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Dynamics and Control of Nonlinear Variable Order Oscillators

Gerardo Diaz and Carlos F.M. Coimbra
University of California, Merced
 U.S.A.

1. Introduction

The denomination Fractional Order Calculus has been widely used to describe the mathematical analysis of differentiation and integration to an arbitrary non-integer order, including irrational and complex orders. First proposed around three hundred years ago, it has attracted much interest during the past three decades (Oldham & Spanier (1974), Miller & Ross (1993), Podlubni (1999)). The increased interest in fractional systems in the past few decades is due mainly to a large body of physical evidence describing fractional order behavior in diverse areas such as fluid mechanics, mechanical systems, rheology, electromagnetism, quantitative finances, electrochemistry, and biology. Fractional order modeling provides exceptional capabilities for analysing memory-intense and delay systems and it has been associated with the exact description of complex transport phenomena such as fractional history effects in the unsteady viscous motion of small particles in suspension (Coimbra et al. 2004, L'Esperance et al. 2005). Although fractional order dynamical and control systems were studied only marginally until a few decades ago, the recent development of effective mathematical methods of integration of non-integer order differential equations (Charef et al. (1992); Coimbra & Kobayashi (2002), Diethelm et al. (2002); Momany (2006), Diethelm et al. (2005)) has resulted in a number of control schemes and algorithms, many of which have shown better performance and disturbance rejection compared to other traditional integer-order controllers (Podlubni (1999); Hartly & Lorenzo (2002), Ladaci & Charef (2006), among others).

Variable order (VO) systems constitute a generalization of fractional order representations to functional order. In VO systems the order of the derivative changes with respect to either the dependent or the independent variables (or both), or parametrically with respect to an external functional behavior (Samko & Ross, 1993). Compared to fractional order applications, VO systems have not received much attention, although the potential to characterize complex behavior by the functional order of differentiation or integration is clear. Variable order formulations have been utilized, among other applications, to describe the mechanics of an oscillating mass subjected to a variable viscoelasticity damper and a linear spring (Coimbra, 2003), to analyze elastoplastic indentation problems (Ingman & Suzdalnitsky (2004)), to interpolate the behavior of systems with multiple fractional terms (Soon et al., 2005), and to develop a statistical mechanics model that yields a macroscopic constitutive relation for a viscoelastic composite material undergoing compression at varying strain rates (Ramirez & Coimbra, 2007). Concerning the dynamics and control of VO

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systems, the authors of this chapter have previously analyzed the dynamics and linear control of a variable viscoelasticity oscillator and have presented a generalization of the van der Pol equation using the VO differential equation formulation (Diaz & Coimbra, 2009). In the present work, we utilize the Coimbra Variable Order Differential Operator (VODOs) to analyze the dynamics of the Duffing equation with a VO damping term. Coimbra's VODO returns the correct value of the p -th derivative for $p < 2$, as can be generalized to any order, positive or negative. The behavior of the variable order differintegrals are shown in variable phase space for different parameters that constitute a pictorial representation of the dynamics of the variable order system, and help understand the transitional regimes between the extreme values of the derivatives. Also, a tracking controller is developed and applied to the oscillator for different expressions of the variable order $q(x(t))$. Finally, a variable order controller is used to eliminate chaotic oscillations of Lorenz-type systems.

2. Fractional and variable order operators

Over the past few centuries, different definitions of a fractional operator have been proposed. For instance the Riemann-Liouville integral is defined as

$$D_{0,t}^{-\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau \quad (1)$$

where $\alpha \in \mathbb{R}^+$ is the order of integration of the function $x(t)$ when the lower limit of integration (initial condition) is chosen to be identically zero. The Riemann-Liouville derivative of order α is given as

$$D_{0,t}^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} x(\tau) d\tau, \quad (2)$$

and the Grunwald-Letnikov differential operation is defined as

$$D_{0,t}^{\alpha}x(t) = \lim_{h \rightarrow 0, nh=t} h^{-\alpha} \sum_{k=0}^n (-1)^k \binom{\alpha}{k} x(t-kh). \quad (3)$$

Finally, the Caputo derivative of fractional order α of $x(t)$ is defined as

$$D_{0,t}^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) d\tau, \quad (4)$$

for which $m-1 < \alpha < m \in \mathbb{Z}^+$. More details about these operators can be found in Li & Deng (2007), Diethelm (2002), and Hartley & Lorenzo (2002).

For variable order systems, Coimbra (2003) defined the canonical differential operator as:

$$D^{q(x(t))}x(t) = \frac{1}{\Gamma(1-q(x(t)))} \int_{0^+}^t (t-\sigma)^{-q(x(t))} D^1x(\sigma) d\sigma + \frac{(x(0^+) - x(0^-)) t^{-q(x(t))}}{\Gamma(1-q(x(t)))} \quad (5)$$

where $q(x(t)) < 1$. The constraint on the upper limit of differentiation can be easily removed, and is adopted here only for convenience. One of the important characteristics of Coimbra's

operator is that it is dynamically consistent with causal behavior in the initial conditions, i.e. the operator returns the appropriate Heaviside contribution to the integral value of $D^q(x(t))x(t)$ when $x(t)$ is not continuous between $t=0^-$ and $t=0^+$ (Coimbra, 2003; Ramirez & Coimbra, 2007; Diaz & Coimbra (2009)). Also of relevance is that all integer and fractional order differentials are returned correctly by the operator, including the upper limit. In this work we used the extended version of this operator that covers the range of $q(x(t)) < 2$. The generalized order differential operator can thus be calculated by the following numerical algorithm:

$$D^q x_n = \frac{1}{\Gamma(4-q)} \sum_{i=0}^n a_{i,n} D^2 x_i + \frac{x(0^+)(1-q)(t_n)^{-q} + D^1 x(0^+) t_n^{1-q}}{\Gamma(2-q)}, \tag{6}$$

with quadrature weights given by

$$a_{i,n} = (3-q)n^{2-q} - n^{3-q} + (n-1)^{3-q} \quad , \text{ if } i=0$$

$$a_{i,n} = (n-i-1)n^{3-q} - 2(n-i)^{3-q} + (n-i+1)^{3-q} \quad , \text{ if } 0 < i < n.$$

$$a_{i,n} = 1 \quad , \text{ if } i=n.$$

As stated earlier, one of the critical properties of this operator for generalized order modeling is that it returns the p -th derivative of $x(t)$ when $q(x(t)) = p$. This can be graphically demonstrated by considering an arbitrary function with known derivatives such as

