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Dynamics of Two-Dimensional Discrete-Time Delayed Hopfield Neural Networks

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1. Introduction

This chapter is devoted to the analysis of the complex dynamics exhibited by two-dimensional discrete-time delayed Hopfield-type neural networks.

Since the pioneering work of (Hopfield, 1982; Tank & Hopfield, 1986), the dynamics of continuous-time Hopfield neural networks have been thoroughly analyzed. In implementing the continuous-time neural networks for practical problems such as image processing, pattern recognition and computer simulation, it is essential to formulate a discrete-time system which is a version of the continuous-time neural network. However, discrete-time counterparts of continuous-time neural networks have only been in the spotlight since 2000.

One of the first problems that needed to be clarified, concerned the discretization technique which should be applied in order to obtain a discrete-time system which preserves certain dynamic characteristics of the continuous-time system. In (Mohamad & Gopalsamy, 2000) a semi-discretization technique has been presented for continuous-time Hopfield neural networks, which leads to discrete-time neural networks which faithfully preserve some characteristics of the continuous-time network, such as the steady states and their stability properties.

In recent years, the theory of discrete-time dynamic systems has assumed a greater importance as a well deserved discipline. In spite of this tendency of independence, there is a striking similarity or even duality between the theories of continuous and discrete dynamic systems. Many results in the theory of difference equations have been obtained as natural discrete analogs of corresponding results from the theory of differential equations. Nevertheless, the theory of difference equations is a lot richer than the corresponding theory of differential equations. For example, a simple difference equation resulting from a first order differential equation may exhibit chaotic behavior which can only happen for higher order differential equations. This is the reason why, when studying discrete-time counterparts of continuous neural networks, important differences and more complicated behavior may also be revealed.

The analysis of the dynamics of neural networks focuses on three directions: discovering equilibrium states and periodic or quasi-periodic solutions (of fundamental importance in biological and artificial systems, as they are associated with central pattern generators (Pasemann et al., 2003)), establishing stability properties and bifurcations (leading to the

discovery of periodic solutions), and identifying chaotic behavior (with valuable applications to practical problems such as optimization (Chen & Aihara, 1995, 1997, 2001; Chen & Shih, 2002), associative memory (Adachi & Aihara, 1997) and cryptography (Yu & Cao, 2006)).

We refer to (Guo & Huang, 2004; Guo et al., 2004) for the study of the existence of periodic solutions of discrete-time Hopfield neural networks with delays and the investigation of exponential stability properties.

In (Yuan et al., 2004, 2005) and in the most general case, in (He & Cao, 2007), a bifurcation analysis of two dimensional discrete neural networks without delays has been undertaken. In (Zhang & Zheng, 2005, 2007), the bifurcation phenomena have been studied, for the case of two- and n-dimensional discrete neural network models with multi-delays obtained by applying the Euler method to a continuous-time Hopfield neural network with no self-connections. In (Kaslik & Balint, 2007a-b), a bifurcation analysis for discrete-time Hopfield neural networks of two neurons with self-connections has been presented, in the case of a single delay and of two delays. In (Guo et al., 2007), a generalization of these results was attempted, considering three delays; however, only two delays were considered independent (the third one is a linear combination of the first two) and the analysis can be reduced to the one presented in (Kaslik & Balint, 2007a).

The latest results concerning chaotic dynamics in discrete-time delayed neural networks can be found in (Huang & Zou, 2005) and (Kaslik & Balint, 2007c).

A general discrete-time Hopfield-type neural network of two neurons with finite delays is defined by:

$$\begin{cases} x_{n+1} = a_1 x_n + T_{11}g_1(x_{n-k_{11}}) + T_{12}g_2(y_{n-k_{12}}) \\ y_{n+1} = a_2 y_n + T_{21}g_1(x_{n-k_{21}}) + T_{22}g_2(y_{n-k_{22}}) \end{cases} \quad \forall n \ge \max(k_{11}, k_{12}, k_{21}, k_{22}) \quad (1)$$

In this system $a_i \in (0,1)$ are the internal decays of the neurons, $T = (T_{ij})_{2\times 2}$ is the interconnection matrix, $g_i : \mathbb{R} \to \mathbb{R}$ represent the neuron input-output activations and $k_{ij} \in \mathbb{N}$ represent the delays. The reason for incorporating delays into the model equations of the network is that, in practice, due to the finite speeds of the switching and transmission of signals in a network, time delays unavoidably exist in a working network.

In order to insure that delays are present, we consider $\max(k_{11}, k_{12}, k_{21}, k_{22}) > 0$. The non-delayed case was extensively studied in (He & Cao, 2007). In the followings, we will denote $k_1 = \max(k_{11}, k_{21})$ and $k_2 = \max(k_{12}, k_{22})$.

We will suppose that the activation functions g_i are of class C^3 in a neighborhood of 0 and that $g_i(0) = 0$. In the followings, let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be the function given by $g(x,y) = (g_1(x), g_2(y))^T$ and

$$B = TDg(0) = \begin{pmatrix} T_{11}g_1'(0) & T_{12}g_2'(0) \\ T_{21}g_1'(0) & T_{22}g_2'(0) \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

We use the notations $2\beta = b_{11} + b_{22} = tr(B)$ and $\delta = b_{11}b_{22} - b_{12}b_{21} = det(B)$.

