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The Quantum Arnold Transformation and its applications

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Summary. — The Quantum Arnold Transformation, a unitary operator mapping the solutions of the Schrödinger equation for time-dependent quadratic Hamiltonians into free-particle solutions, is revisited. Possible applications and extensions are also outlined: the analytic construction of harmonic states for the free particle, the Quantum Arnold-Ermakov-Pinney transformation and the description of the Release & Recapture method.

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1. – Introduction

In this paper we shall review the Quantum Arnold Transformation (QAT), introduced in [1] as a unitary map that relates the space of solutions of the one-dimensional time-dependent Schrödinger equation (TDSE) for an arbitrary, time-dependent, quadratic Hamiltonian (TDQH), into the corresponding one for the free particle. The QAT is an extension to the quantum domain of the (Classical) Arnold Transformation (CAT), introduced by V.I. Arnold in [2]. In that reference, it was observed that the family of graphs of solutions of any linear second-order differential equation (LSODE), with arbitrary time-dependent coefficients, is locally diffeomorphic to the family of graphs of solutions of the simplest one-dimensional equation of motion, *i.e.*, the equation of motion of a free particle. Making use of that fact, it is possible to import the Lie point symmetries of the free particle (which has the largest possible Lie point symmetry group, $SL(3,\mathbb{R})$) into those of an arbitrary system whose equation of motion is a LSODE. The CAT can be seen as a particular class of more general Lie transformations relating systems defined by second order differential equations to the free particle [3].

In order to fully benefit from the properties and implications of the CAT at the quantum level, the explicit formalization of the QAT is key. However, many authors had already taken advantage of such properties, at least partially. Lewis and Riesenfeld (1969) [4] introduced a technique to obtain solutions of the TDSE for a TDQH as eigenfunctions of quadratic invariants. For that purpose they wrote the solutions in terms of auxiliary variables that satisfy the classical equations of motion (something that resembles very much the CAT). Dodonov and Man'ko (1979) [5] constructed invariant operators for the damped harmonic oscillator and introduced coherent states, using a similar method to that of Lewis and Riesenfeld. Jackiw (1980) [6] gave (implicitly) the quantum transformation from the harmonic oscillator (even with a $1/x^2$ term) to the free particle when studying the symmetries of the magnetic monopole. Junker and Inomata (1985) [7] gave the transformation of the propagator, in a path integral approach, for a TDQH, into the free one (the equivalent of the QAT, but in terms propagators). Takagi (1990) [8] gave the quantum transformation from the harmonic oscillator to the free particle, interpreted as the change to comoving coordinates. Bluman and Shtelen (1996) [9] gave the (non-local) transformation of the TDSE for a TDQH plus a non-linear term into the free particle one, in the context of transformations of PDEs. Kagan *et al.* and independently Castin and Dum (1996) [10] introduced a scaling transformation in the Gross-Pitaevskii equation describing Bose-Einstein Condensates (BEC) which is very related to the QAT. Suslov *et al.* (2010) [11] computed the propagator for a TDQH using the classical equations.

The paper is organized as follows: sect. **2** reviews the CAT transforming solutions of an arbitrary LSODE into free particle solutions, and studies its properties. Section **3** introduces the QAT as the quantum version of the CAT. Finally, sect. **4** describes several relevant applications of the QAT: the analytic construction of harmonic states for the free particle, the Quantum Arnold-Ermakov-Pinney transformation (an extension of the QAT to connect two arbitrary LSODE systems) and the description of the Release & Recapture method in the framework of the QAT.

2. – Classical Arnold Transformation

In the context of Lie point symmetries of ordinary differential equations (ODE) [3], a second-order differential equation $y'' = F(x, y, y')$ has the maximal number of Lie point symmetries ($SL(3,\mathbb{R})$) if it can be transformed into the free equation by a point transformation:

$$y'' = F(x, y, y') \overset{\substack{\tilde{x}=\tilde{x}(x,y)\\ \tilde{y}=\tilde{y}(x,y)}}{\Longrightarrow} \tilde{y}'' = 0. \tag{1}$$

V.I. Arnold studied the case of Linear Second-Order Differential Equations (LSODE), giving explicitly the point transformation to the free case [2]. Given

$$\ddot{x} + \dot{f}\dot{x} + \omega^2 x = \Lambda, \tag{2}$$

where f, ω and Λ are functions of t, and the dots indicate derivatives with respect to t, the Arnold transformation, here named Classical Arnold Transformation (CAT), is a local diffeomorphism $A : \mathbb{R} \times T \to \mathbb{R} \times \mathcal{T}$ (T and $\mathcal{T}$ are open intervals) given by $(x, t) \to (\kappa, \tau) = A(x, t)$, where

$$\kappa = \frac{x - u_p(t)}{u_2(t)}, \qquad \tau = \frac{u_1(t)}{u_2(t)}. \tag{3}$$

Here u_1 and u_2 are independent solutions of the homogeneous LSODE corresponding to (2) and u_p is a particular solution of (2). For convenience, we impose u_1, u_2 and u_p to satisfy the initial conditions:

$$u_1(t_0) = \dot{u}_2(t_0) = u_p(t_0) = \dot{u}_p(t_0) = 0, \qquad \dot{u}_1(t_0) = u_2(t_0) = 1, \tag{4}$$

where t_0 is an arbitrary time, conveniently chosen to be $t_0 = 0$ (see [1] for details). With this condition, $(0, 0)$ is a fixed point of A.

