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On Stability of Multivariate Polynomials

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1. Introduction

In the univariate polynomial case there are only two notions of stability: Hurwitz stability for continuous polynomials, and Schur stability for discrete polynomials. However, in the multivariate polynomial case there exists a more complex situation since there are more classes of stability: Wide Sense Stable (*WSS*), Scattering Hurwitz Stable (*SHS*) and Strict Sense Stable (*SSS*) for continuous polynomials (Fettweis & Basu, 1987), and Wide Sense Schur Stable (*WSSS*), Scattering Schur (*SS*) and Strict Sense Schur Stable (*SSSS*) for discrete polynomials (Basu & Fettweis, 1987). These classes have different properties, for example some classes reduce to the Hurwitz or antiSchur univariate notion and some polynomials from some classes may lose their stability property in the presence of arbitrary small coefficient variations. Besides, between these classes has not been possible to establish a similar relationship as it does for Hurwitz and Schur univariate polynomials by the Moebius transformation (Bose, 1982).

For a long time, *SSS* and *SSSS* polynomials have been employed to obtain key properties of stability and robust stability in their own domain because they have more coincident characteristics with Hurwitz and Schur univariate notions than the other multivariate classes have (Basu & Fettweis, 1987; Fettweis & Basu, 1987). Despite of this, in this work the interest is focused in two different notions of stability: Stable class for the continuous case (Kharitonov & Torres-Muñoz, 1999), and Schur Stable class for the discrete case (Torres-Muñoz et al., 2006). The reason is twofold: firstly, both classes have the property of being the largest classes preserving stability when faced to arbitrary small coefficient variations, and secondly, it has been recently shown that any member of the Stable class is associated, by a bilinear transformation, to one member of the Schur Stable class in the same way that Hurwitz and Schur univariate polynomials are related by the Moebius transformation (Torres-Muñoz et al., 2006). Besides, both classes are the natural extension of their univariate counterpart: Hurwitz and Schur univariate classes.

In general, in the analysis and control of any system is important to have efficient, from the computational point of view, criteria to test the stability of its characteristic polynomial. For the univariate case, there is a big variety of well-known efficient algorithms to deal with the Hurwitz and Schur stabilities (Barnett, 1983; Parks & Hahn, 1992; Bhattacharyya, 1995). However, in the multivariate case this problem is more complex: in the m-variate ($m > 2$) case there are few algorithms reported and they have the problem of their efficiency (Bose, 1982). Despite of this, in the bivariate ($m = 2$) case there are a lot of algorithms to deal with the Schur Stable bivariate issue and some of them are efficient (Anderson & Jury, 1973;

Maria & Fahmy, 1973; Siljak, 1975; Bose, 1977; Jury, 1988; Yang & Unbehauen, 1998; Bistritz, 2002; Xu et al., 2004; Dumitrescu, 2006). In contrast, in the continuous bivariate case the reported algorithms are devoted to the *SSS* class (Zeheb & Walach, 1981; Bose, 1982), i.e. there are no reported algorithms dealing with the Stable bivariate class. In this work an attempt is made to give a simple and efficient criterion to the Stable bivariate class.

In the univariate case, the fact that the Moebius transformation of any Hurwitz polynomial gives a Schur polynomial and viceversa has allowed extending stability and robust stability results from one domain to the other (Parks & Hahn, 1992). In this work it is used a similar fact between the Stable and Schur Stable bivariate classes in the following way: firstly it is obtain the discrete counterpart of a given continuous bivariate polynomial, next its Schur stability is proved, and finally the Schur stability of this polynomial implies the stability of the continuous polynomial. For this, a new Schur Stable bivariate test is developed by constructing a reduced order polynomial array for univariate Schur polynomials with literal coefficients in such a way that the Schur stability of these polynomials together with specific coefficient conditions implies the stability of the original polynomial.

This work begins with the introduction of some preliminaries notions and notation of multivariate polynomials and a summary of some key properties of the Stable and Schur Stable classes. Next the problem is clearly defined, following with the presentation of the Schur Stable and Stable tests. Finally, some examples and conclusions remarks are given.

2. Preliminaries of multivariate polynomials

(Bose, 1982) A multivariate polynomial in the variable vector $\mathbf{s} = (s_1, s_2, \ldots, s_m)$ is a finite sum of the form

$$p(\mathbf{s}) = \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} \cdots \sum_{i_m=0}^{n_m} a_{i_1 i_2 \cdots i_m} s_1^{i_1} s_2^{i_2} \cdots s_m^{i_m} \tag{1}$$

where s_k, $k = 1,2,\ldots,m$ are the independent variables of partial degree $n_k = \deg_k\{p(\mathbf{s})\}$, $k = 1,2,\ldots,m$. The coefficients $a_{i_1 i_2 \cdots i_m}$ are given real (or complex numbers). One may define, following the lexicographic order of the indices, the coefficient vector

$$\mathbf{a} = (a_{00\cdots 0}, a_{10\cdots 0}, a_{20\cdots 0}, \ldots, a_{n_1 n_2 \cdots n_m}).$$

In the analysis of multivariate polynomials is very useful to write the polynomial $p(\mathbf{s})$ as an univariate polynomial with polynomial coefficients, i.e.

$$p(\mathbf{s}) = \sum_{k=0}^{n_i} a_k^{(i)}(s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m)\, s_i^k \tag{2}$$

for $i = 1,2,\ldots,m$, and where the coefficients

$$a_k^{(i)}(s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m) = \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} \cdots \sum_{i_{i-1}=0}^{n_{i-1}} \sum_{i_{i+1}=0}^{n_{i+1}} \cdots \sum_{i_m=0}^{n_m} a_{i_1 i_2 \cdots i_{i-1} k i_{i+1} \cdots i_m} s_1^{i_1} s_2^{i_2} \cdots s_{i-1}^{i_{i-1}} s_{i+1}^{i_{i+1}} \cdots s_m^{i_m}$$

are $(m-1)$-variate polynomials. In this case, the free and the main polynomial coefficients with respect to the variable s_i correspond to $k = 0$ and $k = n_i$ respectively.

