Stability Results for Uncertain Stochastic High-Order Hopfield Neural Networks with Time Varying Delays[1]

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1. Introduction

Neural networks have been widely applied in image processing, pattern recognition, optimization solvers, fixed-point computation and other engineering areas. It has been known that these applications heavily depend on the dynamic behaviors of neural networks. The stability of neural networks has been extensively studied over the past years because it is one of the most important behaviors of neural networks. On the other hand, time delays are frequently encountered in neural networks due to the finite switching speed of amplifiers and the inherent communication time of neurons. Since the existence of time delay is often a source of instability for neural networks, the stability study for delayed neural networks is of both theoretical and practical importance.

Hopfield [9, 10] has proposed Hopfield neural networks (HNNs) which have found applications in a broad range of disciplines where the targeted problems can reduce to optimization problems. In recent years, HNNs and their various generalizations have attracted the great attention of many scientists including mathematicians, physicists, computer scientists due to their potential for the tasks of classification, associative memory, parallel computation and their ability to solve difficult optimization problems, see for example [4, 10, 13]. HNNs characterized by first-order interactions, [1, 14] presented their intrinsic limitations. Recently, the study of high-order neural networks has received much attention due to that they have stronger approximation property, faster convergence rate, greater storage capacity and higher fault tolerance than lower-order neural networks [17]. In [3, 5, 6, 8, 11, 12, 15, 16, 18, 19, 22], the authors have been studied the stability analysis of high-order neural networks with constant time delays or time varying delays. In this paper, we are concerned with the global stability for a class of uncertain stochastic high-order neural networks with time varying delays. The structure of the stochastic neural networks under consideration is more general than some previous ones existed in the literature. Based on the Lyapunov stability theory, new global asymptotic stability criteria are presented in

[1] The work of the authors was supported by UGC, New Delhi under SAP(DRS) sanctioned No. F510/6/DRS/2004 (SAP-1).

Source: Recurrent Neural Networks, Book edited by: Xiaolin Hu and P. Balasubramaniam, ISBN 978-953-7619-08-4, pp. 400, September 2008, I-Tech, Vienna, Austria

terms of LMIs . Finally, we also provide a numerical example to demonstrate the effectiveness of the proposed stability results.

2. Problem description and preliminaries

Throughout this chapter we will use the notation $A > 0$ (or $A < 0$) to denote that the matrix A is a symmetric and positive definite (or negative definite) matrix. The notation A^T and A^{-1} mean the transpose of A and the inverse of a square matrix. If A,B are symmetric matrices $A > B$ ($A \geq B$) means that $A - B$ is positive definite (positive semi-definite).
Consider the following high-order Hopfield neural networks with time varying delays described by

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j(t))) \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} T_{ijl} f_j(x_j(t-\tau_j(t))) f_l(y_l(t-\tau_l(t))) + J_i, \qquad i = 1, 2, ..., n. \end{aligned} \tag{1}$$

where $i \in \{1, 2, , ..., n\}$, $t \geq t_0$, $x_i(t)$ is the neuron state; c_i is positive constant, it denotes the rate with which the cell resets its potential to the resting state; a_{ij}, b_{ij} are the first-order synaptic weights of the neural networks; T_{ijl} is the second-order synaptic weights of the neural networks; $\tau_j(t)$ ($j = 1, 2, ..., n$) is the transmission delay of the jth neuron such that $0 < \tau_j(t) \leq \tau_j^*$ and $\dot{\tau}_j(t) \leq \eta_j < 1$, where τ_j^*, η_j are constants; the activation function f_j is continuous on $[t_0 - \tau*, +\infty)$; J_i is the external input.
Assume that
(H1) In the neuron activation function $f(y) = (f_1(y_1), f_2(y_2), \ldots, f_n(y_n))^T$, each function f_i is continuously differentiable with $f_i(0) = 0$ and there exists a positive scalars L_i and $\mathcal{X}_i$ such that for any α_i, $\beta_i \in \mathbb{R}$,

$$|f_i(\alpha_i)| \leq \chi_i \qquad 0 \leq [f_i(\alpha_i) - f_i(\beta_i)](\alpha_i - \beta_i) \leq L_i(\alpha_i - \beta_i)^2.$$

Due to the boundedness of the activation function f_i, by employing the well known Brouwer's fixed point theorem, we can easily obtain that there exists an equilibrium point of the system (1). The uniqueness of the equilibrium point can be deduced from the asymptotic stability which will be proved subsequently.
Let x^* be an equilibrium point of (1) and $y(t) = x(t) - x^*$. Set $g_j(y_j(t)) = f_j(x_j(t)) - f_j(x^*_j)$, $g_j(y_j(t - \tau_j(t))) = f_j(x_j(t - \tau_j(t))) - f_j(x^*_j)$. Apparently, for each $i = 1, 2, ..., n$, we have

$$|g_j(z)| \leq L_j |z|, \qquad \forall z \in \mathbb{R}$$

Consider the following high-order HNNs with time varying delay is given by

$$\frac{dy(t)}{dt} = -Cy(t) + Ag(y(t)) + (B + \Gamma^T T_H) g(y(t - \tau(t))) \tag{2}$$

where

$$
\begin{aligned}
C &= diag(c_1, c_2, ..., c_n),\\
A &= (a_{ij})_{n\times n}, \quad B = (b_{ij})_{n\times n},\\
T_i &= (T_{ijl})_{n\times n},\\
T_H &= (T_1 + T_1^T, T_2 + T_2^T, ..., T_n + T_n^T)^T,\\
y(t-\tau(t)) &= \Big(y_1(t-\tau_1(t)), y_2(t-\tau_2(t)), ..., y_n(t-\tau_n(t))\Big)^T,\\
g(y(t)) &= \Big(g_1(y_1(t)), g_2(y_2(t)), ..., g_n(y_n(t))\Big)^T,\\
g(y(t-\tau(t))) &= \Big(g_1(y_1(t-\tau_1(t))), g_2(y_2(t-\tau_2(t))), ..., g_n(y_n(t-\tau_n(t)))\Big)^T,\\
\zeta &= (\zeta_1, \zeta_2, ..., \zeta_n)^T,\\
\Gamma &= diag(\zeta, \zeta, ..., \zeta).
\end{aligned}
$$

In this paper the following high-order HNN with parameter uncertainties and stochastic perturbations is considered

$$
\begin{aligned}
dy(t) &= \Big[-(C+\Delta C(t))y(t) + (A+\Delta A(t))g(y(t)) + ((B+\Delta B(t)) + \Gamma^T T_H)g(y(t-\tau(t)))\Big]\\
&\quad + \sigma(t, y(t), y(t-\tau(t)))dw(t), \qquad (3)
\end{aligned}
$$

where $w(t) = (w_1(t), w_2(t), ..., w_m(t))^T$ is an m-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let $\sigma(t, x, y) : \mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ is locally Lipschitz continuous and satisfies the linear growth condition. The uncertainties $\Delta C(t)$, $\Delta A(t)$, $\Delta B(t)$ are defined by

$$\Delta C(t) = MF(t)N_C, \quad \Delta A(t) = MF(t)N_A, \quad \Delta B(t) = MF(t)N_B,$$

where $\Delta C(t)$ is a diagonal matrix and M, N_C, N_A and N_B are known real constant matrices with appropriate dimensions, which characterize how the deterministic uncertain parameter in $F(t)$ enters the nominal matrices C, A and B. The matrix $F(t)$, which is time varying unknown and satisfies

$$F(t)^T F(t) \leq I.$$

Let $x(t;\xi)$ denote the state trajectory of the neural network (3) from the initial data $x(\theta) = \xi(\theta)$ on $-\tau^* \le \theta \le 0$ in $L^2_{\mathcal{F}_0}([-\tau^*, 0], \mathbb{R}^n)$. It can be easily seen that the system (3) admits a trivial solution $x(t; 0) \equiv 0$ corresponding to the initial data $\xi = 0$, see [2, 7].