$$y = t^2 \sin(t) \tag{7}$$

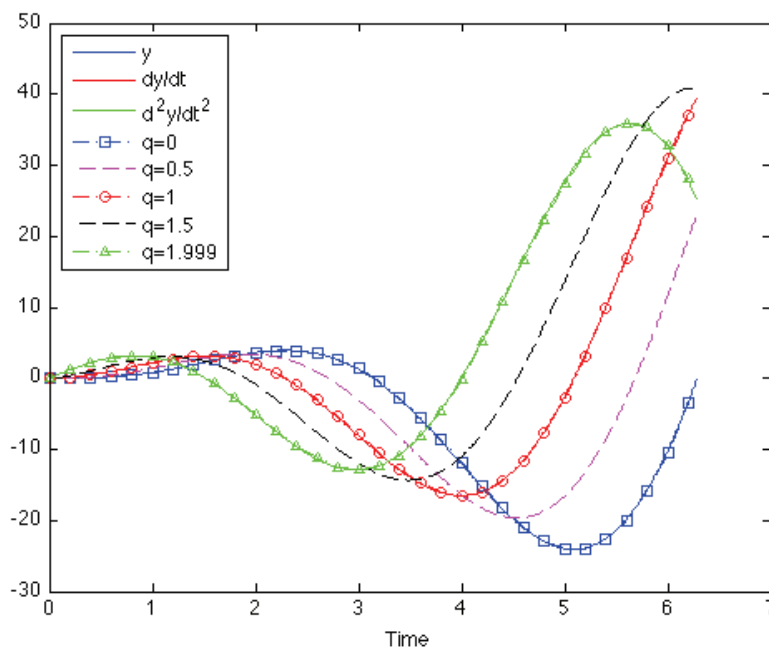


Fig. 1. Comparison of values of function $y=t^2\sin(t)$ and its derivatives with the results obtained with operator described by Eq. (6) for several values of the order q .

Figure 1 shows the values of function y (Eq. 7) and its derivatives dy/dt , and d^2y/dt^2 calculated analytically. The figure also shows that the operator described by Eq. (6) returns values that match the functions y for $q=0$, dy/dt for $q=1$, and d^2y/dt^2 for $q\approx 2$, respectively. The values of $q=0.5$ and $q=1.5$ are also shown to indicate the matching of the rational order derivatives with the values calculated using the VO operator.

3. Dynamics of the Duffing equation with variable order damping

Together with the van de Pol equation, the Duffing equation represents the behavior of one of the most studied oscillators in the field of nonlinear dynamics (Guckenheimer & Holmes (1983), Drazin (1994)). First introduced in 1918 by G. Duffing, different variations of the equation have been used to analyze its dynamics for the autonomous and forced cases. Moon and Holmes (1979, 1980) considered a negative linear stiffness term to analyse the forced vibrations of a cantilever beam near two magnets. Vincent & Kenfack (2008) recently studied the bifurcation structure and synchronization of a double-well Duffing oscillator. They were able to show regions of chaos and quasiperiodicity and they found threshold parameters for which synchronization occurred. With respect to fractional order systems, Sheu et al. (2007) analyzed the Duffing equation with negative linear stiffness and a fractional damping term. They reported a period doubling route to chaos in their study.

3.1 Forced oscillations

We generalize the concept of fractional damping to include a variable order term as:

$$D^2x + \delta D^q x - x + x^3 = \gamma \sin(\omega t). \quad (8)$$

The main difference with respect to the work by Sheu et al. (2007) is that they studied the dynamics of Eq. (8) for a range of values of the fractional order q where this parameter was kept constant for every case analyzed. Here, the oscillator is generalized to include a damping term where the order of the derivative reacts to the effect of the forcing function over time, thus $q = q(t)$. In our analysis, we choose the value of parameters δ and ω to be 0.1 and 2, respectively.

Case $\gamma = 1.5$:

The first case considered in this work relates to the behavior of the oscillator given by Eq. (8) for $\gamma = 1.5$ for two different conditions, i.e. $q = 1$ and $q = (99/100) + \sin(\omega t)$. We note that the operator described by Eq. 6 is valid for $q(t) < 2$, thus the expression used for the change in q with respect to time ensures that this condition is met.

Figure 2 shows the dynamics of the oscillator given by Eq. (8) for $q = 1$ as the order of the derivative in the damping term. The simulations cover the time range $t \in [0, 700]$ where only the results for $t > 200$ are plotted to exclude the initial transients. Chaotic behavior is observed and a strange attractor is depicted in Fig. 2(a). The Poincaré map is shown in Fig. 2(b).

The effect of the variable order derivative on the damping term of Eq. (8) significantly changes the dynamics of the oscillator. This can be observed in Figs. 3(a) and 3(b) where it is seen that after removing the initial transients, the dynamics of the oscillators are confined to a narrower region in the phase space.

The dynamics of the VO oscillators can also be analyzed utilizing a modified version of the phase diagram where the variable order derivative, $D^q x(t)$, is plotted on the ordinate axis

and the position, $x(t)$, is plotted on the abscissa axis. Figure 4(a) shows the variable order phase space (a plot of the value of the VO derivative, $D^q x(t)$, as a function position), whereas Fig. 4(b) shows the behavior of $D^q x(t)$ as a function of the order of the derivative, $q(t)$. It is seen in Fig. 4(b) that $q(t) < 2$, thus meeting the upper limit of differentiation mandated by the numerical algorithm used here (Eq. 6).

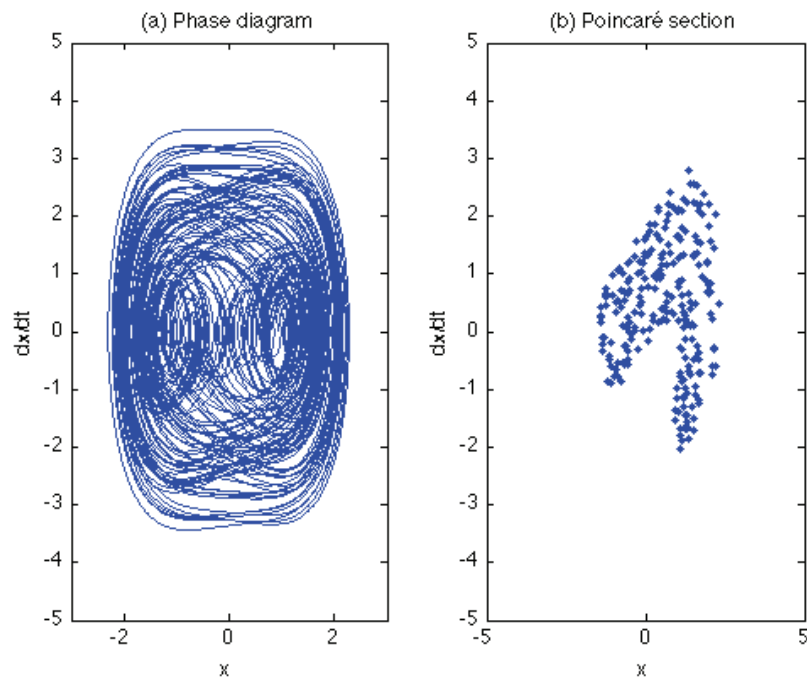


Fig. 2. Phase diagram and Poincaré map for $\gamma = 1.5$ and $q = 1$.