The aim of this chapter is to present a complete stability and bifurcation analysis in a neighborhood of the null solution of (1), choosing the characteristic parameters (β, δ) for the

system. Considering equal internal decays $a_1 = a_2 = a$ and delays satisfying $k_{11} + k_{22} = k_{12} + k_{21}$, two complementary situations are discussed:

- $k_{11} = k_{22}$
- $k_{11} \neq k_{22}$ (with the supplementary hypothesis $b_{11} = b_{22}$)

To the best of our knowledge, these are generalizations of all cases considered so far in the existing literature. This analysis allows the description of the stability domain of the null solution and the types of bifurcation occurring at its boundary, in terms of the characteristic parameters. By applying the center manifold theorem and the normal form theory, the Neimark-Sacker bifurcations are analyzed. A numerical example is presented to substantiate the theoretical findings. Moreover, the numerical example shows that the dynamics become more and more complex as the characteristic parameters leave the stability domain, eventually leading to the installation of chaotic behavior. The route from stability towards chaos passes through several stages of strange attractors and periodic solutions.

2. Preliminary results

We will start by giving two results that have particular importance for the bifurcation analysis to follow, namely for the study of the distribution of the roots of the characteristic polynomial associated to sysem (1) with respect to the unit circle.

The first result concerns the distribution of the roots of a polynomial function with respect to the unit circle, and can be proved using Rouché's theorem.

Proposition 1. (see (Zhang & Zheng, 2005, 2007)) Suppose that $S \subset R$ is a compact and connected set, and the polynomial $P(\lambda, \alpha) = \lambda^m + p_1(\alpha)\lambda^{n-1} + p_2(\alpha)\lambda^{n-2} + ... + p_m(\alpha)$ is continuous on $C \times S$. Then, as the parameter α varies, the sum of the order of the zeros of $P(\lambda, \alpha)$ out of the unit circle, i.e. $card(\{\lambda \in C : P(\lambda, \alpha) = 0, |\lambda| > 1\})$, can change only if a zero appears on or crossed the unit circle. ∎

The second result concerns the existence of the roots of a special equation which plays an important role in the analysis of the characteristic polynomial associated to system (1).

Proposition 2. (see (Kaslik & Balint, 2007b)) Let be $m \geq 0$, $[m]$ the integer part of m and $a \in (0,1)$. The equation

$$\sin(m+1)\phi - a \sin m\phi = 0 \tag{2}$$

has exactly $[m] + 2$ solutions in the interval $[0, \pi]$. More precisely:

- $\phi_0 = 0$ is a solution;
- if $m \geq 1$, there is one solution ϕ_j in every interval $\left(\dfrac{(2j-1)\pi}{2m+1}, \dfrac{j\pi}{m+1} \right) \subset \left(\dfrac{(j-1)\pi}{m}, \dfrac{j\pi}{m} \right)$, $j \in \{1, 2, ..., [m]\}$;
- if $m \in N$ then $\phi_{[m]+1} = \pi$ is a solution and if $m \notin N$ then there is one solution $\phi_{[m]+1} \in \left(\dfrac{[m]\pi}{m}, \pi \right)$. ∎

3. Stability and bifurcation analysis

We transform system (1) into the following system of $k_1 + k_2 + 2$ equations without delays:

$$\begin{cases} x_{n+1}^{(0)} = a_1 x_n^{(0)} + T_{11} g_1(x_n^{(k_{11})}) + T_{12} g_2(y_n^{(k_{12})}) \\ x_{n+1}^{(j)} = x_n^{(j-1)} \qquad \forall j = \overline{1, k_1} \\ y_{n+1}^{(0)} = a_2 y_n^{(0)} + T_{21} g_1(x_n^{(k_{21})}) + T_{22} g_2(y_n^{(k_{22})}) \\ y_{n+1}^{(j)} = y_n^{(j-1)} \qquad \forall j = \overline{1, k_2} \end{cases} \qquad \forall n \in \mathsf{N} \qquad (3)$$

where $x^{(j)} \in \mathsf{R}$, $j = \overline{0, k_1}$ and $y^{(j)} \in \mathsf{R}$, $j = \overline{0, k_2}$.

Let be the function $F : \mathsf{R}^{k_1 + k_2 + 2} \to \mathsf{R}^{k_1 + k_2 + 2}$ given by the right hand side of system (3). The jacobian matrix of system (3) at the fixed point $\overline{0} \in \mathsf{R}^{k_1 + k_2 + 2}$ is $\hat{A} = DF(\overline{0})$.

The following characteristic equation is obtained:

$$(z - a_1 - b_{11} z^{-k_{11}})(z - a_2 - b_{22} z^{-k_{22}}) - b_{12} b_{21} z^{-(k_{12} + k_{21})} = 0 \qquad (4)$$

Studying the stability and bifurcations occurring at the origin in system (1) reduces to the analysis of the distribution of the roots of the characteristic equation (4) with respect to the unit circle. The difficulty of this analysis is due to the large number of parameters appearing in the characteristic equation.

In the followings, considering equal internal decays $a_1 = a_2 = a$ and delays satisfying $k_{11} + k_{22} = k_{12} + k_{21}$, we will analyze the roots of equation (4) in two particular situations, depicting information about the stability and bifurcations occurring at the origin in system (1).

3.1 Situation 1: $k_{11} = k_{22}$

We will denote $k_{11} = k_{22} = k$ and therefore, we have $k_{12} + k_{21} = 2k$.

A particular case of this situation is the one studied in (Kaslik & Balint, 2007a), where in addition, it was considered that $k_{12} = k_{21} = k$, that is, all four delays are equal. Another particular case of this situation is the one analyzed in (Guo et al., 2007), considering the supplementary hypothesis $b_{11} = b_{22}$ (but without assuming that all four delays are equal).