3. Main results

Let $C_{2,1}(\mathbb{R}^n \times \mathbb{R}_+ : \mathbb{R}_+)$ denote the family of all non-negative functions $V(y, t)$ on $\mathbb{R}^n \times \mathbb{R}_+$ which are continuously twice differentiable in x and once differentiable in t. For each $V \in C_{2,1}([-\tau^*, \infty] \times \mathbb{R}^n, \mathbb{R}_+)$, define an operator $\mathrm{L}V(y(t), t)$ associated with stochastic high order neural networks (3) from $\mathbb{R}_+ \times C([-\tau^*, 0]; \mathbb{R}^n)$ to $\mathbb{R}$ by

$$\begin{aligned}\mathcal{L}V(y(t),t) &= V_t(y,t) + V_y(y,t)\Big[-(C+\Delta C(t))y(t) + (A+\Delta A(t))g(y(t)) + ((B+\Delta B(t)) \\ &\quad +\Gamma^T T_H)g(y(t-\tau(t)))\Big] + \frac{1}{2}trace\Big[\sigma^T V_{yy}(y,t)\sigma\Big]\end{aligned}$$

where

$$V_t(y,t) = \frac{\partial V(y,t)}{\partial t}, V_y(y,t) = \Big(\frac{\partial V(y,t)}{\partial y_1}, \frac{\partial V(y,t)}{\partial y_2}, \ldots, \frac{\partial V(y,t)}{\partial y_n}\Big),$$

and

$$V_{yy}(y,t) = \Big(\frac{\partial^2 V(y,t)}{\partial y_i \partial y_j}\Big)_{n\times n}$$

where $i, j = 1, 2, \ldots, n$. In order to prove our results, we need to state the following definitions and Lemma.

Lemma 3.1. *Given any real matrices Σ_1, Σ_2, Σ_3 of appropriate dimensions and a scalar $\epsilon > 0$ such that $0 < \Sigma_3 = \Sigma_3^T$. Then, the following inequality holds:*

$$\Sigma_1^T\Sigma_2 + \Sigma_2^T\Sigma_1 \le \epsilon\Sigma_1^T\Sigma_3\Sigma_1 + \epsilon^{-1}\Sigma_2^T\Sigma_3^{-1}\Sigma_2.$$

We also recall some basic facts about norms of vectors and matrices. Let $y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$. Three commonly used vector norms are given as $\|y\|_1 = \sum_{i=1}^n |y_i|, \|y\|_2 = (\sum_{i=1}^n y_i^2)^{1/2}$ and $\|y\|_\infty = \max_{1\le i\le n} |y_i|$. It is also known that $\|y\|_\infty \le \|y\|_1$. The vector $|y|$ will denote $|y| = (|y_1|, |y_2|, \ldots, |y_n|)^T$. For any matrix $V = (v_{ij})_{n\times n}$, $\lambda_m(V)$ and $\lambda_M(V)$ will denote respectively the minimum and maximum eigenvalues of V. For the matrix V, $\|V\|_2^2 = \lambda_M(V^T V)$.

Now we will prove the following theorem on global asymptotic stability in the mean square for equation (3).

Theorem 3.2. *Assume that there exist matrices $P > 0$, $D_0 \ge 0$ and $D_1 \ge 0$ such that*

$$trace\Big[\sigma^T\Big(t, y(t), y(t-\tau(t))\Big)P\sigma\Big(t, y(t), y(t-\tau(t))\Big)\Big] \le y^T(t)D_0y(t) + y^T(t-\tau(t))D_1y(t-\tau(t)).$$

System (3) is globally asymptotically stable in the mean square, if there exist positive definite matrices Σ_1, Σ_2 and the scalars $\epsilon_k > 0$ $(k = 1, 2)$ such that

$$\Pi_1 = \begin{bmatrix} \psi_1 & PM & PA & PM & PB & PM & \epsilon_1 N_C & \epsilon_2 N_A^T L & L\Sigma_2 N_B \\ * & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\epsilon_2 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\epsilon_3 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Sigma_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -(1-\eta)\Sigma_2 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\epsilon_1 I & 0 & 0 \\ * & * & * & * & * & * & * & -\epsilon_2 I & 0 \\ * & * & * & * & * & * & * & * & -\Sigma_2 \end{bmatrix} < 0 \quad (4)$$

where $L = diag(L_i), i = 1, 2, ..., n,\ \chi = (\chi_1, \chi_2, ..., \chi_n)^T$ *and* $\psi_1 = -PC - C^T P + \epsilon_2 L^2 + D_1 + LT_H T_H^T L + \frac{1}{1-\eta}\|\chi\|^2 P^2$.

Proof: We use the following Lyapunov functional to derive the stability result

$$V(y,t) = y^T(t)Py(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t} q_{ij} g_j^2(y(s))ds.$$

By Ito's formula, we can calculate $\mathcal{L}V_1(y(t),t), \mathcal{L}V_2(y(t),t)$ along the trajectories of the system (3), then we have

$$\mathcal{L}V_1(y(t),t) = 2y^T(t)P[-(C+\Delta C(t))y(t) + (A+\Delta A(t))g(y(t)) + ((B+\Delta B(t)) + \Gamma^T T_H)g(y(t-\tau(t)))] + trace[\sigma^T(t,y(t),y(t-\tau(t)))P\sigma(t,y(t),y(t-\tau(t)))] \quad (5)$$

$$\mathcal{L}V_2(y(t),t) = g^T(y(t))Qg(y(t)) - (1-\eta)g^T(y(t-\tau(t)))Qg(y(t-\tau(t))). \quad (6)$$

From (5)-(6), we get

$$\begin{aligned}\mathcal{L}V(y(t),t) &= \mathcal{L}V_1(y(t),t) + \mathcal{L}V_2(y(t),t) \\ &\le 2y^T(t)P[-(C+\Delta C(t))y(t) + (A+\Delta A(t))g(y(t)) + ((B+\Delta B(t)) \\ &\quad + \Gamma^T T_H)g(y(t-\tau(t)))] + trace[\sigma^T(t,y(t),y(t-\tau(t)))P\sigma(t,y(t),y(t-\tau(t)))] \\ &\quad + g^T(y(t))Qg(y(t)) - (1-\eta)g^T(y(t-\tau(t)))Qg(y(t-\tau(t))). \quad (7)\end{aligned}$$

By Lemma 3.1 we get,

$$-2y^T(t)P(\Delta C(t))y(t) \leq \epsilon_1^{-1}y^T(t)PMM^TPy(t) + \epsilon_1 y^T(t)N_CN_Cy(t) \quad (8)$$

$$2y^T(t)PAg(y(t)) \leq \epsilon_2^{-1}y^T(t)PAA^TPy(t) + \epsilon_2 y^T(t)L^2y(t) \quad (9)$$

$$2y^T(t)P(\Delta A(t))g(y(t)) \leq \epsilon_3^{-1}y^T(t)PMM^TPy(t) + \epsilon_3 y^T(t)LN_A^TN_ALy(t) \quad (10)$$

$$\begin{aligned} 2y^T(t)PBg(y(t-\tau(t))) \leq\ & \frac{1}{1-\eta}y^T(t)PB\Sigma_1^{-1}B^TPy(t) \\ & +(1-\eta)g^T(y(t-\tau(t)))\Sigma_1 g(y(t-\tau(t))) \end{aligned} \quad (11)$$

$$\begin{aligned} 2y^T(t)P(\Delta B(t))g(y(t-\tau(t))) \leq\ & \frac{1}{1-\eta}y^T(t)PM\Sigma_2^{-1}M^TPy(t) \\ & +(1-\eta)g^T(y(t-\tau(t)))N_B^T\Sigma_2N_Bg(y(t-\tau(t))) \end{aligned} \quad (12)$$

$$\begin{aligned} 2y^T(t)P\Gamma^TT_Hg(y(t-\tau(t))) \leq\ & \frac{1}{1-\eta}y^T(t)P\Gamma^T\Gamma Py(t) \\ & +(1-\eta)g^T(y(t-\tau(t)))T_H^TT_Hg(y(t-\tau(t))). \end{aligned} \quad (13)$$

