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# Enhanced Fuzzy Controller for Nonlinear Systems: Membership-Function-Dependent Stability Analysis Approach

H.K. Lam and Mohammad Narimani

*Division of Engineering, The King's College London, Strand, London  
United Kingdom*

## 1. Introduction

Fuzzy-model-based (FMB) control approach provides a systematic and effective way to control nonlinear systems. It has been shown by various applications (Lam et al., 1998; Lian et al., 2006; (b)Tanaka et al., 1998) that FMB control approach performs superior to some traditional control approaches. Based on the T-S fuzzy model (Sugeno & Kang, 1988; Takagi & Sugeno, 1985) the system dynamics of the nonlinear can be represented by some local linear models in the form of linear state-space equations. With the fuzzy logic technique, the overall system dynamics of the nonlinear plant is a fuzzy combination of the local linear models. Consequently, the fuzzy model offers a systematic way and general framework to represent the nonlinear plants in the form of averaged weighted sum of local linear systems. This particular structure exhibits favourable property to facilitate the system analysis and control synthesis.

In general, the stability analysis for FMB control systems can be classified into two categories, i.e., membership function (MF)-independent (Chen et al.,1993; Tanaka & Sugeno, 1992) and MF-dependent (Fang et al., 2006; Feng, 2006; Kim & Lee 2000; Liu & Zhang, 2003a; Liu & Zhang, 2003b; Tanaka et al., 1998a; Teixeira et al.,2003) stability analysis approaches. Under the MF-independent stability analysis, the membership functions of both fuzzy model and fuzzy controller are not considered during stability analysis. The system stability is guaranteed to be asymptotically stable if there exists a common positive definite matrix to a set of stability conditions in the form of Lyapunov inequalities (Chen et al.,1993; Tanaka & Sugeno, 1992). The main advantages under the MF-independent analysis approach are 1). The membership functions of the fuzzy controller can be designed freely. Some simple and easy-to-implement membership functions can be employed to realize the fuzzy controller to lower the implementation cost. 2). The grades of membership functions of the fuzzy model are not necessarily known which implies parameter uncertainties of the nonlinear plant are allowed. Consequently, the fuzzy controller exhibits an inherent robustness property for nonlinear plant subject to parameter uncertainties. However, the membership function mismatch (both fuzzy model and fuzzy controller do not share the same membership functions) leads to very conservative stability analysis results. Furthermore, it can be shown that the fuzzy controller designed based on the stability conditions in (Chen et al.,1993; Tanaka & Sugeno, 1992) can be replaced by a liner controller.

Under the MF-dependent stability analysis approach, the membership functions of both fuzzy model and fuzzy controller are considered. In (Wang et al., 1996), the importance of the membership functions to the stability analysis was shown. By sharing the same premise membership functions between fuzzy model and fuzzy controller which leads to perfect match of membership functions, relaxed stability conditions were achieved. Further relaxed MF-independent stability conditions were reported (Fang et al., 2006; Feng, 2006; Kim & Lee 2000; Liu & Zhang, 2003a; Liu & Zhang, 2003b; Tanaka et al., 1998a; Teixeira et al., 2003). As to achieve perfect match of membership functions between fuzzy model and fuzzy controller which implies the membership functions of the fuzzy model must be known, the design flexibility and inherent robustness property of the fuzzy controller are lost. In both MF-independent and dependent stability analysis approaches, the stability conditions can be represented in the form of linear matrix inequalities (Boyd et al., 1994), some convex programming techniques (e.g., MATLAB LMI toolbox) can be employed to solve the solution to the stability conditions numerically and effectively.

It can be seen that fuzzy controller designed based on stability conditions under MF-dependent or -independent stability analysis approaches offers different favourable and undesired properties. It is a good idea to get the most out of these two analysis approaches by combining the advantages and alleviate the disadvantages of them, thus, to widen the applicability of the FMB control approach. This idea motivates the investigation in this chapter. In this chapter, MF-dependent stability analysis approach is employed to investigate the stability of the FMB control systems under the condition of imperfect match of membership functions. As a result, the design flexibility and inherent robustness of the fuzzy controller can be retained (due to the imperfect match of the membership functions) and the stability conditions can be relaxed (due to the MF-dependent stability analysis approach). In order to strengthen the stabilization ability of the fuzzy controller, fuzzy feedback gains, which enhance the nonlinearity compensation ability, are introduced. In order to carry out stability analysis under MF-dependent Lyapunov-based approach, membership function conditions are proposed to guide the design of the membership functions. Based on membership function conditions, some free matrices can be introduced to the stability analysis and relax the stability conditions.

This chapter is organized as follows. In section II, the fuzzy model and the proposed fuzzy controller are introduced. In section III, system stability of FMB control system is investigated using Lyapunov's stability theory under MF-dependent stability analysis approach. LMI-based stability conditions are derived to aid the design of the fuzzy controller for the nonlinear plant. In section IV, simulation examples are given to illustrate the effectiveness of the proposed approach. In section V, a conclusion is drawn.

## 2. Fuzzy Model and Enhanced Fuzzy Controller

A fuzzy-model-based control system comprising a nonlinear plant represented by a fuzzy model and an enhanced fuzzy controller connected in a closed loop is considered.

### 2.1 Fuzzy Model

Let  $p$  be the number of fuzzy rules describing the nonlinear plant. The  $i$ -th rule is of the following format:

Rule  $i$ : IF  $f_1(\mathbf{x}(t))$  is  $M_1^i$  AND ... AND  $f_\Psi(\mathbf{x}(t))$  is  $M_\Psi^i$

$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t) \quad (1)$$

where  $M_\alpha^i$  is a fuzzy term of rule  $i$  corresponding to the known function  $f_\alpha(\mathbf{x}(t))$ ,  $\alpha = 1, 2, \dots, \Psi$ ;  $i = 1, 2, \dots, p$ ;  $\Psi$  is a positive integer;  $\mathbf{A}_i \in \mathfrak{R}^{n \times n}$  and  $\mathbf{B}_i \in \mathfrak{R}^{n \times m}$  are known constant system and input matrices respectively;  $\mathbf{x}(t) \in \mathfrak{R}^{n \times 1}$  is the system state vector and  $\mathbf{u}(t) \in \mathfrak{R}^{m \times 1}$  is the input vector. The system dynamics are described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^p w_i(\mathbf{x}(t)) (\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t)) \quad (2)$$

where,

$$\sum_{i=1}^p w_i(\mathbf{x}(t)) = 1, \quad w_i(\mathbf{x}(t)) \in [0 \ 1] \quad \text{for all } i \quad (3)$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{M_1^i}(f_1(\mathbf{x}(t))) \times \mu_{M_2^i}(f_2(\mathbf{x}(t))) \times \dots \times \mu_{M_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^p \left( \mu_{M_1^k}(f_1(\mathbf{x}(t))) \times \mu_{M_2^k}(f_2(\mathbf{x}(t))) \times \dots \times \mu_{M_\Psi^k}(f_\Psi(\mathbf{x}(t))) \right)} \quad (4)$$

is a nonlinear function of  $\mathbf{x}(t)$  and  $\mu_{M_\alpha^i}(f_\alpha(\mathbf{x}(t)))$ ,  $\alpha = 1, 2, \dots, \Psi$ , is the grade of membership corresponding to the fuzzy term of  $M_\alpha^i$ .

## 2.2. Enhanced Fuzzy Controller

A fuzzy controller with  $p$  rules is considered. The  $j$ -th rule of the fuzzy controller is defined as follows.