A root of $p(\mathbf{s})$ is a vector $\mathbf{s_0} = (s_{10}, s_{20}, \ldots, s_{m0})$ such that $p(\mathbf{s_0}) = 0$. If $(s_2, s_3, \ldots, s_m) = (s_{20}, s_{30,\ldots}, s_{m0})$ are fixed to some arbitrary value, then $p(s_1, s_{20}, \ldots, s_{m0})$ is an univariate polynomial in the variable s_1 of degree n_1. In conclusion, and in contrast with the univariate case, a multivariate $p(\mathbf{s})$ has a finite number of root manifolds in a n-dimensional complex space.
Besides, in contrast with the univariate case, two multivariate polynomials may be coprime but possessing common roots (Kharitonov & Torres-Muñoz, 1999).
Let us denote the set of constant degree m-variate polynomials by

$$P_{\mathbf{n}} = \{p(\mathbf{s}) \,|\, \deg\{p(\mathbf{s})\} = \mathbf{n}\}$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_m)$, $n_i \in \mathcal{N}$, is the vector of constant partial degrees. Similar definitions will hold for univariate polynomials.
In the analysis of the continuous multivariate polynomials is often used the notion of the conjugate polynomial. The conjugate polynomial of $p(\mathbf{s})$ with respect to the variable s_1, using $p(\mathbf{s})$ as in the decomposition (2) with respect to the variable s_1, is given by

$$p^*(\mathbf{s}) = \overline{p}(-s_1, s_2, \ldots, s_m) = \sum_{k=0}^{n_1} \overline{a_k^{(1)}}(s_2, s_3, \ldots, s_m)\,(-s_1)^k \tag{3}$$

where $\overline{a_k^{(1)}}(s_2, s_3, \ldots, s_m)$ means that all coefficients and variables $s_2, s_3, \ldots, s_m$ are changed by their complex conjugates. Clearly, the conjugate polynomial can be taken from one until m variables. Hereafter it will be considered, unless otherwise stated, the conjugate $p^*(\mathbf{s})$ with respect to the variable s_1.
To distinguish the discrete polynomials from the continuous case, and for tradition, a discrete multivariate polynomial is notated as $q(\mathbf{z})$, the variable vector and the coefficient vector used are $\mathbf{z} = (z_1, z_2, \ldots, z_m)$ and $\mathbf{b} = (b_{00\cdots0}, b_{10\cdots0}, b_{20\cdots0}, \ldots, b_{n_1 n_2 \cdots n_m})$ respectively. Besides, the structure of a discrete polynomial is the same as (1) and it is also possible to write it as in the decomposition (2).
In the analysis of the discrete multivariate polynomials is often used the notion of the reciprocal conjugate polynomial. The reciprocal conjugate polynomial of $q(\mathbf{z})$ with respect to the variable z_1, using $q(\mathbf{z})$ as in the decomposition (2) with respect to the variable z_1, is given by

$$q^{\otimes}(\mathbf{z}) = z_1^{n_1}\,\overline{q}(z_1^{-1}, z_2, \ldots, z_m) = z_1^{n_1} \sum_{k=0}^{n_1} \overline{b_k^{(1)}}(z_2, z_3, \ldots, z_m)\, z_1^{-k} \tag{4}$$

where $\overline{b_k^{(1)}}(z_2, z_3, \ldots, z_m)$ means that all coefficients and variables $z_2, z_3, \ldots, z_m$ are changed by their complex conjugates. Clearly, the reciprocal conjugate polynomial can be taken from one until m variables. Hereafter it will be considered, unless otherwise stated, the conjugate $q^{\otimes}(\mathbf{z})$ with respect to the variable z_1.

2.1 Stable multivariate polynomials

In the continuous case consider the following polydomain

$$\Gamma_m^{(0)} = \{(s_1, s_2, \ldots, s_m) \mid \mathrm{Re}(s_i) \geq 0\,,\ i = 1,2,\ldots,m\}$$

together with its essential boundary

$$\Omega^{(m)} = \{(s_1, s_2, \ldots, s_m) \mid \mathrm{Re}(s_i) = 0\,,\ i = 1,2,\ldots,m\}.$$

Definition 1: *A multivariate polynomial $p(\mathbf{s}) \in P_{\mathbf{n}}$ is called Stable if it satisfies the following conditions*

1) $p(\mathbf{s}) \neq 0 \quad \forall \mathbf{s} \in \Gamma_m^{(0)}$.

2) Main $(m-1)$-variate polynomials $a_{n_i}^{(i)}(s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m)$, for $i = 1,2,\ldots,m$, are Stable polynomials, according to this definition, with degree $(n_1, n_2, \ldots, n_{i-1}, n_{i+1}, \ldots, n_m)$.

A polynomial satisfying just condition 1 is called Strict Sense Stable (*SSS*). Such class of polynomials has played an important role in stability and robust stability analysis (Feetweis & Basu, 1987). This class reduces to the standard notion of Hurwitz stability in the univariate case, but may lose its stability property in the presence of small arbitrary coefficients perturbations (Kharitonov & Torres-Muñoz, 1999). This fragility is very undesirable when one studies the robustness issue.

Note that the stable class is a proper subclass of the *SSS* class, as it is the largest multivariate class preserving stability under small coefficient variations (Kharitonov & Torres-Muñoz, 1999). Besides, the stable class reduces to the traditional Hurwitz class in the univariate case, so the stable class also preserves several useful properties of univariate Hurwitz polynomials too (Kharitonov & Torres-Muñoz, 1999; Kharitonov & Torres-Muñoz, 2002). A summary of some of properties of the stable multivariate class is the following.

Lemma 1: *Let $p(\mathbf{s}) \in P_{\mathbf{n}}$ be a stable multivariate polynomial. Let s_1 be fixed at some value s_{10} such that $\mathrm{Re}(s_{10}) \geq 0$. Then the $(m-1)$-variate polynomial $p(s_{10}, s_2, \ldots, s_m)$ is a stable polynomial of degree $(n_2, n_3, \ldots, n_m)$.*

Lemma 2: *Let $p(\mathbf{s}) \in P_{\mathbf{n}}$ be a stable multivariate polynomial. Assume that $n_1 > 0$, then the polynomial $\hat{p}(\mathbf{s}) = s_1^{n_1} p(s_1^{-1}, s_2, \ldots, s_m)$ is a stable multivariate polynomial of the same degree as $p(\mathbf{s})$.*

Notice that, by successive aplication, in Lemma 1 and Lemma 2 can be taken from one until m variables.

Next result is the extension of the Lucas' Theorem for the Hurwitz univariate polynomials (Marden, 1949), i.e. it shows the invariance of the stability property under differentiation that can be taken, by successive application, from one until m variables.

Theorem 3: *Let $p(\mathbf{s}) \in P_{\mathbf{n}}$ be a stable multivariate polynomial. Assume that $n_1 > 0$, then the polynomial $\tilde{p}(\mathbf{s}) = \dfrac{\partial p(s_1, s_2, \ldots, s_m)}{\partial s_1}$ is a stable multivariate polynomial of degree $(n_1 - 1, n_2, \ldots, n_m)$.*

Lemma 4: *Let $p(\mathbf{s}) \in P_{\mathbf{n}}$ be a stable multivariate polynomial. Then all $(m-1)$-variate polynomial coefficients $a_k^{(i)}(s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m)$ for $k = 0,1,\ldots,n_i$, in the decomposition (2) with respect to the variable s_i for $i = 1,2,\ldots,m$, are stable polynomials of degree $(n_1, n_2, \ldots, n_{i-1}, n_{i+1}, \ldots, n_m)$.*

Next property is the extension of the classical Stodola's Condition for Hurwitz univariate polynomials (Gantmacher, 1959).