Since $\Gamma^T\Gamma = \|\zeta\|^2 I$ and $\|\zeta\| \leq \|\chi\|$, it is clear that

$$y^T(t)P\Gamma^T\Gamma Py(t) \leq \|\chi\|^2 y^T(t)P^2y(t).$$

Since $Q = L^{-1}D_1L^{-1} + N_B^T\Sigma_2N_B + T_H^TT_H$, and from (7)-(13), it follows that

$$\begin{aligned} \mathcal{L}V(y(t),t) \leq\ & y^T(t)(-PC - C^TP)y(t) + y^T(t)(\epsilon_1^{-1}PMM^TP + \epsilon_1 N_C^TN_C)y(t) \\ & +y^T(t)(\epsilon_2^{-1}PAA^TP + \epsilon_2L^2)y(t) + y^T(t)(\epsilon_3^{-1}PMM^TP + \epsilon_3LN_A^TN_AL)y(t) \\ & +\frac{1}{1-\eta}y^T(t)PB\Sigma_1^{-1}B^TPy(t) + (1-\eta)g^T(y(t-\tau(t)))\Sigma_1g(y(t-\tau(t))) \\ & +\frac{1}{1-\eta}y^T(t)PM\Sigma_2^{-1}M^TPy(t) + (1-\eta)g^T(y(t-\tau(t)))N_B^T\Sigma_2N_Bg(y(t-\tau(t))) \\ & +\frac{1}{1-\eta}y^T(t)P\Gamma^T\Gamma Py(t) + (1-\eta)g^T(y(t-\tau(t)))T_H^TT_Hg(y(t-\tau(t))) \\ & +g^T(y(t))Qg(y(t)) - (1-\eta)g^T(y(t-\tau(t)))Qg(y(t-\tau(t))) \end{aligned}$$

$$+trace[\sigma^T(t, y(t), y(t-\tau(t)))P\sigma(t, y(t), y(t-\tau(t)))],$$

Then we have $\mathcal{L}V(y(t), t) < 0$ when $\Pi_1 < 0$, that is the inequality (4) holds, which completes the proof of the theorem.

By constructing another Lyapunov functional, we can obtain the following result.

Theorem 3.3. *Assume that there exist matrices $D_0 \geq 0$ and $D_1 \geq 0$ such that*

$$\begin{aligned} trace[\sigma^T(t, y(t), y(t-\tau(t))) \quad D \quad diag\Big\{\dot{g}_1(y_1(t)), \dot{g}_2(y_2(t)), \cdots, \dot{g}_n(y_n(t))\Big\}\sigma(t, y(t), y(t-\tau(t)))] \\ \leq \quad y^T(t)D_0y(t) + y^T(t-\tau(t))D_1y(t-\tau(t)). \end{aligned}$$

System (3)is globally asymptotically stable in the mean square, if there exist positive definite matrices Σ_1 and the scalars $\epsilon_k > 0$ (k = 1, 2, 3) such that

$$\Pi_2 = \begin{bmatrix} \psi_2 & DM & \epsilon_1 N_C L^{-1} & DM & \epsilon_2 N_A & DM & DB & \epsilon_3 N_B^T \\ * & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\epsilon_2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\epsilon_2 I & 0 & 0 & 0 \\ * & * & * & * & * & -\epsilon_3 I & 0 & 0 \\ * & * & * & * & * & * & \Sigma_1 & 0 \\ * & * & * & * & * & * & * & -\epsilon_3(1-\eta)I \end{bmatrix} < 0, \quad (14)$$

where $L = diag(L_i), i = 1, 2, ..., n,\ \chi = (\chi_1, \chi_2, ..., \chi_n)^T,\ D = diag[d_1, d_2, ..., d_n] > 0\ i = 1, 2, ..., n$ *and* $\psi_2 = -DCL^{-1} - L^{-1}CD + DA + A^TD + \|\chi\|^2D^2 + \frac{1}{1-\eta}\Sigma_1 + \frac{1}{1-\eta}T_H^TT_H + D_1 + L^{-1}D_0L^{-1}$.

Proof: We use the following positive definite Lyapunov functional to derive the stability result,

$$V(y, t) \quad = \quad 2\sum_{i=1}^{n} d_i \int_0^{y_i} g_i(s)ds + \frac{1}{1-\eta}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t} q_{ij}g_j^2(y(s))ds$$

where $Q = (q_{ij})_{n\times n} = N_B^TN_B + T_H^TT_H + D_1 + \Sigma_1,\ d_i > 0,\ i = 1, 2, ..., n$. Define

$$G(y) = \min\Big\{\min\Big\{\int_0^{y_i} g_i(\theta)d\theta, \int_0^{-y_i} g_i(\theta)d\theta\Big\}, i = 1, 2, ..., n\Big\},$$

which satisfies

$$G(r) > 0, \quad r > 0, \qquad G(r) \to +\infty, \quad r \to +\infty,$$

and $G(0)=0, G(y)=G(|y|)$, for $x \in \mathbb{R}^n_+$. We have

$$V(y(t),t) \geq 2\sum_{i=1}^{n} d_i \int_0^{y_i} g_i(s)ds \geq 2\lambda_m(D)G(|y|),$$

which gives a lower by a positive radially unbounded function.
It is to verify that

$$2\lambda_m(D)G(|y|) \leq V(y(t),t) \leq \Big[2q\lambda_M(DL) + \frac{\tau(t)}{1-\eta} q\lambda_M(LQL)\Big]\|y(t)\|^2, \quad q > 1.$$

By Ito' s formula, we can calculate $\mathcal{L}V_1(y(t), t)$,$\mathcal{L}V_2(y(t), t)$ along the trajectories of the system (3), then we have

$$\begin{aligned} \mathcal{L}V_1(y(t),t) &= 2y^T(t)D[-(C+\Delta C(t))y(t) + (A+\Delta A(t))g(y(t)) + ((B+\Delta B(t)) \\ &\quad +\Gamma^T T_H)g(y(t-\tau(t)))] + trace[\sigma^T(t,y(t),y(t-\tau(t)))D\sigma(t,y(t),y(t-\tau(t)))] \\ &\quad diag\Big\{\dot{g}_1(y_1(t)), \dot{g}_2(y_2(t)), \cdots, \dot{g}_n(y_n(t))\Big\}, \end{aligned} \tag{15}$$

$$\mathcal{L}V_2(y(t),t) = \frac{1}{1-\eta} g^T(y(t))Qg(y(t)) - g^T(y(t-\tau(t)))Qg(y(t-\tau(t))). \tag{16}$$

Then it follows from Lemma 3.1 that

$$\begin{aligned} -2g^T(y(t))D(\Delta C(t))y(t) &\leq \epsilon_1^{-1} g^T(y(t))DMM^T Dy(t) \\ &\quad +\epsilon_1 g^T(y(t))L^{-1}N_C^T N_C L^{-1} g(y(t)) \end{aligned} \tag{17}$$

$$\begin{aligned} 2g^T(y(t))D(\Delta A(t))g(y(t)) &\leq \epsilon_2^{-1} y^T(t)DMM^T Dy(t) \\ &\quad +\epsilon_2 g^T(y(t))LN_A^T N_A Lg(y(t)) \end{aligned} \tag{18}$$

$$\begin{aligned} 2g^T(y(t))D(\Delta B(t))g(y(t-\tau(t))) &\leq \epsilon_3^{-1} g^T(y(t))DMM^T Dg(y(t)) \\ &\quad +\epsilon_3 g^T(y(t-\tau(t)))N_B^T N_B g(y(t-\tau(t))) \end{aligned} \tag{19}$$

$$\begin{aligned} g^T(y(t))(t)DBg(y(t-\tau(t))) + g^T(y(t-\tau(t)))B^T Dg(y(t)) &\leq g^T(y(t))DB\Sigma_1^{-1}B^T Dg(y(t)) \\ &\quad +g^T(y(t-\tau(t)))\Sigma_1 g(y(t-\tau(t))) \end{aligned} \tag{20}$$

$$\begin{aligned} g^T(y(t))(t)D\Gamma^T T_H g(y(t-\tau(t))) \quad &+ \quad g^T(y(t-\tau(t)))T_H^T \Gamma g(y(t)) \leq g^T(y(t))D\Gamma^T \Gamma D g(y(t)) \\ &+ g^T(y(t-\tau(t)))T_H^T T_H g(y(t-\tau(t))) \\ &\leq \|\chi\|^2 g^T(y(t))D^2 g(y(t)) \\ &+ g^T(y(t-\tau(t)))T_H^T T_H g(y(t-\tau(t))). \end{aligned} \quad (21)$$

Since $Q = N_B^T N_B + T_H^T T_H + D_1 + \Sigma_1$, and from (15)-(21) it follows that

$$\mathcal{L}V(y(t),t) \quad \leq \quad g^T(y(t))\Pi g(y(t)).$$

Then we have $\mathcal{L}V(y(t), t) < 0$ when $\Pi_2 < 0$, that is the inequality (14) holds, which completes the proof of the theorem.