Rule  $j$ : IF  $g_1(\mathbf{x}(t))$  is  $N_1^j$  AND ... AND  $g_\Omega(\mathbf{x}(t))$  is  $N_\Omega^j$

$$\text{THEN } \mathbf{u}(t) = \mathbf{G}_j(\mathbf{x}(t)) \mathbf{x}(t), \quad j = 1, 2, \dots, p \quad (5)$$

where  $N_\beta^j$  is a fuzzy term of rule  $j$  corresponding to the function  $g_\beta(\mathbf{x}(t))$ ,  $\beta = 1, 2, \dots, \Omega$ ;  $j = 1, 2, \dots, p$ ;  $\Omega$  is a positive integer;  $\mathbf{G}_j(\mathbf{x}(t)) \in \mathfrak{R}^{m \times n}$  is the constant and time-varying feedback gains of rule  $j$ . The time-varying feedback gain is defined as,

$$\mathbf{G}_j(\mathbf{x}(t)) = \sum_{k=1}^p m_k(\mathbf{x}(t)) \mathbf{G}_{jk} \quad (6)$$

where  $\mathbf{G}_{jk} \in \mathfrak{R}^{m \times n}$  are constant feedback gains. The inferred enhanced fuzzy controller is defined as,

$$\mathbf{u}(t) = \sum_{j=1}^p \sum_{k=1}^p m_j(\mathbf{x}(t)) m_k(\mathbf{x}(t)) \mathbf{G}_{jk} \mathbf{x}(t) \quad (7)$$

where

$$\sum_{j=1}^p m_j(\mathbf{x}(t)) = 1, \quad m_j(\mathbf{x}(t)) \in [0 \quad 1], \quad j = 1, 2, \dots, p \quad (8)$$

$$m_j(\mathbf{x}(t)) = \frac{\mu_{N_1^j}(g_1(\mathbf{x}(t))) \times \mu_{N_2^j}(g_2(\mathbf{x}(t))) \times \dots \times \mu_{N_\Omega^j}(g_\Omega(\mathbf{x}(t)))}{\sum_{k=1}^p \left( \mu_{N_1^k}(g_1(\mathbf{x}(t))) \times \mu_{N_2^k}(g_2(\mathbf{x}(t))) \times \dots \times \mu_{N_\Omega^k}(g_\Omega(\mathbf{x}(t))) \right)} \quad (9)$$

is the normalized grade of membership which is a nonlinear function of  $\mathbf{x}(t)$ .  $\mu_{N_\beta^j}(g_\beta(\mathbf{x}(t)))$ ,  $j = 1, 2, \dots, p$ , is the grade of membership corresponding to the fuzzy term  $N_\beta^j$ .

*Remark 1:* It should be noted that the proposed enhanced fuzzy controller of (7) is reduced to the traditional fuzzy controller (Chen et al., 1993; Fang et al., 2006; Feng, 2006; Liu & Zhang, 2003a; Liu & Zhang, 2003b; Tanaka et al., 1998a; Teixeira et al., 2003; Wang et al., 1996) when  $\mathbf{G}_{jk} = \mathbf{F}_j$  for all  $j$  where  $\mathbf{F}_j \in \mathfrak{R}^{m \times n}$  are the constant feedback gains.

### 3. Stability Analysis

In this section, the system stability of fuzzy-model-based control system formed by a nonlinear plant in the form of (2) and the enhanced fuzzy controller of (7) connected in a closed loop, is considered. Based on the Lyapunov stability theory, LMI-based stability conditions are derived to guarantee the asymptotic stability of the fuzzy-model-based control systems. For brevity,  $w_i(\mathbf{x}(t))$  and  $m_j(\mathbf{x}(t))$  are denoted as  $w_i$  and  $m_j$  respectively. The property of the membership functions in (3) and (8), i.e.  $\sum_{i=1}^p w_i = \sum_{i=1}^p m_i = \sum_{i=1}^p \sum_{j=1}^p w_i m_j = 1$  is utilized to facilitate the stability analysis. From (2) and (7), we have,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \sum_{i=1}^p w_i \left( \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \sum_{j=1}^p \sum_{k=1}^p m_j m_k \mathbf{G}_{jk} \mathbf{x}(t) \right) \\ &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p w_i m_j m_k (\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_{jk}) \mathbf{x}(t) \end{aligned} \quad (10)$$

As the membership functions of  $w_i$  and  $m_j$  do not match, which is one of the sources of conservativeness, the MF-dependent stability analysis approach in (Fang et al., 2006; Feng, 2006; Liu & Zhang, 2003a; Liu & Zhang, 2003b; Tanaka et al., 1998a; Teixeira et al., 2003) cannot be applied. In the following, membership function conditions are proposed to alleviate the conservativeness of stability analysis due to the imperfect match of membership functions. To proceed with the stability analysis, the following Lyapunov function candidate is employed to investigate the fuzzy-model-based control system of (10).

$$V(t) = \mathbf{x}(t)^T \mathbf{P} \mathbf{x}(t) \quad (11)$$

where  $\mathbf{P} = \mathbf{P}^T \in \mathfrak{R}^{n \times n} > 0$ . From (10) and (11),

$$\begin{aligned}
 \dot{V}(t) &= \dot{\mathbf{x}}(t)^T \mathbf{P} \mathbf{x}(t) + \mathbf{x}(t)^T \mathbf{P} \dot{\mathbf{x}}(t) \\
 &= \left( \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p w_i m_j m_k (\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_{jk}) \mathbf{x}(t) \right)^T \mathbf{P} \mathbf{x}(t) \\
 &\quad + \mathbf{x}(t)^T \mathbf{P} \left( \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p w_i m_j m_k (\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_{jk}) \mathbf{x}(t) \right) \\
 &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p w_i m_j m_k \mathbf{x}(t) \left( (\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_{jk})^T \mathbf{P} + \mathbf{P} (\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_{jk}) \right) \mathbf{x}(t) \\
 &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p w_i m_j m_k \mathbf{z}(t)^T \left( \mathbf{X} (\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_{jk})^T + (\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_{jk}) \mathbf{X} \right) \mathbf{z}(t) \tag{12}
 \end{aligned}$$

where  $\mathbf{z}(t) = \mathbf{X}^{-1} \mathbf{x}(t)$  and  $\mathbf{X} = \mathbf{P}^{-1} \in \mathfrak{R}^{n \times n}$ . Let  $\mathbf{G}_{jk} = \mathbf{N}_{jk} \mathbf{X}^{-1}$  where  $\mathbf{N}_{jk} \in \mathfrak{R}^{m \times n}$ . From (12), we have,

$$\dot{V}(t) = \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p w_i m_j m_k \mathbf{z}(t)^T \boldsymbol{\Theta}_{ijk} \mathbf{z}(t) \tag{13}$$

where  $\boldsymbol{\Theta}_{ijk} = \mathbf{A}_i \mathbf{X} + \mathbf{X} \mathbf{A}_i^T + \mathbf{B}_i \mathbf{N}_{jk} + \mathbf{N}_{jk}^T \mathbf{B}_i^T$ . To facilitate the stability analysis, the membership functions of the fuzzy controllers are designed as follows.

$$\bar{w}_i = w_i - (\rho_i m_i + \sigma_i) > 0, \quad i = 1, 2, \dots, p \tag{14}$$

where  $\rho_i$  and  $\sigma_i$  are scalars to be determined. From (12), we have,

$$\begin{aligned}
 \dot{V}(t) &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p (w_i + (\rho_i m_i + \sigma_i) - (\rho_i m_i + \sigma_i)) m_j m_k \mathbf{z}(t)^T \boldsymbol{\Theta}_{ijk} \mathbf{z}(t) \\
 &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p m_i m_j m_k \mathbf{z}(t)^T \rho_i \boldsymbol{\Theta}_{ijk} \mathbf{z}(t) + \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p m_j m_k \mathbf{z}(t)^T \sigma_i \boldsymbol{\Theta}_{ijk} \mathbf{z}(t) \\
 &\quad + \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \bar{w}_i m_j m_k \mathbf{z}(t)^T \boldsymbol{\Theta}_{ijk} \mathbf{z}(t) \\
 &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p m_i m_j m_k \mathbf{z}(t)^T \left( \rho_i \boldsymbol{\Theta}_{ijk} + \sum_{l=1}^p \sigma_l \boldsymbol{\Theta}_{ljk} \right) \mathbf{z}(t) \\
 &\quad + \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \bar{w}_i m_j m_k \mathbf{z}(t)^T \boldsymbol{\Theta}_{ijk} \mathbf{z}(t) \tag{15}
 \end{aligned}$$

To alleviate the conservativeness of the stability analysis, from (14), we consider the following conditions.

$$w_i - (\rho_i m_i + \sigma_i) - (\gamma_i m_i + \zeta_i) = \bar{w}_i - (\gamma_i m_i + \zeta_i) < 0, \quad i = 1, 2, \dots, p \tag{16}$$

where  $\gamma$  and  $\zeta_i$  are scalars to be determined. The membership function conditions of (14) and (16) offer the condition to guide the design of the membership functions of fuzzy controller. Under these conditions, some free matrices can be introduced to relax the conservativeness of stability analysis due to the imperfect match of membership functions. From (16), we consider the following condition to introduce some free matrices to (15).

