixed interests and motives; especially in the case of less stable crowds (expressive, acting and protest crowds) norms may be vague and changing; people in crowds make their own rules as they go along. According to currently popular convergence theory, crowd behavior is not a product of the crowd itself, but is carried into the crowd by particular individuals, thus crowds amount to a convergence of like–minded individuals.

We propose that the contagion and convergence theories may be unified by acknowledging that both factors may coexist, even within a single scenario: we propose to refer to this third approach as *behavioral composition*. It represents a substantial philosophical shift from traditional analytical approaches, which have assumed either reduction of a whole into parts or the emergence of the whole from the parts. In particular, both contagion and convergence are related to social entropy, which is the natural decay of structure (such as law, organization, and convention) in a social system [16]. Thus, social entropy provides an entry point into realizing a behavioral–compositional theory of crowd dynamics.

Thus, while all mentioned psycho-social theories of crowd behavior are explanatory only, in this paper we attempt to formulate a geometrically predictive model–theory of crowd psychophysical behavior.

In this chapter we attempt to formulate a geometrically predictive model–theory of crowd behavioral dynamics, based on the previously formulated individual Life Space Foam concept [54].[3]

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[3] General nonlinear stochastic dynamics, developed in a framework of Feynman path integrals, have recently [54] been applied to Lewinian field–theoretic psychodynamics [67], resulting in the development of a new concept of life–space foam (LSF) as a natural medium for motivational and cognitive psychodynamics. According to the LSF–formalism, the classic Lewinian life space can be macroscopically represented as a smooth manifold with steady force–fields and behavioral paths, while at the microscopic level it is more realistically represented as a collection of wildly fluctuating force–fields, (loco)motion paths and local geometries (and topologies with holes).

It is today well known that massive crowd movements can be precisely observed/monitored from satellites and all that one can see is crowd physics. Therefore, all involved psychology of individual crowd agents: cognitive, motivational and emotional – is only a

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A set of least–action principles is used to model the smoothness of global, macro–level LSF paths, fields and geometry, according to the following prescription. The action $S[\Phi]$, with dimensions of *Energy* $\times$ *Time* = *Effort* and depending on macroscopic paths, fields and geometries (commonly denoted by an abstract field symbol Φ^i) is defined as a temporal integral from the initial time instant t_{ini} to the final time instant t_{fin},

$$S[\Phi] = \int_{t_{ini}}^{t_{fin}} \mathfrak{L}[\Phi]dt, \tag{1}$$

with Lagrangian density given by

$$\mathfrak{L}[\Phi] = \int d^n x \mathcal{L}(\Phi_i, \partial_{x^j}\Phi^i),$$

where the integral is taken over all n coordinates $x^j = x^j(t)$ of the LSF, and $\partial_{x^j}\Phi^i$ are time and space partial derivatives of the Φ^i-variables over coordinates. The standard least action principle

$$\delta S[\Phi] = 0, \tag{2}$$

gives, in the form of the so–called Euler–Lagrangian equations, a shortest (loco)motion path, an extreme force–field, and a life–space geometry of minimal curvature (and without holes). In this way, we have obtained macro–objects in the global LSF: a single path described by Newtonian–like equation of motion, a single force–field described by Maxwellian–like field equations, and a single obstacle–free Riemannian geometry (with global topology without holes).

To model the corresponding local, micro–level LSF structures of rapidly fluctuating MD & CD, an adaptive path integral is formulated, defining a multi–phase and multi–path (multi–field and multi– geometry) transition amplitude from the motivational state of *Intention* to the cognitive state of *Action*,

$$\langle Action \mid Intention \rangle_{total} := \int \mathcal{D}[w\Phi]e^{iS[\Phi]}, \tag{3}$$

where the Lebesgue integration is performed over all continuous Φ^i_{con} = *paths* + *fields* + *geometries*, while summation is performed over all discrete processes and regional topologies Φ^j_{dis}. The symbolic differential $\mathcal{D}[w\Phi]$ in the general path integral (24), represents an adaptive path measure, defined as a weighted product

$$\mathcal{D}[w\Phi] = \lim_{N\to\infty} \prod_{s=1}^{N} w_s d\Phi^i_s, (i = 1,...,n = con + dis). \tag{4}$$

The adaptive path integral (3)–(11) represents an ∞–dimensional neural network, with weights w updating by the general rule [57]

$$new\ value(t + 1) = old\ value(t) + innovation(t).$$

non-transparent input (a hidden initial switch) for the fully observable crowd physics. In this paper we will label this initial switch as 'mental preparation' or 'loading', while the manifested physical action is labeled 'hitting'.

We propose the entropy formulation of crowd dynamics as a three–step process involving individual behavioral dynamics and collective behavioral dynamics. The chaotic behavioral phase transitions embedded in crowd dynamics may give a formal description for a phenomenon called *crowd turbulence* by D. Helbing, depicting crowd disasters caused by the panic stampede that can occur at high pedestrian densities and which is a serious concern during mass events like soccer championship games or annual pilgrimage in Makkah (see [37; 38; 39; 62]).

In this paper we propose the entropy formulation of crowd dynamics as a three–step process involving individual dynamics and collective dynamics.

2. Generic three–step crowd psycho–physical behavior

In this section we model a generic crowd dynamics (see e.g., [36; 69]) as a three–step process based on a general partition function formalism. Note that the number of variables X_i in the standard partition function from statistical mechanics (see equation (59) in Appendix) need not be countable, in which case the set of coordinates $\{x^i\}$ becomes a field $\phi = \phi(x)$, so the sum is to be replaced by the *Euclidean path integral* (that is a Wick–rotated Feynman transition amplitude in imaginary time, see subsection 3.4), as

$$Z(\phi) = \int \mathcal{D}[\phi]\exp\big[-H(\phi)\big],$$

More generally, in quantum field theory, instead of the field Hamiltonian $H(\phi)$ we have the action $S(\phi)$ of the theory. Both Euclidean path integral,

$$Z(\phi) = \int \mathcal{D}[\phi]\exp\big[-S(\phi)\big], \qquad \text{real path integral in imaginary time} \qquad (5)$$

and Lorentzian one,

$$Z(\phi) = \int \mathcal{D}[\phi]\exp\big[iS(\phi)\big], \qquad \text{complex path integral in real time} \qquad (6)$$

–r epresent quantum field theory (QFT) partition functions. We will give formal definitions of the above path integrals (i.e., general partition functions) in section 3. For the moment, we only remark that the Lorentzian path integral (6) represents a QFT generalization of the (nonlinear) Schrödinger equation, while the Euclidean path integral (5) in the (rectified) real time represents a statistical field theory (SFT) generalization of the Fokker–Planck equation.

Now, following the framework of the Extended Second Law of Thermodynamics (see Appendix), $\partial_t S \geq 0$, for entropy S in any complex system described by its partition function, we formulate a generic crowd dynamics, based on above partition functions, as the following three–step process:

1. Individual dynamics ($\mathcal{ID}$) is a transition process from an entropy–growing "loading" phase of mental preparation, to the entropy–conserving "hitting/executing" phase of physical action. Formally, $\mathcal{ID}$ is given by the phase–transition map:

$$\mathcal{ID} : \overbrace{\text{MENTAL PREPARATION}}^{\text{"LOADING"}: \partial_t S > 0} \Rightarrow \overbrace{\text{PHYSICAL ACTION}}^{\text{"HITTING"}: \partial_t S = 0} \qquad (7)$$

defined by the individual (chaotic) phase–transition amplitude

$$\left\langle \overset{\partial_t S=0}{\text{PHYS. ACTION}} \middle| CHAOS \middle| \overset{\partial_t S>0}{\text{MENTAL PREP.}} \right\rangle_{\text{ID}} := \int \mathcal{D}[\Phi] e^{iS_{\text{ID}}[\Phi]},$$

where the right-hand-side is the Lorentzian path-integral (or complex path-integral in real time, see Appendix), with the individual action

$$S_{\text{ID}}[\Phi] = \int_{t_{ini}}^{t_{fin}} L_{\text{ID}}[\Phi] dt,$$

where $L_{ID}[\Phi]$ is the behavioral Lagrangian, consisting of mental cognitive potential and physical kinetic energy.

2. Aggregate dynamics ($\mathcal{AD}$) represents the behavioral composition–transition map:

$$\mathcal{AD}: \sum_{i \in \text{AD}} \overset{\text{"LOADING":}\partial_t S>0}{\overbrace{\text{MENTAL PREPARATION}}} \Rightarrow \sum_{i \in \text{AD}} \overset{\text{"HITTING":}\partial_t S=0}{\overbrace{\text{PHYSICAL ACTION}_i}} \tag{8}$$

where the (weighted) aggregate sum is taken over all individual agents, assuming equipartition of the total energy. It is defined by the aggregate (chaotic) phase–transition amplitude

$$\left\langle \overset{\partial_t S=0}{\text{PHYS. ACTION}} \middle| CHAOS \middle| \overset{\partial_t S>0}{\text{MENTAL PREP.}} \right\rangle_{\text{AD}} := \int \mathcal{D}[\Phi] e^{-S_{\text{AD}}[\Phi]},$$

with the Euclidean path-integral in real time, that is the SFT–partition function, based on the aggregate behavioral action

$$S_{\text{AD}}[\Phi] = \int_{t_{ini}}^{t_{fin}} L_{\text{AD}}[\Phi] dt, \qquad \text{with} \qquad L_{\text{AD}}[\Phi] = \sum_{i \in \text{AD}} L_{\text{ID}}^i[\Phi].$$

3. Crowd dynamics ($\mathcal{CD}$) represents the cumulative transition map:

$$\mathcal{CD}: \sum_{i \in \text{CD}} \overset{\text{"LOADING":}\partial_t S>0}{\overbrace{\text{MENTAL PREPARATION}}} \Rightarrow \sum_{i \in \text{CD}} \overset{\text{"HITTING":}\partial_t S=0}{\overbrace{\text{PHYSICAL ACTION}_i}} \tag{9}$$

where the (weighted) cumulative sum is taken over all individual agents, assuming equipartition of the total behavioral energy. It is defined by the crowd (chaotic) phase–transition amplitude

$$\left\langle \overset{\partial_t S=0}{\text{PHYS. ACTION}} \middle| CHAOS \middle| \overset{\partial_t S>0}{\text{MENTAL PREP.}} \right\rangle_{\text{CD}} := \int \mathcal{D}[\Phi] e^{iS_{\text{CD}}[\Phi]},$$

with the general Lorentzian path-integral, that is, the QFT–partition function), based on the crowd behavioral action

$$S_{\text{CD}}[\Phi] = \int_{t_{ini}}^{t_{fin}} L_{\text{CD}}[\Phi] dt, \qquad \text{with} \qquad L_{\text{CD}}[\Phi] = \sum_{i \in \text{CD}} L_{\text{ID}}^i[\Phi] = \sum_{k=\#\text{ofADsinCD}} L_{\text{AD}}^k[\Phi].$$

All three entropic phase–transition maps, $\mathcal{ID}$, $\mathcal{AD}$ and $\mathcal{CD}$, are spatio–temporal biodynamic cognition systems, evolving within their respective configuration manifolds (i.e., sets of their respective degrees-of-freedom with equipartition of energy), according to biphasic action–functional formalisms with behavioral Lagrangian functions L_{ID}, L_{AD} and L_{CD}, each consisting of:

1. Cognitive mental potential (which is a mental preparation for the physical action), and
2. Physical kinetic energy (which describes the physical action itself).

To develop $\mathcal{ID}$, $\mathcal{AD}$ and $\mathcal{CD}$ formalisms, we extend into a physical (or, more precisely, biodynamic) crowd domain a purely–mental individual Life–Space Foam (LSF) framework for motivational cognition [54], based on the quantum–probability concept.[4]

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[4] The quantum probability concept is based on the following physical facts [58; 59]

1. The time–dependent Schrödinger equation represents a complex–valued generalization of the real–valued Fokker-Planck equation for describing the spatio–temporal probability density function for the system exhibiting continuous–time Markov stochastic process.
2. The Feynman path integral (including integration over continuous spectrum and summation over discrete spectrum) is a generalization of the time–dependent Schrödinger equation, including both continuous–time and discrete–time Markov stochastic processes.
3. Both Schrödinger equation and path integral give 'physical description' of any system they are modelling in terms of its physical energy, instead of an abstract probabilistic description of the Fokker–Planck equation.

Therefore, the Feynman path integral, as a generalization of the (nonlinear) time–dependent Schrödinger equation, gives a unique physical description for the general Markov stochastic process, in terms of the physically based generalized probability density functions, valid both for continuous–time and discrete–time Markov systems. Its basic consequence is this: a different way for calculating probabilities. The difference is rooted in the fact that *sum of squares is different from the square of sums*, as is explained in the following text. Namely, in Dirac–Feynman quantum formalism, each possible route from the initial system state *A* to the final system state *B* is called a history. This history comprises any kind of a route, ranging from continuous and smooth deterministic (mechanical–like) paths to completely discontinues and random Markov chains (see, e.g., [23]). Each history (labelled by index *i*) is quantitatively described by a complex number.

In this way, the overall probability of the system's transition from some initial state *A* to some final state *B* is given not by adding up the probabilities for each history–route, but by 'head–to–tail' adding up the sequence of amplitudes making–up each route first (i.e., performing the sum–over–histories) – to get the total amplitude as a 'resultant vector', and then squaring the total amplitude to get the overall transition probability.

Here we emphasize that the domain of validity of the 'quantum' is not restricted to the microscopic world [87]. There are macroscopic features of classically behaving systems, which cannot be explained without recourse to the quantum dynamics. This field theoretic model leads to the view of the phase transition as a condensation that is comparable to the formation of fog and rain drops from water vapor, and that might serve to model both the gamma and beta phase transitions. According to such a model, the production of activity with long–range correlation in the brain takes place through the mechanism of spontaneous

The behavioral dynamics approach to $\mathcal{ID}$, $\mathcal{AD}$ and $\mathcal{CD}$ is based on *entropic motor control* [41; 42], which deals with neuro-physiological feedback information and environmental uncertainty. The probabilistic nature of human motor action can be characterized by entropies at the level of the organism, task, and environment. Systematic changes in motor adaptation are characterized as task–organism and environment–organism tradeoffs in entropy. Such compensatory adaptations lead to a view of goal–directed motor control as the product of an underlying conservation of entropy across the task–organism–environment system. In particular, an experiment conducted in [42] examined the changes in entropy of the coordination of isometric force output under different levels of task demands and feedback from the environment. The goal of the study was to examine the hypothesis that human motor adaptation can be characterized as a process of entropy conservation that is reflected in the compensation of entropy between the task, organism motor output, and environment. Information entropy of the coordination dynamics relative phase of the motor output was made conditional on the idealized situation of human movement, for which the goal was always achieved. Conditional entropy of the motor output decreased as the error tolerance and feedback frequency were decreased. Thus, as the likelihood of meeting the task demands was decreased increased task entropy and/or the amount of information from the environment is reduced increased environmental entropy, the subjects of this experiment employed fewer coordination patterns in the force output to achieve the goal. The conservation of entropy supports the view that context dependent adaptations in human goal–directed action are guided fundamentally by natural law and provides a novel means of examining human motor behavior. This is fundamentally related to the *Heisenberg uncertainty principle* [59] and further supports the argument for the primacy of a probabilistic approach toward the study of biodynamic cognition systems.[5]

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breakdown of symmetry (SBS), which has for decades been shown to describe longrange correlation in condensed matter physics. The adoption of such a field theoretic approach enables modelling of the whole cerebral hemisphere and its hierarchy of components down to the atomic level as a fully integrated macroscopic quantum system, namely as a macroscopic system which is a quantum system not in the trivial sense that it is made, like all existing matter, by quantum components such as atoms and molecules, but in the sense that some of its macroscopic properties can best be described with recourse to quantum dynamics (see [22] and references therein). Also, according to Freeman and Vitielo, *many–body quantum field theory* appears to be the only existing theoretical tool capable to explain the dynamic origin of long–range correlations, their rapid and efficient formation and dissolution, their interim stability in ground states, the multiplicity of coexisting and possibly non–interfering ground states, their degree of ordering, and their rich textures relating to sensory and motor facets of behaviors. It is historical fact that many–body quantum field theory has been devised and constructed in past decades exactly to understand features like ordered pattern formation and phase transitions in condensed matter physics that could not be understood in classical physics, similar to those in the brain.