Theorem 3.4. *Assume that there exist matrices* $C > 0$, $D_0 \geq 0$ *and* $D_1 \geq 0$ *such that*

$$trace[\sigma^T(t, y(t), y(t-\tau(t)))(C + DL)\sigma(t, y(t), y(t-\tau(t)))] \leq y^T(t)D_0 y(t) + y^T(t-\tau(t))D_1 y(t-\tau(t)).$$

System (3)is globally asymptotically stable in the mean square, if the condition (H_1) is satisfied and there exists positive constants β, ϵ_i, i = 4, 5, 6 such that

$$\Omega = \begin{bmatrix} \psi_3 & D & DM & DM & DM & \epsilon_4 L^{-1}N_C & \epsilon_5 N_A^T & \epsilon_6 N_B^T \\ * & -\beta P & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\epsilon_4 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\epsilon_5 I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\epsilon_6 I & 0 & 0 & 0 \\ * & * & * & * & * & -\epsilon_4 I & 0 & 0 \\ * & * & * & * & * & * & -\epsilon_5 I & 0 \\ * & * & * & * & * & * & * & -\epsilon_6 I \end{bmatrix} < 0, \quad (22)$$

where $L = diag(L_i), i = 1, 2, ..., n,\ \chi = (\chi_1, \chi_2, ..., \chi_n)^T,\ D = diag[d_1, d_2, ..., d_n] > 0$ *where* $i = 1, 2, ..., n$ *and* $\psi_3 = -2DCL^{-1} + DA + A^T D + \frac{1+\|\chi\|^2}{1-\eta}\beta\lambda_M(P)(B^T B + T_H^T T_H)$.

Proof: We use the following positive definite Lyapunov functional to derive the stability result,

$$\begin{aligned} V(y,t) \quad = \quad & y^T(t)Cy(t) + 2\sum_{i=1}^{n} d_i \int_0^{y_i} g_i(s)ds + \frac{1}{1-\eta}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t} q_{ij} g_j^2(y(s))ds \\ & + \alpha\beta\frac{1}{1-\eta}\int_{t-\tau(t)}^{t} g^T(y(s))W^T P W g(y(s))ds + \frac{1}{1-\eta}\int_{t-\tau(t)}^{t} g^T(y(s))W^T W g(y(s))ds, \end{aligned}$$

where $W = B + \Gamma^T T_H$ and $Q = (q_{ij})_{n\times n} = (\in_3^{-1} + \alpha \in_6^{-1})N_B^T N_B + L^{-1}D_1L^{-1}$. By Ito's formula, we can calculate $\mathcal{L}V^1, \mathcal{L}V^2, \mathcal{L}V^3, \mathcal{L}V^4$ and $\mathcal{L}V^5$ along the trajectories of the system (3), then we have

$$\begin{aligned}\mathcal{L}V_1(y(t),t) \leq\ & -y^T(t)C^2y(t) - y^T(t)C^2y(t) - 2y^T(t)C(\Delta C(t))y(t) + 2y^T(t)CAy(t) \\ & +2y^T(t)C(\Delta A(t))y(t) + 2y^T(t)CWg(y(t-\tau(t))) + 2y^T(t)C(\Delta B(t))g(y(t-\tau(t))) \\ & +trace[\sigma^T(t,y(t),y(t-\tau))C\sigma(t,y(t),y(t-\tau))]. \end{aligned} \quad (23)$$

Using the inequality technique, we have

$$\begin{aligned}-y^T(t)C^2y(t) + 2y^T(t)CAy(t) =\ & -(Cy(t) - Ag(y(t)))^T(Cy(t) - Ag(y(t))) \\ & +g^T(y(t))A^TAg(y(t)), \end{aligned} \quad (24)$$

$$\begin{aligned}-y^T(t)C^2y(t) + 2y^T(t)CWg(y(t-\tau(t))) =\ & -(Cy(t) - Wg(y(t-\tau(t))))^T(Cy(t) - Wg(y(t-\tau(t)))) \\ & +g^T(y(t-\tau(t)))W^TWg(y(t-\tau(t))). \end{aligned} \quad (25)$$

From Lemma 3.1, it follows that

$$\begin{aligned}2y^T(t)C(\Delta C(t))y(t) \leq\ & \epsilon_1^{-1}g^T(y(t))L^{-1}CMM^TCL^{-1}g(y(t)) \\ & +\epsilon_1 g^T(y(t))L^{-1}N_C^TN_CL^{-1}g(y(t)) \end{aligned} \quad (26)$$

$$\begin{aligned}2y^T(t)C(\Delta A(t))y(t) \leq\ & \epsilon_2^{-1}g^T(y(t))L^{-1}CMM^TCL^{-1}g(y(t)) \\ & +\epsilon_2 g^T(y(t))N_A^TN_Ag(y(t)) \end{aligned} \quad (27)$$

$$\begin{aligned}2y^T(t)C(\Delta B(t))g(y(t-\tau(t))) \leq\ & \epsilon_3 g^T(y(t))L^{-1}CMM^TCL^{-1}g(y(t)) \\ & +\epsilon_3^{-1}g^T(y(t-\tau(t)))N_B^TN_Bg(y(t-\tau(t))). \end{aligned} \quad (28)$$

Since the first term of the equations (24) and (25) are non-positive, we can write the following inequalities:

$$-y^T(t)C^2y(t) + 2y^T(t)CAy(t) \leq g^T(y(t))A^TAg(y(t)) \quad (29)$$

$$-y^T(t)C^2y(t) + 2y^T(t)CWg(y(t-\tau(t))) \leq g^T(y(t-\tau(t)))W^TWg(y(t-\tau(t))). \quad (30)$$

Substitute (26)-(30) in (23), we get

$$\begin{aligned}\mathcal{L}V_1(y(t),t) &\le g^T(y(t))A^TAg(y(t)) + g^T(y(t-\tau(t)))W^TWg(y(t-\tau(t)))\\ &+\epsilon_1^{-1}g^T(y(t))L^{-1}CMM^TCL^{-1}g(y(t)) + \epsilon_1 g^T(y(t))L^{-1}N_C^TN_CL^{-1}g(y(t))\\ &+\epsilon_2^{-1}g^T(y(t))L^{-1}CMM^TCL^{-1}g(y(t)) + \epsilon_2 g^T(y(t))N_A^TN_Ag(y(t))\\ &+\epsilon_3^{-1}g^T(y(t))L^{-1}CMM^TCL^{-1}g(y(t)) + \epsilon_3 g^T(y(t-\tau(t)))N_B^TN_Bg(y(t-\tau(t)))\\ &+trace[\sigma^T(t,y(t),y(t-\tau(t)))C\sigma(t,y(t),y(t-\tau(t)))]. \end{aligned} \tag{31}$$

Also,

$$\begin{aligned}\mathcal{L}V_2(y(t),t) &= 2\alpha g^T(y(t))D\Big[-(C+\Delta C(t))y(t) + (A+\Delta A(t))g(y(t)) + ((B+\Delta B(t))\\ &+\Gamma^T T_H)g(y(t-\tau(t)))\Big] + trace[\sigma^T(t,y(t),y(t-\tau(t)))DL\sigma(t,y(t),y(t-\tau(t)))].\end{aligned}$$

Adding and subtracting $\alpha\beta^{-1}g^T(y(t))DP^{-1}Dg(y(t))$ in the above equation, then we have

$$\begin{aligned}\mathcal{L}V_2(y(t),t) &\le -2\alpha g^T(y(t))DCy(t) - 2\alpha g^T(y(t))D(\Delta C(t))y(t) + 2\alpha g^T(y(t))DAg(y(t))\\ &+2\alpha g^T(y(t))D(\Delta A(t))g(y(t)) + 2\alpha g^T(y(t))DWg(y(t-\tau(t)))\\ &+2\alpha g^T(y(t))D(\Delta B(t))g(y(t-\tau(t))) + \alpha\beta^{-1}g^T(y(t))DP^{-1}Dg(y(t))\\ &-\alpha\beta^{-1}g^T(y(t))DP^{-1}Dg(y(t)).\end{aligned} \tag{32}$$