$$-\sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \bar{w}_i (\bar{w}_j - (\gamma_j m_j + \zeta_j)) m_k (\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^T) \geq 0 \quad (17)$$

where  $\mathbf{S}_{ijk} = \mathbf{S}_{jik}^T \in \mathfrak{R}^{n \times n}$  and  $\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^T \geq 0, j, k = 1, 2, \dots, p; i < j$ . Furthermore, considering the property of the membership functions in (3) and (8), we have  $\sum_{i=1}^p (w_i - m_i) = 0$  which leads to  $\sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p (w_i - m_i) m_j m_k (\mathbf{V}_{jk} + \mathbf{V}_{jk}^T) = \mathbf{0}$  for arbitrary matrix of  $\mathbf{V}_{jk} \in \mathfrak{R}^{n \times n}$ . From (15) and (17), we have,

$$\begin{aligned} \dot{V}(t) &\leq \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p m_i m_j m_k \mathbf{z}(t)^T \left( \rho_i \Theta_{ijk} + \sum_{l=1}^p \sigma_l \Theta_{ljk} \right) \mathbf{z}(t) \\ &+ \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \bar{w}_i m_j m_k \mathbf{z}(t)^T \Theta_{ijk} \mathbf{z}(t) + \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p (w_i - m_i) m_j m_k \mathbf{z}(t)^T (\mathbf{V}_{jk} + \mathbf{V}_{jk}^T) \mathbf{z}(t) \\ &- \sum_{i=1}^p \sum_{j=1}^p \bar{w}_i (\bar{w}_j - (\gamma_j m_j + \zeta_j)) m_k \mathbf{z}(t)^T (\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^T) \mathbf{z}(t) \\ &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p m_i m_j m_k \mathbf{z}(t)^T \left( \rho_i \Theta_{ijk} + \sum_{l=1}^p \sigma_l \Theta_{ljk} - (\mathbf{V}_{jk} + \mathbf{V}_{jk}^T) \right) \mathbf{z}(t) \\ &+ \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \bar{w}_i m_j m_k \mathbf{z}(t)^T \Theta_{ijk} \mathbf{z}(t) + \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p w_i m_j m_k \mathbf{z}(t)^T (\mathbf{V}_{jk} + \mathbf{V}_{jk}^T) \mathbf{z}(t) \\ &- \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \bar{w}_i (\bar{w}_j - (\gamma_j m_j + \zeta_j)) m_k \mathbf{z}(t)^T (\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^T) \mathbf{z}(t) \\ &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p m_i m_j m_k \mathbf{z}(t)^T \left( \rho_i \Theta_{ijk} + \sum_{l=1}^p \sigma_l \Theta_{ljk} - (\mathbf{V}_{jk} + \mathbf{V}_{jk}^T) \right) \mathbf{z}(t) \\ &+ \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \bar{w}_i m_j m_k \mathbf{z}(t)^T \Theta_{ijk} \mathbf{z}(t) \\ &+ \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p (w_i + (\rho_i m_i + \sigma_i) - (\rho_i m_i + \sigma_i)) m_j m_k \mathbf{z}(t)^T (\mathbf{V}_{jk} + \mathbf{V}_{jk}^T) \mathbf{z}(t) \\ &- \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \bar{w}_i (\bar{w}_j - (\gamma_j m_j + \zeta_j)) m_k \mathbf{z}(t)^T (\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^T) \mathbf{z}(t) \end{aligned}$$



$$\begin{aligned}
 &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p m_i m_j m_k \mathbf{z}(t)^\top \left( \rho_i \Theta_{ijk} + \sum_{l=1}^p \sigma_l \Theta_{ljk} - (\mathbf{V}_{jk} + \mathbf{V}_{jk}^\top) \right) \mathbf{z}(t) \\
 &+ \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \bar{w}_i m_j m_k \mathbf{z}(t)^\top (\Theta_{ijk} + \mathbf{V}_{jk} + \mathbf{V}_{jk}^\top) \mathbf{z}(t) \\
 &+ \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p m_i m_j m_k \mathbf{z}(t)^\top \left( \rho_i (\mathbf{V}_{jk} + \mathbf{V}_{jk}^\top) + \sum_{l=1}^p \sigma_l (\mathbf{V}_{jk} + \mathbf{V}_{jk}^\top) \right) \mathbf{z}(t) \\
 &- \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \bar{w}_i (\bar{w}_j - (\gamma_j m_j + \zeta_j)) m_k \mathbf{z}(t)^\top (\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^\top) \mathbf{z}(t) \\
 &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p m_i m_j m_k \mathbf{z}(t)^\top \left( \rho_i \Theta_{ijk} + \sum_{l=1}^p \sigma_l (\Theta_{ljk} + \mathbf{V}_{jk} + \mathbf{V}_{jk}^\top) + (\rho_i - 1) (\mathbf{V}_{jk} + \mathbf{V}_{jk}^\top) \right) \mathbf{z}(t) \\
 &+ \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \bar{w}_i m_j m_k \mathbf{z}(t)^\top (\Theta_{ijk} + \mathbf{V}_{jk} + \mathbf{V}_{jk}^\top) \mathbf{z}(t) \\
 &- \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p (\bar{w}_i \bar{w}_j - \gamma_j \bar{w}_i m_j - \bar{w}_i \zeta_j) m_k \mathbf{z}(t)^\top (\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^\top) \mathbf{z}(t)
 \end{aligned} \tag{18}$$

We consider the first term of (18) shown as follows.

$$\Xi = \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p m_i m_j m_k \Xi_{ijk} \tag{19}$$

where

$$\Xi_{ijk} = \rho_i \Theta_{ijk} + \sum_{l=1}^p \sigma_l (\Theta_{ljk} + \mathbf{V}_{jk} + \mathbf{V}_{jk}^\top) + (\rho_i - 1) (\mathbf{V}_{jk} + \mathbf{V}_{jk}^\top) \tag{20}$$

From (19), by employing similar analysis procedure in (Fang et al., 2006), we have,

$$\begin{aligned}
 \Xi &= \sum_{i=1}^p m_i^3 \Xi_{iii} + \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p m_i^2 m_j (\Xi_{ijj} + \Xi_{jii} + \Xi_{jii}) \\
 &+ \sum_{i=1}^{p-2} \sum_{j=i+1}^{p-1} \sum_{k=j+1}^p m_i m_j m_k (\Xi_{ijk} + \Xi_{ikj} + \Xi_{jik} + \Xi_{jki} + \Xi_{kij} + \Xi_{kji})
 \end{aligned} \tag{21}$$

Define  $\mathbf{Y}_{ijk} \in \mathfrak{R}^{n \times n}$ ,  $\mathbf{Y}_{iii} = \mathbf{Y}_{iii}^\top$ ,  $i = 1, 2, \dots, p$ ,  $\mathbf{Y}_{ijj} = \mathbf{Y}_{jii}^\top$ ,  $i, j = 1, 2, \dots, p$ ;  $i \neq j$ ,  $\mathbf{Y}_{ijk} = \mathbf{Y}_{ikj}^\top$ ,  $\mathbf{Y}_{jik} = \mathbf{Y}_{kji}^\top$  and  $\mathbf{Y}_{kij} = \mathbf{Y}_{kji}^\top$ ,  $i = 1, 2, \dots, p - 2$ ;  $j = 1, 2, \dots, p - 1$ ;  $k = 1, 2, \dots, p$ . Let

$$\mathbf{Y}_{iii} > \Xi_{iii}, i = 1, 2, \dots, p \tag{22}$$

$$\mathbf{Y}_{ijj} + \mathbf{Y}_{jii} + \mathbf{Y}_{jii} \geq \Xi_{ijj} + \Xi_{jii} + \Xi_{jii}, i, j = 1, 2, \dots, p; i \neq j \tag{23}$$

$$\begin{aligned}
 \mathbf{Y}_{ijk} + \mathbf{Y}_{ikj} + \mathbf{Y}_{jik} + \mathbf{Y}_{jki} + \mathbf{Y}_{kij} + \mathbf{Y}_{kji} &\geq \Xi_{ijk} + \Xi_{ikj} + \Xi_{jik} + \Xi_{jki} + \Xi_{kij} + \Xi_{kji} \\
 , i = 1, 2, \dots, p - 2; j = 1, 2, \dots, p - 1; k = 1, 2, \dots, p
 \end{aligned} \tag{24}$$



From (21) to (24), we have,

$$\begin{aligned} \Xi \leq & \sum_{i=1}^p m_i^3 \mathbf{Y}_{iii} + \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^r m_i^2 m_j (\mathbf{Y}_{ijj} + \mathbf{Y}_{jii} + \mathbf{Y}_{jii}) \\ & + \sum_{i=1}^{p-2} \sum_{j=i+1}^{p-1} \sum_{k=j+1}^p m_i m_j m_k (\mathbf{Y}_{ijk} + \mathbf{Y}_{ikj} + \mathbf{Y}_{jik} + \mathbf{Y}_{jki} + \mathbf{Y}_{kij} + \mathbf{Y}_{kji}) \end{aligned} \quad (25)$$