Theorem 5: *Let $p(\mathbf{s}) \in P_{\mathbf{n}}$ be a stable multivariate polynomial with real coefficients. Then all coefficients $a_{i_1 i_2 \cdots i_m}$ of the polynomial have the same sign: either all of them are positive, or all of them are negative.*

Lemma 6: *Let $p(\mathbf{s}) \in P_{\mathbf{n}}$ be a stable multivariate polynomial. Then the main coefficient $a_{n_1 n_2 \cdots n_m}$ is not zero.*

Proof: *For the case $m = 1$ the statement is obvious.*

For $m = 2$, consider a stable bivariate polynomial $p(s_1, s_2)$ of degree (n_1, n_2). By Definition 1, its main univariate polynomial coefficient $a_{n_1}^{(1)}(s_2)$, in the decomposition (2) with respect to the variable s_1, is a Hurwitz stable polynomial of degree n_2, i.e. coefficient $a_{n_1 n_2} \neq 0$.

Assume that the statement is true for $(m-1)$-variate polynomials and consider the m-variate case. Given a stable multivariate polynomial $p(\mathbf{s})$ of degree $(n_1, n_2, \ldots, n_m)$, from Definition 1 follows that its main $(m-1)$-variate polynomial coefficient $a_{n_1}^{(1)}(s_2, s_3, \ldots, s_m)$, in the decomposition (2) with respect to the variable s_1, is a stable $(m-1)$-variate polynomial of degree $(n_2, n_3, \ldots, n_m)$, then, by the induction hypothesis, coefficient $a_{n_1 n_2 \cdots n_m} \neq 0$.

Next results show that for stable multivariate polynomials the robust stability can be considered without structural restrictions on uncertain parameters, and that a stable multivariate polynomial has no roots close to the essential boundary.

Theorem 7: *Let $p(\mathbf{s}) \in P_{\mathbf{n}}$ be a stable multivariate polynomial. Then there always exists ε>0 such that every multivariate polynomial with a coefficient vector lying in the ε-neighbourhood of the coefficient vector of $p(\mathbf{s})$ is stable too.*

Theorem 8: *Let $p(\mathbf{s}) \in P_{\mathbf{n}}$ be a stable multivariate polynomial. Then there always exists ε>0 such that it has no roots in the ε-neighbourhood of the essential boundary $\Omega^{(m)}$.*

2.2 Schur Stable multivariate polynomials

In the discrete domain consider the polydisc given by

$$U_m^{(0)} = \{(z_1, z_2, \ldots, z_m) \mid |z_i| \geq 1,\ i = 1,2,\ldots,m\}$$

and its essential boundary given by

$$T^{(m)} = \{(z_1, z_2, \ldots, z_m) \mid |z_i| = 1,\ i = 1,2,\ldots,m\}.$$

Definition 2: *A multivariate polynomial $q(\mathbf{z}) \in P_{\mathbf{n}}$ is called Schur Stable if $q(\mathbf{z}) \neq 0 \quad \forall \mathbf{z} \in U_m^{(0)}$.*

The so-called Strict Sense Schur Stable (*SSSS*) class is often employed in the literature and it considers a $q(\mathbf{z}) \in P_{\mathbf{n}}$ *SSSS* if $q(\mathbf{z}) \neq 0$ for all $\mathbf{z} \in \{(z_1, z_2, \ldots, z_m) \mid |z_i| \leq 1,\ i = 1,2,\ldots,m\}$, (Basu & Feetweis, 1987). Despite of *SSSS* class preserves stability under small coefficient variations, it reduces to the standard antiSchur polynomials notion in the univariate case, so some key properties of Schur univariate polynomials can not be extended to the multivariate case as the invariance of the Schur stability property under differentiation (Torres-Muñoz et al., 2006).

The Schur stable class, in the sense of Definition 2, is also used in the literature (Huang, 1972; Kaczorek, 1985). Actually, it is the reciprocal class of the *SSSS* class and also preserves stability under small coefficient variations. Besides, the Schur stable class reduces to the standard Schur class in the univariate case, so several useful properties may be extended from the univariate Schur polynomials to the multivariate case (Torres-Muñoz et al., 2006). A summary of some of the properties of the Schur stable multivariate class is the following.

Lemma 9: *Let* $q(\mathbf{z}) \in P_{\mathbf{n}}$ *be a Schur stable multivariate polynomial. Let* z_1 *be fixed at some value* z_{10} *such that* $|z_{10}| \geq 1$. *Then the* $(m-1)$*-variate polynomial* $q(z_{10}, z_2, \ldots, z_m)$ *is a Schur stable polynomial of degree* $(n_2, n_3, \ldots, n_m)$.

Lemma 10: *Let* $q(\mathbf{z}) \in P_{\mathbf{n}}$ *be a Schur stable multivariate polynomial. Assume that* $n_1 > 0$, *then the polynomial* $\tilde{q}(\mathbf{z}) = q(-z_1, z_2, \ldots, z_m)$ *is a Schur stable multivariate polynomial of the same degree as* $q(\mathbf{z})$.

Theorem 11: *Let* $q(\mathbf{z}) \in P_{\mathbf{n}}$ *be a Schur stable multivariate polynomial. Assume that* $n_1 > 0$, *then the polynomial* $\tilde{q}(\mathbf{z}) = \dfrac{\partial q(z_1, z_2, \ldots, z_m)}{\partial z_1}$ *is a Schur stable multivariate polynomial of degree* $(n_1 - 1, n_2, \ldots, n_m)$.

Notice that Lemma 9, Lemma 10 and Theorem 11 are the discrete version of Lemma 1, Lemma 2 and Theorem 3 respectively, then it can be also taken from one until m variables.

Lemma 12: *Let* $q(\mathbf{z}) \in P_{\mathbf{n}}$ *be a Schur stable multivariate polynomial. Then the main* $(m-1)$*-variate polynomial coefficients* $b_{n_i}^{(i)}(z_1, z_2, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)$, *for* $i = 1, 2, \ldots, m$, *in the decomposition (2) with respect to the variable* z_i, *are Schur stable polynomials of degree* $(n_1, n_2, \ldots, n_{i-1}, n_{i+1}, \ldots, n_m)$.

It is worth to mention that other polynomial coefficients, different from the main polynomial coefficients, in the decomposition (2), are not necessarily Schur stable (Torres-Muñoz et al., 2006).