In this situation, the characteristic equation (4) can be written as:

$$z^{2k}(z - a)^2 - 2\beta z^k(z - a) + \delta = 0 \qquad (5)$$

The distribution of the roots of the characteristic equation (5) has been thoroughly analyzed in (Kaslik & Balint, 2007a). This analysis provides us with the following results concerning the stability and bifurcations occurring at the origin in system (1):

Considering the following notations and associated basic results:

- ϕ_1 the unique solution of the equation $\sin(k+1)\phi - a \sin k\phi = 0$ from the interval $(0, \dfrac{\pi}{k+1})$;

- the strictly decreasing function $c : [0, \phi_1] \to \mathsf{R}$, $c(\theta) = \cos(k+1)\theta - a \cos k\theta$;

- $c(\phi_1) = -(a^2 + 1 - 2a \cos \phi_1)^{\frac{1}{2}} < 0$;

- the strictly decreasing function $U : [c(\phi_1), 1-a] \to (0, \infty)$ defined by $U(\beta) = 1 + a^2 - 2a\cos(c^{-1}(\beta))$;

- the function $\lambda_0 : R \to R$, $\lambda_0(\beta) = 2(1-a)\beta - (1-a)^2$;

- the function $\lambda_1 : R \to R$, $\lambda_1(\beta) = 2c(\phi_1)\beta - c(\phi_1)^2$;

- the function $L : [c(\phi_1), 1-a] \to R$, $L(\beta) = \max(\lambda_j(\beta)/j \in \{0,1\})$;

- $\beta_0 = \dfrac{1}{2}[c(\phi_1) + 1 - a]$;

The following theorem holds:

Theorem 1. The null solution of (1) is asymptotically stable if and only if β and δ satisfy the following inequalities:

$$c(\phi_1) < \beta < 1 - a \quad \text{and} \quad L(\beta) < \delta < U(\beta). \qquad (6)$$

On the boundary of the set $D_S = \{(\beta, \delta) \in R^2 : c(\phi_1) < \beta < 1 - a \text{ and } L(\beta) < \delta < U(\beta)\}$ the following bifurcation phenomena causing the loss of asymptotical stability of the null solution of (1) take place:

i.   Let be $\beta \in (\beta_0, 1-a)$. When $\delta = L(\beta) = \lambda_0(\beta)$ system (1) has a Fold bifurcation at the origin.

ii. Let be $\beta \in (c(\phi_1), \beta_0)$. When $\delta = L(\beta) = \lambda_1(\beta)$ a Neimark-Sacker bifurcation occurs in system (1), i.e. a unique closed invariant curve bifurcates from the origin near $\delta = \lambda_1(\beta)$.

iii. Let be $\beta \in (c(\phi_1), 1-a)$. When $\delta = U(\beta)$, system (1) has a Neimark-Sacker bifurcation at the origin. That is, system (1) has a unique closed invariant curve bifurcating from the origin near $\delta = U(\beta)$.

iv. For $\beta = \beta_0$ and $\delta = L(\beta_0) = c(\phi_1)(1-a)$ a Fold-Neimark-Sacker bifurcation occurs at the origin in system (1).

v.   For $\beta = c(\phi_1)$ and $\delta = c(\phi_1)^2$, the null solution of (1) is a double Neimark-Sacker bifurcation point.

vi. For $\beta = (1-a)$ and $\delta = (1-a)^2$, the system (1) has a strong 1:1 resonant bifurcation at the origin. ∎

The set D_S given by Theorem 1 is the stability domain of the null solution of (1) with respect to the characteristic parameters β and δ .

3.2 Situation 2: $k_{11} \neq k_{22}$ and $b_{11} = b_{22}$

A particular case of this situation has been studied in (Kaslik & Balint, 2007b), where in addition, it was considered that $k_{11} = k_{21}$ and $k_{12} = k_{22}$.

In this situation, the characteristic equation (4) can be written as:

$$z^{k_{11}+k_{22}}(z-a)^2 - \beta z^{k_{22}}(z-a) - \beta z^{k_{11}}(z-a) + \delta = 0 \qquad (7)$$

This equation is the same as the one obtained and analyzed in (Kaslik & Balint, 2007b). The conclusions of this analysis will be presented below.

First, a list of notations will be introduced and some mathematical results will be presented, which can be proved using basic mathematical tools:

- $m = \dfrac{1}{2}(k_{11} + k_{22})$ and $l = \dfrac{1}{2}|k_{11} - k_{22}|$; remark: $l \geq \dfrac{1}{2}$, $m > 1$;

- $S_1 = \{\phi_0 = 0, \phi_1, \phi_2, ..., \phi_{[m]+1}\}$ the set of all solutions of the equation (2) from the interval $[0, \pi]$;

- $S_2 = \{\psi_j = \dfrac{(2j-1)\pi}{2l} / j \in \{1, 2, ..., \left[\dfrac{2l+1}{2}\right]\}\}$;

- $\theta_1 = \min(\phi_1, \psi_1)$;

- the function $c : [0, \pi] \to R$, $c(\theta) = \cos(m+1)\theta - a\cos m\theta$;

- the function $s : [0, \pi] \to R$, $s(\theta) = \sin(m+1)\theta - a\sin m\theta$;

- the strictly decreasing function $h : [0, \theta_1) \to R$, $h(\theta) = c(\theta)\sec(l\theta)$;

- $\alpha = \lim\limits_{\theta \to \theta_1} h(\theta) = \begin{cases} c(\phi_1)\sec(l\phi_1) < 0 & \text{if } \phi_1 < \psi_1 \\ -\infty & \text{if } \phi_1 \geq \psi_1 \end{cases}$

- $h^{-1} : (\alpha, 1-a] \to [0, \theta_1)$ the inverse of the function h;

- the strictly decreasing function $U : (\alpha, 1-a] \to (0, \infty)$, $U(\beta) = 1 + a^2 - 2a\cos(h^{-1}(\beta))$;

- the functions $\lambda_j : R \to R$, $\lambda_j(\beta) = 2c(\phi_j)\cos(l\phi_j)\beta - c(\phi_j)^2$;

- the function $L : (\alpha, 1-a] \to R$, $L(\beta) = \max\left(\lambda_j(\beta) / j \in \{0, 1, ..., [m]+1\}\right)$;

- β_{ij} the solution of the equation $\lambda_i(\beta) = \lambda_j(\beta)$, $i \neq j$;

- $\beta_0 = \max(\beta_{0j} / j \in \{1, 2, ..., [m]+1\}, \beta_{0j} < 0)$;

- remark: $L(\beta) = \lambda_0(\beta) = 2(1-a)\beta - (1-a)^2$ for any $\beta \in [\beta_0, 1-a]$;

- if the equation $U(\beta) = L(\beta)$ has some roots in the interval (α, β_0), then β_1 is the largest of these roots; otherwise, $\beta_1 = \alpha$.