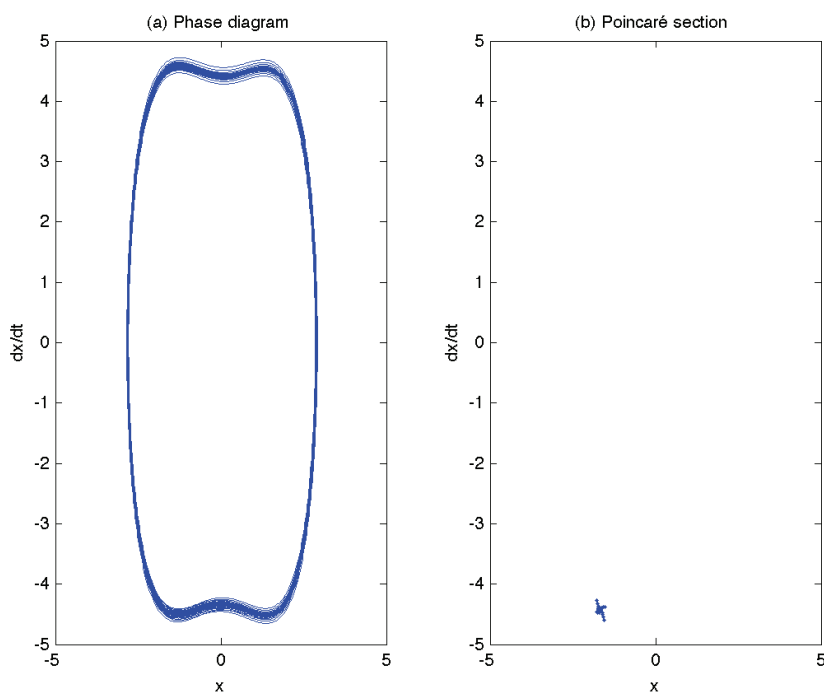


Fig. 3. Phase diagram and Poincaré map for $\gamma = 1.5$ and $q = (99/100) + \sin(\omega t)$.

Figure 5(a) shows the change of $x(t)$ and $D^q x(t)$ as a function of time. Figures 6(a) and 6(b) show that $q(t)$ also has an oscillatory behavior with $D^q x(t)$ having a minimum value when $x(t)$ and $q(t)$ approach their maximum value. This is also depicted in the VO phase diagrams shown in Figs. 4(a) and 4(b).

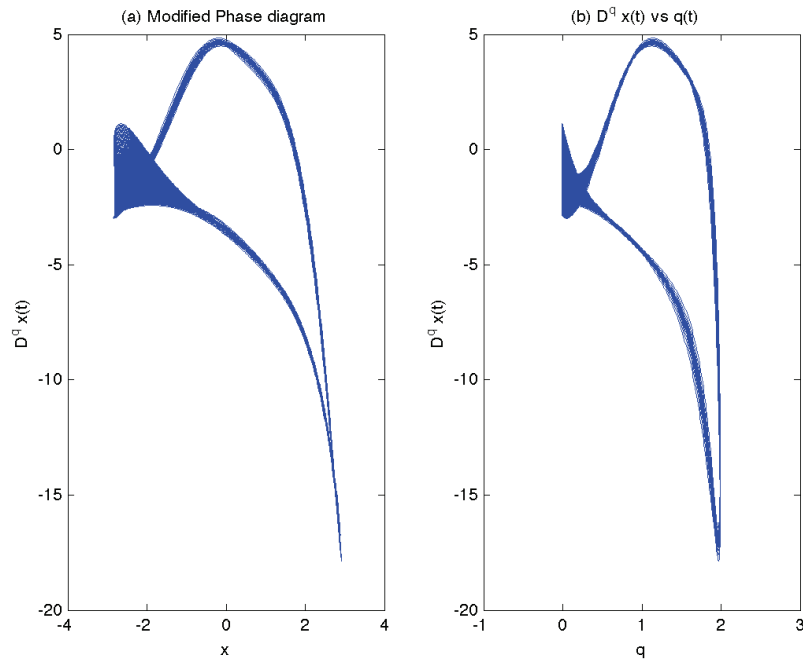


Fig. 4. Modified phase diagram and $D^q x(t)$ vs. $q(t)$ plots for $\gamma=1.5$.

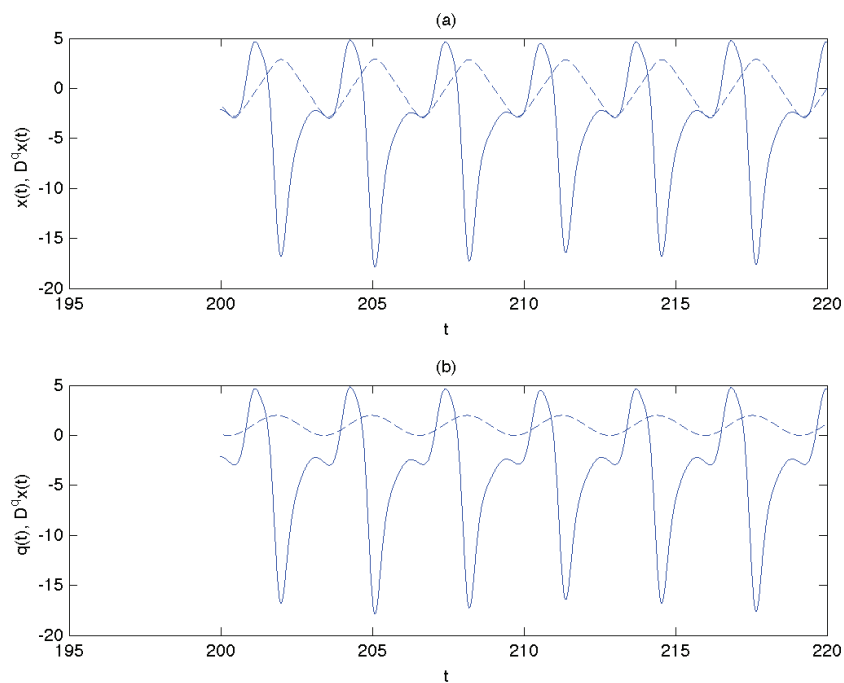


Fig. 5. Dynamics of VO Duffing equation with respect to time for $\gamma=1.5$. (a) - - - $x(t)$, _____ = $D^q x(t)$; (b) - - - $q(t)$, _____ = $D^q x(t)$;