Next property is the extension of the classical coefficient condition for Schur univariate polynomials (Bhattacharyya, 1995).

Lemma 13: *Let* $q(\mathbf{z}) \in P_{\mathbf{n}}$ *be a Schur stable multivariate polynomial. Then the coefficient condition* $|b_{00\cdots0}| < |b_{n_1 n_2 \cdots n_m}|$ *is hold.*

Corollary 14: *Let* $q(\mathbf{z}) \in P_{\mathbf{n}}$ *be a Schur stable multivariate polynomial. Then the main coefficient* $b_{n_1 n_2 \cdots n_m}$ *is not zero.*

Proof*: It directly follows from Lemma 13 and the fact that* $0 \leq |b_{00\cdots0}|$.

Next results show that the Schur stable multivariate class is suitable to study the robustness issue.

Theorem 15: *Let* $q(\mathbf{z}) \in P_{\mathbf{n}}$ *be a Schur stable multivariate polynomial. Then there always exists* $\varepsilon > 0$ *such that every multivariate polynomial with a coefficient vector lying in the* ε*-neighbourhood of the coefficient vector of* $q(\mathbf{z})$ *is Schur stable too.*

Theorem 16: *Let* $q(\mathbf{z}) \in P_{\mathbf{n}}$ *be a Schur stable multivariate polynomial. Then there always exists* $\varepsilon > 0$ *such that it has no roots in the* ε*-neighbourhood of the essential boundary* $T^{(m)}$.

3. Problem statement

From a practical point of view is essential to dispose of computationally feasible polynomial stability criteria. For the univariate case, there are some very well-known efficient stability criteria (Barnett, 1983; Parks & Hahn, 1992; Bhattacharyya, 1995), but for the m-variate case there aren't. However, there are some criteria for the bivariate case (Jury, 1988; Bistritz, 2002; Xu et al., 2004; Dumitrescu, 2006), but their implementation in the multivariate case is not easy. In this work the main goal is to tackle the following.

Problem: *Given a continuous bivariate polynomial $p(s_1, s_2)$, find an efficient polynomial coefficients dependent criterion allowing to conclude whether or not it belongs to the stable class, in the sense of Definition 1. This criterion must be also potentially suitable for its extension to the multivariate case.*

At first glance, by nature of the continuous stable class, trying to obtain non-recursive criteria might be a hard task. This contrasts with the discrete Schur case where research efforts leaded to reliable algorithms allowing to analyze stability depending on the polynomial coefficients in a finite number of steps.

In the univariate polynomial case, Hurwitz stability implies Schur stability and viceversa. This correspondence has allowed to translate stability results between continuous and discrete domains. For instance, translation of Routh-Hurwitz stability criterion inspired the development of coefficient-based algorithms for Schur stability (Parks & Hahn, 1992).

In such a vein, the belief that *SSS* bivariate polynomials are in strict equivalence with Schur stable bivariate polynomials was in the center of earlier attempts to develope a bivariate stability theory. In these attempts were used a different transformation of transformation (5). Unfortunately, the early conclusion was only *SSS* stability is implied by Schur stability and not in the reverse sense (Bose, 1982). The same conclusion is obtained using transformation (5): consider the *SSS* polynomial $p(\mathbf{s}) = s_1 s_2 + s_2 + 1$, it turns out that the transformed discrete polynomial, using transformation (5), $q(\mathbf{z}) = 3z_1 z_2 - z_1 + z_2 + 1$ is not Schur stable as it has the root $(-1,1) \in U_2^{(0)}$, (Torres-Muñoz et al., 2006) . Therefore, there is no way to infer stability results between *SSS* and Schur stable classes.

However, recently was shown that the multivariate stable class in the sense of Definition 1 is the counterpart of the multivariate Schur stable class in the sense of Definition 2.

Theorem 17: *(Torres-Muñoz et al., 2006) The polynomial $p(\mathbf{s})$ of degree $(n_1, n_2, \ldots, n_m)$ is stable if and only if the polynomial*

$$q(\mathbf{z}) = (z_1 - 1)^{n_1} (z_2 - 1)^{n_2} \cdots (z_m - 1)^{n_m} \, p\left(\frac{z_1 + 1}{z_1 - 1}, \frac{z_2 + 1}{z_2 - 1}, \ldots, \frac{z_m + 1}{z_m - 1} \right) \tag{5}$$

is a Schur stable polynomial of degree $(n_1, n_2, \ldots, n_m)$.

Observe that this transformation is the natural extension of the Moebius univariate transformation. This result was used as a bridge to translate properties and stability results from the continuous domain to the discrete one and viceversa (Torres-Muñoz et al., 2006).

4. An indirect criterion for continuous bivariate polynomial stability

On the basis of Theorem 17, an indirect bivariate continuous stability algorithm can be stated as follows:

Given a continuous bivariate polynomial

$$p(\mathbf{s}) = \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} a_{i_1 i_2} s_1^{i_1} s_2^{i_2}$$

1. Construct the discrete polynomial $q(\mathbf{z})$ using the transformation (5). If $q(\mathbf{z})$ has the same degree as $p(\mathbf{s})$, then continue. If it is not the case, then the polynomial $p(\mathbf{s})$ is not stable.
2. Apply any Schur bivariate stability test to $q(\mathbf{z})$.
3. If $q(\mathbf{z})$ is Schur stable, then the polynomial $p(\mathbf{s})$ is stable. If it is not the case, then the polynomial $p(\mathbf{s})$ is not stable.

It is worth noticing that there exists a variety of criteria for the Schur bivariate stability case (Jury, 1988; Bistritz 2002) that might be potentially adapted to the case of continuous bivariate stable polynomials in the step 2. However, a new simple Schur stability test was introduced recently as one alternative way to tackle the problem of giving a reliable criterion for the continuous stable class (Rodriguez-Angeles et al., 2007). The underlying philosophy is, inspired on the univariate case, to try to find an array of reduced degree polynomials whose stability will imply stability of the original polynomial (Bhattacharyya et al., 1995).

Theorem 18: *(Rodriguez-Angeles et al., 2007) The polynomial* $q(z_1, z_2) \in P_{\mathbf{n}}$ *is Schur stable if and only if*

i) $q(z_{10}, z_2) \neq 0$ *for all* $|z_2| \geq 1$ *and for a fixed* $z_1 = z_{10}$ *such that* $|z_{10}| \geq 1$.

ii) Given the following polynomial sequence

$$q^{(j+1)}(z_1, z_{20}) = \frac{1}{z_1}\left\{b_{n_1-j}^{(1,j)}(z_{20})\, q^{(j)}(z_1, z_{20}) - b_0^{(1,j)}(z_{20})\left[q^{(j)}(z_1, z_{20})\right]^{\otimes}\right\} \tag{6}$$

the following inequality holds

$$\left|b_{n_1-j}^{(1,j)}(z_{20})\right| > \left|b_0^{(1,j)}(z_{20})\right| \tag{7}$$

for all $z_2 = z_{20}$ *such that* $|z_{20}| = 1$, *where* $q^{(0)}(z_1, z_{20}) = q(z_1, z_{20})$ *and* $b_k^{(1,j)}(z_{20})$ *is the* k*-th coefficient of the* j*–th polynomial* $q^{(j)}(z_1, z_{20})$ *for* $j = 0,1,\ldots,n_1 - 1$.