We will consider the following two cases:

(c1) At least one of the delays k_{11} or k_{22} is odd.

(c2) Both delays k_{11} and k_{22} are even.

Theorem 2. The null solution of (1) is asymptotically stable if β and δ satisfy the following inequalities:

$$\beta_1 < \beta < 1 - a \quad \text{and} \quad L(\beta) < \delta < U(\beta). \tag{8}$$

On the boundary of the set $D_S = \{(\beta, \delta) \in R^2 : \beta_1 < \beta < 1-a \text{ and } L(\beta) < \delta < U(\beta)\}$ the following bifurcation phenomena causing the loss of asymptotical stability of the null solution of (1) take place:

i.   Let be $\beta \in (\beta_1, 1-a)$. When $\delta = U(\beta)$, system (1) has a Neimark-Sacker bifurcation at the origin. That is, system (1) has a unique closed invariant curve bifurcating from the origin near $\delta = U(\beta)$.

ii. Let be $\beta \in (\beta_1, \beta_0)$ such that the function L is differentiable at β. When $\delta = L(\beta)$:

(c1) system (1) has a Neimark-Sacker bifurcation at the origin.

(c2) system (1) has a Flip or a Neimark-Sacker bifurcation at the origin.

iii. Let be $\beta \in (\beta_0, 1-a)$. When $\delta = L(\beta) = 2(1-a)\beta - (1-a)^2$ system (1) has a Fold bifurcation at the origin.

iv. For $\beta = (1-a)$ and $\delta = (1-a)^2$, system (1) has a strong $1:1$ resonant bifurcation at the origin.

v. For $\beta = \beta_0$ and $\delta = L(\beta_0) = 2(1-a)\beta_0 - (1-a)^2$, system (1) has a Fold-Neimark-Sacker bifurcation at the origin.

vi. For $\beta = \beta_1$ and $\delta = U(\beta_1)$:

(c1) system (1) has a double Neimark-Sacker bifurcation at the origin.

(c2) system (1) has a double Neimark-Sacker or a Flip-Neimark-Sacker bifurcation at the origin.

vii. If there exists $\beta^* \in (\beta_1, \beta_0)$ such that the function L is not differentiable at β^*, then for

$\beta = \beta^*$ and $\delta = L(\beta^*)$:

(c1) system (1) has a double Neimark-Sacker bifurcation at the origin.

(c2) system (1) has a double Neimark-Sacker or a Flip-Neimark-Sacker bifurcation at the origin. ∎

We underline that Theorems 1 and 2 completely characterize the stability domain (in the (β, δ)-plane) of the null solution of (1) and the bifurcations occurring at its boundary, in the considered situations.

4. Direction and stability of Neimark-Sacker bifurcations

Let be the function $F: \mathbb{R}^{k_1+k_2+2} \to \mathbb{R}^{k_1+k_2+2}$ given by the right hand side of system (3). Let be the operators $\hat{A} = DF(\bar{0})$, $\hat{B} = D^2F(\bar{0})$ and $\hat{C} = D^3F(\bar{0})$.

In the cases ii. and iii. of Theorem 1 and i. and ii. of Theorem 2, Neimark-Sacker bifurcations occur at the origin in system (1). That is, matrix $\hat{A}$ has a simple pair $(z, \bar{z})$ of eigenvalues on the unit circle, such that z is not a root of order $1,2,3,4$ of the unity.

The restriction of system (3) to its two dimensional center manifold at the critical parameter values can be transformed into the normal form written in complex coordinates (see (Kuznetsov, 2004)):

$$w \mapsto zw(1 + \frac{1}{2}d \mid w \mid^2) + O(\mid w \mid^4), \qquad w \in \mathbb{C} \tag{9}$$

with

$$d = \bar{z}\langle p, \hat{C}(q,q,\bar{q}) + 2\hat{B}(q,(I-\hat{A})^{-1}\hat{B}(q,\bar{q})) + \hat{B}(\bar{q},(z^2I-\hat{A})^{-1}\hat{B}(q,q))\rangle$$

where $\hat{A}q = zq$, $\hat{A}^Tp = \bar{z}p$ and $\langle p,q\rangle = 1$ (with $\langle p,q\rangle = \bar{p}^Tq$)

Direct computations provide the following result:

Proposition 3. Suppose that $k_{11} + k_{22} = k_{12} + k_{21}$ and $a_1 = a_2 = a$. Consider $P(z) = [z^{k_{11}}(z-a) - b_{11}][z^{k_{22}}(z-a) - b_{22}]$. The vectors q and p of $C^{k_1 + k_2 + 2}$ which verify

$$\hat{A}q = zq \quad ; \quad \hat{A}^T p = \bar{z}p \quad ; \quad \langle p,q \rangle = 1$$

are given by:

$$q = (z^{k_1}q_1, z^{k_1 - 1}q_1, ..., zq_1, q_1, z^{k_2}q_2, z^{k_2 - 1}q_2, ..., zq_2, q_2)^T$$

$$p = (p_1, (\bar{z} - a)p_1, \bar{z}(\bar{z} - a)p_1, ..., \bar{z}^{k_1 - 1}(\bar{z} - a)p_1, p_2, (\bar{z} - a)p_2, \bar{z}(\bar{z} - a)p_2, ..., \bar{z}^{k_2 - 1}(\bar{z} - a)p_2)^T$$

where $q_1 = z^{k_{22}}(z-a) - \beta$; $q_2 = b_{21}$; $\bar{p}_1 = \dfrac{1}{P'(z)}$; $\bar{p}_2 = \dfrac{z^{k_{11}}(z-a) - \beta}{b_{21}P'(z)}$. ∎

The following result gives us information about the direction and stability of Neimark-Sacker bifurcations.