[5] Our entropic action–amplitude formalism represents a kind of a generalization of the Haken-Kelso- Bunz (HKB) model of self-organization in the individual's motor system [24; 65], including: multistability, phase transitions and hysteresis effects, presenting a contrary view to the purely feedback driven systems. HKB uses the concepts of synergetics (order

On the other hand, it is well known that humans possess more degrees of freedom than are needed to perform any defined motor task, but are required to co-ordinate them in order to reliably accomplish high-level goals, while faced with intense motor variability. In an attempt to explain how this takes place, Todorov and Jordan have formulated an alternative theory of human motor co-ordination based on the concept of stochastic optimal feedback control [84]. They were able to conciliate the requirement of goal achievement (e.g., grasping an object) with that of motor variability (biomechanical degrees of freedom). Moreover, their theory accommodates the idea that the human motor control mechanism uses internal 'functional synergies' to regulate task–irrelevant (redundant) movement.

Also, a developing field in coordination dynamics involves the theory of social coordination, which attempts to relate the DC to normal human development of complex social cues following certain patterns of interaction. This work is aimed at understanding how human social interaction is mediated by meta-stability of neural networks. fMRI and EEG are particularly useful in mapping thalamocortical response to social cues in experimental studies. In particular, a new theory called the *Phi complex* has been developed by S. Kelso and collaborators, to provide experimental results for the theory of social coordination dynamics (see the recent nonlinear dynamics paper discussing social coordination and EEG dynamics [85]). According to this theory, a pair of phi rhythms, likely generated in the mirror neuron system, is the hallmark of human social coordination. Using a dual–EEG recording system, the authors monitored the interactions of eight pairs of subjects as they moved their fingers with and without a view of the other individual in the pair.

Finally, the chaotic behavioral phase transitions embedded in $\mathcal{CD}$ may give a formal description for a phenomenon called *crowd turbulence* by D. Helbing, depicting crowd disasters caused by the panic stampede that can occur at high pedestrian densities and

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parameters, control parameters, instability, etc) and the mathematical tools of nonlinearly coupled (nonlinear) dynamical systems to account for self-organized behavior both at the cooperative, coordinative level and at the level of the individual coordinating elements. The HKB model stands as a building block upon which numerous extensions and elaborations have been constructed. In particular, it has been possible to derive it from a realistic model of the cortical sheet in which neural areas undergo a reorganization that is mediated by intra- and inter-cortical connections. Also, the HKB model describes phase transitions ('switches') in coordinated human movement as follows: (i) when the agent begins in the anti-phase mode and speed of movement is increased, a spontaneous switch to symmetrical, in-phase movement occurs; (ii) this transition happens swiftly at a certain critical frequency; (iii) after the switch has occurred and the movement rate is now decreased the subject remains in the symmetrical mode, i.e. she does not switch back; and (iv) no such transitions occur if the subject begins with symmetrical, in-phase movements. The HKB dynamics of the order parameter relative phase as is given by a nonlinear first-order ODE:

$$\dot{\phi} = (\alpha + 2\beta r^2)\sin\phi - \beta r^2 \sin 2\phi,$$

where ϕ is the phase relation (that characterizes the observed patterns of behavior, changes abruptly at the transition and is only weakly dependent on parameters outside the phase transition), r is the oscillator amplitude, while α, β are coupling parameters (from which the critical frequency where the phase transition occurs can be calculated).

which is a serious concern during mass events like soccer championship games or annual pilgrimage in Makkah (see [37; 38; 39; 62]).

3. Formal crowd dynamics

In this section we formally develop a three–step crowd behavioral dynamics, conceptualized by transition maps (7)–(8)–(9), in agreement with Haken's synergetics [25; 26]. We first develop a macro–level individual behavioral dynamics $\mathcal{ID}$. Then we generalize $\mathcal{ID}$ into an 'orchestrated' behavioral–compositional crowd dynamics $\mathcal{CD}$, using a quantum–like micro–level formalism with individual agents representing 'crowd quanta'. Finally we develop a meso–level aggregate statistical–field dynamics $\mathcal{AD}$, such that composition of the aggregates $\mathcal{AD}$ makes–up the crowd.

3.1 Individual behavioral dynamics ($\mathcal{ID}$)

$\mathcal{ID}$ transition map (7) is developed using the following action–amplitude formalism (see [53; 54]):

1. Macroscopically, as a smooth Riemannian n–manifold M_{ID} (see Appendix) with steady force–fields and behavioral paths, modelled by a real–valued classical action functional $S_{ID}[\Phi]$, of the form

$$S_{\mathrm{ID}}[\Phi] = \int_{t_{ini}}^{t_{fin}} L_{\mathrm{ID}}[\Phi]dt,$$

    (where macroscopic paths, fields and geometries are commonly denoted by an abstract field symbol Φ^i) with the potential–energy based Lagrangian L given by

$$L_{\mathrm{ID}}[\Phi] = \int d^n x \mathcal{L}_{\mathrm{ID}}(\Phi_i, \partial_{x^j}\Phi^i),$$

    where $\mathcal{L}$ is Lagrangian density, the integral is taken over all n local coordinates $x^j = x^j(t)$ of the ID, and $\partial_{x^j}\Phi^i$ are time and space partial derivatives of the Φ^i –variables over coordinates. The standard least action principle

$$\delta S_{\mathrm{ID}}[\Phi] = 0,$$

    gives, in the form of the Euler–Lagrangian equations, a shortest path, an extreme force–field, with a geometry of minimal curvature and topology without holes. We will see below that high Riemannian curvature generates chaotic behavior, while holes in the manifold produce topologically induced phase transitions.

2. Microscopically, as a collection of wildly fluctuating and jumping paths (histories), force–fields and geometries/topologies, modelled by a complex–valued adaptive path integral, formulated by defining a multi–phase and multi–path (multi–field and multi–geometry) transition amplitude from the entropy–growing state of Mental Preparation to the entropy–conserving state of Physical Action,

$$\langle \text{Physical Action}|\text{Mental Preparation}\rangle_{\mathrm{ID}} := \int_{\mathrm{ID}} \mathcal{D}[\Phi]e^{iS_{\mathrm{ID}}[\Phi]} \qquad (10)$$

    where the functional ID–measure $\mathcal{D}[w\Phi]$ is defined as a weighted product

$$\mathcal{D}[w\Phi] = \lim_{N \to \infty} \prod_{s=1}^{N} w_s d\Phi_s^i, \qquad (i = 1,...,n = con + dis), \tag{11}$$

representing an ∞–dimensional neural network [54], with weights w_s updating by the general rule

new value(t + 1) = *old value*(t) + *innovation*(t).

More precisely, the weights $w_s = w_s(t)$ in (11) are updated according to one of the two standard neural learning schemes, in which the micro–time level is traversed in discrete steps, i.e., if $t = t_0, t_1, ..., t_s$ then $t + 1 = t_1, t_2, ..., t_{s+1}$: [6]

a.   A *self–organized*, *unsupervised* (e.g., Hebbian–like [35]) learning rule:

$$w_s(t+1) = w_s(t) + \frac{\sigma}{\eta}(w_s^d(t) - w_s^a(t)), \tag{12}$$

where $\sigma = \sigma(t)$, $\eta = \eta(t)$ denote *signal* and *noise*, respectively, while superscripts *d* and *a* denote *desired* and *achieved* micro–states, respectively; or

b.   A certain form of a *supervised gradient descent learning*:

$$w_s(t+1) = w_s(t) - \eta \nabla J(t), \tag{13}$$

where η is a small constant, called the *step size*, or the *learning rate*, and $\nabla J(n)$ denotes the gradient of the 'performance hyper–surface' at the *t*–th iteration.

(Note that we could also use a reward–based, reinforcement learning rule [83], in which system learns its optimal policy: *innovation*(t) = | *reward*(t) – *penalty*(t) | .)

In this way, we effectively derive a unique and globally smooth, causal and entropic phase–transition map (7), performed at a macroscopic (global) time–level from some initial time t_{ini} to the final time t_{fin}. Thus, we have obtained macro–objects in the ID: a single path described by Newtonian–like equation of motion, a single force–field described by Maxwellian–like field equations, and a single obstacle–free Riemannian geometry (with global topology without holes).

In particular, on the macro–level, we have the ID–paths, that is biodynamical trajectories generated by the Hamilton action principle

$$\delta S_{ID}[x] = 0,$$

with the Newtonian action $S_{ID}[x]$ given by (Einstein's summation convention over repeated indices is always assumed)

$$S_{ID}[x] = \int_{t_{ini}}^{t_{fin}} [\varphi + \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j] dt, \tag{14}$$

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[6] The traditional neural networks approaches are known for their classes of functions they can represent. Here we are talking about functions in an *extensional* rather than merely *intensional* sense; that is, function can be read as input/output behavior [5; 6; 19; 34]. This limitation has been attributed to their low-dimensionality (the largest neural networks are limited to the order of 10^5 dimensions [61]). The proposed path integral approach represents a new family of function-representation methods, which potentially offers a basis for a fundamentally more expansive solution.
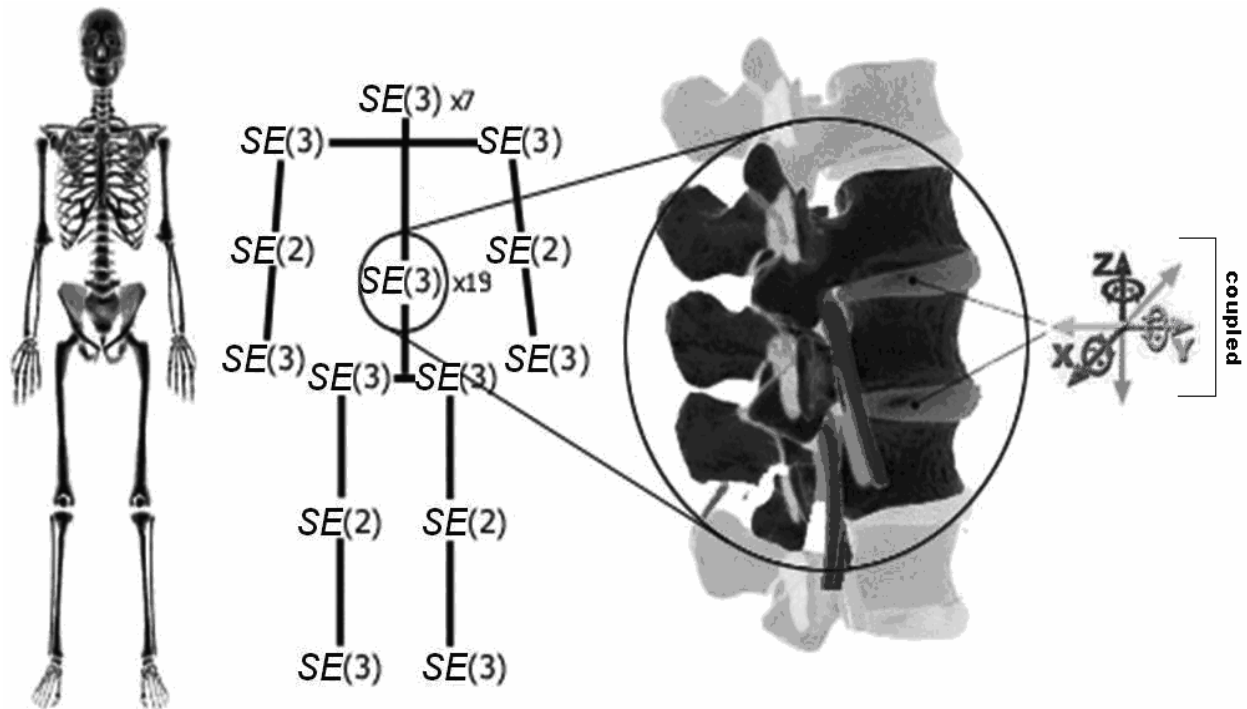
Fig. 1. Riemannian configuration manifold M_{ID} of human biodynamics is defined as a topological product $M = \prod_i SE(3)^i$ of constrained Euclidean $SE(3)$–groups of rigid body motion in 3D Euclidean space (see [49; 52]), acting in all major (synovial) human joints. The manifold M is a dynamical structure activated/controlled by potential covariant forces (16) produced by a synergetic action of about 640 skeletal muscles [47].

where $\varphi = \varphi(t, x^i)$ denotes the mental LSF–potential field, while the second term,

$$T = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j,$$

represents the physical (biodynamic) kinetic energy generated by the Riemannian inertial metric tensor g_{ij} of the configuration biodynamic manifold M_{ID} (see Figure 1). The corresponding Euler-Lagrangian equations give the Newtonian equations of human movement

$$\frac{d}{dt} T_{\dot{x}^i} - T_{x^i} = F_i, \tag{15}$$

where subscripts denote the partial derivatives and we have defined the covariant muscular forces $F_i = F_i(t, x^i, \dot{x}^i)$ as negative gradients of the mental potential $\varphi(x^i)$,

$$F_i = -\varphi_{x^i}. \tag{16}$$

Equation (15) can be put into the standard Lagrangian form as

$$\frac{d}{dt} L_{\dot{x}^i} = L_{x^i}, \quad \text{with} \quad L = T - \varphi(x^i), \tag{17}$$

or (using the Legendre transform) into the forced, dissipative Hamiltonian form [44; 47]

$$\dot{x}^i = \partial_{p_i} H + \partial_{p_i} R, \qquad \dot{p}_i = F_i - \partial_{x^i} H + \partial_{x^i} R, \tag{18}$$

where p_i are the generalized momenta (canonically–conjugate to the coordinates x^i), $H = H(p, x)$ is the Hamiltonian (total energy function) and $R = R(p, x)$ is the general dissipative function.

The human motor system possesses many independently controllable components that often allow for more than a single movement pattern to be performed in order to achieve a goal.