The Schur bivariate stability algorithm, based on Theorem 18, can be stated as follows:
Given a discrete bivariate polynomial in the decomposition (2) with respect to the variable z_1

$$q(\mathbf{z}) = \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} b_{i_1 i_2} z_1^{i_1} z_2^{i_2} = \sum_{k=0}^{n_1} b_k^{(1)}(z_2)\, z_1^k$$

1. Verify if the univariate polynomial $q(1, z_2)$ is Schur stable. If it is Schur stable, then continue. If it is not the case, then the bivariate polynomial $q(\mathbf{z})$ is not Schur stable.
2. Verify step by step if the inequality

$$\left|b_{n_1-j}^{(1,j)}\left(e^{j\theta}\right)\right| > \left|b_0^{(1,j)}\left(e^{j\theta}\right)\right|$$

holds for all $\theta \in [0, 2\pi]$ and for $j = 0,1,\ldots,n_1 - 1$, and where coefficients $b_{n_1-j}^{(1,j)}\left(e^{j\theta}\right)$ and $b_0^{(1,j)}\left(e^{j\theta}\right)$ are obtained from sequence (6) with $z_2 = e^{j\theta}$. If all inequalities hold, then the bivariate polynomial $q(\mathbf{z})$ is Schur stable. If one of the coefficient conditions fails, then stop and the bivariate polynomial $q(\mathbf{z})$ is not Schur stable.

Actually, the step 2 can be implemented in a numerical and graphical way providing a simple test for Schur bivariate stability. Besides, notice that the graphical testing is just needed in a bounded interval independently of the polynomial vector degree.
Example 1: *Determine the stability of the continuous bivariate polynomial*

$$p(\mathbf{s}) = \left(0.75 + s_2 + 1.25 s_2^2\right) + \left(1 + 2s_2 + 3s_2^2\right)s_1 + \left(1.25 + 3s_2 + 2.75 s_2^2\right)s_1^2 .$$

According to Theorem 17 the stability of $p(\mathbf{s})$ is equivalent to the Schur stability of the transformed polynomial $q(\mathbf{z})$ given by

$$q(\mathbf{z}) = \left(0.25 z_2^2\right) + \left(0.25 z_2 + 0.5 z_2^2\right)z_1 + \left(0.25 + 0.5 z_2 + z_2^2\right)z_1^2 .$$

Hence $q(\mathbf{z})$ has the same degree as $p(\mathbf{s})$, one has to check the Schur stability of $q(\mathbf{z})$. Following the algorithm for Schur bivariate stability, one may verify the step 1. Then let us tackle the step 2.
The first polynomial of the sequence (6) is

$$q^{(0)}\left(z_1, e^{j\theta}\right) = \left(0.25 e^{j2\theta}\right) + \left(0.25 e^{j\theta} + 0.5 e^{j2\theta}\right)z_1 + \left(0.25 + 0.5 e^{j\theta} + e^{j2\theta}\right)z_1^2 .$$

From Figure 1 one can see that the inequality $\left|b_2^{(1,0)}\left(e^{j\theta}\right)\right| > \left|b_0^{(1,0)}\left(e^{j\theta}\right)\right|$ holds for all $\theta \in [0, 2\pi]$. Then, let us continue with the test.

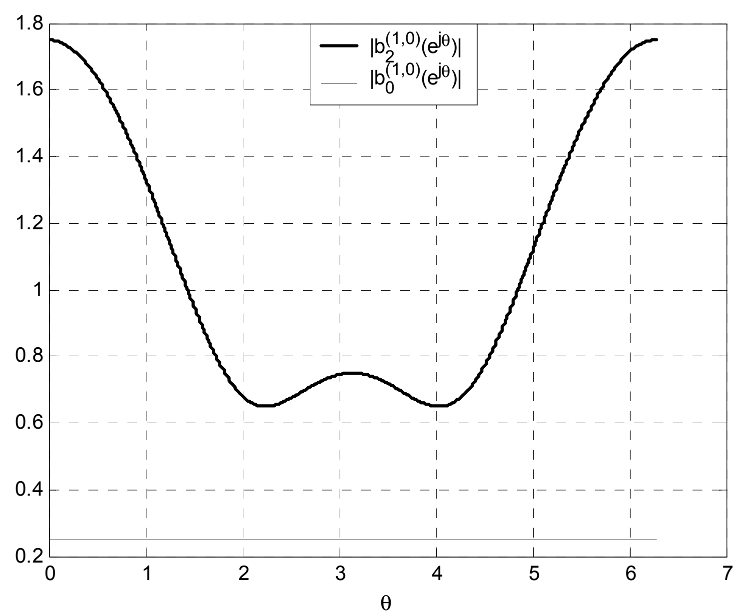

Figure 1. Coefficient condition for the 1st polynomial $q^{(0)}\left(z_1, e^{j\theta}\right)$ of the polynomial sequence

The second polynomial of the sequence (6) is

$$q^{(1)}\left(z_1, e^{j\theta}\right) = \left(0.5 + 0.25 e^{-j\theta} + 0.25 e^{j\theta} + 0.125 e^{j2\theta}\right) + \left(1.25 + 0.625 e^{-j\theta} + 0.625 e^{j\theta} + 0.25 e^{-j2\theta} + 0.25 e^{j2\theta}\right)z_1 .$$

From Figure 2 one can see that the inequality $\left|b_1^{(1,1)}\left(e^{j\theta}\right)\right| > \left|b_0^{(1,1)}\left(e^{j\theta}\right)\right|$ holds for all $\theta \in [0, 2\pi]$. Then the discrete bivariate polynomial $q(\mathbf{z})$ is Schur stable as it is reported in several papers

(Huang, 1972; Jury, 1988; Bistritz, 2002). Therefore, the continuous bivariate polynomial $p(\mathbf{s})$ is stable.