From Lemma 3.1, it follows that

$$\begin{aligned}-2\alpha g^T(y(t))D(\Delta C(t))y(t) &\le \alpha\epsilon_4^{-1}g^T(y(t))DMM^TDg(y(t))\\ &+\alpha\epsilon_4 g^T(y(t))L^{-1}N_C^TN_CL^{-1}g(y(t))\end{aligned} \tag{33}$$

$$\begin{aligned}2\alpha g^T(y(t))D(\Delta A(t))g(y(t)) &\le \alpha\epsilon_5^{-1}g^T(y(t))DMM^TDg(y(t))\\ &+\alpha\epsilon_5 g^T(y(t))N_A^TN_Ag(y(t))\end{aligned} \tag{34}$$

$$\begin{aligned}2\alpha g^T(y(t))D(\Delta B(t))g(y(t-\tau(t))) &\le \alpha\epsilon_6^{-1}g^T(y(t))DMM^TDg(y(t))\\ &+\alpha\epsilon_6 g^T(y(t-\tau(t)))N_B^TN_Bg(y(t-\tau(t)))\end{aligned} \tag{35}$$

and

$$-2\alpha g^T(y(t))DCy(t) \leq -2\alpha g^T(y(t))DCL^{-1}g(y(t)). \tag{36}$$

Using the inequality technique, we have

$$\begin{aligned}
-\alpha\beta^{-1}g^T(y(t))DP^{-1}Dg(y(t)) \quad + \quad & 2\alpha g^T(y(t))DWg(y(t-\tau(t))) \\
= \quad & -\alpha[\beta^{\frac{1}{2}}DP^{\frac{1}{2}}g(y(t)) - \beta^{\frac{1}{2}}P^{\frac{1}{2}}Wg(y(t-\tau(t)))]^T \\
& \times[\beta^{\frac{1}{2}}DP^{\frac{1}{2}}g(y(t)) - \beta^{\frac{1}{2}}P^{\frac{1}{2}}Wg(y(t-\tau(t)))] \\
& +\alpha\beta g^T(y(t-\tau(t)))PWg(y(t-\tau(t))).
\end{aligned}$$

Since the first term of the above equation is non-positive, we can write the following inequality

$$\begin{aligned}
-\alpha\beta^{-1}g^T(y(t))DP^{-1}Dg(y(t)) \quad + \quad & 2\alpha g^T(y(t))DWg(y(t-\tau(t))) \\
\leq \quad & \alpha\beta g^T(y(t-\tau(t)))PWg(y(t-\tau(t))).
\end{aligned} \tag{37}$$

Substitute (33)-(37) in (32), we get

$$\begin{aligned}
\mathcal{L}V_2(y(t),t) \quad \leq \quad & -2\alpha g^T(y(t))DCL^{-1}g(y(t)) + 2\alpha g^T(y(t))DAg(y(t)) + \alpha\epsilon_4^{-1}g^T(y(t))DMM^TDg(y(t)) \\
& +\alpha\epsilon_4 g^T(y(t))L^{-1}N_C^TN_Cg(y(t)) + \alpha\epsilon_5^{-1}g^T(y(t))DMM^TDg(y(t)) \\
& +\alpha\epsilon_5 g^T(y(t))N_A^TN_Ag(y(t)) + \alpha\epsilon_6^{-1}g^T(y(t))DMM^TDg(y(t)) \\
& +\alpha\epsilon_6 g^T(y(t-\tau(t)))N_B^TN_Bg(y(t-\tau(t))) + \alpha\beta^{-1}g^T(y(t))DP^{-1}Dg(y(t)) \\
& +\alpha\beta g^T(y(t-\tau(t)))W^TPWg(y(t-\tau(t))) \\
& +trace[\sigma^T(t,y(t),y(t-\tau(t)))DL\sigma(t,y(t),y(t-\tau(t)))],
\end{aligned} \tag{38}$$

$$\mathcal{L}V_3(y(t),t) \quad \leq \quad \frac{1}{1-\eta}g^T(y(t))Qg(y(t)) - g^T(y(t-\tau(t)))Qg(y(t-\tau(t))), \tag{39}$$

$$\mathcal{L}V_4(y(t),t) \quad \leq \quad \alpha\beta\frac{1}{1-\eta}g^T(y(t))W^TPWg(y(t)) - \alpha\beta g^T(y(t-\tau(t)))W^TPWg(y(t-\tau(t))), \tag{40}$$

$$\mathcal{L}V_5(y(t),t) \quad \leq \quad \frac{1}{1-\eta}g^T(y(t))W^TWg(y(t)) - g^T(y(t-\tau(t)))W^TWg(y(t-\tau(t))). \tag{41}$$

From (31) and (38)-(41), it follows that

$$\begin{aligned}
\mathcal{L}V(y(t),t) &= \mathcal{L}V_1(y(t),t)+\mathcal{L}V_2(y(t),t)+\mathcal{L}V_3(y(t),t)+\mathcal{L}V_4(y(t),t)+\mathcal{L}V_5(y(t),t)\\
\leq\ & g^T(y(t))A^TAg(y(t))+g^T(y(t-\tau(t)))W^TWg(y(t-\tau(t)))\\
&+(\epsilon_1^{-1}+\epsilon_2^{-1}+\epsilon_3^{-1})g^T(y(t))L^{-1}CMM^TCL^{-1}g(y(t))+\epsilon_1 g^T(y(t))L^{-1}N_C^TN_CL^{-1}g(y(t))\\
&+\epsilon_2 g^T(y(t))N_A^TN_Ag(y(t))+\epsilon_3 g^T(y(t-\tau(t)))N_B^TN_Bg(y(t-\tau(t)))\\
&-2\alpha g^T(y(t))DCL^{-1}g(y(t))+2\alpha g^T(y(t))DAg(y(t))\\
&+\alpha(\epsilon_4^{-1}+\epsilon_5^{-1}+\epsilon_6^{-1})g^T(y(t))DMM^TDg(y(t))+\alpha\epsilon_4 g^T(y(t))L^{-1}N_C^TN_CL^{-1}g(y(t))\\
&+\alpha\epsilon_5 g^T(y(t))N_A^TN_Ag(y(t))+\alpha\epsilon_6 g^T(y(t-\tau(t)))N_B^TN_Bg(y(t-\tau(t)))\\
&+\alpha\beta^{-1}g^T(y(t))DP^{-1}Dg(y(t))+\alpha\beta g^T(y(t-\tau(t)))W^TPWg(y(t-\tau(t)))\\
&+g^T(y(t))L^{-1}D_0L^{-1}g(y(t))+g^T(y(t-\tau(t)))L^{-1}D_1L^{-1}g(y(t-\tau(t)))\\
&+\frac{1}{1-\eta}g^T(y(t))Qg(y(t))-g^T(y(t-\tau(t)))Qg(y(t-\tau(t)))\\
&+\alpha\beta\frac{1}{1-\eta}g^T(y(t))W^TPWg(y(t))-\alpha\beta g^T(y(t-\tau(t)))W^TPWg(y(t-\tau(t)))\\
&+\frac{1}{1-\eta}g^T(y(t))W^TWg(y(t))-g^T(y(t-\tau(t)))W^TWg(y(t-\tau(t))).
\end{aligned}$$