From (18) and (25), we have,

$$\begin{aligned} \dot{V}(t) \leq & \sum_{i=1}^p m_i^3 \mathbf{Y}_{iii} + \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^r m_i^2 m_j (\mathbf{Y}_{ijj} + \mathbf{Y}_{jii} + \mathbf{Y}_{jii}) \\ & + \sum_{i=1}^{p-2} \sum_{j=i+1}^{p-1} \sum_{k=j+1}^p m_i m_j m_k (\mathbf{Y}_{ijk} + \mathbf{Y}_{ikj} + \mathbf{Y}_{jik} + \mathbf{Y}_{jki} + \mathbf{Y}_{kij} + \mathbf{Y}_{kji}) \\ & + \frac{1}{2} \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \bar{w}_i m_j m_k \mathbf{z}(t)^T \left( \begin{aligned} & \Theta_{ijk} + \mathbf{V}_{jk} + \mathbf{V}_{jk}^T + \gamma_j (\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^T) + \sum_{l=1}^p \zeta_l (\mathbf{S}_{ilk} + \mathbf{S}_{ilk}^T) \\ & + \Theta_{ikj} + \mathbf{V}_{kj} + \mathbf{V}_{kj}^T + \gamma_k (\mathbf{S}_{ikj} + \mathbf{S}_{ikj}^T) + \sum_{l=1}^p \zeta_l (\mathbf{S}_{ilj} + \mathbf{S}_{ilj}^T) \end{aligned} \right) \mathbf{z}(t) \\ & - \sum_{i=1}^p \sum_{j=1}^p \bar{w}_i^2 m_k \mathbf{z}(t)^T (\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^T) \mathbf{z}(t) - \sum_{j=1}^p \sum_{i < j} \sum_{k=1}^p \bar{w}_i \bar{w}_j m_k \mathbf{z}(t)^T (\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^T + \mathbf{S}_{jik} + \mathbf{S}_{jik}^T) \mathbf{z}(t) \end{aligned} \quad (26)$$

Define  $\mathbf{R}_{ijk} = \mathbf{R}_{jik}^T \in \mathfrak{R}^{n \times n}$ ,  $i, j, k = 1, 2, \dots, p$ . Let

$$\mathbf{R}_{ijk} + \mathbf{R}_{jik}^T \geq \frac{1}{2} \left( \begin{aligned} & \Theta_{ijk} + \mathbf{V}_{jk} + \mathbf{V}_{jk}^T + \gamma_j (\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^T) + \sum_{l=1}^p \zeta_l (\mathbf{S}_{ilk} + \mathbf{S}_{ilk}^T) \\ & + \Theta_{ikj} + \mathbf{V}_{kj} + \mathbf{V}_{kj}^T + \gamma_k (\mathbf{S}_{ikj} + \mathbf{S}_{ikj}^T) + \sum_{l=1}^p \zeta_l (\mathbf{S}_{ilj} + \mathbf{S}_{ilj}^T) \end{aligned} \right) \quad (27)$$

From (26) and (27), we have,

$$\begin{aligned} \dot{V}(t) \leq & \sum_{i=1}^p m_i^3 \mathbf{Y}_{iii} + \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^r m_i^2 m_j (\mathbf{Y}_{ijj} + \mathbf{Y}_{jii} + \mathbf{Y}_{jii}) \\ & + \sum_{i=1}^{p-2} \sum_{j=i+1}^{p-1} \sum_{k=j+1}^p m_i m_j m_k (\mathbf{Y}_{ijk} + \mathbf{Y}_{ikj} + \mathbf{Y}_{jik} + \mathbf{Y}_{jki} + \mathbf{Y}_{kij} + \mathbf{Y}_{kji}) \\ & + \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \bar{w}_i m_j m_k \mathbf{z}(t)^T (\mathbf{R}_{ijk} + \mathbf{R}_{jik}^T) \mathbf{z}(t) \\ & - \sum_{i=1}^p \sum_{j=1}^p \bar{w}_i^2 m_k \mathbf{z}(t)^T (\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^T) \mathbf{z}(t) - \sum_{j=1}^p \sum_{i < j} \sum_{k=1}^p \bar{w}_i \bar{w}_j m_k \mathbf{z}(t)^T (\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^T + \mathbf{S}_{jik} + \mathbf{S}_{jik}^T) \mathbf{z}(t) \\ & \leq \sum_{i=1}^p m_k \begin{bmatrix} \mathbf{r}(t) \\ \mathbf{s}(t) \end{bmatrix}^T \begin{bmatrix} \mathbf{Y}_k & \mathbf{R}_k^T \\ \mathbf{R}_k & \mathbf{S}_k \end{bmatrix} \begin{bmatrix} \mathbf{r}(t) \\ \mathbf{s}(t) \end{bmatrix} \end{aligned} \quad (28)$$

where

$$\mathbf{r}(t) = \begin{bmatrix} m_1 \mathbf{z}(t) \\ m_2 \mathbf{z}(t) \\ \vdots \\ m_p \mathbf{z}(t) \end{bmatrix}, \quad \mathbf{s}(t) = \begin{bmatrix} \bar{w}_1 \mathbf{z}(t) \\ \bar{w}_2 \mathbf{z}(t) \\ \vdots \\ \bar{w}_p \mathbf{z}(t) \end{bmatrix}, \quad \mathbf{Y}_k = \begin{bmatrix} \mathbf{Y}_{1k1} & \mathbf{Y}_{1k2} & \cdots & \mathbf{Y}_{1kp} \\ \mathbf{Y}_{2k1} & \mathbf{Y}_{2k2} & \cdots & \mathbf{Y}_{2kp} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Y}_{pk1} & \mathbf{Y}_{pk2} & \cdots & \mathbf{Y}_{pkp} \end{bmatrix},$$

$$\mathbf{R}_k = \begin{bmatrix} \mathbf{R}_{11k} & \mathbf{R}_{12k} & \cdots & \mathbf{R}_{1pk} \\ \mathbf{R}_{21k} & \mathbf{R}_{22k} & \cdots & \mathbf{R}_{2pk} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{R}_{p1k} & \mathbf{R}_{p2k} & \cdots & \mathbf{R}_{ppk} \end{bmatrix} \text{ and } \mathbf{S}_k = -2 \begin{bmatrix} \mathbf{S}_{11k} & \mathbf{S}_{12k} & \cdots & \mathbf{S}_{1pk} \\ \mathbf{S}_{21k} & \mathbf{S}_{22k} & \cdots & \mathbf{S}_{2pk} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{S}_{p1k} & \mathbf{S}_{p2k} & \cdots & \mathbf{S}_{ppk} \end{bmatrix}, k = 1, 2, \dots, p.$$

It can be seen from (28) that, if  $\begin{bmatrix} \mathbf{Y}_k & \mathbf{R}_k^T \\ \mathbf{R}_k & \mathbf{S}_k \end{bmatrix} < 0$  for all  $k$ ,  $\dot{V}(t) \leq 0$  (equality holds for

$\begin{bmatrix} \mathbf{r}(t) \\ \mathbf{s}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$ ) which implies the asymptotic stability of the fuzzy-model-based control system

of (10). The stability analysis results are summarized in the following theorem.

**Theorem 1:** The fuzzy-model-based control system of (10), formed by the nonlinear system in the form of (2) and the proposed enhanced fuzzy controller of (7) connected in closed loop, is guaranteed to be asymptotically stable if the membership functions are designed to satisfy the membership function conditions of  $\bar{w}_i(\mathbf{x}(t)) = w_i(\mathbf{x}(t)) - (\rho_i m_i(\mathbf{x}(t)) + \sigma_i) > 0$  and  $\bar{w}_i(\mathbf{x}(t)) - (\gamma_i m_i(\mathbf{x}(t)) + \zeta_i) < 0$  for all  $i$  and  $\mathbf{x}(t)$  where  $\rho_i > 0$ ,  $\sigma_i$ ,  $\gamma_i$  and  $\zeta_i$  are scalars, and there exist matrices of  $\mathbf{N}_{jk} \in \mathfrak{R}^{m \times n}$ ,  $j, k = 1, 2, \dots, p$ ;  $\mathbf{R}_{ijk} = \mathbf{R}_{jik}^T \in \mathfrak{R}^{n \times n}$ ,  $\mathbf{S}_{ijk} = \mathbf{S}_{jik}^T \in \mathfrak{R}^{n \times n}$ ,  $j, k = 1, 2, \dots, p$ ;  $i < j$ ;  $\mathbf{V}_{jk} \in \mathfrak{R}^{n \times n}$ ,  $j, k = 1, 2, \dots, p$ ;  $\mathbf{X} = \mathbf{X}^T \in \mathfrak{R}^{n \times n}$ ,  $\mathbf{Y}_{iii} = \mathbf{Y}_{iii}^T \in \mathfrak{R}^{n \times n}$ ,  $i = 1, 2, \dots, p$ ,  $\mathbf{Y}_{ijj} = \mathbf{Y}_{jii}^T \in \mathfrak{R}^{n \times n}$ ,  $i, j = 1, 2, \dots, p$ ;  $i \neq j$ ,  $\mathbf{Y}_{ijk} = \mathbf{Y}_{ikj}^T \in \mathfrak{R}^{n \times n}$ ,  $\mathbf{Y}_{jik} = \mathbf{Y}_{kji}^T \in \mathfrak{R}^{n \times n}$  and  $\mathbf{Y}_{kjj} = \mathbf{Y}_{jki}^T \in \mathfrak{R}^{n \times n}$ ,  $i = 1, 2, \dots, p - 2$ ;  $j = 1, 2, \dots, p - 1$ ;  $k = 1, 2, \dots, p$  such that the following LMIs hold.