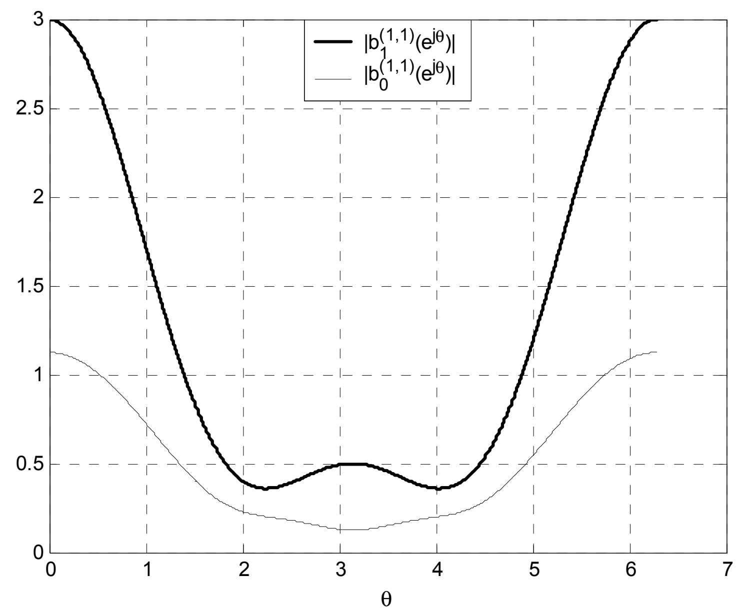

Figure 2. Coefficient condition for the 2nd polynomial $q^{(1)}(z_1, e^{j\theta})$ of the polynomial sequence

5. Numerical examples

The aim is to show the potential applicability of the indirect algorithm presented in the previous section when dealing with bivariate polynomials of relatively high degree. From a computational point of view, it is instrumental to take into account the relationship between the coefficients of a continuous multivariate polynomial $p(\mathbf{s})$ and those of its discrete counterpart $q(\mathbf{z})$. Actually, the coefficients of the bivariate polynomials $p(s_1, s_2)$ and $q(z_1, z_2)$ are related by a linear transformation as it is expressed in the following.

Theorem 19: *Let* $p(\mathbf{s}) = \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} a_{i_1 i_2} s_1^{i_1} s_2^{i_2}$ *be a given bivariate polynomial of degree* (n_1, n_2)*. Consider its transformed discrete bivariate polynomial*

$$q(\mathbf{z}) = (z_1 - 1)^{n_1}(z_2 - 1)^{n_2}\, p\left(\frac{z_1 + 1}{z_1 - 1}, \frac{z_2 + 1}{z_2 - 1}\right) \tag{8}$$

then $q(\mathbf{z})$ *can be expressed as*

$$q(\mathbf{z}) = \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} b_{i_1 i_2} z_1^{i_1} z_2^{i_2}$$

where the coefficients $a_{i_1 i_2}$ *and* $b_{i_1 i_2}$ *are related as follows*

$$b_{i_1 i_2} = \sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2} a_{k_1 k_2}\left[\sum_{l_1=m_1}^{r_1}\sum_{l_2=m_2}^{r_2}\binom{n_1 - j_1}{l_1}\binom{n_2 - j_2}{l_2}\binom{j_1}{i_1 - l_1}\binom{j_2}{i_2 - l_2}(-1)^{\phi}\right]$$

where

$$\begin{aligned}
k_1 &= n_1 - j_1,\\
k_2 &= n_2 - j_2,\\
m_1 &= i_1 - \min(i_1, j_1),\\
m_2 &= i_2 - \min(i_2, j_2),\\
r_1 &= \min(i_1, n_1 - j_1),\\
r_2 &= \min(i_2, n_2 - j_2),\\
\phi &= j_1 + j_2 + l_1 + l_2 - i_1 - i_2.
\end{aligned}$$

Corollary 20: *Let $p(\mathbf{s})$ and $q(\mathbf{z})$ two bivariate polynomials related as in (8) with coefficient vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The coefficient vectors are related through the matrix equation $\mathbf{b} = \mathbf{T}\mathbf{a}$. This relationship can be expressed as*

$$\begin{bmatrix} \mathbf{b}_{n_1} \\ \mathbf{b}_{n_1-1} \\ \vdots \\ \mathbf{b}_0 \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathbf{a}_{n_1} \\ \mathbf{a}_{n_1-1} \\ \vdots \\ \mathbf{a}_0 \end{bmatrix}$$

where $\mathbf{a}_i = [a_{in_2}, a_{in_2-1}, \ldots, a_{i0}]^T$, $\mathbf{b}_i = [b_{in_2}, b_{in_2-1}, \ldots, b_{i0}]^T$ *and* $\mathbf{T}$ *is a constant nonsingular matrix given by*

$$\mathbf{T} = \begin{bmatrix} T_{1,1} & T_{2,1} & \cdots & T_{n_1+1,1} \\ T_{1,2} & T_{2,2} & \cdots & T_{n_1+1,2} \\ \vdots & \vdots & \ddots & \vdots \\ T_{1,n_1+1} & T_{2,n_1+1} & \cdots & T_{n_1+1,n_1+1} \end{bmatrix}$$

where

$$T_{i,j} = \begin{bmatrix} t_{1,1}^{(i,j)} & t_{1,2}^{(i,j)} & \cdots & t_{1,n_2+1}^{(i,j)} \\ t_{2,1}^{(i,j)} & t_{2,2}^{(i,j)} & \cdots & t_{2,n_2+1}^{(i,j)} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n_2+1,1}^{(i,j)} & t_{n_2+1,2}^{(i,j)} & \cdots & t_{n_2+1,n_2+1}^{(i,j)} \end{bmatrix}$$

with

$$t_{k,l}^{(i,j)} = \sum_{p=m_1}^{r_1} \sum_{q=m_2}^{r_2} \binom{n_1 - i + 1}{p} \binom{n_2 - l + 1}{q} \binom{i-1}{n_1 - j - p + 1} \binom{l-1}{n_2 - k - q + 1} (-1)^{\phi}$$

where

$$\begin{aligned}
m_1 &= n_1 - j - \min(n_1 - j + 1, i - 1) + 1,\\
m_2 &= n_2 - k - \min(n_2 - k + 1, l - 1) + 1,\\
r_1 &= \min(n_1 - j + 1, n_1 - i + 1),\\
r_2 &= \min(n_2 - k + 1, n_2 - l + 1),\\
\phi &= i + j + k + l + p + q - n_1 - n_2.
\end{aligned}$$

Notice that previous statements may be deduced by straightforward matrix calculations from the transformation (8).