Hence, the motor system is endowed with a high level of adaptability to different tasks and also environmental contexts [42]. The multiple SE(3)–dynamics applied to human musculo–skeletal system gives the fundamental law of biodynamics, which is the *covariant force law*:

$$\text{Force co – vector field} = \text{Mass distribution} \times \text{Acceleration vector – field}, \tag{19}$$

which is formally written:

$$F_i = g_{ij} a^j, \qquad (i, j = 1, ..., n = \dim(M))$$

where F_i are the covariant force/torque components, g_{ij} is the inertial metric tensor of the configuration Riemannian manifold $M = \prod_i SE(3)^i$ (g_{ij} defines the mass–distribution of the human body), while a^j are the contravariant components of the linear and angular acceleration vector-field. (This fundamental biodynamic law states that contrary to common perception, acceleration and force are not quantities of the same nature: while acceleration is a non-inertial vector-field, force is an inertial co-vector-field. This apparently insignificant difference becomes crucial in injury prediction/prevention, especially in its derivative form in which the 'massless jerk' (= $\dot{a}$) is relatively benign, while the 'massive jolt' (= $\dot{F}$) is deadly.) Both Lagrangian and (topologically equivalent) Hamiltonian development of the covariant force law is fully elaborated in [47; 48; 49; 52]. This is consistent with the postulation that human action is guided primarily by natural law [66].

On the micro–ID level, instead of each single trajectory defined by the Newtonian equation of motion (15), we have an ensemble of fluctuating and crossing paths on the configuration manifold M with weighted probabilities (of the unit total sum). This ensemble of micro–paths is defined by the simplest instance of our adaptive path integral (10), similar to the Feynman's original sum over histories,

$$\langle Physical\ Action | Mental\ Preparation \rangle_M = \int_{ID} \mathcal{D}[wx] e^{iS[x]}, \tag{20}$$

where $\mathcal{D}[wx]$ is the functional ID–measure on the space of all weighted paths, and the exponential depends on the action $S_{ID}[x]$ given by (14).

3.2 Crowd behavioral–compositional dynamics ($\mathcal{CD}$)

In this subsection we develop a generic crowd $\mathcal{CD}$, as a unique and globally smooth, causal and entropic phase–transition map (9), in which agents (or, crowd's individual entities) can

be both humans and robots. This crowd behavioral action takes place in a crowd smooth Riemannian $3n$-manifold M. Recall from Figure 1 that each individual segment of a human body moves in the Euclidean 3–space $\mathbb{R}^3$ according to its own constrained SE(3)–group. Similarly, each individual agent's trajectory, $x^i = x^i(t)$, $i = 1, \dots n$, is governed by the Euclidean SE(2)–group of rigid body motions in the plane. (Recall that a Lie group $SE(2) \equiv SO(2) \times \mathbb{R}$ is a set of all 3×3– matrices of the form:

$$\begin{bmatrix} \cos\theta & \sin\theta & x \\ -\sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix},$$

including both rigid translations (i.e., Cartesian x,y–coordinates) and rotation matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ in Euclidean plane $\mathbb{R}^2$ (see [49; 52]). The crowd configuration manifold M is defined as a union of Euclidean SE(2)–groups for all n individual agents in the crowd, that is crowd's configuration $3n$–manifold is defined as a set

$$M = \sum_{k=1}^{n} SE(2)^k \equiv \sum_{k=1}^{n} SO(2)^k \times \mathbb{R}^k , \tag{21}$$

coordinated by $\mathbf{x}^k = \{x^k, y^k, \theta^k\}$, (for $k = 1, 2, \dots, n$).

In other words, the crowd configuration manifold M is a *dynamical planar graph* with individual agents' SE(2)–groups of motion in the vertices and time-dependent inter-agent distances $I_{ij} = \left[x^i(t_i) - x^j(t_j) \right]$ as edges.

Similarly to the individual case, the crowd action functional includes mental cognitive potential and physical kinetic energy, formally given by (with $i, j = 1, \dots, 3n$):

$$A[x^i, x^j; t_i, t_j] = \frac{1}{2} \int_{t_i} \int_{t_j} \delta(I_{ij}^2) \, \dot{x}^i(t_i) \dot{x}^j(t_j) \, dt_i dt_j + \frac{1}{2} \int_t g_{ij} \dot{x}^i(t) \dot{x}^j(t) dt, \tag{22}$$

with $I_{ij}^2 = \left[x^i(t_i) - x^j(t_j) \right]^2$, where $IN \leq t_i, t_j, t \leq OUT$.

The first term in (22) represents the mental potential for the interaction between any two agents x^i and x^i within the total crowd matrix x^{ij}. (Although, formally, this term contains cognitive velocities, it still represents 'potential energy' from the physical point of view.) It is defined as a double integral over a delta function of the square of interval I^2 between two points on the paths in their individual cognitive LSFs. Interaction occurs only when this LSF– distance between the two agents x^i and x^j vanishes. Note that the cognitive intentions of any two agents generally occur at different times t_i and t_j unless $t_i = t_j$, when cognitive synchronization occurs. This term effectively represents the *crowd cognitive controller* (see [53]).

The second term in (22) represents kinetic energy of the physical interaction of agents. Namely, after the above cognitive synchronization is completed, the second term of physical kinetic energy is activated in the common CD manifold, reducing it to just one of the agents' individual manifolds, which is equivalent to the center-of-mass segment in the human musculo-skeletal system. Therefore, from (22) we can derive a generic Euler–Lagrangian dynamics that is a composition of (17), which also means that we have in place a generic Hamiltonian dynamics that is a amalgamate of (18), as well as the crowd covariant force law (19), the governing law of crowd biodynamics:

Crowd force co – vector field = Crowd mass distribution × Crowd acceleration vector – field,

$$\text{formally:} \quad F_i = g_{ij}a^j, \quad \text{where} \quad g_{ij} \quad \text{is the inertial metric tensor of crowd manifold } M. \quad (23)$$

The left-hand side of this equation defines forces acting on the crowd, while right-hand defines its mass distribution coupled to the crowd kinematics ($\mathcal{CK}$, described in the next subsection).

At the slave level, the adaptive path integral, representing an ∞–dimensional neural network, corresponding to the crowd behavioral action (22), reads

$$\langle Physical\ Action | Mental\ Preparation \rangle_{\mathrm{CD}} = \int_{CD} \mathcal{D}[w,x,y]\mathrm{e}^{iA[x,y;t_i,t_j]}, \quad (24)$$

where the Lebesgue-type integration is performed over all continuous paths $x^i = x^i(t_i)$ and $y^j = y^j(t_j)$, while summation is performed over all associated discrete Markov fluctuations and jumps. The symbolic differential in the path integral (24) represents an adaptive path measure, defined as the weighted product

$$\mathcal{D}[w,x,y] = \lim_{N\to\infty} \prod_{s=1}^{N} w_{ij}^s dx^i dy^j, \quad (i,j = 1,...,n). \quad (25)$$

The quantum–field path integral (24)–(25) defines the microstate $\mathcal{CD}$–level, an ensemble of fluctuating and crossing paths on the crowd $3n$–manifold M.

The crowd manifold M itself has quite a sophisticated topological structure defined by its macrostate Euler–Lagrangian dynamics. As a Riemannian smooth n–manifold, M gives rise to its fundamental n–groupoid, or n–category $\prod_n(M)$ (see ([49; 52])). In $\prod_n(M)$, 0–cells are points in M; 1–cells are paths in M(i.e., parameterized smooth maps $f : [0,1]\to M$); 2–cells are smooth homotopies (denoted by $\simeq$) of paths relative to endpoints (i.e., parameterized smooth maps $h : [0,1] \times [0,1]\to M$); 3–cells are smooth homotopies of homotopies of paths in M (i.e., parameterized smooth maps $j : [0,1] \times [0,1] \times [0,1]\to M$). Categorical composition is defined by pasting paths and homotopies. In this way, the following recursive homotopy dynamics emerges on the crowd $3n$–manifold M:
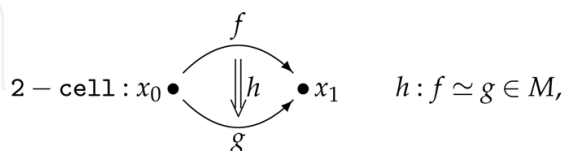
0 − cell : $x_0 \bullet$ $x_0 \in M$;   in the higher cells below: $t,s \in [0,1]$;

1 − cell : $x_0 \bullet \xrightarrow{\quad f \quad} \bullet x_1$ $f : x_0 \simeq x_1 \in M$,

$f : [0,1] \to M, f : x_0 \mapsto x_1, x_1 = f(x_0), f(0) = x_0, f(1) = x_1$;

e.g., linear path: $f(t) = (1-t)x_0 + t x_1$;   or

Euler–Lagrangian f − dynamics with endpoint conditions (x_0, x_1) :

$$\frac{d}{dt} f_{\dot{x}^i} = f_{x^i}, \quad \text{with} \quad x(0) = x_0, \quad x(1) = x_1, \quad (i = 1, ..., n);$$

2 − cell : $x_0 \bullet \quad \Downarrow h \quad \bullet x_1$ $h : f \simeq g \in M$,

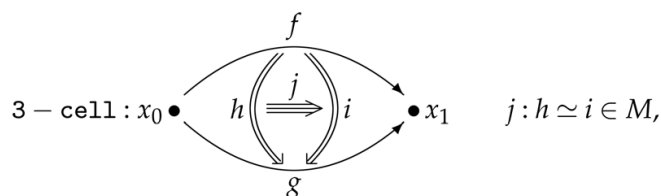$h : [0,1] \times [0,1] \to M, h : f \mapsto g, g = h(f(x_0))$,

$h(x_0, 0) = f(x_0), h(x_0, 1) = g(x_0), h(0, t) = x_0, h(1, t) = x_1$

e.g., linear homotopy: $h(x_0, t) = (1-t)f(x_0) + t g(x_0)$;   or

homotopy between two Euler–Lagrangian (f, g) − dynamics

with the same endpoint conditions (x_0, x_1) :

$$\frac{d}{dt} f_{\dot{x}^i} = f_{x^i}, \quad \text{and} \quad \frac{d}{dt} g_{\dot{x}^i} = g_{x^i} \quad \text{with} \quad x(0) = x_0, \quad x(1) = x_1;$$

3 − cell : $x_0 \bullet \quad h \left(\!\! \begin{array}{c} j \\ \Rightarrow \end{array} \!\! \right) i \quad \bullet x_1$ $j : h \simeq i \in M$,

$j : [0,1] \times [0,1] \times [0,1] \to M, j : h \mapsto i, i = j(h(f(x_0)))$

$j(x_0, t, 0) = h(f(x_0)), j(x_0, t, 1) = i(f(x_0))$,

$j(x_0, 0, s) = f(x_0), j(x_0, 1, s) = g(x_0)$,

$j(0, t, s) = x_0, j(1, t, s) = x_1$

e.g., linear composite homotopy: $j(x_0, t, s) = (1-t)h(f(x_0)) + t i(f(x_0))$;

or, homotopy between two homotopies between above two Euler-

Lagrangian (f, g) − dynamics with the same endpoint conditions (x_0, x_1).

3.3 Dissipative crowd kinematics ($\mathcal{CD}$)

The crowd action (22) with its amalgamate Lagrangian dynamics (17) and amalgamate Hamiltonian dynamics (18), as well as the crowd force law (23) define the macroscopic crowd dynamics, $\mathcal{CD}$. Suppose, for a moment, that $\mathcal{CD}$ is force–free and dissipation free, therefore conservative. Now, the basic characteristic of the conservative Lagrangian/Hamiltonian systems evolving in the phase space spanned by the system coordinates and their velocities/momenta, is that their *flow* φ_t^L (explained below) preserves the phase–space volume, as proposed by the Liouville theorem, which is the well known fact in statistical mechanics. However, the preservation of the phase volume causes structural instability of the conservative system, i.e., the phase–space spreading effect by which small phase regions R_t will tend to get distorted from the initial one R_o during the conservative system evolution. This problem, governed by entropy growth ($\partial_t S > 0$), is much

more serious in higher dimensions than in lower dimensions, since there are so many 'directions' in which the region can locally spread (see [49; 74]). This phenomenon is related to *conservative Hamiltonian chaos* (see section 4 below).

However, this situation is not very frequent in case of 'organized' human crowd. Its self-organization mechanisms are clearly much stronger than the conservative statistical mechanics effects, which we interpret in terms of Prigogine's dissipative structures (see Appendix). Formally, if dissipation of energy in a system is much stronger then its inertial characteristics, then instead of the second-order Newton–Lagrangian dynamic equations of motion, we are actually dealing with the first-order driftless (non-acceleration, non-inertial) kinematic equations of motion (see Appendix, eq. (64)), which is related to *dissipative chaos* [71]. Briefly, the dissipative crowd flow can be depicted like this: from the set of initial conditions for individual agents, the crowd evolves in time towards the set of the corresponding *entangled attractors*,[7] which are mutually separated by fractal (non-integer dimension) separatrices.

In this subsection we elaborate on the dissipative crowd kinematics ($\mathcal{CK}$), which is self–controlled and dominates the $\mathcal{CD}$ if the crowd's inertial forces are much weaker then the crowd's dissipation of energy, presented here in the form of nonlinear velocity controllers.

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[7] Recall that quantum entanglement is a quantum mechanical phenomenon in which the quantum states of two or more objects are linked together so that one object can no longer be adequately described without full mention of its counterpart – even though the individual objects may be spatially separated. This interconnection leads to correlations between observable physical properties of remote systems. The related phenomenon of wave-function collapse gives an impression that measurements performed on one system instantaneously influence the other systems entangled with the measured system, even when far apart.

Entanglement has many applications in quantum information theory. Mixed state entanglement can be viewed as a resource for quantum communication. A common measure of entanglement is the entropy of a mixed quantum state (see, e.g. [59]). Since a mixed quantum state ρ is a probability distribution over a quantum ensemble, this leads naturally to the definition of the *von Neumann entropy*, $S(\rho) = -\mathrm{Tr}\,(\rho \log_2 \rho)$, which is obviously similar to the classical *Shannon entropy* for probability distributions $(p_1, \dots , p_n)$, defined as $S(p_1, \dots , p_n) = -\Sigma_i\, p_i \log_2 p_i$. As in statistical mechanics, one can say that the more uncertainty (number of microstates) the system should possess, the larger is its entropy. Entropy gives a tool which can be used to quantify entanglement. If the overall system is pure, the entropy of one subsystem can be used to measure its degree of entanglement with the other subsystems.

The most popular issue in a research on dissipative quantum brain modelling has been *quantum entanglement* between the *brain* and its *environment* [77; 78], where the brain–environment system has an entangled 'memory' state, identified with the ground (vacuum) state $|0>_\mathcal{N}$, that cannot be factorized into two single–mode states. (In the Vitiello–Pessa dissipative quantum brain model [77; 78], the evolution of the $\mathcal{N}$–coded memory system was represented as a trajectory of given initial condition running over time–dependent states $|0(t)>_\mathcal{N}$, each one minimizing the free energy functional.) Similar to this microscopic brain–environment entanglement, we propose a kind of *macroscopic entanglement* between the operating modes of the crowd behavioral controller and its biodynamics, which can be considered as a 'long–range correlation'.

Applied externally to the dimension of the crowd $3n$–manifold M, entanglement effectively reduces the number of active degrees of freedom in (21).