Proposition 4. (see (Kuznetsov, 2004)) The direction and stability of the Neimark-Sacker bifurcation is determined by the sign of $Re(d)$. If $Re(d) < 0$ then the bifurcation is supercritical, i.e. the closed invariant curve bifurcating from the origin is asymptotically stable. If $Re(d) > 0$, the bifurcation is subcritical, i.e. the closed invariant curve bifurcating from the origin is unstable. ∎

5. Example

In the following example, we will consider the delays $k_{11} = 1$, $k_{22} = 5$, $k_{12} = 4$ and $k_{21} = 2$. We will also choose $a = 0.5$ and $b_{11} = b_{22} = \beta$. In this case, using Mathematica, we compute:

- $S_1 = \{0, 0.667561, 1.44928, 2.28703, \pi\}$ (rad), $S_2 = \{\dfrac{\pi}{4}, \dfrac{3\pi}{4}\}$;

- $\theta_1 = \phi_1 = 0.667561$ (rad), $\alpha = -2.91934$, $\beta_1 = -0.723816$, $\beta_0 = -0.162831$;

- $\beta^* = -0.380779$.

The bifurcations occurring at the boundary of D_S (provided by Theorem 2) are:

- For $\beta \in (\beta_1, \beta_0)$ and $\delta = U(\beta)$ a Neimark-Sacker bifurcation occurs, with the multipliers $e^{\pm i h^{-1}(\beta)}$;

- For $\beta \in (\beta_0, 1-a)$ and $\delta = \lambda_0(\beta) = 2(1-a)\beta - (1-a)^2$ a Fold bifurcation occurs;

- For $\beta \in (\beta_1, \beta^*)$ and $\delta = L(\beta) = \lambda_2(\beta)$ a Neimark-Sacker bifurcation occurs, with the multipliers $e^{\pm i\phi_2}$;

- For $\beta \in (\beta^*, \beta_0)$ and $\delta = L(\beta) = \lambda_1(\beta)$ a Neimark-Sacker bifurcation occurs, with the multipliers $e^{\pm i\phi_1}$;

- For $\beta = 1-a$ and $\delta = (1-a)^2$ a 1:1 resonant bifurcation occurs;

- For $\beta = \beta_1$ and $\delta = U(\beta_1)$ a double Neimark-Sacker bifurcation occurs;

- For $\beta = \beta_0$ and $\delta = L(\beta_0) = \lambda_0(\beta_0)$ a Fold-Neimark-Sacker bifurcation occurs.

- For $\beta = \beta^*$ and $\delta = L(\beta^*)$ a double Neimark-Sacker bifurcation occurs.

The stability domain in the (β, δ)-plane for this network is the one presented in Figure 1. More precisely, we consider the delayed discrete-time Hopfield neural network:

$$\begin{cases} x_{n+1} = 0.5x_n + \beta\tanh(x_{n-1}) - \sin(y_{n-4}) \\ y_{n+1} = 0.5y_n + (\delta - \beta^2)\tanh(x_{n-2}) + \beta\sin(y_{n-5}) \end{cases} \quad \forall n \geq 5 \quad (10)$$

Choosing $\beta = -0.25$, we obtain that the origin is asymptotically stable if $\delta \in (-0.385082, 0.324255)$ and supercritical Neimark-Sacker bifurcations occur at $\delta = L(\beta) = -0.385082$ and $\delta = U(\beta) = 0.324255$ respectively (see Figures 4-5). The bifurcation diagram for $\delta \in (-2.5, 2.5)$ is presented in Figure 2 and the values of the Largest Lyapunov Characteristic Exponent are presented in Figure 3. It can be seen that as δ leaves the stability domain D_S, the dynamics in a neighborhood of the origin become more and more complex, eventually leading to the occurrence of chaotic behavior. The phase portraits presented in Figures 6-7 sillustrate the changes which appear on the route from stable dynamics to chaotic dynamics, in a neighborhood of the origin, as $|\delta|$ increases from 0 to 2.5.
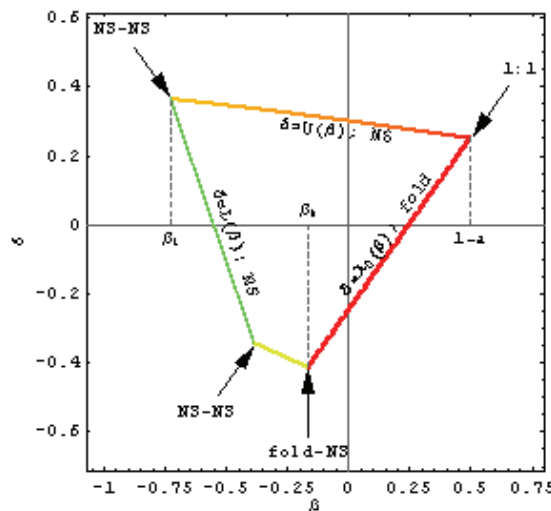
Fig. 1. Stability domain for the null solution when $k_{11} = 1$, $k_{22} = 5$, $k_{12} = 4$, $k_{21} = 2$
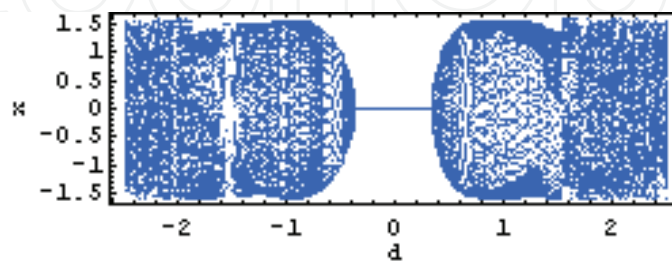