Besides, for the computational implementation it is useful to write a polynomial $p(\mathbf{s}) = \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} a_{i_1 i_2} s_1^{i_1} s_2^{i_2}$ in a matrix form, i.e.

$$p(\mathbf{s}) = \mathbf{s}_1^T \mathbf{A} \mathbf{s}_2 \tag{9}$$

where $\mathbf{s}_i = \left[1, s_i, \ldots, s_i^{n_i}\right]^T$ for $i = 1,2$ and

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n_2} \\ a_{10} & a_{11} & \cdots & a_{1n_2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_1 0} & a_{n_1 1} & \cdots & a_{n_1 n_2} \end{bmatrix}.$$

Example 2: *Check the stability of the* $(11,6)$*-degree continuous polynomial* $p(\mathbf{s})$ *expressed in the form (9) with*

$$\mathbf{A} = \begin{bmatrix} 31.48 & 204.27 & 208.83 & 539.64 & 303.39 & 195.02 & 25.37 \\ 309.64 & 1627.41 & 4783.46 & 6549.36 & 6401.40 & 2713.10 & 573.71 \\ 1262.99 & 6618.74 & 19317.08 & 20982.39 & 18707.68 & 6668.39 & 1126.03 \\ 4709.04 & 27049.89 & 80827.85 & 97595.24 & 93651.45 & 34208.89 & 7844.98 \\ 7542.04 & 39201.83 & 108598.78 & 120303.72 & 99540.80 & 34799.89 & 6280.04 \\ 15449.72 & 84868.56 & 250832.28 & 295166.17 & 271691.44 & 94090.95 & 23230.21 \\ 10114.02 & 49422.82 & 146547.85 & 155032.07 & 128235.49 & 42495.42 & 8600.23 \\ 11403.69 & 61815.44 & 192019.44 & 223116.98 & 214378.74 & 71019.48 & 20629.75 \\ 3268.07 & 16771.33 & 45043.39 & 52752.20 & 41378.44 & 14103.27 & 3229.63 \\ 1869.30 & 9736.73 & 34143.05 & 39425.71 & 41892.57 & 13418.74 & 4809.57 \\ 170.10 & 771.73 & 2855.26 & 2569.22 & 2729.21 & 610.5541 & 268.0381 \\ 59.20 & 131.84 & 942.98 & 482.68 & 1371.35 & 260.29 & 229.77 \end{bmatrix}.$$

Applying Theorem 19 or Corollary 20 it is possible to find its discrete counterpart $q(\mathbf{z}) = \mathbf{z}_1^T \mathbf{B} \mathbf{z}_2$ with

$$\mathbf{B} = \begin{bmatrix} 1.0104 & 1.2885 & 1.0728 & 1.1404 & 1.3005 & 4.0107 & 8.6281 \\ 1.3895 & 1.1343 & 1.6465 & 2.1553 & 1.1160 & 1.6695 & 5.6868 \\ 1.5998 & 1.0654 & 2.1054 & 1.6767 & 2.9511 & 6.8432 & 1.6960 \\ 1.2380 & 1.3298 & 1.3324 & 1.1543 & 2.8162 & 1.9281 & 1.0090 \\ 1.6711 & 1.1183 & 2.1497 & 1.4480 & 5.0204 & 1.0897 & 2.8963 \\ 1.1488 & 4.7785 & 2.1156 & 1.0795 & 9.2660 & 2.7439 & 1.6748 \\ 7.7516 & 1.8247 & 1.0305 & 5.4882 & 2.4008 & 3.3660 & 0.64643 \\ 3.0988 & 1.5180 & 3.0174 & 1.3338 & 6.1266 & 1.2426 & 1.9972 \\ 3.1737 & 1.9988 & 1.3906 & 3.7818 & 1.4017 & 1.1261 & 0.083623 \\ 1.4275 & 2.4748 & 2.0228 & 1.4274 & 4.4073 & 10.103 & 1.4442 \\ 1.1867 & 1.8995 & 4.3172 & 3.6410 & 1.1026 & 1.3655 & 1.4409 \\ 2.3584 & 1.0134 & 1.3097 & 2.0179 & 1.6222 & 2.0122 & 27.189 \end{bmatrix}.$$

Hence $q(\mathbf{z})$ has the same degree as $p(\mathbf{s})$, one has to check the Schur stability of $q(\mathbf{z})$. Following the Schur bivariate stability algorithm, it is easy to check that step 1 holds. Then let us proceed to check step 2 of the algorithm.

Actually, the sequence of polynomials (6) and the coefficient conditions (7) of the Schur bivariate stability algorithm may be easily implemented in a numerical way. Indeed one may generate step by step a sequence of graphics allowing to decide if such conditions are satisfied. In this example we have the following graphics.

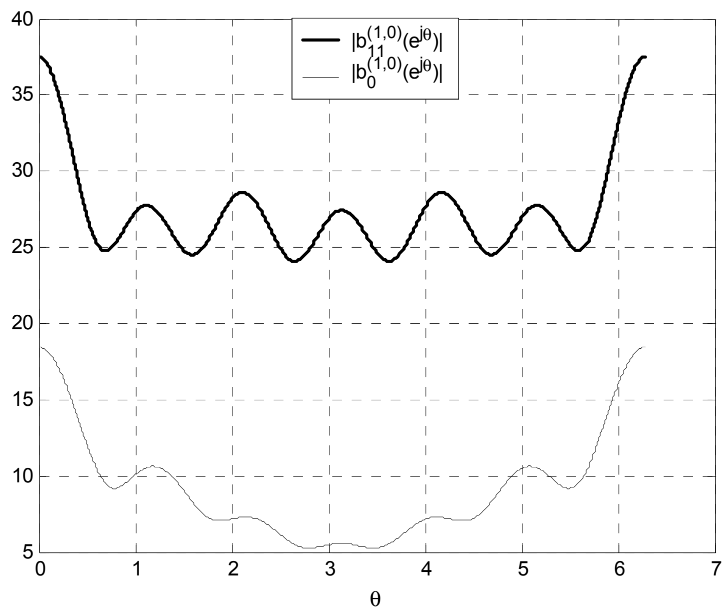

Figure 3. Coefficient condition for the 1st polynomial $q^{(0)}(z_1, e^{j\theta})$ of the polynomial sequence