Recall that the essential concept in dynamical systems theory is the notion of a *vector–field* (that we will denote by a boldface symbol), which assigns a tangent vector to each point p in the manifold in case. In particular, $\mathbf{v}$ is a gradient vector–field if it equals the gradient of some scalar function. A *flow–line* of a vector–field $\mathbf{v}$ is a path $\mathbf{fl}(t)$ satisfying the vector ODE, $\dot{\mathbf{fl}}(t) = \mathbf{v}(\mathbf{fl}(t))$, that is, $\mathbf{v}$ yields the velocity field of the path $\mathbf{fl}(t)$. The set of all flow lines of a vector–field $\mathbf{v}$ comprises its flow φ_t that is (technically, see e.g., [49; 52]) a one–parameter Lie group of diffeomorphisms (smooth bijective functions) generated by a vector-field $\mathbf{v}$ on M, such that

$$\varphi_t \circ \varphi_s = \varphi_{t+s}, \qquad \varphi_0 = \text{identity}, \qquad \text{which gives:} \quad \gamma(t) = \varphi_t(\gamma(0)).$$

Analytically, a vector-field $\mathbf{v}$ is defined as a set of autonomous ODEs. Its solution gives the flow φ_t, consisting of integral curves (or, flow lines) $\mathbf{fl}(t)$ of the vector–field, such that all the vectors from the vector-field are tangent to integral curves at different representative points $p \in M$. In this way, through every representative point $p \in M$ passes both a curve from the flow and its tangent vector from the vector-field. Geometrically, vector-field is defined as a cross-section of the tangent bundle TM of the manifold M.

In general, given an nD frame $\{\partial_i\} \equiv \{\partial/\partial x^i\}$ on a smooth n–manifold M (that is, a basis of tangent vectors in a local coordinate chart $x^i = (x^1, ..., x^n) \subset M$), we can define any vector-field $\mathbf{v}$ on M by its components $v^i = v^i(t)$ as

$$\mathbf{v} = v^i \partial_i = v^i \frac{\partial}{\partial x^i} = v^1 \frac{\partial}{\partial x^1} + ... + v^n \frac{\partial}{\partial x^n}.$$

Thus, a vector-field $\mathbf{v} \in \mathcal{X}(M)$ (where $\mathcal{X}(M)$ is the set of all smooth vector-fields on M) is actually a differential operator that can be used to differentiate any smooth scalar function $f = f(x^1, ..., x^n)$ on M, as a *directional derivative* of f in the direction of $\mathbf{v}$. This is denoted simply $\mathbf{v}f$, such that

$$\mathbf{v}f = v^i \partial_i f = v^i \frac{\partial f}{\partial x^i} = v^1 \frac{\partial f}{\partial x^1} + ... + v^n \frac{\partial f}{\partial x^n}.$$

In particular, if $\mathbf{v} = \dot{\gamma}(t)$ is a velocity vector-field of a space curve $\gamma(t) = (x^1(t), ..., x^n(t))$, defined by its components $v^i = \dot{x}^i(t)$, directional derivative of $f(x^i)$ in the direction of $\mathbf{v}$ becomes

$$\mathbf{v}f = \dot{x}^i \partial_i f = \frac{dx^i}{dt} \frac{\partial f}{\partial x^i} = \frac{df}{dt} = \dot{f},$$

which is a rate-of-change of f along the curve $\gamma(t)$ at a point $x^i(t)$.

Given two vector-fields, $\mathbf{u} = u^i \partial_i, \mathbf{v} = v^i \partial_i \in \mathcal{X}(M)$, their Lie bracket (or, commutator) is another vector-field $[\mathbf{u}, \mathbf{v}] \in \mathcal{X}(M)$, defined by

$$[\mathbf{u}, \mathbf{v}] = \mathbf{u}\mathbf{v} - \mathbf{v}\mathbf{u} = u^i \partial_i v^j \partial_j - v^j \partial_j u^i \partial_i,$$

which, applied to any smooth function f on M, gives

$$[\mathbf{u}, \mathbf{v}](f) = \mathbf{u}\big(\mathbf{v}(f)\big) - \mathbf{v}\big(\mathbf{u}(f)\big).$$

The Lie bracket measures the failure of 'mixed directional derivatives' to commute. Clearly, mixed partial derivatives *do* commute, $[\partial_i, \partial_j] = 0$, while in general it is *not* the case, $[\mathbf{u},\mathbf{v}] \neq 0$. In addition, suppose that $\mathbf{u}$ generates the flow φ_t and $\mathbf{v}$ generates the flow φ_s. Then, for any smooth function f on M, we have at any point p on M,

$$[\mathbf{u},\mathbf{v}](f)(p) = \frac{\partial^2}{\partial t \partial s}\big(f(\varphi_s(\varphi_t(p))) - f(\varphi_t(\varphi_s(p)))\big),$$

which means that in $f(\varphi_s(\varphi_t(p)))$ we are starting at p, flowing along $\mathbf{v}$ a little bit, then along $\mathbf{u}$ a little bit, and then evaluating f, while in $f(\varphi_t(\varphi_s(p)))$ we are flowing first along $\mathbf{u}$ and then $\mathbf{v}$. Therefore, the Lie bracket infinitesimally measures how these flows fail to commute.

The Lie bracket satisfies the following three properties (for any three vector-fields $\mathbf{u},\mathbf{v},\mathbf{w} \in M$ and two constants a, b – thus forming a Lie algebra on the crowd manifold M):

i. $[\mathbf{u},\mathbf{v}] = -[\mathbf{v},\mathbf{u}]$ skew-symmetry;

ii. $[\mathbf{u}, a\mathbf{v} + b\mathbf{w}] = a[\mathbf{u},\mathbf{v}] + b[\mathbf{u},\mathbf{w}]$ – bilinearity; and

iii. $[\mathbf{u},[\mathbf{v},\mathbf{w}]] + [\mathbf{v},[\mathbf{w},\mathbf{u}]] + [\mathbf{w},[\mathbf{u},\mathbf{v}]]$ – Jacobi identity.

A new set of vector-fields on M can be generated by repeated Lie brackets of $\mathbf{u}, \mathbf{v}, \mathbf{w} \in M$.

The Lie bracket is a standard tool in geometric nonlinear control theory (see, e.g. [49; 52]). Its action on vector-fields can be best visualized using the popular car parking example, in which the driver has two different vector–field transformations at his disposal. They can turn the steering wheel, or they can drive the car forward or backward. Here, we specify the state of a car by four coordinates: the (x, y) coordinates of the center of the rear axle, the direction θ of the car, and the angle ϕ between the front wheels and the direction of the car. l is the constant length of the car. Therefore, the 4D configuration manifold of a car is a set $M \equiv SO(2) \times \mathbb{R}^2$, coordinated by $\mathbf{x} \equiv \{x, y, \theta, \phi\}$, which is slightly more complicated than the individual crowd agent's 3D configuration manifold $SE(2) \equiv SO(2) \times \mathbb{R}$, coordinated by $\mathbf{x} = \{x, y, \theta\}$. The driftless car kinematics can be defined as a vector ODE:

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x})c_1 + \mathbf{v}(\mathbf{x})c_2, \tag{26}$$

with two vector–fields, $\mathbf{u},\mathbf{v} \in \mathcal{X}(M)$, and two scalar control inputs, c_1 and c_2. The infinitesimal car–parking transformations will be the following vector–fields

$$\mathbf{u}(\mathbf{x}) \equiv \mathrm{DRIVE} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} + \frac{\tan\phi}{l}\frac{\partial}{\partial\theta} \equiv \begin{pmatrix} \cos\theta \\ \sin\theta \\ \dfrac{1}{l}\tan\phi \\ 0 \end{pmatrix},$$

$$\text{and} \qquad \mathbf{v}(\mathbf{x}) \equiv \mathrm{STEER} = \frac{\partial}{\partial\phi} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The car kinematics (26) therefore expands into a matrix ODE:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \text{DRIVE} \cdot c_1 + \text{STEER} \cdot c_2 \equiv \begin{pmatrix} \cos\theta \\ \sin\theta \\ \dfrac{1}{l}\tan\phi \\ 0 \end{pmatrix} \cdot c_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot c_2.$$

However, STEER and DRIVE do not commute (otherwise we could do all your steering at home before driving of on a trip). Their combination is given by the Lie bracket

$$[\mathbf{v},\mathbf{u}] \equiv [\text{STEER},\text{DRIVE}] = \frac{1}{l\cos^2\phi}\frac{\partial}{\partial\theta} \equiv \text{WRIGGLE}.$$

The operation $[\mathbf{v},\mathbf{u}] \equiv \text{WRIGGLE} \equiv [\text{STEER},\text{DRIVE}]$ is the infinitesimal version of the sequence of transformations: steer, drive, steer back, and drive back, i.e.,

$$\{\text{STEER},\text{DRIVE},\text{STEER}^{-1},\text{DRIVE}^{-1}\}.$$

Now, WRIGGLE can get us out of some parking spaces, but not tight ones: we may not have enough room to WRIGGLE out. The usual tight parking space restricts the DRIVE transformation, but not STEER. A truly tight parking space restricts STEER as well by putting your front wheels against the curb.
Fortunately, there is still another commutator available:

$$[-\mathbf{u},[\mathbf{v},\mathbf{u}]] \equiv [\text{DRIVE},[\text{STEER},\text{DRIVE}]] = [[\mathbf{u},\mathbf{v}],\mathbf{u}] \equiv$$

$$[\text{DRIVE},\text{WRIGGLE}] = \frac{1}{l\cos^2\phi}\left(\sin\theta\frac{\partial}{\partial x} - \cos\theta\frac{\partial}{\partial y}\right) \equiv \text{SLIDE}$$

The operation $[[\mathbf{u},\mathbf{v}],\mathbf{u}] \equiv \text{SLIDE} \equiv [\text{DRIVE},\text{WRIGGLE}]$ is a displacement at right angles to the car, and can get us out of any parking place. We just need to remember to steer, drive, steer back, drive some more, steer, drive back, steer back, and drive back:

$$\{\text{STEER},\text{DRIVE},\text{STEER}^{-1},\text{DRIVE},\text{STEER},\text{DRIVE}^{-1},\text{STEER}^{-1},\text{DRIVE}^{-1}\}.$$

We have to reverse steer in the middle of the parking place. This is not intuitive, and no doubt is part of a common problem with parallel parking.
Thus, from only two controls, c_1 and c_2, we can form the vector–fields DRIVE $\equiv \mathbf{u}$, STEER $\equiv \mathbf{v}$, WRIGGLE $\equiv [\mathbf{v},\mathbf{u}]$, and SLIDE $\equiv [[\mathbf{u},\mathbf{v}],\mathbf{u}]$, allowing us to move anywhere in the car configuration manifold $M \equiv SO(2) \times \mathbb{R}^2$. All above computations are straightforward in *Mathematica*[TM8] if we define the following three symbolic functions:

1.   Jacobian matrix: JacMat[v_List, x_List] := Outer[D, v, x];
2.   Lie bracket: LieBrc[u_List, v_List, x_List] := JacMat[v, x] . u - JacMat[u, x] . v;
3.   Repeated Lie bracket: Adj[u_List, v_List, x_List, k_] :=
                              If[k == 0, v, LieBrc[u, Adj[u, v, x, k - 1], x]];

---

[8] The above computations could instead be done in other available packages, such as Maple, by suitably translating the provided example code.

In case of the human crowd, we have a slightly simpler, but multiplied problem, i.e., superposition of n individual agents' motions. So, we can define the dissipative crowd kinematics as a system of n vector ODEs:

$$\dot{\mathbf{x}}^k = \mathbf{u}^k(\mathbf{x})c_1^k + \mathbf{v}^k(\mathbf{x})c_2^k, \qquad \text{where} \tag{27}$$

$$\mathbf{u}^k(\mathbf{x}) \equiv \mathrm{DRIVE}^k = \cos^k\theta\frac{\partial}{\partial x^k} + \sin^k\theta\frac{\partial}{\partial y^k} \equiv \begin{pmatrix} \cos^k\theta \\ \sin^k\theta \\ 0 \end{pmatrix}, \qquad \text{and}$$

$$\mathbf{v}^k(\mathbf{x}) \equiv \mathrm{STEER}^k = \frac{\partial}{\partial \theta^k} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{while } c_1^k \text{ and } c_2^k \text{ are crowd controls.}$$

Thus, the crowd kinematics (27) expands into the matrix ODE:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \mathrm{DRIVE}^k \cdot c_1^k + \mathrm{STEER}^k \cdot c_2^k \equiv \begin{pmatrix} \cos^k\theta \\ \sin^k\theta \\ 0 \end{pmatrix} \cdot c_1^k + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot c_2^k. \tag{28}$$

A 3D simulation of random, dissipative crowd kinematics (27)–(28) of 120 penguin-like $SE(2)$–robots, developed in C++/DirX is presented in Figure 2.
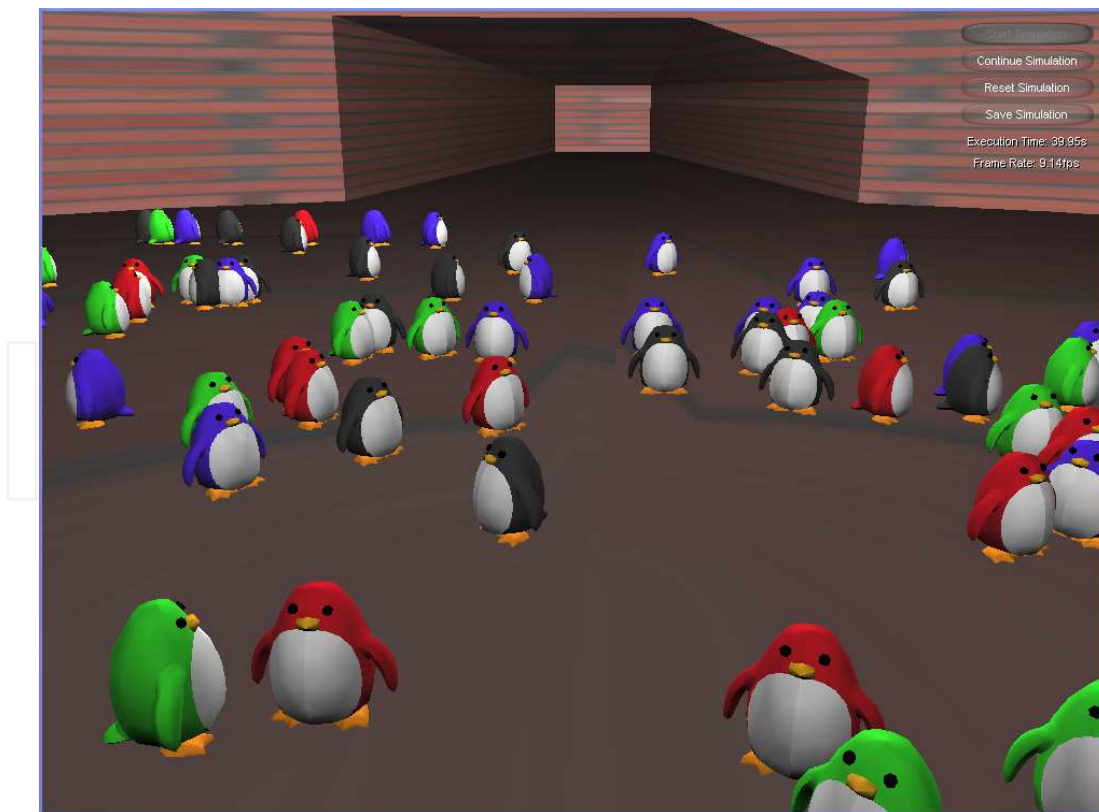

Fig. 2. Driving and steering random $SE(2)$–dynamics of 120 penguin-like robots (with embedded collision-detection). Compare with [2].