$$\mathbf{X} > 0; \mathbf{S}_{ijk} + \mathbf{S}_{jik}^T \geq 0, j, k = 1, 2, \dots, p; i < j; \mathbf{Y}_{iii} > \Xi_{iii}, i = 1, 2, \dots, p;$$

$$\mathbf{Y}_{ijj} + \mathbf{Y}_{jii} + \mathbf{Y}_{jii} \geq \Xi_{ijj} + \Xi_{jii} + \Xi_{jii}, i, j = 1, 2, \dots, p; i \neq j;$$

$$\mathbf{Y}_{ijk} + \mathbf{Y}_{ikj} + \mathbf{Y}_{jik} + \mathbf{Y}_{jki} + \mathbf{Y}_{kij} + \mathbf{Y}_{kji} \geq \Xi_{ijk} + \Xi_{ikj} + \Xi_{jik} + \Xi_{jki} + \Xi_{kij} + \Xi_{kji},$$

$$i = 1, 2, \dots, p - 2; j = 1, 2, \dots, p - 1; k = 1, 2, \dots, p;$$

$$\mathbf{R}_{ijk} + \mathbf{R}_{jik}^T \geq \frac{1}{2} \begin{pmatrix} \Theta_{ijk} + \mathbf{V}_{jk} + \mathbf{V}_{jk}^T + \gamma_j (\mathbf{S}_{ijk} + \mathbf{S}_{ijk}^T) + \sum_{l=1}^p \zeta_l (\mathbf{S}_{ilk} + \mathbf{S}_{ilk}^T) \\ + \Theta_{ikj} + \mathbf{V}_{kj} + \mathbf{V}_{kj}^T + \gamma_k (\mathbf{S}_{ikj} + \mathbf{S}_{ikj}^T) + \sum_{l=1}^p \zeta_l (\mathbf{S}_{ilj} + \mathbf{S}_{ilj}^T) \end{pmatrix}, i, j, k = 1, 2, \dots, p;$$

$$\begin{bmatrix} \mathbf{Y}_k & \mathbf{R}_k^T \\ \mathbf{R}_k & \mathbf{S}_k \end{bmatrix} < 0, k = 1, 2, \dots, p \text{ where}$$

$$\mathbf{Y}_k = \begin{bmatrix} \mathbf{Y}_{1k1} & \mathbf{Y}_{1k2} & \cdots & \mathbf{Y}_{1kp} \\ \mathbf{Y}_{2k1} & \mathbf{Y}_{2k2} & \cdots & \mathbf{Y}_{2kp} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Y}_{pk1} & \mathbf{Y}_{pk2} & \cdots & \mathbf{Y}_{pkp} \end{bmatrix}, \mathbf{R}_k = \begin{bmatrix} \mathbf{R}_{11k} & \mathbf{R}_{12k} & \cdots & \mathbf{R}_{1pk} \\ \mathbf{R}_{21k} & \mathbf{R}_{22k} & \cdots & \mathbf{R}_{2pk} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{R}_{p1k} & \mathbf{R}_{p2k} & \cdots & \mathbf{R}_{ppk} \end{bmatrix} \text{ and } \mathbf{S}_k = -2 \begin{bmatrix} \mathbf{S}_{11k} & \mathbf{S}_{12k} & \cdots & \mathbf{S}_{1pk} \\ \mathbf{S}_{21k} & \mathbf{S}_{22k} & \cdots & \mathbf{S}_{2pk} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{S}_{p1k} & \mathbf{S}_{p2k} & \cdots & \mathbf{S}_{ppk} \end{bmatrix}$$

and  $\Xi_{ijk} = \rho_i \Theta_{ijk} + \sum_{l=1}^p \sigma_l (\Theta_{ljk} + \mathbf{V}_{jk} + \mathbf{V}_{jk}^T) + (\rho_i - 1)(\mathbf{V}_{jk} + \mathbf{V}_{jk}^T)$ ,  $i, j, k = 1, 2, \dots, p$  and the feedback gains are designed as  $\mathbf{G}_{jk} = \mathbf{N}_{jk} \mathbf{X}^{-1}$ ,  $j, k = 1, 2, \dots, p$ .

#### 4. Simulation Examples

Two simulation examples are given in this section to illustrate the effectiveness of the proposed approach.

##### 4.1 Simulation Example 1

A numerical example is given in this sub-section to investigate the stability region of the fuzzy-model-based control systems. Consider the following fuzzy model similar to that in (Fang et al., 2006).

$$\text{Rule } i: \text{ IF } x_1(t) \text{ is } M_i^i \text{ THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i u(t), i = 1, 2, 3 \quad (29)$$

where  $\mathbf{A}_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}$ ,  $\mathbf{A}_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}$  and  $\mathbf{A}_3 = \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix}$ ;  $\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{B}_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$  and  $\mathbf{B}_3 = \begin{bmatrix} -b+6 \\ -3 \end{bmatrix}$  where  $\mathbf{x}(t) = [x_1(t) \ x_2(t)]^T$ ,  $4 \leq a \leq 12$  and  $4 \leq b \leq 12$ . From (2), the inferred fuzzy model is given as follows.

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^3 w_i(x_1(t)) (\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i u(t)) \quad (30)$$

It is assumed that the fuzzy model works in the operating domain of  $x_1(t) = [-\pi \ \pi]$ . The membership functions of the fuzzy model are chosen arbitrarily as

$$w_1(x_1(t)) = \mu_{M_1^1}(x_1(t)) = 0.1e^{\frac{-(x_1(t)-\pi)^2}{5}} + 0.02 \cos(x_1(t)^2)^2 + 0.25$$

$$w_2(x_1(t)) = \mu_{M_1^2}(x_1(t)) = 0.1e^{\frac{-(x_1(t)+\pi)^2}{5}} - 0.02 \sin(x_1(t)^2)^2 + 0.25$$

and

$$w_3(x_1(t)) = \mu_{M_1^3}(x_1(t)) = 1 - w_1(x_1(t)) - w_2(x_1(t))$$

A three-rule fuzzy controller with the following rules is designed for the fuzzy model of (30).

$$\text{Rule } i: \text{ IF } x_1(t) \text{ is } N_1^i \text{ THEN } u(t) = \mathbf{G}_j(\mathbf{x}(t))\mathbf{x}(t), i = 1, 2, 3 \quad (31)$$

From (7), the inferred enhanced fuzzy controller is given as follows.

$$u(t) = \sum_{j=1}^3 \sum_{k=1}^3 m_j(x_1(t))m_k(x_1(t))\mathbf{G}_{jk}\mathbf{x}(t) \quad (32)$$

The membership functions of the enhanced fuzzy controller are chosen as

$$m_1(x_1(t)) = \mu_{N_1^1}(x_1(t)) = \frac{3(x_1(t) + \pi)^2}{100\pi^2} + 0.255,$$

$$m_2(x_1(t)) = \mu_{N_1^2}(x_1(t)) = \frac{3(x_1(t) - \pi)^2}{100\pi^2} + 0.255$$

and

$$m_3(x_1(t)) = \mu_{N_1^3}(x_1(t)) = 1 - m_1(x_1(t)) - m_2(x_1(t))$$

It can be shown that the membership conditions of (14) and (16) are satisfied with  $\rho_1 = \rho_2 = 0.945$ ,  $\rho_3 = 0.915$ ;  $\sigma_1 = \sigma_2 = 0.0005$ ,  $\sigma_3 = 0.009$ ;  $\gamma_1 = \gamma_2 = 0.11$ ,  $\gamma_3 = 0.12$ ;  $\zeta_1 = \zeta_2 = \zeta_3 = 0$ .