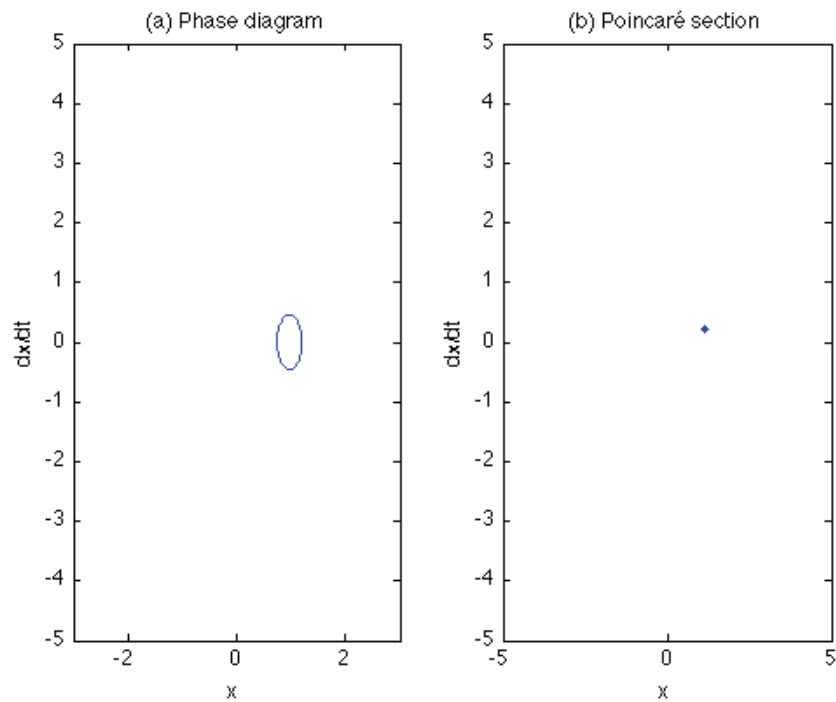


Fig. 6. Phase diagram and Poincaré map for $\gamma=0.5$ and $q=1$.

Case $\gamma = 0.5$:

We now analyze the case where parameter $\gamma = 0.5$. After the initial transient, the standard configuration ($q = 1$) shows an oscillatory behavior as depicted in Fig. 6(a) with a single point appearing in the Poincaré map, Fig. 6(b).

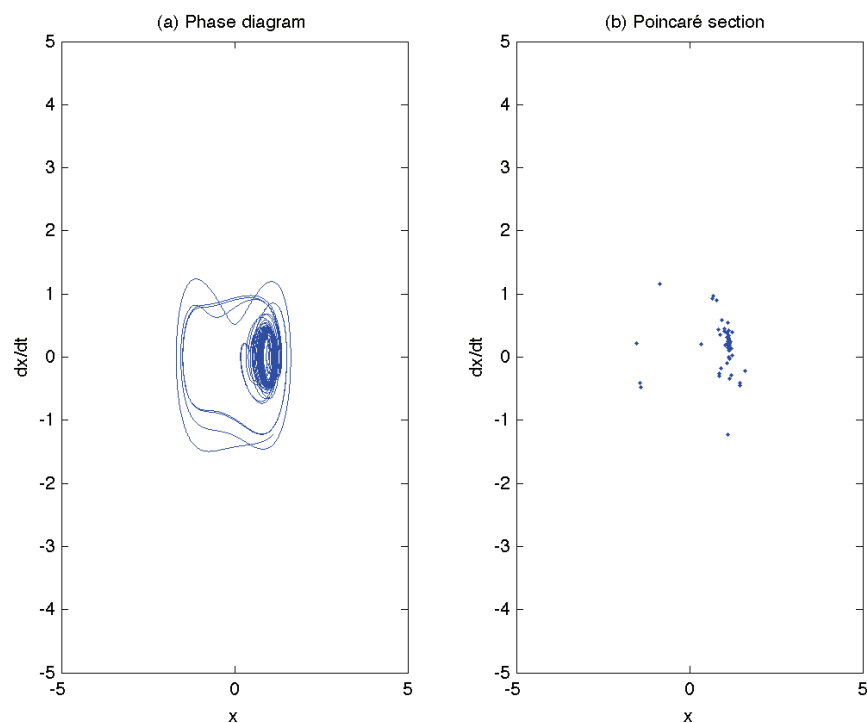


Fig. 7. Phase diagram and Poincaré map for $\gamma = 0.5$ and $q = (99/100) + \sin(\omega t)$ for $t > 200$.

Figures 7(a) and 7(b) show the results of the simulations for $\gamma = 0.5$ and a variable order of the derivative given by $q(t) = (99/100) + \sin(\omega t)$. It is seen that the phase diagram and Poincaré maps differ significantly from the case $q = 1$. However, plotting $x(t)$ as a function of time, as depicted in Fig. 8, shows the transient effects seem to last longer than for the case of $q = 1$. After $t \sim 400$, the system settles to an oscillatory behavior with a smaller amplitude.