The dissipative crowd kinematics (27)–(28) obeys the set of n-tuple integral rules of motion that are similar (though slightly simpler) to the above rules of the car kinematics, including the following derived vector-fields:

$$\text{WRIGGLE}^k \equiv [\text{STEER}^k, \text{DRIVE}^k] \equiv [\mathbf{v}^k, \mathbf{u}^k] \text{ and}$$

$$\text{SLIDE}^k \equiv [\text{DRIVE}^k, \text{WRIGGLE}^k] \equiv [[\mathbf{u}^k, \mathbf{v}^k], \mathbf{u}^k]$$

Thus, controlled by the two vector controls c_1^k and c_2^k, the crowd can form the vector–fields: DRIVE $\equiv \mathbf{u}^k$, STEER $\equiv \mathbf{v}^k$, WRIGGLE $\equiv [\mathbf{v}^k, \mathbf{u}^k]$, and SLIDE $\equiv [[\mathbf{u}^k, \mathbf{v}^k], \mathbf{u}^k]$, allowing it to move anywhere within its configuration manifold M given by (21). Solution of the dissipative crowd kinematics (27)–(28) defines the dissipative crowd flow, ϕ_t^K.

Now, the general $\mathcal{CD}$–$\mathcal{CK}$ crowd behavior can be defined as a amalgamate flow (behavioral Lagrangian flow, ϕ_t^L, plus dissipative kinematic flow, ϕ_t^K) on the crowd manifold M defined by (21),

$$C_t = \phi_t^L + \phi_t^K : t \mapsto (M(t), g(t)),$$

which is a one-parameter family of homeomorphic (topologically equivalent) Riemannian manifolds[9] (M, $g = g_{ij}$), parameterized by a 'time' parameter t. That is, C_t can be used for

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[9] Proper differentiation of vector and tensor fields on a smooth Riemannian manifold (like the crowd $3n$–manifold M) is performed using the *Levi–Civita covariant derivative* (see, e.g., [49; 52]). Formally, let M be a Riemannian N–manifold with the tangent bundle TM and a local coordinate system $\{x^i\}_{i=1}^N$ defined in an open set $U \subset M$. The covariant derivative operator, $\nabla_X : C^\infty(TM) \to C^\infty(TM)$, is the unique linear map such that for any vector-fields X, Y, Z, constant c, and scalar function f the following properties are valid:

$$\nabla_{X+cY} = \nabla_X + c\nabla_Y, \qquad \nabla_X(Y + fZ) = \nabla_X Y + (Xf)Z + f\nabla_X Z, \qquad \nabla_X Y - \nabla_Y X = [X, Y],$$

where $[X, Y]$ is the Lie bracket of X and Y. In local coordinates, the metric g is defined for any orthonormal basis ($\partial_i = \partial/\partial x^i$) in $U \subset M$ by $g_{ij} = g(\partial_i, \partial_j) = \delta_{ij}$, $\partial_k g_{ij} = 0$. Then the affine *Levi–Civita connection* is defined on M by

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k, \quad \text{where} \quad \Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \text{ are the Christoffel symbols.}$$

Now, using the covariant derivative operator ∇_X we can define the *Riemann curvature* (3,1)–tensor $\mathfrak{Rm}$ by

$$\mathfrak{Rm}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

which measures the curvature of the manifold by expressing how noncommutative covariant differentiation is. The (3,1)–components R_{ijk}^l of $\mathfrak{Rm}$ are defined in $U \subset M$ by

$$\mathfrak{Rm}\left(\partial_i, \partial_j\right)\partial_k = R_{ijk}^l \partial_l, \quad \text{or} \quad R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{jk}^m \Gamma_{im}^l - \Gamma_{ik}^m \Gamma_{jm}^l.$$

Also, the Riemann (4,0)–tensor $R_{ijk}^l = g_{lm} R_{ijk}^m$ is defined as the g–based inner product on M,

$$R_{ijkl} = \left\langle \mathfrak{Rm}\left(\partial_i, \partial_j\right)\partial_k, \partial_l \right\rangle.$$

The first and second Bianchi identities for the Riemann (4,0)–tensor R_{ijkl} hold,

describing smooth deformations of the crowd manifold M over time. The manifold family $(M(t), g(t))$ at time t determines the manifold family $(M(t + dt), g(t + dt))$ at an infinitesimal time $t + dt$ into the future, according to some presecribed geometric flow, like the celebrated *Ricci flow* [30; 31; 32; 33] (that was an instrument for a proof of a 100–year old Poincaré conjecture),

$$\partial_t g_{ij}(t) = -2R_{ij}(t), \tag{29}$$

where R_{ij} is the Ricci curvature tensor (see Appendix) of the crowd manifold M and $\partial_t g(t)$ is defined as

$$\partial_t g(t) \equiv \frac{d}{dt} g(t) := \lim_{dt \to 0} \frac{g(t + dt) - g(t)}{dt}. \tag{30}$$

3.4 Aggregate behavioral–compositional dynamics ($\mathcal{AD}$)

To formally develop the meso-level aggregate behavioral–compositional dynamics ($\mathcal{AD}$), we start with the crowd path integral (24), which can be redefined if we Wick–rotate the time variable t to imaginary values, $t \to \tau = it$, thereby transforming the Lorentzian path integral in real time into the Euclidean path integral in imaginary time. Furthermore, if we rectify the time axis back to the real line, we get the adaptive SFT–partition function as our proposed $\mathcal{AD}$ –model:

$$\langle \text{Physical Action} | \text{Mental Preparation} \rangle_{\text{AD}} = \int_{\text{CD}} \mathcal{D}[w, x, y] e^{-A[x,y;t_i,t_j]}. \tag{31}$$

The adaptive $\mathcal{AD}$ –transition amplitude $\langle \text{Physical Action} | \text{Mental Preparation} \rangle_{\text{AD}}$ as defined by the SFT-partition function (31) is a general model of a *Markov stochastic process*. Recall that Markov process is a random process characterized by a *lack of memory*, i.e., the statistical properties of the immediate future are uniquely determined by the present, regardless of the past (see, e.g. [23; 49]). The N–dimensional Markov process can be defined by the Ito stochastic differential equation,

$$dx_i(t) = A_i[x^i(t), t]dt + B_{ij}[x^i(t), t]dW^j(t), \tag{32}$$

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0, \qquad \nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0,$$

while the twice contracted second Bianchi identity reads: $2\nabla_j R_{ij} = \nabla_i R$.

The $(0,2)$ *Ricci tensor* $\mathfrak{Rc}$ is the trace of the Riemann $(3,1)$ –tensor $\mathfrak{Rm}$,

$$\mathfrak{Rc}(Y, Z) + \text{tr}(X \to \mathfrak{Rm}(X, Y)Z), \quad \text{so that} \quad \mathfrak{Rc}(X, Y) = g(\mathfrak{Rm}(\partial_i, X)\partial_i, Y),$$

Its components $R_{jk} = \mathfrak{Rc}(\partial_j, \partial_k)$ are given in $U \subset M$ by the contraction

$$R_{jk} = R^i_{ijk}, \quad \text{or} \quad R_{jk} = \partial_i \Gamma^i_{jk} - \partial_k \Gamma^i_{ji} + \Gamma^i_{mi} \Gamma^m_{jk} - \Gamma^i_{mk} \Gamma^m_{ji}.$$

Finally, the scalar curvature R is the trace of the Ricci tensor $\mathfrak{Rc}$, given in $U \subset M$ by: $R = g^{ij} R_{ij}$.

$$x^i(0) = x_{i0}, \qquad (i, j = 1, \ldots, N) \tag{33}$$

or corresponding *Ito stochastic integral equation*

$$x^i(t) = x^i(0) + \int_0^t ds\, A_i[x^i(s), s] + \int_0^t dW^j(s) B_{ij}[x^i(s), s], \tag{34}$$

in which $x^i(t)$ is the variable of interest, the vector $A_i[x(t), t]$ denotes deterministic *drift*, the matrix $B_{ij}[x(t), t]$ represents continuous stochastic *diffusion fluctuations*, and $W^j(t)$ is an N–variable *Wiener process* (i.e., generalized Brownian motion [23]) and

$$dW^j(t) = W^j(t + dt) - W^j(t).$$

The two Ito equations (33)–(34) are equivalent to the general *Chapman–Kolmogorov probability equation* (see equation (35) below). There are three well known special cases of the Chapman– Kolmogorov equation (see [23]):

1. When both $B_{ij}[x(t), t]$ and $W(t)$ are zero, i.e., in the case of pure deterministic motion, it reduces to the *Liouville equation*

$$\partial_t P(x', t' \mid x'', t'') = -\sum_i \frac{\partial}{\partial x^i} \{ A_i[x(t), t] P(x', t' \mid x'', t'') \}.$$

2. When only $W(t)$ is zero, it reduces to the *Fokker–Planck equation*

$$\partial_t P(x', t' \mid x'', t'') = -\sum_i \frac{\partial}{\partial x^i} \{ A_i[x(t), t] P(x', t' \mid x'', t'') \}$$

$$+ \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial x^i \partial x^j} \{ B_{ij}[x(t), t] P(x', t' \mid x'', t'') \}.$$

3. When both $A_i[x(t), t]$ and $B_{ij}[x(t), t)$ are zero, i.e., the state–space consists of integers only, it reduces to the *Master equation* of discontinuous jumps

$$\partial_t P(x', t' \mid x'', t'') = \int dx\, W(x' \mid x'', t) P(x', t' \mid x'', t'') - \int dx\, W(x'' \mid x', t) P(x', t' \mid x'', t'').$$

The *Markov assumption* can now be formulated in terms of the conditional probabilities $P(x^i, t_i)$: if the times t_i increase from right to left, the conditional probability is determined entirely by the knowledge of the most recent condition. Markov process is generated by a set of conditional probabilities whose probability–density $P = P(x', t' \mid x'', t'')$ evolution obeys the general *Chapman–Kolmogorov integro–differential equation*

$$\partial_t P = -\sum_i \frac{\partial}{\partial x^i} \{ A_i[x(t), t] P \} + \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial x^i \partial x^j} \{ B_{ij}[x(t), t] P \}$$

$$+ \int dx \{ W(x' \mid x'', t) P - W(x'' \mid x', t) P \}$$

including *deterministic drift*, *diffusion fluctuations* and *discontinuous jumps* (given respectively in the first, second and third terms on the r.h.s.). This general Chapman–Kolmogorov integro-differential equation (35), with its conditional probability density evolution, $P = P(x', t' \mid x'', t'')$, is represented by our SFT-partition function (31).

Furthermore, discretization of the adaptive SFT–partition function (31) gives the standard *partition function* (see Appendix)

$$Z = \sum_j e^{-w_j E^j / T},$$

(35)

where E^j is the motion energy eigenvalue (reflecting each possible motivational energetic state), T is the temperature–like environmental control parameter, and the sum runs over all ID energy eigenstates (labelled by the index j). From (35), we can calculate the *transition entropy*, as $S = k_B \ln Z$ (see the next section).

4. Entropy, chaos and phase transitions in the crowd manifold

Recall that nonequilibrium phase transitions [25; 26; 27; 28; 29] are phenomena which bring about qualitative physical changes at the macroscopic level in presence of the same microscopic forces acting among the constituents of a system. In this section we extend the $\mathcal{CD}$ formalism to incorporate both algorithmic and geometrical entropy as well as dynamical chaos [50; 58; 60] between the entropy–growing phase of Mental Preparation and the entropy– conserving phase of Physical Action, together with the associated topological phase transitions.

4.1 Algorithmic entropy

The Boltzmann and Shannon (hence also Gibbs entropy, which is Shannon entropy scaled by $k \ln 2$, where k is the Bolzmann constant) entropy definitions involve the notion of *ensembles*. Membership of microscopic states in ensembles defines the probability density function that underpins the entropy function; the result is that the entropy of a definite and completely known microscopic state is precisely zero. Bolzmann entropy defines the probabilistic model of the system by effectively discarding part of the information about the system, while the Shannon entropy is concerned with measuring the ignorance of the observer – the amount of missing information – about the system.

Zurek proposed a new physical entropy measure that can be applied to individual microscopic system states and does not use the ensemble structure. This is based on the notion of a fixed individually random object provided by Algorithmic Information Theory and Kolmogorov Complexity: put simply, the randomness $K(x)$ of a binary string x is the length in terms of number of bits of the smallest program p on a universal computer that can produce x.

While this is the basic idea, there are some important technical details involved with this definition. The randomness definition uses the prefix complexity $K(.)$ rather than the older Kolmogorov complexity measure $C(.)$: the prefix complexity $K(x|y)$ of x given y is the Kolmogorov complexity $C_{\phi_u}(x|y) = \min\{p | x = \phi_u(\langle y, p \rangle)\}$ (with the convention that $C_{\phi_u}(x|y) = \infty$ if there is no such p) that is taken with respect to a reference universal partial recursive function ϕ_u that is a universal prefix function. Then the prefix complexity $K(x)$ of x is just $K(x|\varepsilon)$ where ε is the empty string. A partial recursive prefix function $\phi : M \to \mathbb{N}$ is a partial recursive function such that if $\phi(p) < \infty$ and $\phi(q) < \infty$ then p is not a proper prefix of q: that is, we restrict the complexity definition to a set of strings (which are descriptions of effective procedures) such that none is a proper prefix of any other. In this way, all effective procedure descriptions are *self-delimiting*: the total length of the description is given within

the description itself. A universal prefix function ϕ_u is a prefix function such that $\forall n \in \mathbb{N}$ $\phi_u(\langle y, \langle n, p \rangle \rangle = \phi_n(\langle y, p \rangle$, where ϕ_n is numbered n according to some Godel numbering of the partial recursive functions; that is, a universal prefix function is a partial recursive function that simulates any partial recursive function. Here, $\langle x, y \rangle$ stands for a total recusive one-one mapping from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$, $\langle x_1, x_2, \ldots, x_n \rangle = \langle x_1, \langle x_2, \ldots, x_n \rangle \rangle, \mathbb{N}$ is the set of natural numbers, and $M = \{0,1\}^*$ is the set of all binary strings.

This notion of entropy circumvents the use of probability to give a concept of entropy that can be applied to a fully specified macroscopic state: the algorithmic randomness of the state is the length of the shortest possible effective description of it. To illustrate, suppose for the moment that the set of microscopic states is countably infinite, with each state identified with some natural number. It is known that the discrete version of the Gibbs entropy (and hence of Shannon's entropy) and the algorithmic entropy are asymptotically consistent under mild assumptions. Consider a system with a countably infinite set of microscopic states X supporting a probability density function $P(.)$ so that $P(x)$ is the probability that the system is in microscopic state $x \in X$. Then the Gibbs entropy is $S_G(P) = -(k \ln 2) \sum_{x \in X} P(x) \log P(x)$

(which is Shannon's information-theoretic entropy $H(P)$ scaled by $k \ln 2$). Supposing that $P(.)$ is recursive, then $S_G(P) = (k \ln 2) \sum_{x \in X} P(x) K(x) + C$, where C_ϕ is a constant depending only on

the choice of the reference universal prefix function ϕ. Hence, as a measure of entropy, the function $K(.)$ manifests the same kind of behavior as Shannon's and Gibbs entropy measures.