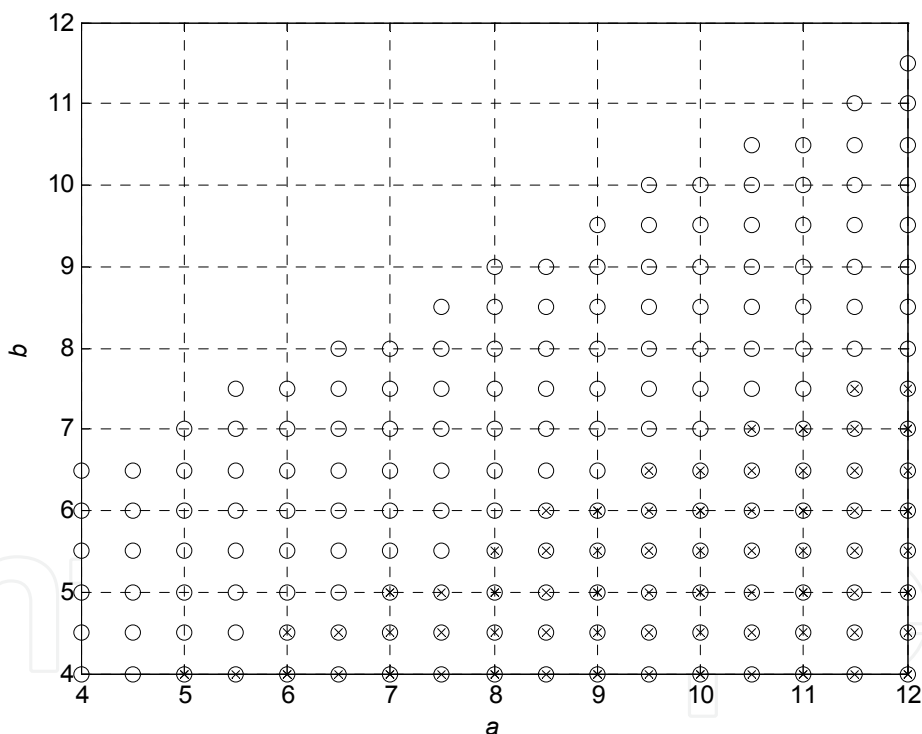


Figure 1. Stability regions for stability conditions in Theorem 1 indicated by “o” and in (Fang et al., 2006) indicated by indicated by “x”

Theorem 1 are employed to investigate the stability region of the fuzzy-model-based control system formed by (30) and (32). Fig. 1 shows the stability region indicated by “o”. For comparison purpose, the stability conditions in (Fang et al., 2006) is employed to investigate the stability region of the fuzzy-model-based control system. Fig. 1 shows the stability region given by the stability conditions in (Fang et al., 2006) indicated by “x”. It should be noted that the stability conditions in (Fang et al., 2006; Kim & Lee 2000; Liu & Zhang, 2003a; Liu &

Zhang, 2003b; Tanaka et al., 1998a; Teixeira et al., 2003; Wang et al., 1996) can be applied to fuzzy-model-based control systems with both fuzzy model and fuzzy controller sharing the same membership functions. It was reported in (Fang et al., 2006) that the stability conditions in (Fang et al., 2006; Kim & Lee 2000; Liu & Zhang, 2003a; Liu & Zhang, 2003b; Tanaka et al., 1998a; Teixeira et al., 2003) are subset of that in (Fang et al., 2006). In this example, in order to obtain the stability region for the stability conditions in (Fang et al., 2006), the fuzzy controller takes the membership functions of the fuzzy model. Referring to figure 1, it can be seen that the proposed enhanced fuzzy controller offers a larger stability region. Furthermore, as the proposed enhanced fuzzy controller does not need to share the same membership functions with the fuzzy model, simple membership functions can be employed to realize the fuzzy controller which can lower the implementation cost.

#### 4.2. Simulation Example 2

In this example, the proposed enhanced fuzzy controller is designed based on Theorem 1 for an inverted pendulum which is described by the following dynamic equations (Ma & Sun, 2001).

$$\dot{x}_1(t) = x_2(t) \quad (33)$$

$$\dot{x}_2(t) = \frac{\left( \begin{array}{l} -F_1(M+m)x_2(t) - m^2 l^2 x_2(t)^2 \sin x_1(t) \cos x_1(t) + F_0 m l x_4(t) \cos x_1(t) \\ + (M+m)mgl \sin x_1(t) - ml \cos x_1(t) u(t) \end{array} \right)}{(M+m)(J+ml^2) - m^2 l^2 (\cos x_1(t))^2} \quad (34)$$

$$\dot{x}_3(t) = x_4(t) \quad (35)$$

$$\dot{x}_4(t) = \frac{\left( \begin{array}{l} F_1 m l x_2(t) \cos x_1(t) + (J+ml^2) m l x_2(t)^2 \sin x_1(t) - F_0 (J+ml^2) x_4(t) \\ - m^2 g l^2 \sin x_1(t) \cos x_1(t) + (J+ml^2) u(t) \end{array} \right)}{(M+m)(J+ml^2) - m^2 l^2 (\cos x_1(t))^2} \quad (36)$$

where  $x_1(t)$  and  $x_2(t)$  denote the angular displacement (rad) and the angular velocity (rad/s) of the pendulum from vertical respectively,  $x_3(t)$  and  $x_4(t)$  denote the displacement (m) and the velocity (m/s) of the cart respectively,  $g = 9.8 \text{ m/s}^2$  is the acceleration due to gravity,  $m = 0.22 \text{ kg}$  is the mass of the pendulum,  $M = 1.3282 \text{ kg}$  is the mass of the cart,  $l = 0.304 \text{ m}$  is the length from the center of mass of the pendulum to the shaft axis,  $J = ml^2/3 \text{ kgm}^2$  is the moment of inertia of the pendulum around the center of mass,  $F_0 = 22.915 \text{ N/m/s}$  and  $F_1 = 0.007056 \text{ N/rad/s}$  are the friction factors of the cart and the pendulum respectively, and  $u(t)$  is the force (N) applied to the cart.

In this example, the control objective is to balance the pole and drive the cart to the origin, i.e.,  $x_i(t) \rightarrow 0$ ,  $k = 1, 2, 3, 4$ , as time tends to infinite. To facilitate the design of fuzzy controller, the following fuzzy model for the inverted pendulum of (33) to (36) is considered (Ma & Sun, 2001).

$$\text{Rule } i: \text{ IF } x_1(t) \text{ is } M_i^i \text{ THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i u(t), \quad i = 1, 2 \quad (37)$$

The inferred system dynamics of the fuzzy model are described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^2 w_i(x_1(t)) (\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i u(t)) \quad (38)$$

where  $\mathbf{x}(t) = [x_1(t) \ x_2(t) \ x_3(t) \ x_4(t)]^T$ ;

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (M+m)mgl/a_1 & -F_1(M+m)/a_1 & 0 & F_0ml/a_1 \\ 0 & 0 & 1 & 0 \\ -m^2gl^2/a_1 & F_1ml/a_1 & 0 & -F_0(J+ml^2)/a_1 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{3\sqrt{3}}{2\pi}(M+m)mgl/a_2 & -F_1(M+m)/a_2 & 0 & F_0ml \cos(\pi/3)/a_2 \\ 0 & 0 & 1 & 0 \\ -\frac{3\sqrt{3}}{2\pi}m^2gl^2 \cos(\pi/3)/a_2 & F_1ml \cos(\pi/3)/a_2 & 0 & -F_0(J+ml^2)/a_1 \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ -ml/a_1 \\ 0 \\ (J+ml^2)/a_1 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ -ml \cos(\pi/3)/a_2 \\ 0 \\ (J+ml^2)/a_2 \end{bmatrix};$$

$a_1 = (M+m)(J+ml^2) - m^2l^2$ ,  $a_2 = (M+m)(J+ml^2) - m^2l^2 \cos(\pi/3)^2$  and the membership functions are defined as

$$w_1(x_1(t)) = \mu_{M_1}(x_1(t)) = \left(1 - \frac{1}{1 + e^{-7(x_1(t) - \pi/6)}}\right) \frac{1}{1 + e^{-7(x_1(t) + \pi/6)}}$$

and

$$w_2(x_1(t)) = \mu_{M_2}(x_1(t)) = 1 - \mu_{M_1}(x_1(t))$$

which are shown as the bell shape in Fig. 2.