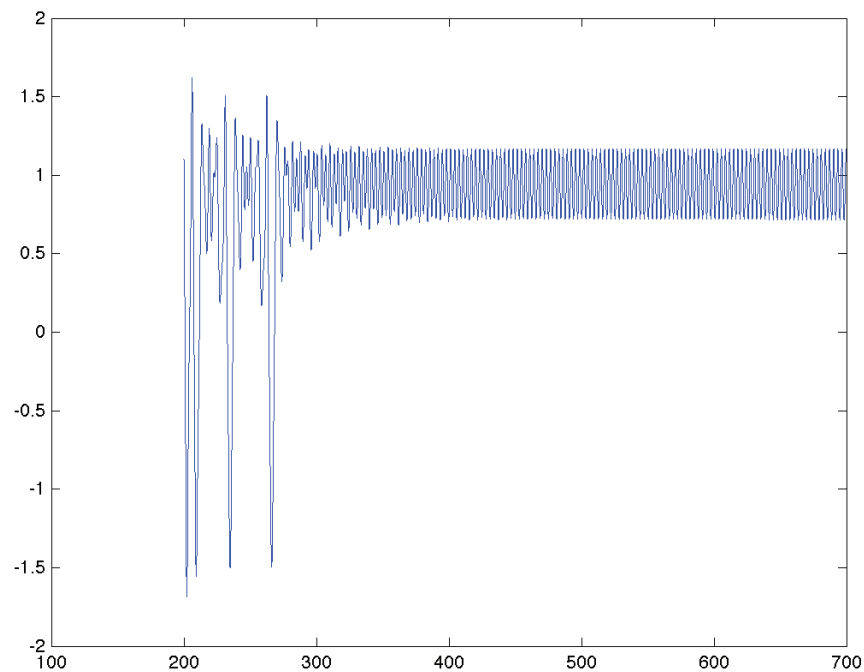


Fig. 8. Phase diagram and Poincaré map for $\gamma = 0.5$ and $q = (99/100) + \sin(\omega t)$ for $t > 200$.

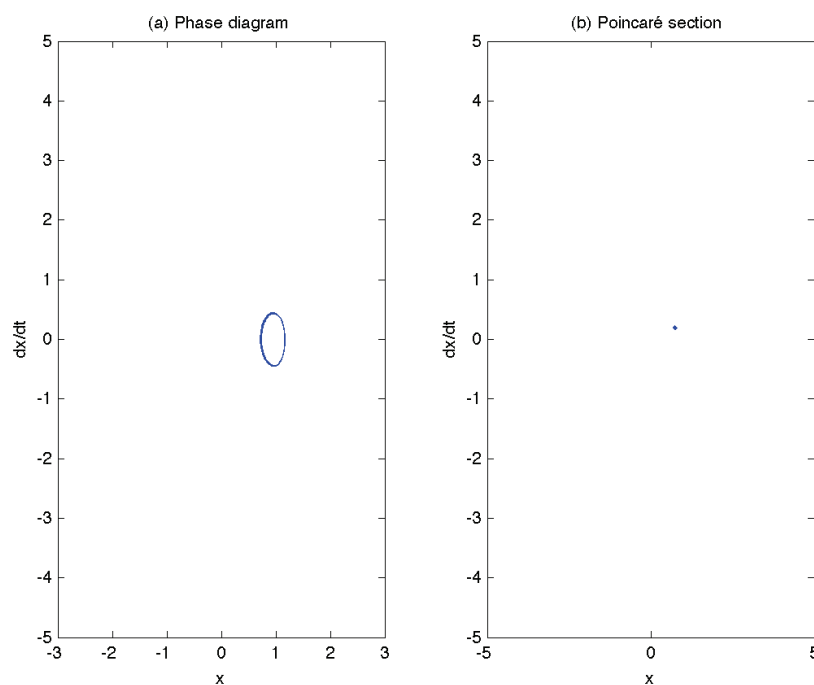


Fig. 9. Phase diagram and Poincaré map for $\gamma = 0.5$ and $q = (99/100) + \sin(\omega t)$ for $t > 400$.

Plots of the phase diagram and the Poincare map for $t > 400$ are shown in Figs. 9(a) and 9(b), respectively. Similar dynamics compared to $q = 1$ are displayed by the system.

3.2 Control of the VO Duffing equation

The dynamics of the variable order Duffing equations were analyzed in the previous section for the cases $\delta = 0.1$, $\omega = 2$, with $\gamma = 1.5$ and $\gamma = 0.5$, respectively. In this section, we study controls aspects of this equation subject to a VO damping term. An exact feedback linearization is performed to obtain a tracking controller that drives the VO Duffing oscillator to follow a periodic reference function, r (Khalil, 1996). The forcing function in Eq. (8) can be replaced by a control action as shown by Eq. 9.

$$D^2x = x - x^3 - \delta D^q x + u. \quad (9)$$

Exact feedback linearization is obtained by choosing the control action

$$u = x^3 + \delta D^q x + v. \quad (10)$$

Thus, Eq. 9 is converted to a linear equation of the form

$$D^2x = x + v. \quad (11)$$

This second order differential equation is transformed to a system of first order differential equations

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= A\bar{x} + Bv = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v, \\ y = C\bar{x} &= [1 \quad 0]\bar{x}. \end{aligned} \quad (12)$$

A control action of form $u = -K\bar{x} + Gr = -k_1x_1 - k_2x_2 + Gr$ is chosen where k_1 and k_2 are constants that are used to select the location of the closed-loop eigenvalues, G is the feedforward gain, and r is the reference. For the controllable system given by Eq. (12) we arbitrarily select closed-loop eigenvalues $\lambda_{1,2} = -5$ to obtain $k_1 = 24$ and $k_2 = 10$. The feedforward gain is obtained with Eq. (13) (Williams & Lawrence, 2007).

$$G = -(C(A - BK)^{-1}B)^{-1}. \quad (13)$$

The tracking scheme is tested with the variable order derivative in the VO damping term having the expression $q = (99/100) + \sin(\omega t)$, where $\gamma = 1.5$ and $\omega = 2$. Figures 10(a) to 10(d) show the behavior of the tracking system for $r(t) = 2 \cos(\omega/10) + \sin(3\omega/10)$. The output of the system, $y(t)$, follows the reference, $r(t)$, consistently, as seen in Fig. 10(a). Figure 10(b) shows the control action, $u(t)$, and the sinusoidal behavior of the order of the VO derivative, $q(t)$, is shown in Fig. 10(c) where the value of the variable order derivative, $D^q y(t)$, is plotted in Fig. 10(d).

Exact feedback linearization can be used for different functions of $q(t)$. Figure 11(a) to 11(d) show the tracking of reference r for $q(t) = r(t)/3$. Scaling of $q(t)$ with respect to r is performed so that the value of $q(t)$ remains smaller than 2.