Zurek's proposal was of a new physical entropy measure that includes contributions from both the randomness of a state and ignorance about it. Assume now that we have determined the macroscopic parameters of the system, and encode this as a string - which can always be converted into an equivalent binary string, which is just a natural number under a standard encoding. It is standard to denote the binary string and its corresponding natural number interchangeably; here let x be the encoded macroscopic parameters. Zurek's definition of *algorithmic entropy* of the macroscopic state is then $K(x) + H_x$, where $H_x = S_B(x)/(k \ln 2)$, where $S_B(x)$ is the Bolzmann entropy of the system constrained by x and k is Bolzmann's constant; the physical version of the algorithmic entropy is therefore defined as $S_A(x) = (k \ln 2)(K(x) + H_x)$. Here H_x represents the level of ignorance about the microscopic state, given the parameter set x; it can decrease towards zero as knowledge about the state of the system increases, at which point the algorithmic entropy reduces to the Bolzmann entropy.

4.2 Ricci flow and Perelman entropy–action on the crowd manifold

Recall that the inertial metric crowd flow, $C_t : t \rightarrow (M(t), g(t))$ on the crowd $3n$–mani-fold (21) is a one-parameter family of homeomorphic Riemannian manifolds (M, g), evolving by the Ricci flow (29)–(30).

Now, given a smooth scalar function $u : M \rightarrow \mathbb{R}$ on the Riemannian crowd $3n$–manifold M, its Laplacian operator Δ is locally defined as

$$\Delta u = g^{ij} \nabla_i \nabla_j u,$$

where ∇_i is the covariant derivative (or, Levi–Civita connection, see Appendix). We say that a smooth function $u : M \times [0,T) \rightarrow \mathbb{R}$, where $T \in (0,\infty]$, is a solution to the heat equation (see Appendix, eq. (60)) on M if

$$\partial_t u = \Delta u. \tag{36}$$

One of the most important properties satisfied by the heat equation is the maximum principle, which says that for any smooth solution to the heat equation, whatever point-wise bounds hold at $t = 0$ also hold for $t > 0$ [13]. This property exhibits the smoothing behavior of the heat diffusion (36) on M.

Closely related to the heat diffusion (36) is the (the Fields medal winning) Perelman entropy–action functional, which is on a $3n$–manifold M with a Riemannian metric g_{ij} and a (temperature-like) scalar function f given by [75]

$$\mathcal{E} = \int_M (R + |\nabla f|^2) e^{-f} d\mu \tag{37}$$

where R is the scalar Riemann curvature on M, while $d\mu$ is the volume $3n$–form on M, defined as

$$d\mu = \sqrt{\det(g_{ij})}\, dx^1 \wedge dx^2 \wedge ... \wedge dx^{3n}. \tag{38}$$

During the Ricci flow (29)–(30) on the crowd manifold (21), that is, during the inertial metric crowd flow, $C_t : t \to (M(t), g(t))$, the Perelman entropy functional (37) evolves as

$$\partial_t \mathcal{E} = 2\int |R_{ij} + \nabla_i \nabla_j f|^2 \, e^{-f} d\mu. \tag{39}$$

Now, the *crowd breathers* are solitonic crowd behaviors, which could be given by localized periodic solutions of some nonlinear soliton PDEs, including the exactly solvable sine–Gordon equation and the focusing nonlinear Schrödinger equation. In particular, the time–dependent crowd inertial metric $g_{ij}(t)$, evolving by the Ricci flow $g(t)$ given by (29)–(30) on the crowd $3n$–manifold M is the *Ricci crowd breather*, if for some $t_1 < t_2$ and $\alpha > 0$ the metrics $\alpha g_{ij}(t_1)$ and $g_{ij}(t_2)$ differ only by a diffeomorphism; the cases $\alpha = 1$, $\alpha < 1$, $\alpha > 1$ correspond to steady, shrinking and expanding crowd breathers, respectively. Trivial crowd breathers, for which the metrics $g_{ij}(t_1)$ and $g_{ij}(t_2)$ on M differ only by diffeomorphism and scaling for each pair of t_1 and t_2, are the *crowd Ricci solitons*. Thus, if we consider the Ricci flow (29)–(30) as a biodynamical system on the space of Riemannian metrics modulo diffeomorphism and scaling, then crowd breathers and solitons correspond to periodic orbits and fixed points respectively. At each time the Ricci soliton metric satisfies on M an equation of the form [75]

$$R_{ij} + c g_{ij} + \nabla_i b_j + \nabla_j b_i = 0,$$

where c is a number and b_i is a 1–form; in particular, when $b_i = \dfrac{1}{2} \nabla_i a$ for some function a on M, we get a gradient Ricci soliton.

Define $\lambda(g_{ij}) = \inf \mathcal{E}\,(g_{ij}, f)$, where infimum is taken over all smooth f, satisfying

$$\int_M e^{-f} d\mu = 1. \tag{40}$$

$\lambda(g_{ij})$ is the lowest eigenvalue of the operator $-4\Delta + R$. Then the entropy evolution formula (39) implies that $\lambda(g_{ij}(t))$ is non-decreasing in t, and moreover, if $\lambda(t_1) = \lambda(t_2)$, then for $t \in [t_1, t_2]$ we have $R_{ij} + \nabla_i \nabla_j f = 0$ for f which minimizes $\mathcal{E}$ on M [75]. Therefore, a steady breather on M is necessarily a steady soliton.

If we define the conjugate heat operator on M as

$$\Box^* = -\partial / \partial t - \Delta + R$$

then we have the conjugate heat equation: $\Box^* u = 0$.

The entropy functional (37) is nondecreasing under the coupled Ricci–diffusion flow on M [56]

$$\partial_t g_{ij} = -2R_{ij}, \qquad \partial_t u = -\Delta u + \frac{R}{2} u - \frac{|\nabla u|^2}{u}, \tag{41}$$

where the second equation ensures $\int_M u^2 d\mu = 1$, to be preserved by the Ricci flow $g(t)$ on M.

If we define $u = e^{-\frac{f}{2}}$, then (41) is equivalent to f–evolution equation on M (the nonlinear backward heat equation),

$$\partial_t f = -\Delta f + |\nabla f|^2 - R,$$

which instead preserves (40). The coupled Ricci–diffusion flow (41) is the most general biodynamic model of the crowd reaction–diffusion processes on M. In a recent study [1] this general model has been implemented for modelling a generic perception–action cycle with applications to robot navigation in the form of a dynamical grid.

Perelman's functional $\mathcal{E}$ is analogous to negative thermodynamic entropy [75]. Recall (see Appendix) that thermodynamic partition function for a generic canonical ensemble at temperature β^{-1} is given by

$$Z = \int e^{-\beta E} d\omega(E), \tag{42}$$

where $\omega(E)$ is a 'density measure', which does not depend on β. From it, the average energy is given by $\langle E \rangle = -\partial_\beta \ln Z$, the entropy is $S = \beta \langle E \rangle + \ln Z$, and the fluctuation is $\sigma = \langle (E - \langle E \rangle)^2 \rangle = \partial_{\beta^2} \ln Z$.

If we now fix a closed $3n$–manifold M with a probability measure m and a metric $g_{ij}(\tau)$ that depends on the temperature τ, then according to equation

$$\partial_\tau g_{ij} = 2(R_{ij} + \nabla_i \nabla_j f),$$

the partition function (42) is given by

$$\ln Z = \int (-f + \frac{n}{2}) dm. \tag{43}$$

From (43) we get (see [75])

$$\langle E \rangle = -\tau^2 \int_M (R + |\nabla f|^2 - \frac{n}{2\tau}) dm, \qquad S = -\int_M (\tau(R + |\nabla f|^2) + f - n) dm,$$

$$\sigma = 2\tau^4 \int_M |R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}|^2 dm, \qquad \text{where } dm = u\, dV, \; u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}.$$

From the above formulas, we see that the fluctuation σ is nonnegative; it vanishes only on a gradient shrinking soliton. $\langle E \rangle$ is nonnegative as well, whenever the flow exists for all sufficiently small $\tau > 0$. Furthermore, if the heat function u: (a) tends to a δ–function as $\tau \to 0$, or (b) is a limit of a sequence of partial heat functions u_i, such that each u_i tends to a δ–function as $\tau \to \tau_i > 0$, and $\tau_i \to 0$, then the entropy S is also nonnegative. In case (a), all the quantities $\langle E \rangle$, S, σ tend to zero as $\tau \to 0$, while in case (b), which may be interesting if $g_{ij}(\tau)$ becomes singular at $\tau = 0$, the entropy S may tend to a positive limit.

4.3 Chaotic inter-phase in crowd dynamics induced by its Riemannian geometry change

Recall that $\mathcal{CD}$ transition map (9) is defined by the chaotic crowd phase–transition amplitude

$$\left\langle \text{PHYS.} \overset{\partial_t S=0}{\text{ACTION}} \middle| CHAOS \middle| \overset{\partial_t S>0}{\text{MENTAL PREP.}} \right\rangle := \int_M \mathcal{D}[x] e^{iA[x]},$$

where we expect the inter-phase chaotic behavior (see [53]). To show that this chaotic interphase is caused by the change in Riemannian geometry of the crowd $3n$–manifold M, we will first simplify the $\mathcal{CD}$ action functional (22) as

$$A[x] = \frac{1}{2} \int_{t_{ini}}^{t_{fin}} [g_{ij} \dot{x}^i \dot{x}^j - V(x, \dot{x})] dt, \tag{44}$$

with the associated standard Hamiltonian, corresponding to the amalgamate version of (18),

$$H(p, x) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + V(x, \dot{x}), \tag{45}$$

where p_i are the SE(2)–momenta, canonically conjugate to the individual agents' SE(2)–coordinates x_i, ($i = 1, ...,3n$). Biodynamics of systems with action (44) and Hamiltonian (45) are given by the set of *geodesic equations* [49; 52]

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \tag{46}$$

where Γ^i_{jk} are the Christoffel symbols of the affine Levi–Civita connection of the Riemannian $\mathcal{CD}$ manifold M (see Appendix). In this geometrical framework, the instability of the trajectories is the instability of the geodesics, and it is completely determined by the curvature properties of the $\mathcal{CD}$ manifold M according to the Jacobi equation of geodesic deviation [49; 52]

$$\frac{D^2 J^i}{ds^2} + R^i_{jkm} \frac{dx^j}{ds} J^k \frac{dx^m}{ds} = 0, \tag{47}$$

whose solution J, usually called Jacobi variation field, locally measures the distance between nearby geodesics; D/ds stands for the covariant derivative along a geodesic and R^i_{jkm} are the components of the Riemann curvature tensor of the $\mathcal{CD}$ manifold M.

The relevant part of the Jacobi equation (47) is given by the tangent dynamics equation [12; 15]

$$\ddot{J}^i + R^i_{0k0} J^k = 0, \qquad (i, k = 1, \ldots, 3n), \tag{48}$$

where the only non-vanishing components of the curvature tensor of the $\mathcal{CD}$ manifold M are

$$R^i_{0k0} = \partial^2 V / \partial x^i \partial x^k. \tag{49}$$

The tangent dynamics equation (48) can be used to define Lyapunov exponents in dynamical systems given by the Riemannian action (44) and Hamiltonian (45), using the formula [14]

$$\lambda_1 = \lim_{t \to \infty} 1 / 2t \log(M^N_{i=1}[J^2_i(t) + \dot{J}^2_i(t)] / M^N_{i=1}[J^2_i(0) + \dot{J}^2_i(0)]). \tag{50}$$

Lyapunov exponents measure the strength of dynamical chaos in the crowd behavior. The sum of positive Lyapunov exponents defines the *Kolmogorov–Sinai entropy* (see Appendix).

4.4 Crowd nonequilibrium phase transitions induced by manifold topology change

Now, to relate these results to topological phase transitions within the $\mathcal{CD}$ manifold M given by (21), recall that any two high–dimensional manifolds M_v and $M_{v'}$ have the same topology if they can be continuously and differentiably deformed into one another, that is if they are diffeomorphic. Thus by topology change the 'loss of diffeomorphicity' is meant [80]. In this respect, the so–called topological theorem [21] says that non–analyticity is the 'shadow' of a more fundamental phenomenon occurring in the system's configuration manifold (in our case the $\mathcal{CD}$ manifold): a topology change within the family of equipotential hypersurfaces

$$M_v = \{(x^1, \ldots, x^{3n}) \in \mathbb{R}^{3n} \mid V(x^1, \ldots, x^{3n}) = v\},$$

where V and x^i are the microscopic interaction potential and coordinates respectively. This topological approach to PTs stems from the numerical study of the dynamical counterpart of phase transitions, and precisely from the observation of discontinuous or cuspy patterns displayed by the largest Lyapunov exponent λ_1 at the transition energy [14]. Lyapunov exponents cannot be measured in laboratory experiments, at variance with thermodynamic observables, thus, being genuine dynamical observables they are only be estimated in numerical simulations of the microscopic dynamics. If there are critical points of V in configuration space, that is points $x_c = [\bar{x}_1, \ldots, \bar{x}_{3n}]$ such that $\nabla V(x)|_{x=x_c} = 0$, according to the Morse Lemma [40], in the neighborhood of any critical point x_c there always exists a coordinate system $x(t) = [x^1(t), \ldots, x^{3n}(t)]$ for which [14]

$$V(x) = V(x_c) - x^2_1 - \ldots - x^2_k + x^2_{k+1} + \ldots + x^2_{3n}, \tag{51}$$

where k is the index of the critical point, i.e., the number of negative eigenvalues of the Hessian of the potential energy V. In the neighborhood of a critical point of the $\mathcal{CD}$–manifold M, equation (51) yields the simplified form of (49), $\partial^2 V / \partial x^i \partial x^j = \pm \delta_{ij}$, giving j unstable directions that contribute to the exponential growth of the norm of the tangent vector J.

This means that the strength of dynamical chaos within the $\mathcal{CD}$–manifold M, measured by the largest Lyapunov exponent λ_1 given by (50), is affected by the existence of critical points x_c of the potential energy $V(x)$. However, as $V(x)$ is bounded below, it is a good Morse

function, with no vanishing eigenvalues of its Hessian matrix. According to Morse theory [40], the existence of critical points of V is associated with topology changes of the hypersurfaces $\{M_v\}_{v\in\mathbb{R}}$. The topology change of the $\{M_v\}_{v\in\mathbb{R}}$ at some v_c is a necessary condition for a phase transition to take place at the corresponding energy value [21]. The topology changes implied here are those described within the framework of Morse theory through 'attachment of handles' [40] to the $\mathcal{CD}$–manifold M.