Based on the fuzzy model of (38), a two-rule enhanced fuzzy controller is employed to realize stabilize the plant. The rule of the fuzzy controller is of the following format.

$$\text{Rule } j: \text{ IF } x_1(t) \text{ is } N_{1i} \text{ THEN } u(t) = \mathbf{G}_j(\mathbf{x}(t))\mathbf{x}(t), j = 1, 2 \quad (39)$$

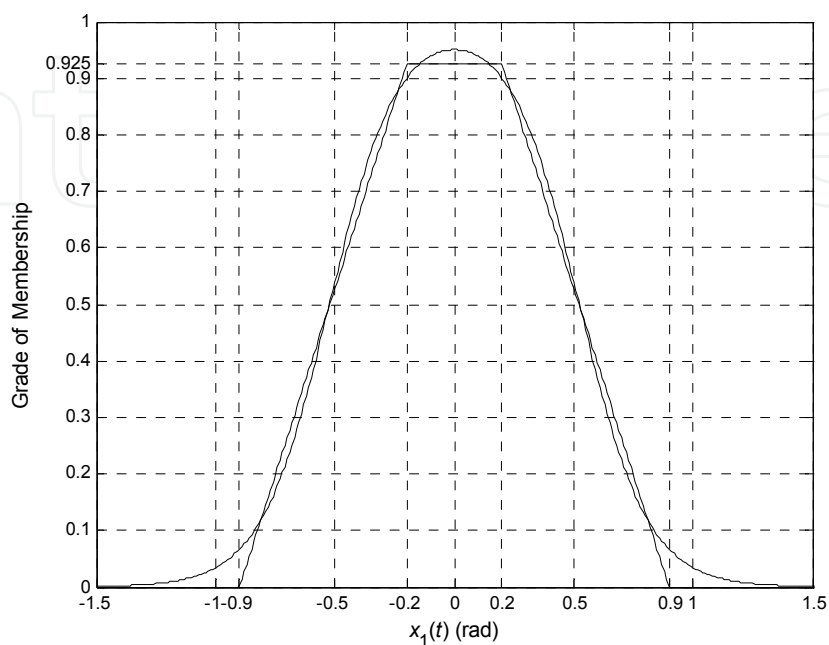
From (7), the inferred enhanced fuzzy controller is given as follows.

$$u(t) = \sum_{j=1}^2 \sum_{k=1}^2 m_j(x_1(t))m_k(x_1(t))\mathbf{G}_{jk}\mathbf{x}(t) \quad (40)$$

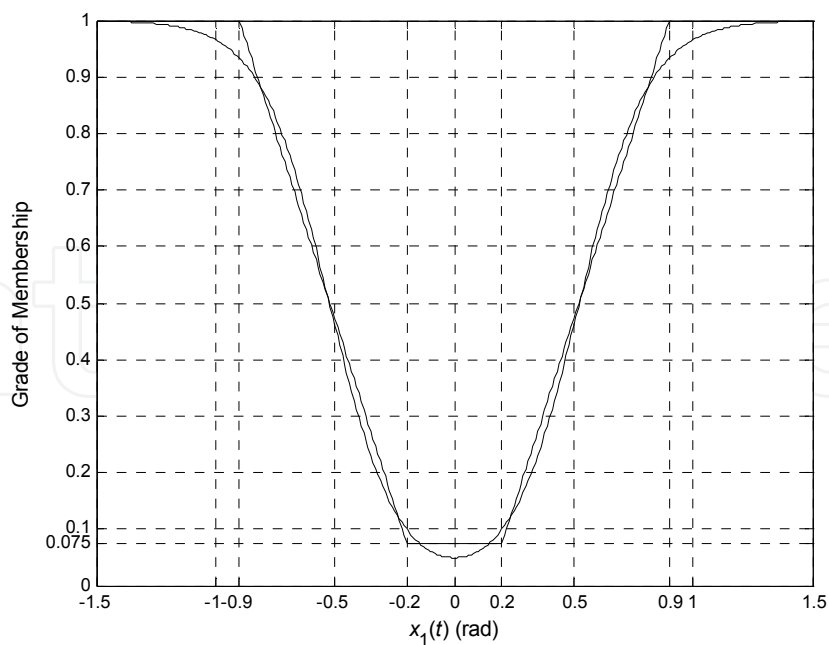
The membership functions for the fuzzy model are chosen as  $m_1(x_1(t)) = \mu_{N_1}(x_1(t)) =$

$$\begin{cases} 0 & \text{for } x_1(t) < -0.9 \\ \frac{37}{28}(x+0.9) & \text{for } -0.9 \leq x_1(t) \leq -0.2 \\ 1 & \text{for } -0.2 < x_1(t) < 0.2 \\ -\frac{37}{28}(x-0.9) & \text{for } 0.2 \leq x_1(t) \leq 0.9 \\ 0 & \text{for } x_1(t) > 0.9 \end{cases} \quad \text{and } m_2(x_1(t)) = \mu_{N_2}(x_1(t)) = 1 - \mu_{M_1}(x_1(t)) \text{ which are}$$

shown as the trapezoids in Fig. 2. It can be shown that the membership conditions of (14) and (16) are satisfied with  $\rho_1 = 0.99$ ,  $\rho_2 = 0.95$ ;  $\sigma_1 = 0.07$ ,  $\sigma_2 = 0.02$ ;  $\gamma_1 = \gamma_2 = 0.05$ ;  $\zeta_1 = 0.1$ ,  $\zeta_2 = 0.085$ .



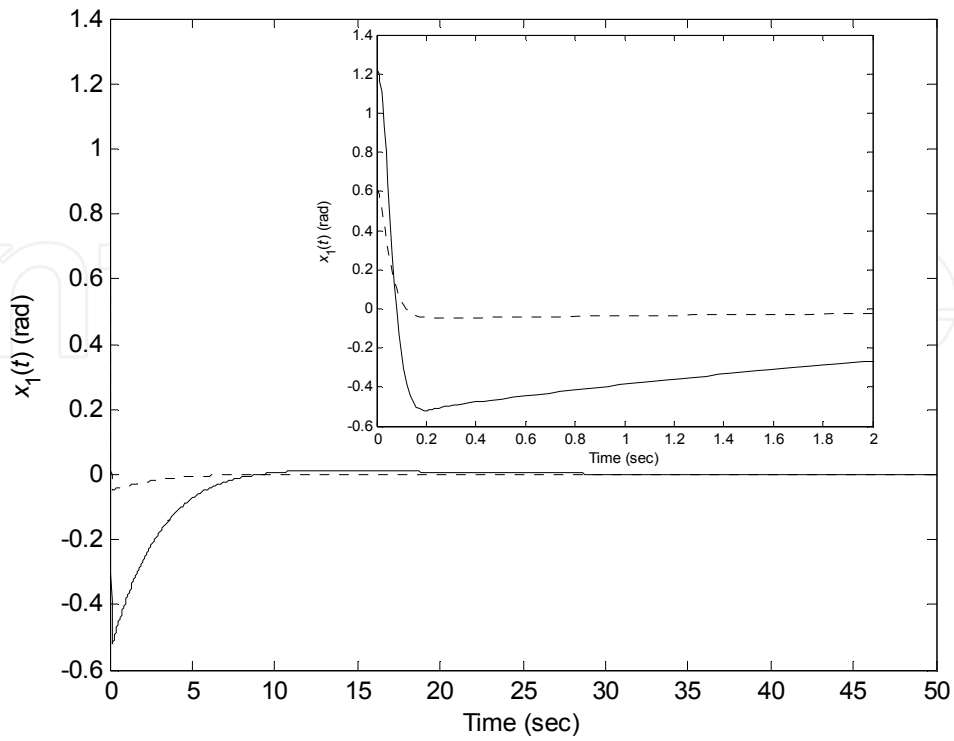
a)  $\mu_{M_1}(x_1(t))$  (bell) and  $\mu_{N_1}(x_1(t))$  (trapezoid)



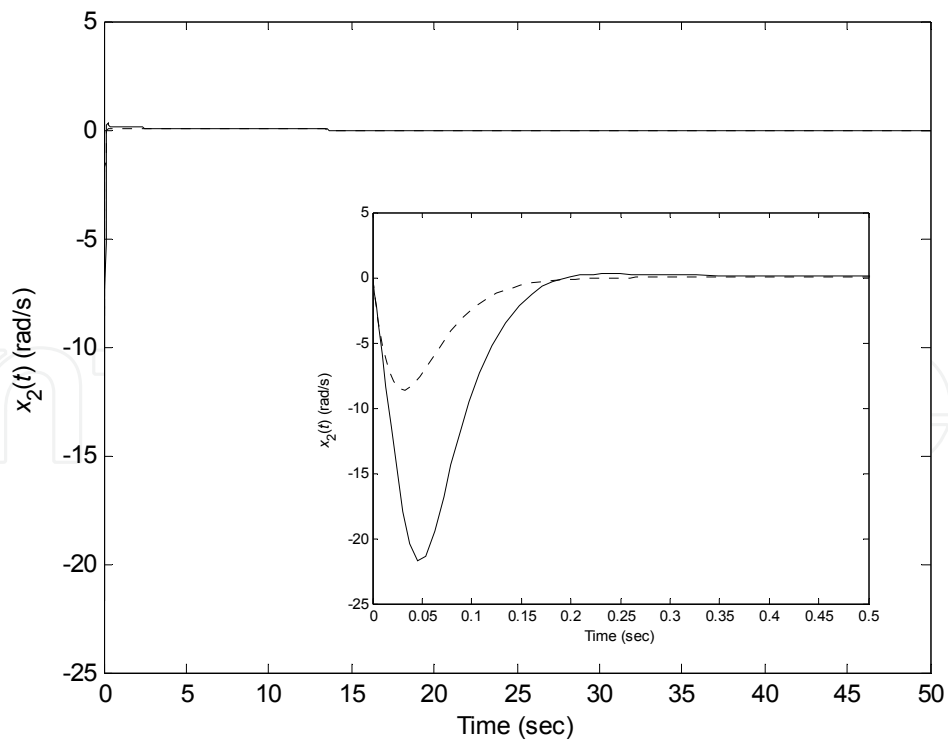
b)  $\mu_{M_2}(x_1(t))$  (bell) and  $\mu_{N_2}(x_1(t))$  (trapezoid)