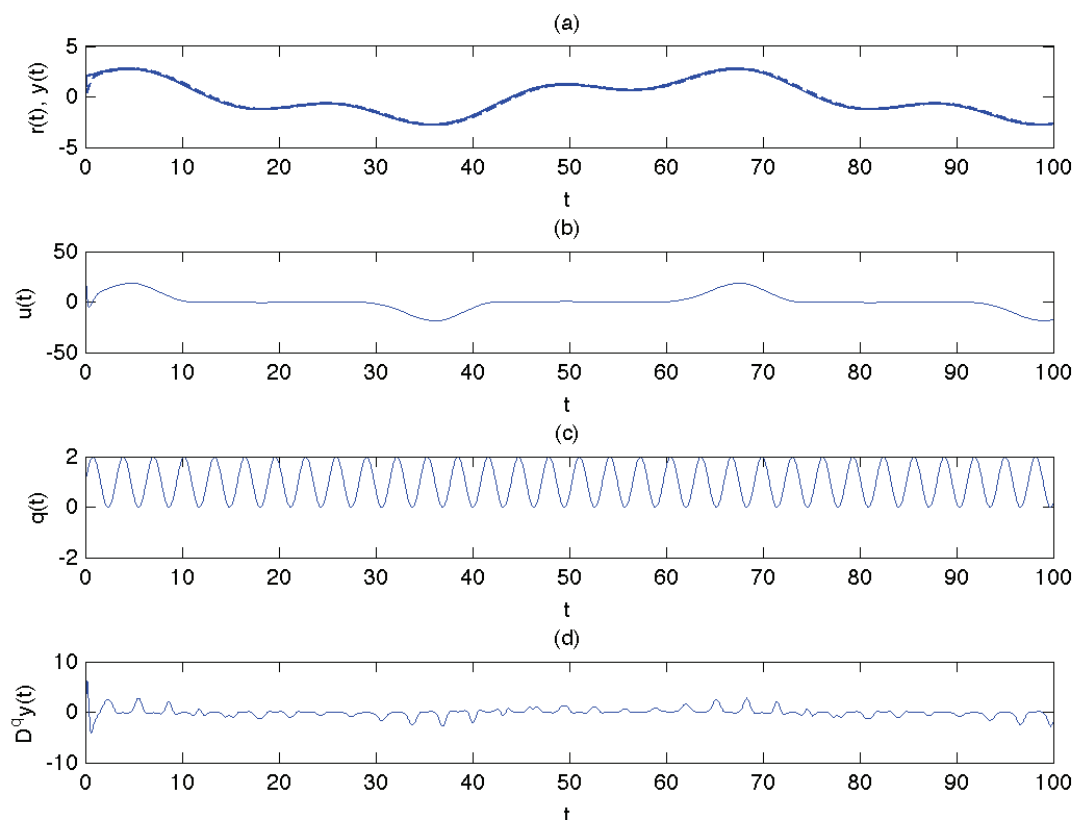


Fig. 10. Tracking control for the VO duffing equation for $q(t) = (99/100) + \sin(\omega t)$. (a) $_ = r(t)$, $\dots = y(t)$; (b) $u(t)$, (c) $q(t)$, and (d) $D^q y(t)$.

We note that if the value of the order of the VO derivative, $q(t)$, is known to remain within the requirement of the operator (i.e. $q(t) < 2$) then an implicit form of the variation of q (i.e. $q=q(x)$) can also be utilized (Diaz & Coimbra, 2009). It is also mentioned that if the closed-loop eigenvalues are chosen to have positive real parts then the system becomes unstable.

4. VO control of the Lorenz system

So far, we have analyzed the dynamics and control of VO systems that have the term $D^q x(t)$ as part of the expression describing their dynamics. We now apply the variable order approach as the control action to stabilize a chaotic dynamical system. First proposed as a way to describe the dynamics of weather systems, the Lorenz system of equations (Lorenz, 1963) has been intensively studied as a dynamical system that displays chaotic behavior where a strange attractor is encountered under certain values of its parameters. Control techniques have been proposed in the past (Vincent & Yu, 1991) but to the best knowledge of the authors, there is no study in the literature that has utilized a variable order controller to stabilize the chaotic dynamics of the Lorenz system.

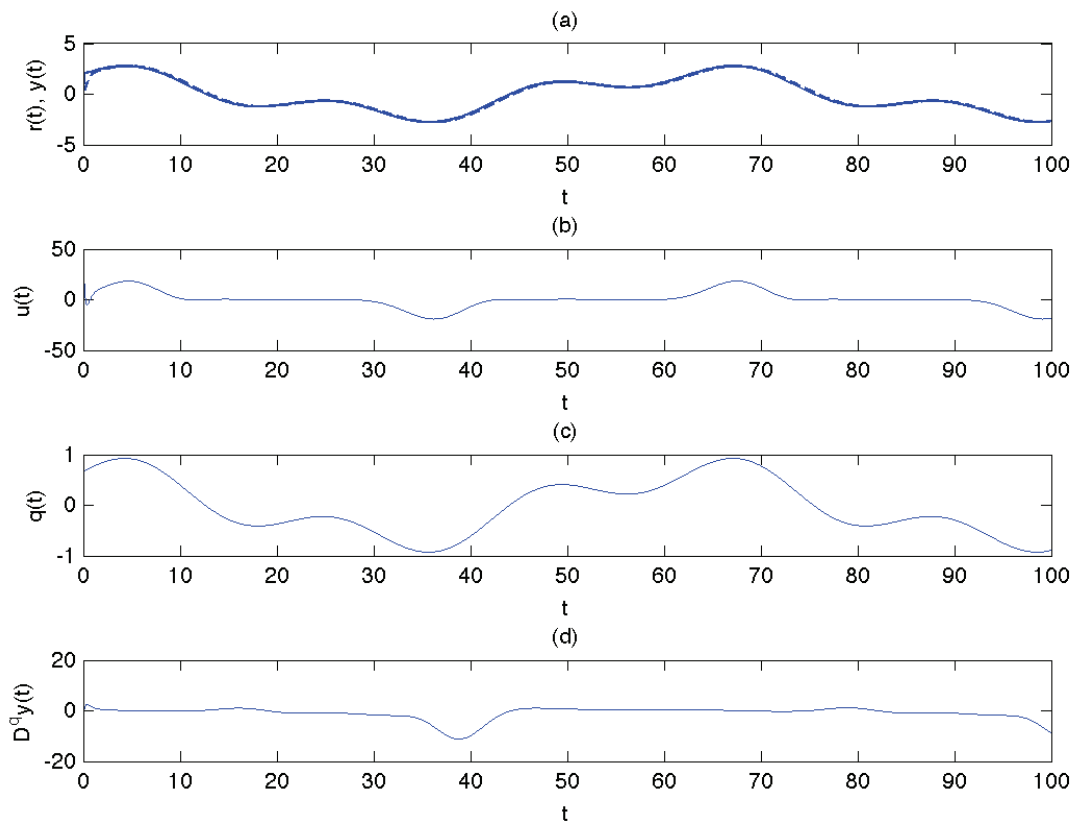


Fig. 11. Tracking control for the VO duffing equation for $q(t)=(1/3) [2\cos(\omega/10)+\sin(3\omega/10)]$. (a) $— = r(t), \dots = y(t)$; (b) $u(t)$, (c) $q(t)$, and (d) $D^q y(t)$.