In our path–integral language this means that suitable topology changes of equipotential submanifolds of the $\mathcal{CD}$–manifold M can entail thermodynamic–like phase transitions [25; 26; 27], according to the general formula:

$$\langle \text{phase out}|\text{phase in}\rangle := \int_{\text{top-ch}} \mathcal{D}[w\Phi]e^{iS[\Phi]}.$$

The statistical behavior of the crowd biodynamics system with the action functional (44) and the Hamiltonian (45) is encompassed, in the canonical ensemble, by its partition function, given by the Hamiltonian path integral [52]

$$Z_{3n} = \int_{\text{top-ch}} \mathcal{D}[p]\mathcal{D}[x]\exp\{i\int_t^{t'}[p_i\dot{x}^i - H(p,x)]d\tau\}, \tag{52}$$

where we have used the shorthand notation

$$\int_{\text{top-ch}} \mathcal{D}[p]\mathcal{D}[x] \equiv \iint \prod_\tau \frac{dx(\tau)dp(\tau)}{2\pi}.$$

The path integral (52) can be calculated as the partition function [20],

$$Z_{3n}(\beta) = \iint \prod_{i=1}^{3n} dp_i\, dx^i e^{-\beta H(p,x)} = \left(\frac{\pi}{\beta}\right)^{\frac{3n}{2}} \int \prod_{i=1}^{3n} dx^i e^{-\beta V(x)}$$

$$= \left(\frac{\pi}{\beta}\right)^{\frac{3n}{2}} \int_0^\infty dv e^{-\beta v} \int_{M_v} \frac{d\sigma}{\|\nabla V\|}, \tag{53}$$

where the last term is written using the so–called co–area formula [18], and v labels the equipotential hypersurfaces M_v of the $\mathcal{CD}$ manifold M,

$$M_v = \{(x^1,\dots,x^{3n}) \in \mathbb{R}^{3n} \mid V(x^1,\dots,x^{3n}) = v\}.$$

Equation (53) shows that the relevant statistical information is contained in the canonical configurational partition function

$$Z_{3n}^C = \int \prod dx^i\, V(x)e^{-\beta V(x)}.$$

Note that Z_{3n}^C is decomposed, in the last term of (53), into an infinite summation of geometric integrals,

$$\int_{M_v} d\sigma / \|\nabla V\|,$$

defined on the $\{M_v\}_{v \in \mathbb{R}}$. Once the microscopic interaction potential $V(x)$ is given, the configuration space of the system is automatically foliated into the family $\{M_v\}_{v \in \mathbb{R}}$ of these equipotential hypersurfaces. Now, from standard statistical mechanical arguments we know that, at any given value of the inverse temperature β, the larger the number $3n$, the closer to $M_v \equiv M_{u_\beta}$ are the microstates that significantly contribute to the averages, computed through $Z_{3n}(\beta)$, of thermodynamic observables. The hypersurface M_{u_β} is the one associated with

$$u_\beta = (Z_{3n}^C)^{-1} \int \prod dx^i V(x) e^{-\beta V(x)},$$

the average potential energy computed at a given β. Thus, at any β, if $3n$ is very large the effective support of the canonical measure shrinks very close to a single $M_v = M_{u_\beta}$. Hence, the basic origin of a phase transition lies in a suitable topology change of the $\{M_v\}$, occurring at some v_c [20]. This topology change induces the singular behavior of the thermodynamic observables at a phase transition. It is conjectured that the counterpart of a phase transition is a breaking of diffeomorphicity among the surfaces M_v, it is appropriate to choose a diffeomorphism invariant to probe if and how the topology of the M_v changes as a function of v. Fortunately, such a topological invariant exists, the Euler characteristic of the crowd manifold M, defined by [49; 52]

$$\chi(M) = \sum_{k=0}^{3n} (-1)^k b_k(M), \tag{54}$$

where the Betti numbers $b_k(M)$ are diffeomorphism invariants (b_k are the dimensions of the de Rham's cohomology groups $H^k(M;\mathbb{R})$; therefore the b_k are integers). This homological formula can be simplified by the use of the Gauss–Bonnet theorem, that relates $\mathcal{X}(M)$ with the total Gauss–Kronecker curvature K_G of the $\mathcal{CD}$–manifold M given by [52; 58]

$$\chi(M) = \int_M K_G \, d\mu, \qquad \text{where } d\mu \text{ is given by (38)}.$$

5. Conclusion

Our understanding of crowd dynamics is presently limited in important ways; in particular, the lack of a geometrically *predictive* theory of crowd behavior restricts the ability for authorities to intervene appropriately, or even to recognize when such intervention is needed. This is not merely an idle theoretical investigation: given increasing population sizes and thus increasing opportunity for the formation of large congregations of people, death and injury due to trampling and crushing – even within crowds that have not formed under common malicious intent – is a growing concern among police, military and emergency services. This paper represents a contribution towards the understanding of crowd behavior for the purpose of better informing decision–makers about the dangers and likely consequences of different intervention strategies in particular circumstances.

In this chapter, we have proposed an entropic geometrical model of crowd dynamics, with dissipative kinematics, that operates across macro–, micro– and meso–levels. This proposition is motivated by the need to explain the dynamics of crowds across these levels simultaneously: we contend that only by doing this can we expect to adequately

characterize the geometrical properties of crowds with respect to regimes of behavior and the changes of state that mark the boundaries between such regimes.

In pursuing this idea, we have set aside traditional assumptions with respect to the separation of mind and body. Furthermore, we have attempted to transcend the long–running debate between contagion and convergence theories of crowd behavior with our multi-layered approach: rather than representing a reduction of the whole into parts or the emergence of the whole from the parts, our approach is build on the supposition that the direction of logical implication can and does flow in both directions simultaneously. We refer to this third alternative, which effectively unifies the other two, as *behavioral composition.*

The most natural statistical descriptor is crowd entropy, which satisfies the extended second thermodynamics law applicable to open systems comprised of many components. Similarities between the configuration manifolds of individual (micro–level) and crowds (macro–level) motivate our claim that goal–directed movement operates under entropy conservation, while natural crowd dynamics operates under monotonically increasing entropy functions. Of particular interest is what happens between these distinct topological phases: the phase transition is marked by chaotic movement.

We contend that backdrop gives us a basis on which we can build a geometrically predictive model–theory of crowd behavior dynamics. This contrasts with previous approaches, which are explanatory only (explanation that is really narrative in nature). We propose an entropy formulation of crowd dynamics as a three step process involving individual and collective psycho-dynamics, and – crucially – non-equilibrium phase transitions whereby the forces operating at the microscopic level result in geometrical change at the macroscopic level. Here we have incorporated both geometrical and algorithmic notions of entropy as well as chaos in studying the topological phase transition between the entropy conservation of physical action and the entropy increase of mental preparation.

6. Appendix

6.1 Extended second law of thermodynamics

According to Boltzmann's interpretation of the second law of thermodynamics, there exists a function of the state variables, usually chosen to be the *physical entropy S* of the system that varies monotonically during the approach to the unique final state of thermodynamic equilibrium:

$$\partial_t S \geq 0 \quad \text{(for any isolated system).} \tag{55}$$

It is usually interpreted as a *tendency to increased disorder*, i.e., an irreversible trend to maximum disorder. The above interpretation of entropy and a second law is fairly obvious for systems of *weakly interacting particles*, to which the arguments developed by Boltzmann referred.

However, according to Prigogine [70], the above interpretation of entropy and a second law is fairly obvious *only* for systems of *weakly interacting particles*, to which the arguments developed by Boltzmann referred. On the other hand, for strongly interacting systems like the crowd, the above interpretation does not apply in a straightforward manner since, we know that for such systems there exists the possibility of evolving to more ordered states through the mechanism of *phase transitions*.

Let us now turn to nonisolated systems (like a human crowd), which exchange energy/matter with the environment. The entropy variation will now be the sum of two terms. One, entropy flux, d_eS, is due to these exchanges; the other, entropy production, d_iS, is due to the phenomena going on within the system. Thus the entropy variation is

$$\partial_t S = \frac{d_i S}{dt} + \frac{d_e S}{dt}. \tag{56}$$

For an isolated system $d_eS = 0$, and (56) together with (55) reduces to $dS = d_iS \geq 0$, the usual statement of the second law. But even if the system is nonisolated, d_iS will describe those (irreversible) processes that would still go on even in the absence of the flux term d_eS. We thus require the following extended form of the second law:

$$\partial_t S \geq 0 \qquad \text{(for any nonisolated system)}. \tag{57}$$

As long as d_iS is strictly positive, irreversible processes will go on continuously within the system.[10] Thus, $d_iS > 0$ is equivalent to the condition of dissipativity as time irreversibility. If, on the other hand, d_iS reduces to zero, the process will be reversible and will merely join neighboring states of equilibrium through a slow variation of the flux term d_eS.

From a computational perspective, we have a related *algorithmic entropy*. Suppose we have a universal machine capable of simulating any effective procedure (i.e., a universal machine that can compute any computable function). There are several models to choose from, classically we would use a Universal Turing Machine but for technical reasons we are more interested in Lambda–type Calculi or Combinatory Logics. Let us describe the system of interest through some encoding as a combinatorial structure (classically this would be a

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[10] Among the most common irreversible processes contributing to d_iS are chemical reactions, heat conduction, diffusion, viscous dissipation, and relaxation phenomena in electrically or magnetically polarized systems. For each of these phenomena two factors can be defined: an appropriate internal *flux*, J_i, denoting essentially its rate, and a driving *force*, X_i, related to the maintenance of the nonequilibrium constraint. A most remarkable feature is that d_iS becomes a *bilinear form* of J_i and X_i. The following table summarizes the fluxes and forces associated with some commonly observed irreversible phenomena (see [48; 70])

Phenomenon	Flux	Force	Rank
Heat conduction	Heat flux, $\mathbf{J}_{th}$	$grad(1/T)$	Vector
Diffusion	Mass flux, $\mathbf{J}_d$	$-[grad(\mu/T) - \mathbf{F}]$	Vector
Viscous flow	Pressure tensor, $\mathbf{P}$	$(1/T)\, grad\,\mathbf{v}$	Tensor
Chemical reaction	Rate of reaction, ω	Affinity of reaction	Scalar

In general, the fluxes J_k are very complicated functions of the forces X_i. A particularly simple situation arises when their relation is linear, then we have the celebrated *Onsager relations*,

$$J_i = L_{ik} X_k, \qquad (i, k = 1, ..., n) \tag{58}$$

in which L_{ik} denote the set of *phenomenological coefficients*. This is what happens near equilibrium where they are also symmetric, $L_{ik} = L_{ki}$. Note, however, that certain states far from equilibrium can still be characterized by a linear dependence of the form of (58) that occurs either accidentally or because of the presence of special types of regulatory processes.

binary string, but again I prefer for technical reasons Normal Forms with respect to alpha/beta/eta, weak, strong reduction, which are basically the Lambda–type Calculi and Combinatory Logic notions roughly akin to a "computational" step). In other words, we have states of our system now represented as sentences in some language. The entropy is simply the minimum effective procedure against our computational model that generates the description of the system state. This is a universal and absolute notion of compression of our data – the entropy is the strongest compression over all possible compression schemes, in effect. Now here is the 'magic': this minimum is absolute in the sense that it does not vary (except by a constant) with respect to our reference choice of machine.

6.2 Thermodynamic partition function

Recall that the partition function Z is a quantity that encodes the statistical properties of a system in thermodynamic equilibrium. It is a function of temperature and other parameters, such as the volume enclosing a gas. Other thermodynamic variables of the system, such as the total energy, free energy, entropy, and pressure, can be expressed in terms of the partition function or its derivatives.

A canonical ensemble is a statistical ensemble representing a probability distribution of microscopic states of the system. Its probability distribution is characterized by the proportion p_i of members of the ensemble which exhibit a measurable macroscopic state i, where the proportion of microscopic states for each macroscopic state i is given by the Boltzmann distribution,

$$p_i = \tfrac{1}{Z} e^{-E_i/(kT)} = e^{-(E_i - A)/(kT)},$$

where E_i is the energy of state i. It can be shown that this is the distribution which is most likely, if each system in the ensemble can exchange energy with a heat bath, or alternatively with a large number of similar systems. In other words, it is the distribution which has *maximum entropy* for a given average energy $\langle E_i \rangle$.

The partition function of a *canonical ensemble* is defined as a sum $Z(\beta) = \sum_j e^{-\beta E_j}$,

where $\beta = 1/(k_B T)$ is the 'inverse temperature', where T is an ordinary temperature and k_B is the Boltzmann's constant. However, as the position x^i and momentum p_i variables of an ith particle in a system can vary continuously, the set of microstates is actually uncountable. In this case, some form of *coarse–graining* procedure must be carried out, which essentially amounts to treating two mechanical states as the same microstate if the differences in their position and momentum variables are 'small enough'. The partition function then takes the form of an integral. For instance, the partition function of a gas consisting of N molecules is proportional to the 6N–dimensional phase–space integral,

$$Z(\beta) \sim \int_{\mathbb{R}^{6N}} d^3 p_i \, d^3 x^i \exp[-\beta H(p_i, x^i)],$$

where $H = H(p_i, x^i)$, ($i = 1, ..., N$) is the classical Hamiltonian (total energy) function.

More generally, the so–called *configuration integral*, as used in probability theory, information science and dynamical systems, is an abstraction of the above definition of a partition function in statistical mechanics. It is a special case of a normalizing constant in probability theory, for the Boltzmann distribution. The partition function occurs in many problems of probability theory because, in situations where there is a natural symmetry, its

associated probability measure, the *Gibbs measure* (see below), which generalizes the notion of the canonical ensemble, has the *Markov property*.

Given a set of random variables X_i taking on values x_i, and purely potential Hamiltonian function $H(x_i)$, ($i = 1, ...,N$), the partition function is defined as

$$Z(\beta) = \sum_{x^i} \exp\left[-\beta H(x^i)\right]. \tag{59}$$

The function H is understood to be a real-valued function on the space of states $\{X_1, X_2 ...\}$ while β is a real-valued free parameter (conventionally, the inverse temperature). The sum over the x^i is understood to be a sum over all possible values that the random variable X_i may take. Thus, the sum is to be replaced by an integral when the X_i are continuous, rather than discrete. Thus, one writes

$$Z(\beta) = \int dx^i \exp\left[-\beta H(x^i)\right],$$

for the case of continuously-varying random variables X_i.

The Gibbs measure of a random variable X_i having the value x^i is defined as the probability density function

$$P(X_i = x^i) = \frac{1}{Z(\beta)} \exp\left[-\beta E(x^i)\right] = \frac{\exp\left[-\beta H(x^i)\right]}{\sum_{x^i} \exp\left[-\beta H(x^i)\right]}.$$

where $E(x^i) = H(x^i)$ is the energy of the configuration x^i. This probability, which is now properly normalized so that $0 \leq P(x^i) \leq 1$, can be interpreted as a likelihood that a specific configuration of values x^i, ($i = 1, 2, ...N$) occurs in the system. $P(x^i)$ is also closely related to Ω, the probability of a *random partial recursive function halting*.

As such, the partition function $Z(\beta)$ can be understood to provide the Gibbs measure on the space of states, which is the unique statistical distribution that maximizes the entropy for a fixed expectation value of the energy,

$$\langle H \rangle = -\frac{\partial \log(Z(\beta))}{\partial \beta}.$$

The associated entropy is given by

$$S = -\sum_{x^i} P(x^i) \ln P(x^i) = \beta \langle H \rangle + \log Z(\beta),$$

representing 'ignorance' + 'randomness'.