Since

$$\begin{aligned}
W^TW &= (B+\Gamma^TT_H)^T(B+\Gamma^TT_H)=B^TB+B^T\Gamma^TT_H+T_H^T\Gamma B+T_H^T\Gamma\Gamma^TT_H\\
&\leq (1+\|\chi\|^2)B^TB+(1+\|\chi\|^2)T_H^TT_H=(1+\|\chi\|^2)(B^TB+T_H^TT_H).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{L}V(y(t),t) \leq\ & g^T(y(t))\Big[A^TA+(\epsilon_1^{-1}+\epsilon_2^{-1}+\epsilon_3^{-1})L^{-1}CMM^TCL^{-1}+\epsilon_1L^{-1}N_C^TN_CL^{-1}+\epsilon_2N_A^TN_A\\
&+L^{-1}D_0L^{-1}+\epsilon_3N_B^TN_B+L^{-1}D_1L^{-1}\Big]g(y(t))\\
&+\alpha g^T(y(t))\Big[-2DCL^{-1}+DA+A^TD+\beta^{-1}DP^{-1}D+(\epsilon_4^{-1}+\epsilon_5^{-1}+\epsilon_6^{-1})DMM^TD
\end{aligned}$$

$$+\epsilon_4 L^{-1}N_C^TN_CL^{-1}+\epsilon_5N_A^TN_A+\epsilon_6N_B^TN_B+\frac{1+\|\chi\|^2}{1-\eta}\beta\lambda_M(P)(B^TB+T_H^TT_H)\Big]g(y(t))$$

$$\leq \lambda_M\Big[A^TA+(\epsilon_1^{-1}+\epsilon_2^{-1}+\epsilon_3^{-1})L^{-1}CMM^TCL^{-1}+\epsilon_1L^{-1}N_C^TN_CL^{-1}+\epsilon_2N_A^TN_A$$

$$+L^{-1}D_0L^{-1}+\epsilon_3N_B^TN_B+L^{-1}D_1L^{-1}\Big]g^T(y(t))g(y(t))-\alpha\lambda_m(-\Omega)g^T(y(t))g(y(t))$$

$$= \Big[\lambda_M\Big[A^TA+(\epsilon_1^{-1}+\epsilon_2^{-1}+\epsilon_3^{-1})L^{-1}CMM^TCL^{-1}+\epsilon_1L^{-1}N_C^TN_CL^{-1}+\epsilon_2N_A^TN_A$$

$$+L^{-1}D_0L^{-1}+\epsilon_3N_B^TN_B+L^{-1}D_1L^{-1}\Big]-\alpha\lambda_m(-\Omega)\Big]\|g(y(t))\|_2^2.$$

The choice

$$\alpha > \frac{\lambda_M\Big[A^TA+(\epsilon_1+\epsilon_2+\epsilon_3)L^{-1}CMM^TCL^{-1}+\epsilon_1^{-1}L^{-1}N_C^TN_CL^{-1}+\epsilon_2^{-1}N_A^TN_A\Big]}{\lambda_m(-\Omega)}$$
$$+\frac{\lambda_M\Big[L^{-1}D_0L^{-1}+\epsilon_3^{-1}N_B^TN_B+L^{-1}D_1L^{-1}\Big]}{\lambda_m(-\Omega)}>0$$

ensures that $\mathcal{L}V(y(t), t) < 0$, for all $g(y(t)) \neq 0$. Thus, for ensuring negativity of $\mathcal{L}V(y(t), t)$ for any possible state, it suffices to require Ω be a negative definite matrix. This implies that the equilibrium point of system (3) is globally asymptotically stable in the mean square. The proof is completed.

Theorem 3.5. *Assume that there exist matrices $D_0 \geq 0$ and $D_1 \geq 0$ such that*

$$trace[\sigma^T(t,y(t),y(t-\tau(t)))(I+L)\sigma(t,y(t),y(t-\tau(t)))]\leq y^T(t)D_0y(t)+y^T(t-\tau(t))D_1y(t-\tau(t)).$$

System (3)is globally asymptotically stable in the mean square, if the condition (H_1) is satisfied and if the following condition hold:

$$\Omega_1=2r^2-2r\|A\|_2-\|W\|_2^2-\frac{r^2}{1-\eta}-r(\epsilon_4+\epsilon_5+\epsilon_6)\|M\|_2^2+\epsilon_4^{-1}r\|N_C\|_2^2+\epsilon_5^{-1}r\|N_A\|_2^2+\frac{\epsilon_6^{-1}r}{1-\eta}\|N_B\|_2^2>0.$$

Proof: We use the following positive definite Lyapunov functional to derive the stability result,

$$V(y,t) = y^T(t)y(t)+2\alpha r\sum_{i=1}^n\int_0^{y_i}g_i(s)ds+\alpha r^2\frac{1}{1-\eta}\sum_{i=1}^n\int_{t-\tau_i(t)}^t g_i^T(y_i(s))ds$$
$$+\beta\frac{1}{1-\eta}\sum_{i=1}^n\int_{t-\tau_i(t)}^t g_i^T(y_i(s))ds,$$

where α and β are some positive constants to be determined later. Let $W = B + \Gamma^T T_H$, by Ito's formula, we can calculate $\mathcal{L}V_1(y(t), t)$,$\mathcal{L}V_2(y(t), t)$,$\mathcal{L}V_3(y(t), t)$ and $\mathcal{L}V_4(y(t), t)$ along the trajectories of the system (3), then we have

$$\begin{aligned}\mathcal{L}V_1(y(t),t) &= -2y^T(t)Cy(t) - 2y^T(t)(\Delta C(t))y(t) + 2y^T(t)Ay(t)\\ &+2y^T(t)(\Delta A(t))y(t) + 2y^T(t)Wg(y(t-\tau(t))) + 2y^T(t)(\Delta B(t))g(y(t-\tau(t)))\\ &+trace[\sigma^T(t,y(t),y(t-\tau(t)))I\sigma(t,y(t),y(t-\tau(t)))]. \end{aligned} \quad (42)$$

Using the inequality technique, we have

$$\begin{aligned}-y^T(t)Cy(t) + 2y^T(t)Ag(y(t)) &= -[C^{\frac{1}{2}}y(t) - C^{-\frac{1}{2}}Ag(y(t))]^T[C^{\frac{1}{2}}y(t) - C^{-\frac{1}{2}}Ag(y(t))]\\ &+g^T(y(t))A^TC^{-1}Ag(y(t)) \end{aligned} \quad (43)$$

$$\begin{aligned}-y^T(t)Cy(t) + 2y^T(t)Wg(y(t-\tau(t))) &= -[C^{\frac{1}{2}}y(t) - C^{-\frac{1}{2}}Wg(y(t-\tau(t)))]^T\\ &[C^{\frac{1}{2}}y(t) - C^{-\frac{1}{2}}Wg(y(t-\tau(t)))]\\ &+g^T(y(t-\tau(t)))W^TC^{-1}Wg(y(t-\tau(t))). \end{aligned} \quad (44)$$

Since the first terms of the equations (43) and (44) are non-positive, we can write the following inequalities

$$-y^T(t)Cy(t) + 2y^T(t)Ag(y(t)) \leq g^T(y(t))A^TC^{-1}Ag(y(t)) \quad (45)$$

$$-y^T(t)Cy(t) + 2y^T(t)Wg(y(t-\tau(t))) \leq g^T(y(t-\tau(t)))W^TC^{-1}Wg(y(t-\tau(t))). \quad (46)$$

From Lemma 3.1, it follows that

$$\begin{aligned}-2y^T(t)(\Delta C(t))y(t) &\leq \epsilon_1^{-1}g^T(y(t))L^{-1}M^TML^{-1}g(y(t))\\ &+\epsilon_1 g^T(y(t))L^{-1}N_C^TN_CL^{-1}g(y(t)) \end{aligned} \quad (47)$$

$$2y^T(t)(\Delta A(t))g(y(t)) \leq \epsilon_2^{-1}g^T(y(t))L^{-1}M^TML^{-1}g(y(t)) + \epsilon_2 g^T(y(t))N_A^TN_Ag(y(t)) \quad (48)$$

$$\begin{aligned}2y^T(t)(\Delta B(t))g(y(t-\tau(t))) &\leq \epsilon_3^{-1}g^T(y(t))L^{-1}M^TML^{-1}g(y(t))\\ &+\epsilon_3 g^T(y(t-\tau(t)))N_B^TN_Bg(y(t-\tau(t))). \end{aligned} \quad (49)$$

From (45)-(49), we get

$$\begin{aligned}\mathcal{L}V_1(y(t),t) &\leq g^T(y(t))A^TC^{-1}Ag(y(t)) + g^T(y(t-\tau(t)))W^TC^{-1}Wg(y(t-\tau(t)))\\ &+\epsilon_1^{-1}g^T(y(t))L^{-1}M^TML^{-1}g(y(t)) + \epsilon_1 g^T(y(t))L^{-1}N_C^TN_CL^{-1}g(y(t))\\ &+\epsilon_2^{-1}g^T(y(t))L^{-1}M^TML^{-1}g(y(t)) + \epsilon_2 g^T(y(t))N_A^TN_Ag(y(t))\\ &+\epsilon_3^{-1}g^T(y(t-\tau(t)))L^{-1}M^TML^{-1}g(y(t)) + \epsilon_3 g^T(y(t-\tau(t)))N_B^TN_Bg(y(t-\tau(t)))\end{aligned}$$

$$+trace[\sigma^T(t,y(t),y(t-\tau(t)))I\sigma(t,y(t),y(t-\tau(t)))]. \quad (50)$$

$$\begin{aligned}\mathcal{L}V_2(y(t),t) &= -2\alpha r g^T(y(t))Cy(t) - 2\alpha r g^T(y(t))(\Delta C(t))y(t) + 2\alpha r g^T(y(t))Ag(y(t)) \\ &+2\alpha r g^T(y(t))(\Delta A(t))g(y(t)) + 2\alpha r g^T(y(t))Wg(y(t-\tau(t))) \\ &+2\alpha r g^T(y(t))(\Delta B(t))g(y(t-\tau(t))) \\ &+trace[\sigma^T(t,y(t),y(t-\tau(t)))I\sigma(t,y(t),y(t-\tau(t)))]. \quad (51)\end{aligned}$$