The Lorenz system is described by the following equations

$$\begin{aligned}
 \frac{dx_1}{dt} &= -\sigma x_1 + \sigma x_2, \\
 \frac{dx_2}{dt} &= rx_1 - x_2 - x_2 x_3, \\
 \frac{dx_3}{dt} &= x_1 x_2 - bx_3 + u.
 \end{aligned}
 \tag{14}$$

For $r > 1$ there are two non-trivial equilibrium points, i.e. $\bar{x}_1 = \bar{x}_2 = \pm (b(r-1))^{1/2}$, $\bar{x}_3 = r-1$. Linearizing the system with respect to the first non-trivial equilibrium point, we obtain

$$\begin{aligned}
 \frac{dz_1}{dt} &= -\sigma z_1 + \sigma z_2, \\
 \frac{dz_2}{dt} &= z_1 - z_2 - \sqrt{b(r-1)} z_3, \\
 \frac{dz_3}{dt} &= \sqrt{b(r-1)} z_1 + \sqrt{b(r-1)} z_2 - bz_3 + u^*,
 \end{aligned}
 \tag{15}$$

which can be written as $\frac{dz}{dt} = Az + Bu^*$, where

$$\begin{aligned} z_1 &= x_1 - \sqrt{b(r-1)}, \\ z_2 &= x_2 - \sqrt{b(r-1)}, \\ z_3 &= x_3 - (r-1). \end{aligned} \quad (16)$$

Tavazoei et al. (2009) developed a control strategy using a fractional order controller with three parameters that is used to suppress chaos. They showed that a chaotic system is stabilized using the single control input $u(t) = J^q y(t)$, where J^q is a fractional integral operator and $y(t) = -(\mu T_1 + \nu T_3)(x(t) - x^*)$, and where T_1 and T_3 are the first and third row of a transformation matrix such that

$$\bar{A} = TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -b & -c \end{bmatrix}, \quad \bar{B} = TB = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (17)$$

where the parameters a, b, c are the coefficients of the characteristic polynomial of the Jacobian matrix A

$$s^3 + cs^2 + bs + a = 0. \quad (18)$$

Tavazoei et al. (2009) also showed that for the integral fractional operator with $-1 < q < 0$ the controller stabilizes the system when

$$0 < \mu < \frac{cb^{(1-q/2)}}{\cos(-q\pi/2)}; \quad \nu > \frac{ab^{(-1-q/2)}}{\cos(-q\pi/2)}. \quad (19)$$

We use the VO operator described by Eq. (5) with a negative value of q (i.e. integral variable order operator) to suppress chaos of the Lorenz system. Choosing $\sigma = 10$, $b = 8/3$, $r = 28$ and $q = -0.2$, we obtain $0 < \mu < 2310.9$ and $\nu > 23.7$. Arbitrary values of $\mu = 23.1$ and $\nu = 26.1$ are chosen that satisfy the constraints given by Eq. (19). Figure 12(a) depicts the chaotic behaviour displayed by the Lorenz system for $t < 25$. At $t = 25$, the controller is turned on and the system is stabilized around the selected equilibrium point. Figure 12(b) shows the values of the control action, $u(t)$. In this case q has been considered constant for the VO operator.

The variable order capability of the controller can be verified by running a similar case where the parameters μ and ν are kept constant and the order of the VO derivative is changed. The controller works until the constraints given by Eq. (19) are no longer met. Fixed values for μ and ν are used. However, for $t > 25$ the order of the VO derivative $q(t)$ is monotonically decreased starting from $q = -0.2$. Figure 13(a) shows the behaviour of the system subject to the control action $u(t)$ shown in Fig. 13(b). It is observed that once the controller is turned on ($t > 25$) stabilization of the chaotic system is obtained for variable q until parameters μ and ν fall outside of the constraints. Figure 13(c) shows the variation of q over time. The controller reaches a point where it no longer stabilizes the chaotic behaviour of the system. This situation is resolved by re-calculating the values of μ and ν for the VO

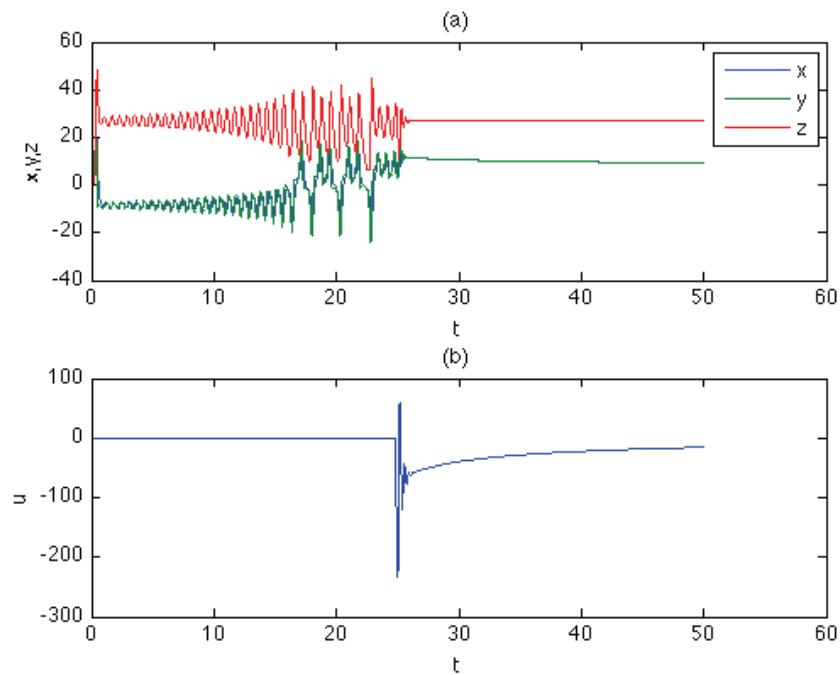


Fig. 12. Chaos suppression in the Lorenz system with $\sigma = 10$, $b = 8/3$, $r = 28$, $q = -0.2$, and fixed values of μ and ν in VO operator in Eq. (5). (a) x, y, z vs t (b) u vs t .