Fig. 2. Bifurcation diagram for system (10) with $\beta = -0.25$, in the (δ, x)-plane, for $\delta \in (-2.5, 2.5)$ (with the step size of 0.02 for δ). For this bifurcation diagram, for each δ value, the initial conditions were reset to $(x_0, y_0) = (0.01, 0.01)$ and 10^5 time steps were iterated before plotting the data (which consists of 10^2 points per δ value).
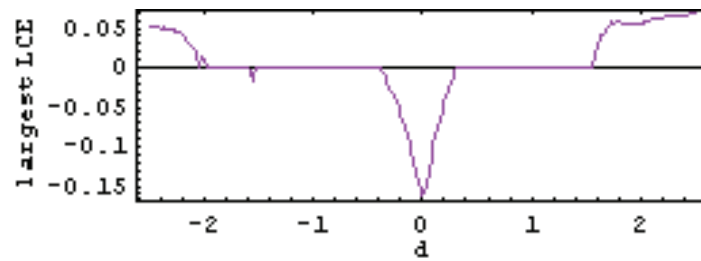
Fig. 3. Largest Lyapunov Characteristic Exponent for system (10) with $\beta = -0.25$. For the computation of the Lyapunov spectrum, for each δ value (step size 0.02 for δ), the initial conditions were reset and 10^5 time-steps were iterated before calculating the LCEs (which were computed over the next 10^5 time steps). The Lyapunov spectrum was computed using the Householder QR based (HQRB) method presented in (Bremen et al., 1997).
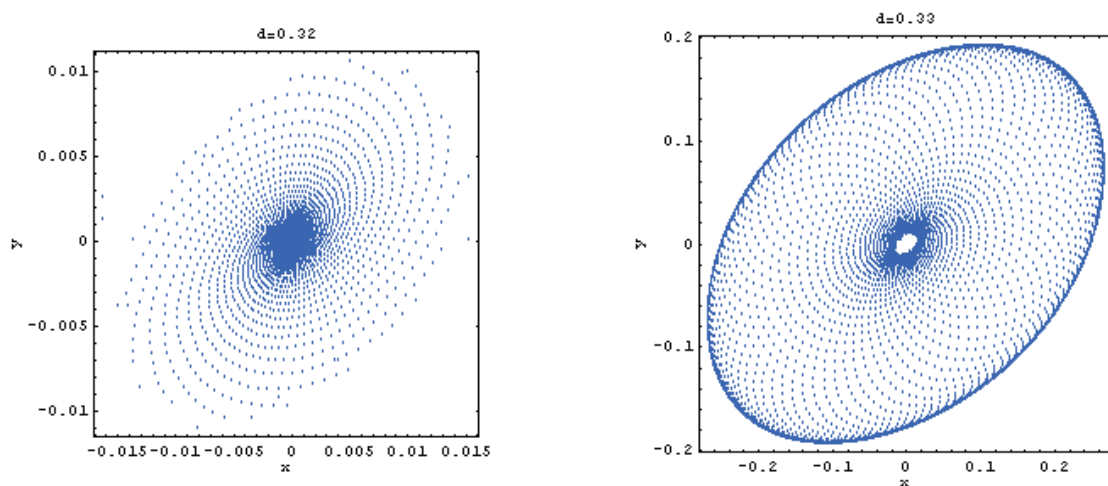