From Lemma 3.1, it follows that

$$\begin{aligned}-2\alpha r g^T(y(t))(\Delta C(t))y(t) &\leq \epsilon_4^{-1}\alpha r g^T(y(t))M^TMg(y(t)) \\ &+\epsilon_4\alpha r g^T(y(t))L^{-1}N_C^TN_CL^{-1}g(y(t)) \quad (52)\end{aligned}$$

$$\begin{aligned}2\alpha r g^T(y(t))(\Delta A(t))g(y(t)) &\leq \epsilon_5^{-1}\alpha r g^T(y(t))M^TMg(y(t)) \\ &+\epsilon_5\alpha r g^T(y(t))N_A^TN_Ag(y(t)) \quad (53)\end{aligned}$$

$$\begin{aligned}2\alpha r g^T(y(t))(\Delta B(t))g(y(t-\tau(t))) &\leq \epsilon_6^{-1}\alpha r g^T(y(t))M^TMg(y(t)) \\ &+\epsilon_6\alpha r g^T(y(t-\tau(t)))N_B^TN_Bg(y(t-\tau(t))) \quad (54)\end{aligned}$$

$$\mathcal{L}V_3(y(t),t) \leq \frac{\alpha r^2}{1-\eta}g^T(y(t)g(y(t)) - \alpha r^2 g^T(y(t-\tau(t)))g(y(t-\tau(t))) \quad (55)$$

$$\mathcal{L}V_4(y(t),t) \leq \frac{\beta}{1-\eta}g^T(y(t)g(y(t)) - \beta g^T(y(t-\tau(t)))g(y(t-\tau(t))). \quad (56)$$

Using the inequality technique, we have

$$\begin{aligned}2\alpha r g^T(y(t))Wg(y(t-\tau(t))) &- \alpha r^2 g^T(y(t-\tau(t)))g(y(t-\tau(t))) \\ = &-\alpha[rg(y(t-\tau(t))) - Wg(y(t))]^T[rg(y(t-\tau(t))) - Wg(y(t))] \\ &+\alpha g^T(y(t)W^TWg(y(t)).\end{aligned}$$

Since the first term of the above equation is non-positive, we can write the following inequality,

$$2\alpha r g^T(y(t))Wg(y(t-\tau(t))) - \alpha r^2 g^T(y(t-\tau(t)))g(y(t-\tau(t))) \leq \alpha g^T(y(t)W^TWg(y(t)). \quad (57)$$

From (42)-(57), it follows that

$$\mathcal{L}V(y(t),t) \leq g^T(y(t))A^TC^{-1}Ag(y(t)) + g^T(y(t-\tau(t)))W^TC^{-1}Wg(y(t-\tau(t)))$$

$$
\begin{aligned}
&+(\epsilon_1^{-1}+\epsilon_2^{-1}+\epsilon_3^{-1})g^T(y(t))L^{-1}M^TML^{-1}g(y(t))+\epsilon_1 g^T(y(t))L^{-1}N_C^TN_CL^{-1}g(y(t))\\
&+\epsilon_2 g^T(y(t))N_A^TN_Ag(y(t))+\epsilon_3 g^T(y(t-\tau(t)))N_B^TN_Bg(y(t-\tau(t)))\\
&+y^T(t)D_0y(t)+y^T(t-\tau(t))D_1y(t-\tau(t))\\
&-2\alpha r g^T(y(t))Cy(t)+2\alpha r g^T(y(t))Ag(y(t))+\alpha g^T(y(t)W^TWg(y(t))\\
&+\alpha r(\epsilon_4+\epsilon_5+\epsilon_6)g^T(y(t))M^TMg(y(t))+\epsilon_4^{-1}\alpha r g^T(y(t))L^{-1}N_C^TN_CL^{-1}g(y(t))\\
&+\epsilon_5^{-1}\alpha r g^T(y(t))N_A^TN_Ag(y(t))+\epsilon_6^{-1}\alpha r g^T(y(t-\tau(t)))N_B^TN_Bg(y(t-\tau(t)))\\
&+\frac{\alpha r^2}{1-\eta}g^T(y(t)g(y(t))-\alpha r^2 g^T(y(t-\tau(t)))g(y(t-\tau(t)))\\
&+\frac{\beta}{1-\eta}g^T(y(t)g(y(t))-\beta g^T(y(t-\tau(t)))g(y(t-\tau(t))),\\
\le\quad & -2\alpha r\sum_{i=1}^n\frac{c_i}{L_i}g_i^2(y_i(t))+\|A\|_2^2\|C^{-1}\|_2\|g(y(t))\|_2^2+2\alpha r\|A\|_2\|g(y(t))\|_2^2\\
&+\|W\|_2^2\|C^{-1}\|_2\|g(y(t-\tau(t)))\|_2^2+(\epsilon_1+\epsilon_2+\epsilon_3)\|M\|_2^2\|g(y(t))\|_2^2+\epsilon_1^{-1}\|N_C\|_2^2\|g(y(t))\|_2^2\\
&+\epsilon_2^{-1}\|N_A\|_2^2\|g(y(t))\|_2^2+\epsilon_3^{-1}\|N_B\|_2^2\|g(y(t-\tau(t)))\|_2^2+\|L^{-1}\|_2^2\|D_0\|_2\|g(y(t))\|_2^2\\
&+\|L^{-1}\|_2^2\|D_1\|_2\|g(y(t-\tau(t)))\|_2^2+\alpha\|W\|_2^2\|g(y(t))\|_2^2+\alpha r(\epsilon_4+\epsilon_5+\epsilon_6)\|M\|_2^2\|g(y(t))\|_2^2\\
&+\epsilon_4^{-1}\alpha r\|N_C\|_2^2\|g(y(t))\|_2^2+\epsilon_5^{-1}\alpha r\|N_A\|_2^2\|g(y(t))\|_2^2+\epsilon_6^{-1}\alpha r\|N_B\|_2^2\|g(y(t-\tau(t)))\|_2^2\\
&+\frac{\alpha r^2}{1-\eta}\|g(y(t))\|_2^2+\frac{\beta}{1-\eta}\|g(y(t))\|_2^2-\beta\|g(y(t-\tau(t)))\|_2^2.
\end{aligned}
$$

Since $r=\min_{1\le i\le n}\left(\frac{c_i}{L_i}\right)$, we have

$$-2\alpha r\sum_{i=1}^n\frac{c_i}{L_i}g_i^2(y_i(t))\le -2\alpha r^2\sum_{i=1}^n g_i^2(y_i(t))=-2\alpha r^2\|g(y(t))\|_2^2.$$

Let $\beta=\|W\|_2^2\|C^{-1}\|_2+\epsilon_3^{-1}\|N_B\|_2^2+\alpha r\epsilon_6^{-1}\|N_B+\|L^{-1}\|_2^2\|D_1\|_2$.Thus, in the light of the above inequality, $\mathcal{L}V$ can now be written as

$$\mathcal{L}V(y(t),t)\quad\le\quad -\alpha\Big(2r^2-2r\|A\|_2-\|W\|_2^2-\frac{r^2}{1-\eta}-r(\epsilon_4+\epsilon_5+\epsilon_6)\|M\|_2^2+\epsilon_4^{-1}r\|N_C\|_2^2$$

$$+\epsilon_5^{-1}r\|N_A\|_2^2+\frac{\epsilon_6^{-1}r}{1-\eta}\|N_B\|_2^2\Big)\|g(y(t))\|_2^2$$

$$+\Big(\|A\|_2^2\|C^{-1}\|_2+\frac{1}{1-\eta}\|W\|_2^2\|C^{-1}\|_2+(\epsilon_1+\epsilon_2+\epsilon_3)\|M\|_2^2+\epsilon_1^{-1}\|N_C\|_2^2$$

$$+\epsilon_2^{-1}\|N_A\|_2^2+\frac{\epsilon_3^{-1}}{1-\eta}\|N_B\|_2^2+\|L^{-1}\|_2^2\|D_0\|_2+\|L^{-1}\|_2^2\|D_1\|_2\Big)\|g(y(t))\|_2^2.$$