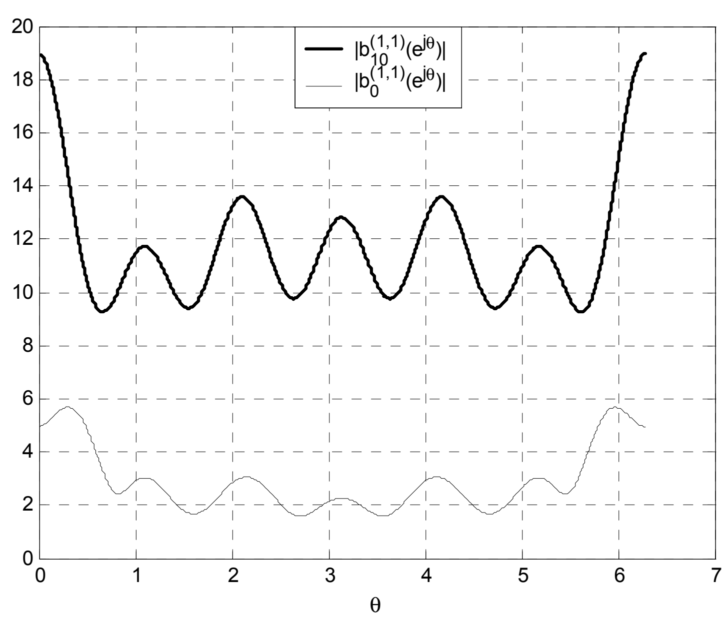

Figure 4. Coefficient condition for the 2nd polynomial $q^{(1)}(z_1, e^{j\theta})$ of the polynomial sequence

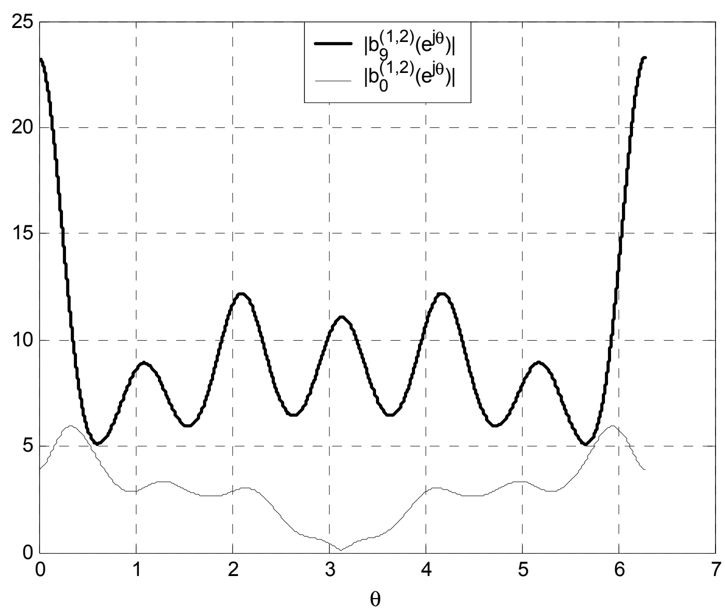

Figure 5. Coefficient condition for the 3rd polynomial $q^{(2)}\left(z_1, e^{j\theta}\right)$ of the polynomial sequence

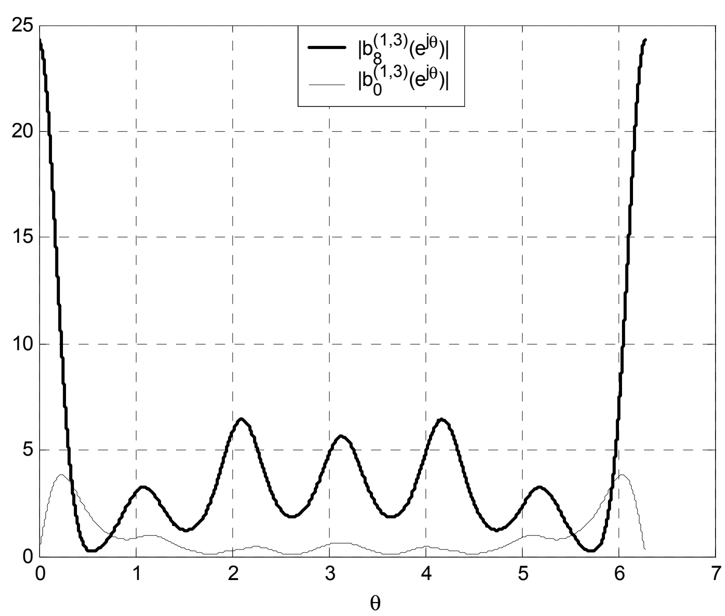

Figure 6. Coefficient condition for the 4th polynomial $q^{(3)}\left(z_1, e^{j\theta}\right)$ of the polynomial sequence

The first graphics, Figure 3, Figure 4 and Figure 5, show that condition (7) is hold. However, the last graphic, Figure 6, shows that condition (7) is not respected. Certainly, polynomial $q(\mathbf{z})$ is not Schur stable and by consequence polynomial $p(\mathbf{s})$ is not stable.

7. Conclusions

In this work, an unified multivariate polynomial stability theory was considered and it is based on the Stable and the Schur stable multivariate classes, for continuous and discrete domains respectively. The main focus was to give feasible criteria to determine whether or not a continuous bivariate polynomial belongs to the Stable class.

In a direct approach, the recursive nature of the continuous Stable class imposes the needing of checking Hurwitz stability of the two main univariate polynomial coefficients, where partial degree preservation is required as well, and the *SSS* stability of the original polynomial. To check the first items one can use every of the well-known univariate criteria, and to check the *SSS* stability there are some criteria that can be used, but them have some efficient problems.

In an indirect approach, the stability of a continuous bivariate polynomial is deduced by analyzing the Schur stability of its discrete bivariate polynomial counterpart. Firstly, the method presented in this work requires of checking Schur stability of a constant coefficients univariate polynomial, and secondly checking Schur stability of an univariate polynomial with literal coefficients. To check the first item there are no problem. To check the last item it is necessary the fulfilment of a sequence of coefficient conditions, of the form (7), in the finite frequency interval $\theta \in [0,2\pi]$. If these two items are satisfied, then the continuous polynomial belongs to the Stable class. Because of its simplicity, coefficient conditions are feasible in a graphical manner and, by construction, the complexity of the algorithm is independent of the polynomial degree.

In a future work, the extension of the proposed indirect bivariate algorithm to the multivariate case can be analyzed, and there are another way to use the relationship between Stable and Schur stable multivariate polynomials: obtain a direct continuous stability criterion by translating, through the relation (5), an existing Schur stable test.

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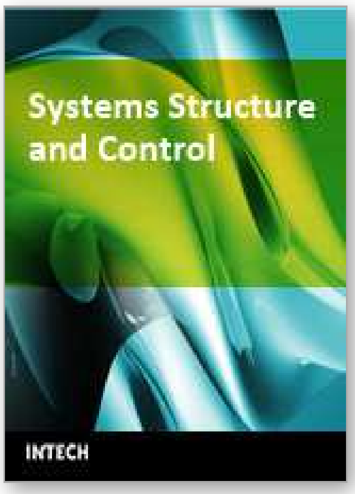

Systems Structure and Control
Edited by Petr Husek

ISBN 978-953-7619-05-3
Hard cover, 248 pages
Publisher InTech
Published online 01, August, 2008
Published in print edition August, 2008

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