Figure 2. Membership functions of fuzzy model and fuzzy controller in simulation example 2





a)  $x_1(t)$



b)  $x_2(t)$

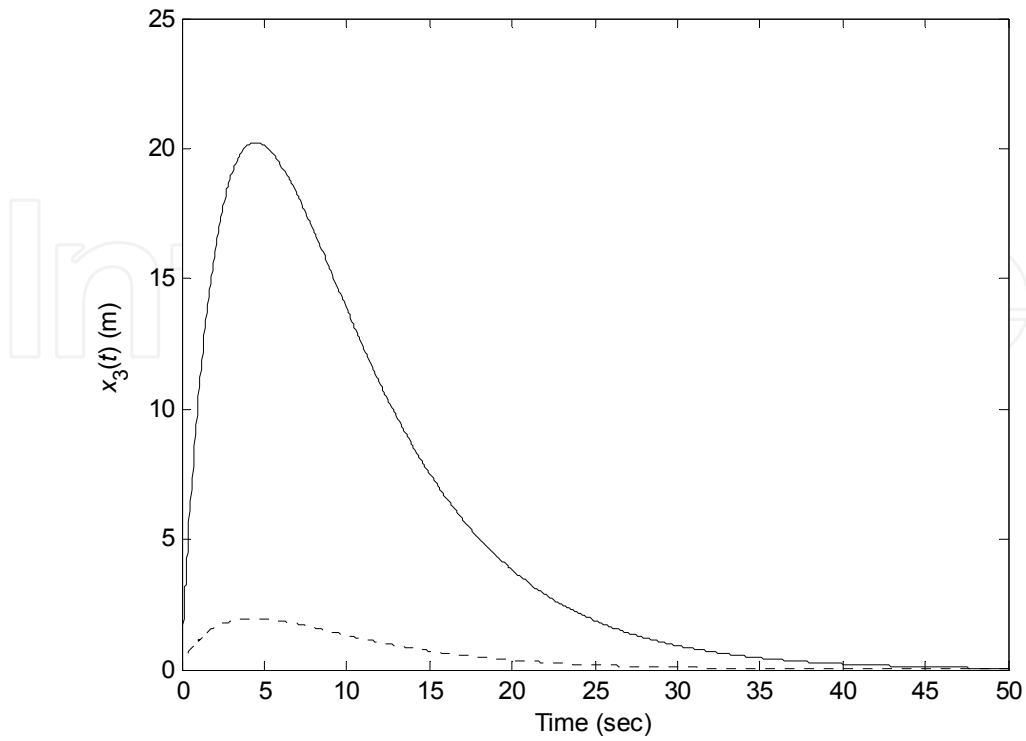
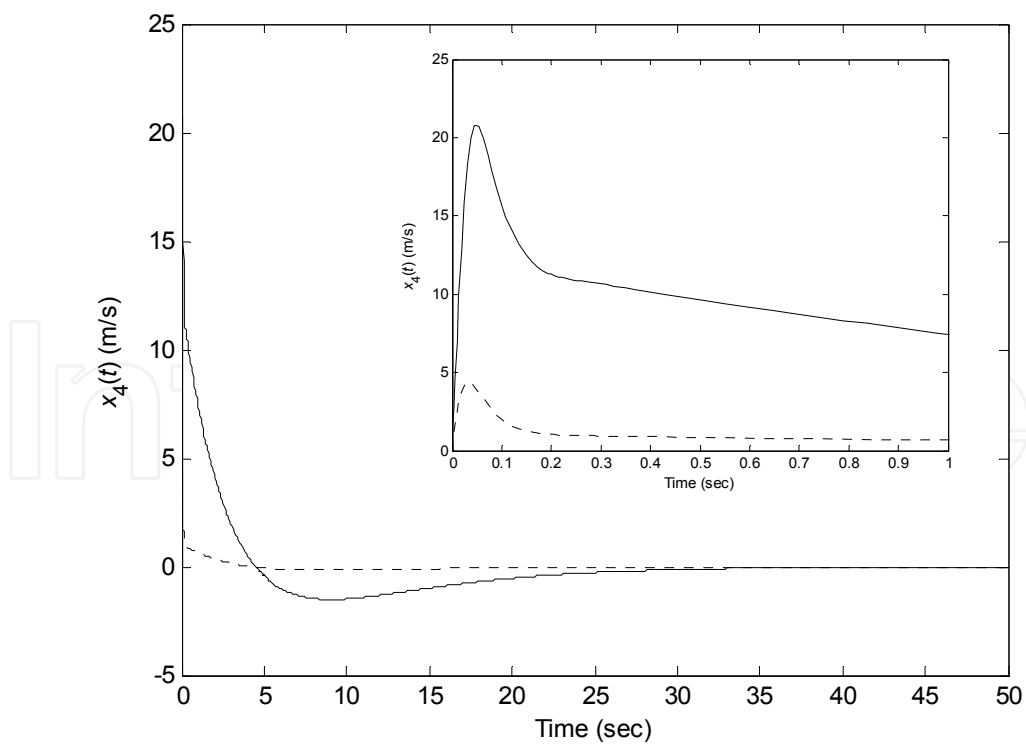
c)  $x_3(t)$ d)  $x_4(t)$ 

Figure 3. System state responses for the fuzzy-model-based control system under different initial system state conditions in simulation example 2

To obtain the feedback gains for the enhanced fuzzy controller of (40), stability conditions in Theorem 1 are employed to achieve a stabilizing fuzzy controller of (40). With the help of Matlab LMI toolbox to solve the solution to the stability conditions in Theorem 1, we have  $\mathbf{G}_{11} = [702.0204 \ 51.4864 \ 3.1163 \ 54.5589]$ ,  $\mathbf{G}_{12} = [707.8873 \ 48.2611 \ 3.0247 \ 53.4503]$ ,  $\mathbf{G}_{21} = [707.8873 \ 48.2611 \ 3.0247 \ 53.4503]$  and  $\mathbf{G}_{22} = [965.2063 \ 63.0629 \ 3.7704 \ 60.9679]$ . Fig. 3 shows the system state responses for the fuzzy-model-based control system with  $x(0) = [7\pi/18 \ 0 \ 0 \ 0]^T$  and  $[7\pi/36 \ 0 \ 0 \ 0]^T$  respectively. It can be seen that the proposed enhanced fuzzy controller is able to stabilize the nonlinear plant. It is worth noting that the stability conditions in (Fang et al., 2006; Kim & Lee 2000; Liu & Zhang, 2003a; Liu Zhang, 2003b; Tanaka et al., 1998a; Teixeira et al., 2003; Wang et al., 1996) cannot apply to design the fuzzy controller as the members functions for both fuzzy model and fuzzy controllers are different in this example.

## 5. Conclusion

The system stability of fuzzy-model-based control systems has been investigated in this chapter. An enhanced fuzzy controller has been proposed for the control process. The stabilization ability of the proposed fuzzy controller is strengthened by the enhanced nonlinear feedback gains. Imperfect match of membership functions between fuzzy model and fuzzy controller has been considered. Compared to the existing approaches, greater design flexibility can be achieved due to the membership functions for the fuzzy controller can be designed freely. Under such a situation, most of the published stability conditions cannot be applied for the design of stable fuzzy-model-based control systems. To alleviate the conservativeness of stability analysis due to imperfect match of membership functions, membership function conditions have been proposed to govern the design of the membership functions of the fuzzy controller. With such membership function conditions, free matrices can be introduced to the Lyapunov-based stability analysis to relax the stability conditions. Simulation examples have been given to illustrate the effectiveness of the proposed approach.

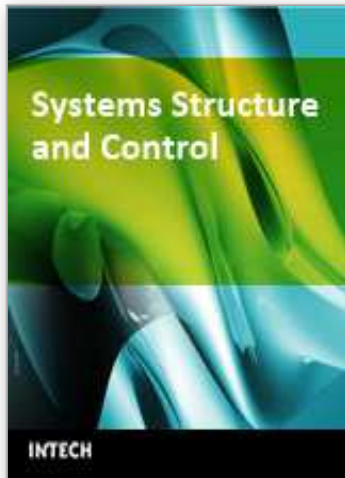
## 6. Acknowledgement

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