Since

$$2r^2-2r\|A\|_2-\|W\|_2^2-\frac{r^2}{1-\eta}-r(\epsilon_4+\epsilon_5+\epsilon_6)\|M\|_2^2+\epsilon_4^{-1}r\|N_C\|_2^2+\epsilon_5^{-1}r\|N_A\|_2^2+\frac{\epsilon_6^{-1}r}{1-\eta}\|N_B\|_2^2>0,$$

the choice

$$\alpha>\frac{N_r}{D_r}>0,$$

ensures that $\mathcal{L}V(y(t),t)<0$, for all $g(y(t))\neq 0$, where

$$\begin{aligned}N_r &= \|A\|_2^2\|C^{-1}\|_2+\frac{1}{1-\eta}\|W\|_2^2\|C^{-1}\|_2+(\epsilon_1+\epsilon_2+\epsilon_3)\|M\|_2^2+\epsilon_1^{-1}\|N_C\|_2^2\\ &\quad+\epsilon_2^{-1}\|N_A\|_2^2+\frac{\epsilon_3^{-1}}{1-\eta}\|N_B\|_2^2+\|L^{-1}\|_2^2\|D_0\|_2+\|L^{-1}\|_2^2\|D_1\|_2\end{aligned}$$

and

$$\begin{aligned}D_r &= 2r^2-2r\|A\|_2-\|W\|_2^2-\frac{r^2}{1-\eta}-r(\epsilon_4+\epsilon_5+\epsilon_6)\|M\|_2^2+\epsilon_4^{-1}r\|N_C\|_2^2\\ &\quad+\epsilon_5^{-1}r\|N_A\|_2^2+\frac{\epsilon_6^{-1}r}{1-\eta}\|N_B\|_2^2.\end{aligned}$$

Thus, for ensuring negativity of $\mathcal{L}V(y(t),t)$ for any possible state, it suffices to require Ω_1 be a positive definite matrix. This implies that the equilibrium point of system (3) is globally asymptotically stable in the mean square. The proof is completed.

Remark 3.6. *In* [12], *stability of equilibrium point of High-order Hopfield neural networks with time varying delays has been considered by means of Lyapunov functional and LMI techniques. We extend this technique to study the stochastic high-order neural networks with time-varying uncertain parameters. In view of this, our results in this chapter extend the results in* [12].

Remark 3.7. *In* [20]*, the authors studied the global stability of stochastic high-order neural networks with discrete and distributed delays. Similarly in [21], the authors studied stability results of stochastic high-order Markovian jumping neural networks with mixed time delays. It should be noted that the uncertain stochastic neural network studied in this chapter is time-varying delays. Therefore, our results and those established in* [20, 21] *are complementary each other.*

4. An illustrative example.

The effectiveness of the theories will be demonstrated through the following example. Consider the following high-order stochastic Hopfield neural network with time varying delays

$$dy_i(t) = [-c_i y_i(t) + \sum_{j=1}^{2} a_{ij} g_j(y_j(t)) + \sum_{j=1}^{2} b_{ij} g_j(y_j(t-\tau_j(t))) + \sum_{j=1}^{2}\sum_{j=1}^{2} T_{ijl} g_j(y_j(t-\tau_j(t))) + J_i]dt$$
$$+\sigma(t, y(t), y(t-\tau(t)))dw(t) \quad (58)$$

where $g_1(y_1) = tanh(0.95y_1)$, $g_2(y_2) = tanh(y_2)$,

$$\sigma(t, y(t), y(t-\tau(t))) = [0.5y(t) + 0.5y(t-\tau(t)), 0.4y(t) + 0.4y(t-\tau(t))], \quad \eta = 0.4,\ J_1 = 1.5,\ J_2 = 2,$$

$$C = \begin{bmatrix} 4.5 & 0 \\ 0 & 4.5 \end{bmatrix}, A = \begin{bmatrix} 0.05 & 0.14 \\ 0.20 & 0.31 \end{bmatrix}, B = \begin{bmatrix} 0.09 & 0.25 \\ 0.21 & 0.45 \end{bmatrix}, T_1 = \begin{bmatrix} 0.05 & 0.14 \\ -0.06 & 0.05 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 0.29 & 0.10 \\ 0.23 & -0.14 \end{bmatrix}, T_3 = \begin{bmatrix} -0.23 & 0.07 \\ 0.09 & -0.02 \end{bmatrix}, M = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.03 \end{bmatrix}, N_C = \begin{bmatrix} 0.06 & 0 \\ 0 & 0.06 \end{bmatrix},$$

$$N_A = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.03 \end{bmatrix}, N_B = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.04 \end{bmatrix}, D_0 = D_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.32 \end{bmatrix}.$$

Thus we have $\mathcal{L} = I$, $\|\mathcal{X}\|_2 = 1$. Now, solving the LMI in Theorem 3.2, using Matlab LMI Control toolbox, we get the following feasible solution

$$S_1 = 10^3 \times \begin{bmatrix} 2.0438 & -0.0233 \\ -0.0233 & 1.9622 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 132.3299 & -191.3967 \\ -191.3967 & 494.7130 \end{bmatrix},$$

$\epsilon_1 = 5.5014$, $\epsilon_2 = 0.2838$, $\epsilon_3 = 21.7583$

It follows from Theorem 3.2 that the equilibrium point of the system (58) is globally asymptotically stable in the mean square.

Now, solving the LMI in Theorem 3.3, using Matlab LMI Control toolbox, we get the following feasible solution

$$S_1 = \begin{bmatrix} 0.5141 & -0.1631 \\ -0.1631 & 0.5591 \end{bmatrix}, \qquad \epsilon_1 = 1.0550,$$
$$\epsilon_2 = 4.3317, \quad \epsilon_3 = 4.0591.$$

Therefore, from Theorem 3.3 that the equilibrium point of the system (58) is globally asymptotically stable in the mean square.

Now we let $D_0 = D_1 = \begin{bmatrix} 1.0 & 0 \\ 0 & 0.64 \end{bmatrix}$. Again solving the LMI in Theorem 3.4, using Matlab LMI Control toolbox, we get the following feasible solution

$$\beta = 0.5690, \quad \epsilon_4 = 2.0315, \quad \epsilon_5 = 8.7405, \quad \epsilon_6 = 5.0297.$$

Therefore, from Theorem 3.4 that the equilibrium point of the system (58) is globally asymptotically stable in the mean square.

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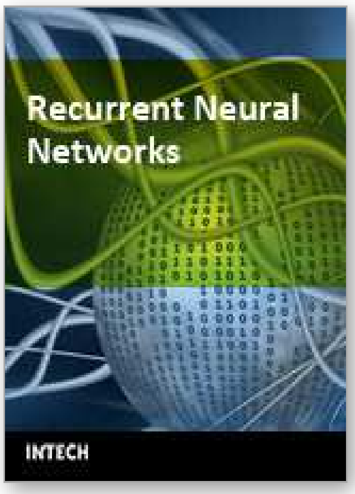

Recurrent Neural Networks

Edited by Xiaolin Hu and P. Balasubramaniam

ISBN 978-953-7619-08-4
Hard cover, 400 pages
Publisher InTech
Published online 01, September, 2008
Published in print edition September, 2008

The concept of neural network originated from neuroscience, and one of its primitive aims is to help us understand the principle of the central nerve system and related behaviors through mathematical modeling. The first part of the book is a collection of three contributions dedicated to this aim. The second part of the book consists of seven chapters, all of which are about system identification and control. The third part of the book is composed of Chapter 11 and Chapter 12, where two interesting RNNs are discussed, respectively.The fourth part of the book comprises four chapters focusing on optimization problems. Doing optimization in a way like the central nerve systems of advanced animals including humans is promising from some viewpoints.