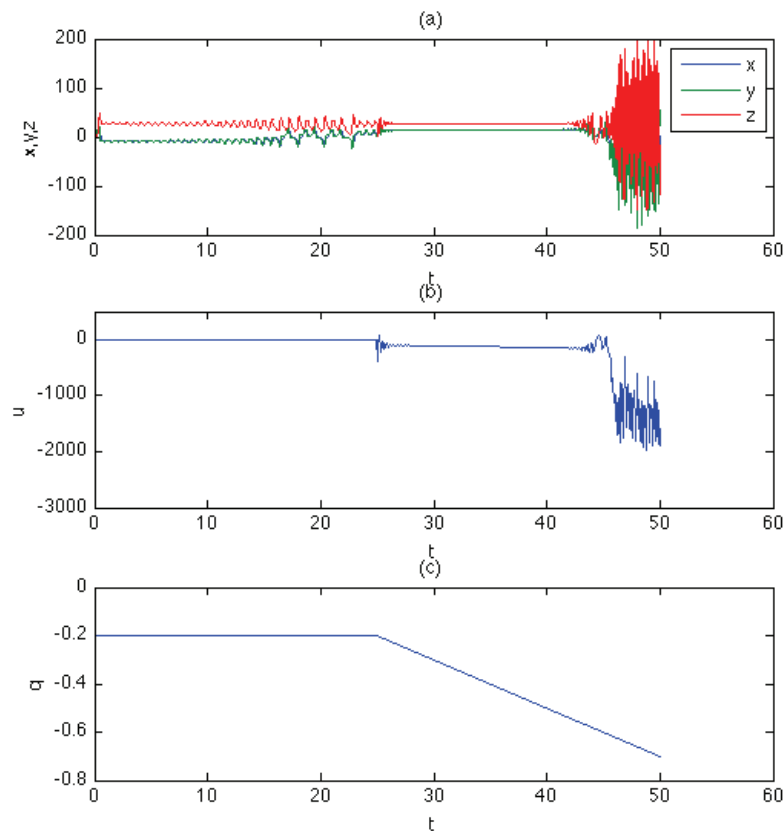


Fig. 13. Performance of controllers for fixed values of μ and ν and decreasing value of $q(t)$. (a) x, y, z vs t (b) u vs t , (c) q vs t .

value of q to remain within the required constraints. Figure 14(a) shows that the controller stabilizes the chaotic system under the variation of q with respect to time shown in Fig. 14(c) that generates the control action displayed in Fig. 14(b). The variation in the values of μ and ν is observed in Fig. 14(d) that shows that as q decreases the values of μ and ν also increase rapidly.

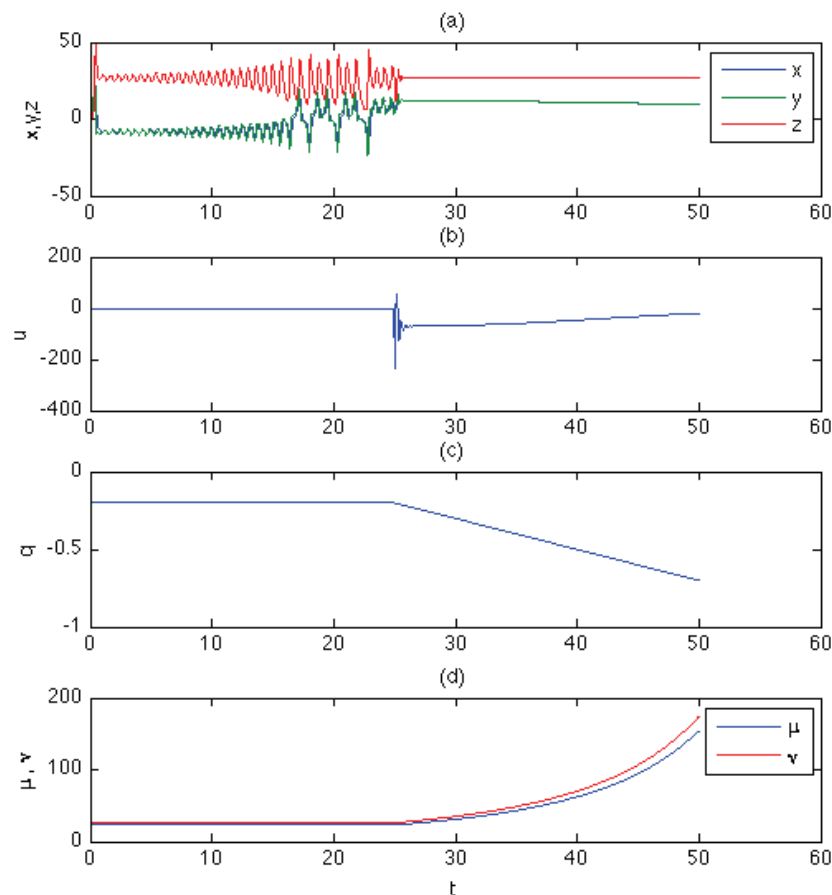


Fig. 14. Performance of controllers for variable values of μ and ν and decreasing value of $q(t)$. (a) x, y, z vs t (b) u vs t , (c) q vs t , (d) μ, ν vs t .

Grigorenko and Grigorenko (2003) have shown that the generalized fractional order Lorenz system also presents chaotic behaviour. Clearly, a VO controller technique as presented here can also be utilized to suppress chaos in such a system.

5. Conclusion

Variable order systems, i.e. systems where the order of the derivative changes with respect to either the dependent or the independent variables have not received as much attention as fractional order systems, despite of the ability of variable order formulations to model continuous spectral behavior in complex dynamics. We illustrate some of the characteristics of variable order systems and controllers through the numerical simulation of nonlinear dynamic oscillators and systems of equations. In this work, we analyze the dynamics of a modified Duffing equation, which includes a variable order derivative as the damping term,

and illustrate its behavior as compared to the classical Duffing equation. Exact feedback linearization is used to derive a linear controller of the Duffing equation with variable order damping. Finally, a variable order controller is used to suppress chaos on the Lorenz system of equations. To the best knowledge of the authors, this is the first time a variable order controller is described.

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University Campus STeP Ri
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51000 Rijeka, Croatia
Phone: +385 (51) 770 447
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Phone: +86-21-62489820
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