Fig. 4. Supercritical Neimark-Sacker bifurcation at $\delta = 0.324255$. For $\delta = 0.32$, the null solution is asymptotically stable, and the trajectory converges to the origin. For $b = 0.33$, an asymptotically stable cycle (1-torus) is present, and the trajectory converges to this cycle.
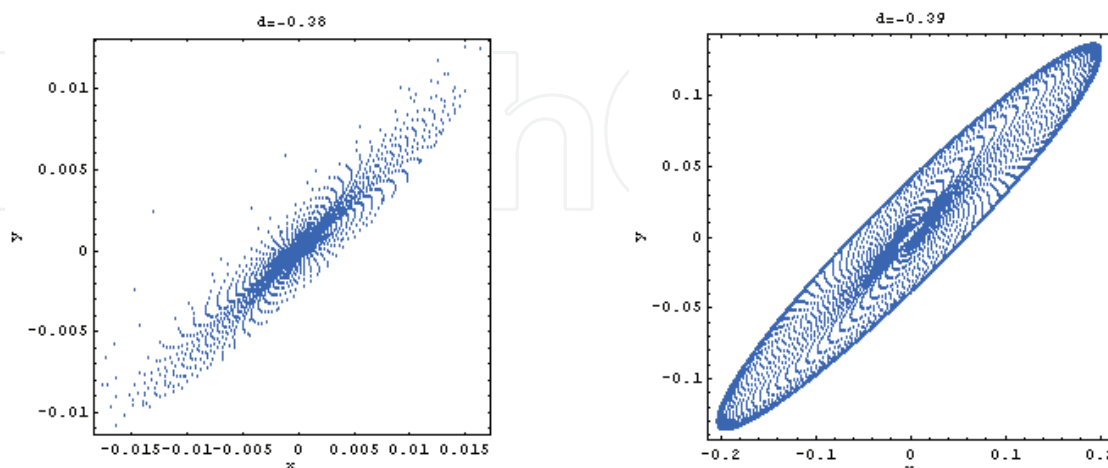
Fig. 5. Supercritical Neimark-Sacker bifurcation at $\delta = -0.385082$. For $\delta = -0.38$, the null solution is asymptotically stable, and the trajectory converges to the origin. For $b = -0.39$, an asymptotically stable cycle (1-torus) is present, and the trajectory converges to this cycle.
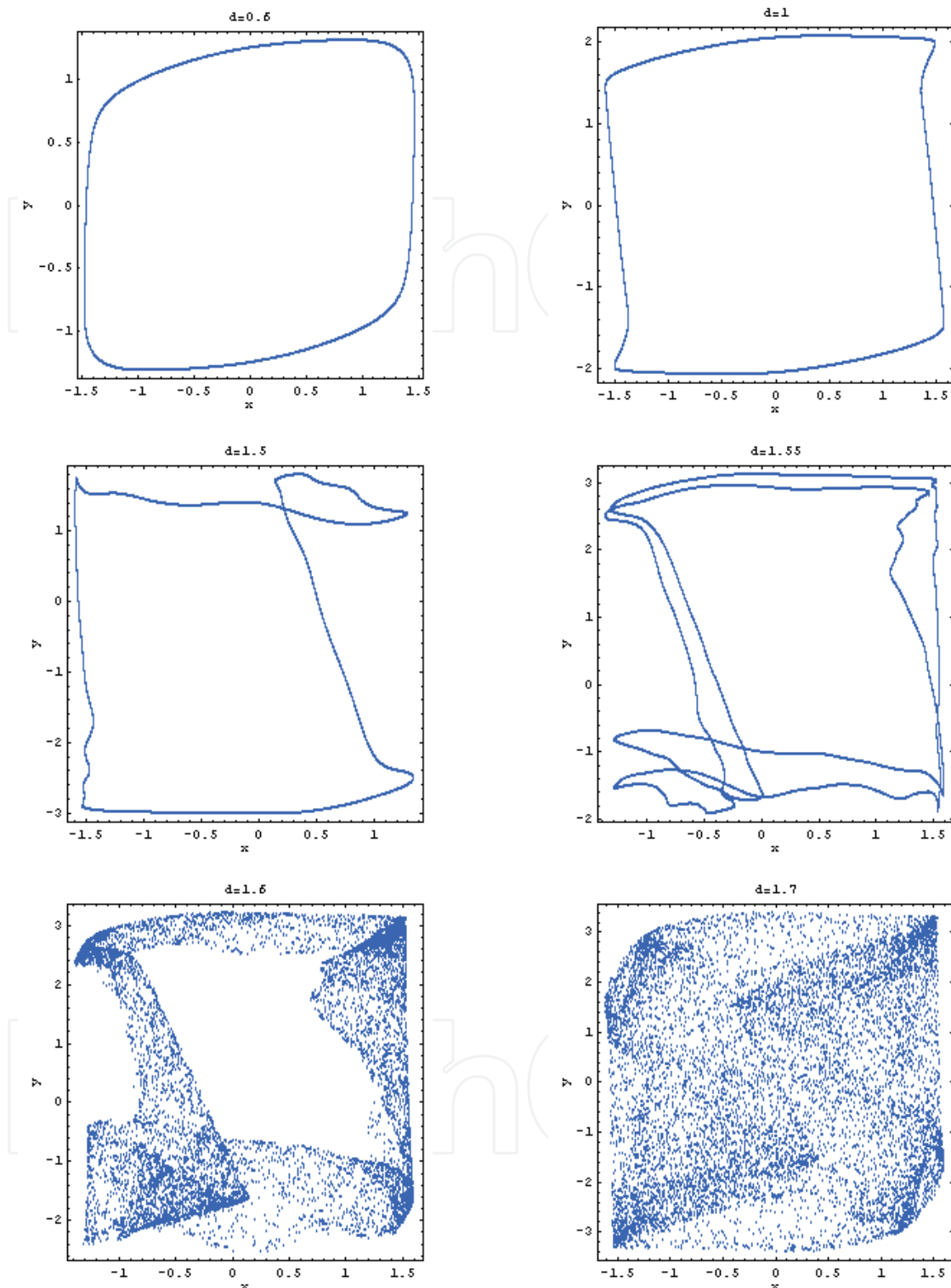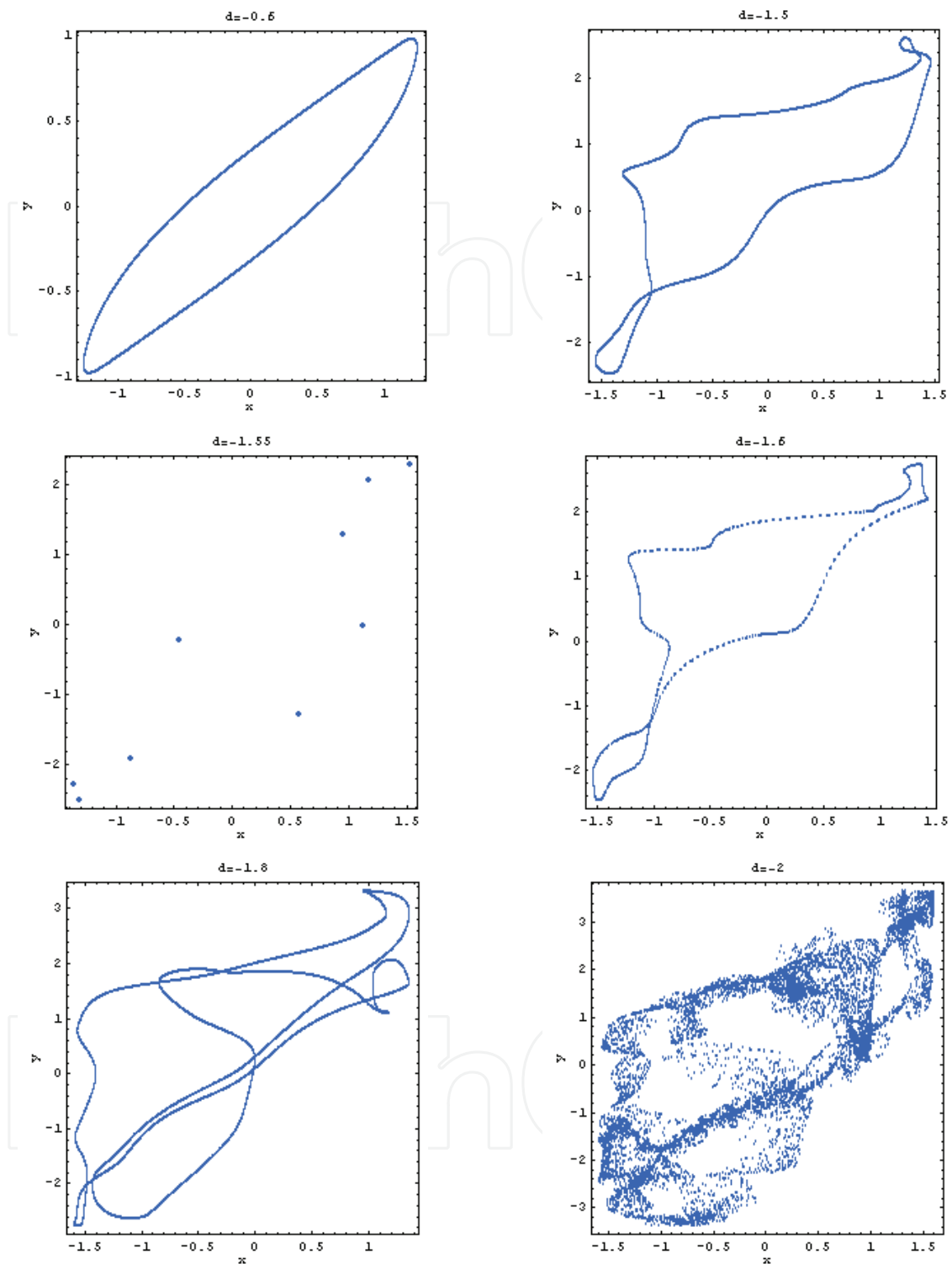
Fig. 6. Phase portraits for various values of $\delta \in (0,2.5)$, at the first step towards chaos. The route towards chaos passes through several stages: $\delta = 0.6$, $\delta = 1$, $\delta = 1.5$: 1-toruses ($\text{LLCE} = 0$); $\delta = 1.55$: 2-torus ($\text{LLCE} = 0$); $\delta = 1.6$: strange attractor ($\text{LLCE} \approx 0$); $\delta = 1.7$: chaos (LLCE>0). For each plot, considering the initial conditions $(x_0,y_0)=(0.01,0.01)$, the first 10^6 iterations of system (10) have been dropped, and the next 10^4 iterations have been plotted.