The principle of maximum entropy related to the expectation value of the energy $\langle H \rangle$, is a postulate about a universal feature of any probability assignment on a given set of propositions (events, hypotheses, indices, etc.). Let some testable information about a probability distribution function be given. Consider the set of all trial probability distributions which encode this information. Then the probability distribution which maximizes the information entropy is the true probability distribution, with respect to the testable information prescribed.

Applied to the crowd dynamics, the Boltzman's theorem of *equipartition of energy* states that the expectation value of the energy $\langle H \rangle$ is uniformly spread among all degrees-of-freedom of the crowd (that is, across the whole crowd manifold M).

6.3 Free energy, Landau's phase transitions and Haken's synergetics

All thermodynamic–like properties of a multi-component system like a human (or robot) crowd may be expressed in terms of its *free energy potential, $\mathcal{F} = -k_B T \ln Z(\beta)$*, and its partial derivatives. In particular, the physical entropy S of the crowd is defined as the (negative) first partial derivative of the free energy $\mathcal{F}$ with respect to the control parameter temperature T, i.e., $S = -\partial_T \mathcal{F}$, while the *specific heat capacity C* is the second derivative, $C = T\partial_T S$.

A *phase* of the crowd denotes a set of its states that have relatively uniform behavioral properties. A *crowd phase transition* represents the its transformation from one phase to another (see e.g., [48; 58]). In general, the crowd phase transitions are divided into two categories:

- The *first–order phase transitions*, or, *discontinuous phase transitions*, are those that involve a latent heat C. During such a transition, a crowd either absorbs or releases a fixed (and typically large) amount of energy. Because energy cannot be instantaneously transferred between the system and its environment, first–order crowd transitions are associated with *mixed–phase regimes* in which some parts of the crowd have completed the transition and others have not. This forms a turbulent spatioi-temporal chaotic interphase, difficult to study, because its dynamics can be violent and hard to control.

- The *second–order phase transitions* are the *continuous phase transitions*, in the entropy S is continuous, without any latent heat C. They are purely entropic crowd transitions, which are at the focus of the present study.

In Landau's theory od phase transitions (see [48; 58]), the probability density function P is exponentially related to the free energy potential $\mathcal{F}$, i.e., $P \approx e^{-\mathcal{F}(T)}$, if $\mathcal{F}$ is considered as a function of some order parameter o. Thus, the most probable order parameter is determined by the requirement $\mathcal{F} = \min$. Therefore, the most natural order parameter for the crowd dynamics would be its entropy S.

The following table gives the analogy between various systems in thermal equilibrium and the corresponding nonequilibrium systems analyzed in Haken's synergetics [25; 26; 27]:

System in thermal equilibrium	Nonequilibrium system
Free energy potential $\mathcal{F}$	Generalized potential V
Order parameters o_i	Order parameters o_i
$\dot{o}_i = -\frac{\partial \mathcal{F}}{\partial o_i}$	$\dot{o}_i = -\frac{\partial V}{\partial o_i}$
Temperature T	Control input u
Entropy S	System output y
Specific Heat c	System efficiency e

In particular, in case of human biodynamics [48; 58], natural control inputs u_i are muscular forces and torques, F_i, natural system outputs y_i are joint coordinates q^i and momenta p_i, while the system efficiencies e_i represent the changes of coordinates and momenta with changes of corresponding muscular torques for the ith active human joint, $e_i^q = \dfrac{\partial q^i}{\partial F_i}$, $\quad e_i^p = \dfrac{\partial p_i}{\partial F_i}$.

6.4 Heat equation, Dirichlet action and gradient flow on a Riemannian manifold

The heat equation

$$\dot{u} = \Delta u, \tag{60}$$

on a compact Riemannian manifold M with static metric ($\partial_t g = 0$), where $u : [0,T] \times M \to \mathbb{R}$ is a scalar field, can be interpreted as the gradient flow for the *Dirichlet action*

$$E(u) := \frac{1}{2} \int_M |\nabla u|_g^2 \, d\mu, \tag{61}$$

using the inner product, $\langle u_1, u_2 \rangle_\mu := \int_M u_1 u_2 \, d\mu$, associated to the volume measure $d\mu$. This can be proved if we evolve u in time at some arbitrary rate u, an application of integration by parts formula,

$$\int_M u \nabla_\alpha X^\alpha \, d\mu = -\int_M (\nabla_\alpha u) X^\alpha \, d\mu$$

(where $\operatorname{div}(X) := \nabla_\alpha X^\alpha$ is the divergence of the vector-field X^α, which validates the Stokes theorem, $\int_M \operatorname{div}(X) \, d\mu = 0$), gives

$$\partial_t E(u) = -\int_M (\Delta u)\dot{u} \, d\mu = \langle -\Delta u, \dot{u} \rangle_\mu, \tag{62}$$

from which we see that (60) is indeed the gradient flow for (62) with respect to the inner product. In particular, if u solves the heat equation (60), we see that the Dirichlet energy is decreasing in time,

$$\partial_t E(u) = -\int_M |\Delta u|^2 \, d\mu. \tag{63}$$

Thus we see that by representing the parabolic PDE (60) as a gradient flow, we automatically gain a controlled quantity of the evolution, namely the energy functional that is generating the gradient flow. This representation also strongly suggests that solutions of (60) should eventually converge to stationary points of the Dirichlet energy (61), which by (62) are harmonic functions (i.e., the functions u with $\Delta u = 0$). As an application of the gradient flow interpretation, we can assert that the only periodic (or, "breather") solutions to the heat equation (60) are the harmonic functions (which must be constant if the manifold M is compact). Indeed, if a solution u was periodic, then the monotone functional E must be constant, which by (63) implies that u is harmonic as claimed.

6.5 Lyapunov exponents and Kolmogorov–Sinai entropy

A branch of nonlinear dynamics has been developed with the aim of formalizing and quantitatively characterizing the general sensitivity to initial conditions. The *largest Lyapunov exponent* λ, together with the related *Kaplan–Yorke dimension* d_{KY} and the *Kolmogorov–Sinai entropy* h_{KS} are the three indicators for measuring the *rate of error growth* produced by a dynamical system [17; 50; 60].

The characteristic Lyapunov exponents are somehow an extension of the linear stability analysis to the case of aperiodic motions. Roughly speaking, they measure the typical rate of

exponential divergence of nearby trajectories. In this sense they give information on the rate of growth of a very small error on the initial state of a system [9; 10].

Consider an nD dynamical system given by the set of ODEs of the form

$$\dot{x} = f(x),\tag{64}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$. Recall that since the r.h.s of equation (64) does not depend on t explicitly, the system is called *autonomous*. We assume that f is smooth enough that the evolution is well defined for time intervals of arbitrary extension, and that the motion occurs in a bounded region R of the system phase space M. We intend to study the separation between two trajectories in M, $x(t)$ and $x'(t)$, starting from two close initial conditions, $x(0)$ and $x'(0) = x(0) + \delta x(0)$ in $R_0 \subset M$, respectively.

As long as the difference between the trajectories, $\delta x(t) = x'(t) - x(t)$, remains infinitesimal, it can be regarded as a vector, $z(t)$, in the tangent space $T_x M$ of M. The time evolution of $z(t)$ is given by the linearized differential equations:

$$\dot{z}_i(t) = \left.\frac{\partial f_i}{\partial x_j}\right|_{x(t)} z_j(t).$$

Under rather general hypothesis, Oseledets [72] proved that for almost all initial conditions $x(0) \in R$, there exists an orthonormal basis $\{e_i\}$ in the tangent space $T_x M$ such that, for large times,

$$z(t) = c_i e_i \exp(\lambda_i t),\tag{65}$$

where the coefficients $\{c_i\}$ depend on $z(0)$. The exponents $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$ are called *characteristic Lyapunov exponents*. If the dynamical system has an ergodic invariant measure on M, the spectrum of LEs $\{\lambda_i\}$ does not depend on the initial conditions, except for a set of measure zero with respect to the natural invariant measure.

Equation (65) describes how an nD spherical region $R = S^n \subset M$, with radius ϵ centered in $x(0)$, deforms, with time, into an ellipsoid of semi–axes $\epsilon_i(t) = \epsilon \exp(\lambda_i t)$, directed along the e_i vectors. Furthermore, for a generic small perturbation $\delta x(0)$, the distance between the reference and the perturbed trajectory behaves as

$$|\delta x(t)| \sim |\delta x(0)| \exp(\lambda_1 t)\left[1 + O\left(\exp{-(\lambda_1 - \lambda_2)t}\right)\right].$$

If $\lambda_1 > 0$ we have a rapid (exponential) amplification of an error on the initial condition. In such a case, the system is chaotic and, unpredictable on the long times. Indeed, if the initial error amounts to $\delta_0 = |\delta x(0)|$, and we purpose to predict the states of the system with a certain tolerance Δ, then the prediction is reliable just up to a *predictability time* given by

$$T_p \sim \frac{1}{\lambda_1}\ln\left(\frac{\Delta}{\delta_0}\right).$$

This equation shows that T_p is basically determined by the *positive leading Lyapunov exponent*, since its dependence on δ_0 and Δ is logarithmically weak. Because of its preeminent role, λ_1 is often referred as 'the leading positive Lyapunov exponent', and denoted by λ.

Therefore, Lyapunov exponents are average rates of expansion or contraction along the principal axes. For the ith principal axis, the corresponding Lyapunov exponent is defined as

$$\lambda_i = \lim_{t \to \infty} \{(1 / t) \ln[L_i(t) / L_i(0)]\}, \tag{66}$$

where $L_i(t)$ is the radius of the ellipsoid along the ith principal axis at time t.
An initial volume V_0 of the phase–space region R_0 evolves on average as

$$V(t) = V_0 e^{(\lambda_1 + \lambda_2 + \cdots + \lambda_{2n})t}, \tag{67}$$

and therefore the rate of change of $V(t)$ is simply

$$\dot{V}(t) = \sum_{i=1}^{2n} \lambda_i V(t).$$

In the case of a 2D phase area A, evolving as $A(t) = A_0 e^{(\lambda_1 + \lambda_2)t}$, a *Lyapunov dimension* d_L is defined as

$$d_L = \lim_{\epsilon \to 0} \left[\frac{d(\ln(N(\epsilon)))}{d(\ln(1 / \epsilon))} \right],$$

where $N(\epsilon)$ is the number of squares with sides of length ϵ required to cover $A(t)$, and d represents an ordinary *capacity dimension*,

$$d_c = \lim_{\epsilon \to 0} \left(\frac{\ln N}{\ln(1 / \epsilon)} \right).$$

Lyapunov dimension can be extended to the case of nD phase–space by means of the *Kaplan–Yorke dimension* [64; 73; 89] as

$$d_{KY} = j + \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_j}{|\lambda_{j+1}|},$$

where the λ_i are ordered (λ_1 being the largest) and j is the index of the smallest nonnegative Lyapunov exponent.
On the other hand, a state, initially determined with an error $\delta x(0)$, after a time enough larger than $1/\lambda$, may be found almost everywhere in the region of motion $R \in M$. In this respect, the *Kolmogorov–Sinai* (KS) *entropy*, h_{KS}, supplies a more refined information. The error on the initial state is due to the maximal resolution we use for observing the system. For simplicity, let us assume the same resolution ϵ for each degree of freedom. We build a partition of the phase space M with cells of volume ϵ^d, so that the state of the system at $t = t_0$ is found in a region R_0 of volume $V_0 = \epsilon^d$ around $x(t_0)$. Now we consider the trajectories starting from V_0 at t_0 and sampled at discrete times $t_j = j \, \tau$ ($j =1, 2, 3, \ldots, t$). Since we are considering motions that evolve in a bounded region $R \subset M$, all the trajectories visit a finite number of different cells, each one identified by a symbol. In this way a unique sequence of symbols $\{s(0), s(1), s(2), \ldots \}$ is associated with a given trajectory $x(t)$. In a chaotic system,

although each evolution $x(t)$ is univocally determined by $x(t_0)$, a great number of different symbolic sequences originates by the same initial cell, because of the divergence of nearby trajectories. The total number of the admissible symbolic sequences, $\tilde{N}(\epsilon, t)$, increases exponentially with a rate given by the topological entropy

$$h_T = \lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{1}{t} \ln \widetilde{N}(\epsilon, t).$$

However, if we consider only the number of sequences $N_{eff}(\epsilon, t) \leq \tilde{N}(\epsilon, t)$ which appear with very high probability in the long time limit – those that can be numerically or experimentally detected and that are associated with the natural measure – we arrive at a more physical quantity called the Kolmogorov–Sinai (or metric) entropy, which is the key entropy notion in ergodic theory [17]:

$$h_{KS} = \lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{1}{t} \ln N_{eff}(\epsilon, t) \leq h_T. \tag{68}$$

h_{KS} quantifies the long time exponential rate of growth of the number of the effective coarse-grained trajectories of a system. This suggests a link with information theory where the Shannon entropy measures the mean asymptotic growth of the number of the typical sequences – the ensemble of which has probability almost one – emitted by a source.

We may wonder what is the number of cells where, at a time $t > t_0$, the points that evolved from R_0 can be found, i.e., we wish to know how big is the coarse–grained volume $V(\epsilon, t)$, occupied by the states evolved from the volume V_0 of the region R_0, if the minimum volume we can observe is $V_{min} = \epsilon^d$. As stated above (67), we have

$$V(t) \sim V_0 \exp(t \sum_{i=1}^{d} \lambda_i).$$

However, this is true only in the limit $\epsilon \to 0$. In this (unrealistic) limit, $V(t) = V_0$ for a conservative system (where $\sum_{i=1}^{d} \lambda_i = 0$) and $V(t) < V_0$ for a dissipative system (where $\sum_{i=1}^{d} \lambda_i < 0$). As a consequence of limited resolution power, in the evolution of the volume $V_0 = \epsilon^d$ the effect of the contracting directions (associated with the negative Lyapunov exponents) is completely lost. We can experience only the effect of the expanding directions, associated with the positive Lyapunov exponents. As a consequence, in the typical case, the coarse grained volume behaves as

$$V(\epsilon, t) \sim V_0 e^{(\Sigma_{\lambda_i > 0} \lambda_i) t},$$

when V_0 is small enough. Since $N_{eff}(\epsilon, t) \propto V(\epsilon, t) / V_0$, one has: $h_{KS} = \sum_{\lambda_i > 0} \lambda_i$. This argument can be made more rigorous with a proper mathematical definition of the metric entropy. In this case one derives the Pesin relation [17; 76]: $h_{KS} \leq \sum_{\lambda_i > 0} \lambda_i$. Because of its relation with the Lyapunov exponents, or by the definition (68), it is clear that also h_{KS} is a fine-grained and global characterization of a dynamical system. $\sum_{\lambda_i > 0}$

The metric entropy is an invariant characteristic quantity of a dynamical system, i.e., given two systems with invariant measures, their KS–entropies exist and they are equal iff the systems are isomorphic [7].

Finally, the *topological entropy* on the manifold M equals the supremum of the Kolmogorov-Sinai entropies,

$$h(u) = \sup\{h_{KS}(u) = h_\mu(u) : \mu \in P_u(M)\},$$

where $u : M \to M$ is a continuous map on M, and μ ranges over all u–invariant (Borel) probability measures on M. Dynamical systems of positive topological entropy are often considered topologically chaotic.

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