Fig. 7: Phase portraits for various values of $\delta \in (-2.5,0)$, at the first step towards chaos. The route towards chaos passes through several stages: $\delta = -0.6$, $\delta = -1.5$: 1-toruses ($LLCE = 0$); $\delta = -1.55$: stable period-9 orbit ($LLCE < 0$); $\delta = -1.6$: 1-torus ($LLCE = 0$); $\delta = -1.8$: 2-torus ($LLCE = 0$), $\delta = -2$: strange attractor ($LLCE \approx 0$). For each plot, considering the initial conditions $(x_0, y_0) = (0.01, 0.01)$, the first 10^6 iterations of system (10) have been dropped, and the next 10^4 iterations have been plotted.

6. Conclusions

A complete bifurcation analysis has been presented for a discrete-time Hopfield-type neural network of two neurons with several delays, uncovering the structure of the stability domain of the null solution, as well as the types of bifurcations occurring at its boundary. The numerical example illustrated the theoretical results and suggested some routes towards chaos as the characteristic parameters of the system leave the stability domain.

A generalization of these results to more complicated networks of two or more neurons may constitute a direction for future research.

7. Acknowledgements

8. References

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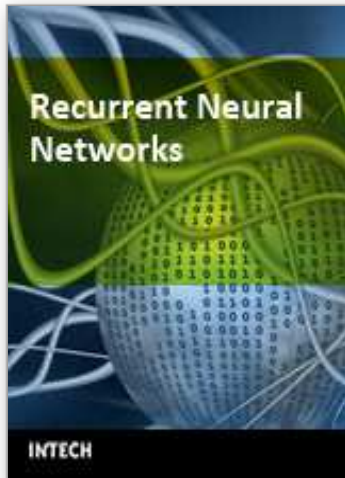
Recurrent Neural Networks

Edited by Xiaolin Hu and P. Balasubramaniam

The concept of neural network originated from neuroscience, and one of its primitive aims is to help us understand the principle of the central nerve system and related behaviors through mathematical modeling. The first part of the book is a collection of three contributions dedicated to this aim. The second part of the book consists of seven chapters, all of which are about system identification and control. The third part of the book is composed of Chapter 11 and Chapter 12, where two interesting RNNs are discussed, respectively.The fourth part of the book comprises four chapters focusing on optimization problems. Doing optimization in a way like the central nerve systems of advanced animals including humans is promising from some viewpoints.

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