The CAT transforms the equation of motion $\ddot{x} + \dot{f}\dot{x} + \omega^2 x = \Lambda$, up to factor, into that of a free particle $\frac{W}{u_2^3}\ddot{\kappa} = 0$, where $W(t) = \dot{u}_1 u_2 - u_1 \dot{u}_2 = e^{-f}$ is the Wronskian of the two solutions. The presence of this factor implies that patches of trajectories of (2) are transformed into patches of straight (free) trajectories. Indeed, an arbitrary trajectory solution of (2) can be written as $x(t) = Au_1(t) + Bu_2(t) + u_p(t)$, and the CAT sends it to $\kappa(\tau) = A\tau + B$. While t varies in the interval T defined by two consecutive zeros of $u_2(t)$ (containing $t_0 = 0$), τ varies in the range of the map defined by $\frac{u_1(t)}{u_2(t)}$. In the case in which $u_2(t)$ has no zeros T is $\mathbb{R}$.

2·1. *The example of the harmonic oscillator*. – Let us include here the expressions corresponding to the simple example of the harmonic oscillator that will be useful in the following. In this case, $\Lambda = 0$, $\dot{f} = 0$ and ω is constant; then, the two solutions verifying (4) are $u_1(t) = \frac{1}{\omega}\sin(\omega t)$ and $u_2(t) = \cos(\omega t)$. The open interval T defined by two consecutive zeros of $u_2(t)$, and containing $t_0 = 0$, is $(-\frac{\pi}{2\omega}, \frac{\pi}{2\omega})$, and the CAT A and its inverse A^{-1} are then written as

$$A : \kappa = \frac{x}{u_2(t)} = \frac{x}{\cos(\omega t)}, \qquad \tau = \frac{u_1(t)}{u_2(t)} = \frac{1}{\omega}\tan(\omega t), \tag{5}$$

$$A^{-1} : x = \cos(\arctan(\omega\tau))\kappa = \frac{\kappa}{\sqrt{1 + \omega^2\tau^2}}, \qquad t = \frac{1}{\omega}\arctan(\omega t). \tag{6}$$

So, $\tau \in \mathbb{R}$. Therefore the CAT maps half a period of the HO trajectories into a complete free trajectory. For the CAT to map other patches of the HO trajectories into the free particle trajectories, different branches of the arctan function in the inverse CAT (6) should be used (and a different $t_0 \neq 0$ for the CAT). See [12, 13] for more details in this case.

3. – The Quantum Arnold Transformation

An arbitrary LSODE system (2) can be derived from the Hamiltonian

$$H = \frac{p^2}{2m}e^{-f} + \left(\frac{1}{2}m\omega^2 x^2 - m\Lambda x\right)e^f, \tag{7}$$

which is known as the generalized Caldirola-Kanai (GCK) Hamiltonian for a damped oscillator (see [1] and references therein). The case in which $\dot{f} = \gamma$ and ω are constant corresponds to the original Caldirola-Kanai Hamiltonian for a damped harmonic oscillator. Canonical quantization of the GCK Hamiltonian leads to the time-dependent Schrödinger equation:

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}e^{-f}\frac{\partial^2\phi}{\partial x^2} + \left(\frac{1}{2}m\omega^2 x^2 - m\Lambda x\right)e^f\phi. \tag{8}$$

The CAT A is a local (in time) diffeomorphism between the space of solutions of the LSODE system (2) and the space of solutions of the free particle. We are interested in extending it to a unitary transformation $\hat{A}$, the Quantum Arnold Transformation (QAT), between the Hilbert space of solutions $\phi(x,t)$ of the time-dependent Schrödinger equation for the GCK oscillator (8) at time t, $\mathcal{H}_t$, into the Hilbert space of solutions $\varphi(\kappa,\tau)$ of the time-dependent Schrödinger equation for the free particle, $i\hbar\frac{\partial\varphi}{\partial\tau} = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial\kappa^2}$, at time τ, $\mathcal{H}^G_\tau$. The desired extension is given by

$$
\begin{aligned}
&\hat{A} : \mathcal{H}_t \longrightarrow \mathcal{H}^G_\tau, \\
&\qquad \phi(x,t) \longmapsto \varphi(\kappa,\tau) = \hat{A}\big(\phi(x,t)\big) = A^*\Big(\sqrt{u_2(t)}\, e^{-\frac{i}{2}\frac{m}{\hbar}\frac{1}{W(t)}\frac{\dot{u}_2(t)}{u_2(t)}x^2}\phi(x,t)\Big).
\end{aligned} \tag{9}
$$

Here A^* is the pullback of the CAT A, acting on functions, and for simplicity of notation we have assumed $\Lambda = 0$, but it can be included in a straightforward way (see [1] for details). The QAT can be diagrammatically represented as

$$
\begin{tikzcd}
\mathcal{H}^G_\tau & \mathcal{H}_t \arrow[l, "\hat{A}"'] \\
\mathcal{H}^G_0 \equiv \mathcal{H} \arrow[u, "\hat{U}_G(\tau)"] \arrow[r, "\hat{1}"'] & \mathcal{H} \equiv \mathcal{H}_0 \arrow[u, "\hat{U}(t)"']
\end{tikzcd} \tag{10}
$$

where $\mathcal{H}_0 \equiv \mathcal{H}^G_0 \equiv \mathcal{H}$ is the common Hilbert space of solutions of the Schrödinger equation for both systems at $t = \tau = 0$, $U(t)$ is the unitary time-evolution operator for the GCK oscillator and $\hat{U}_G(\tau)$ is the corresponding one for the free particle. The map at the bottom of the diagram is the identity due to conditions (4), otherwise a non-trivial unitary transformation appears (see [1]).

The QAT inherits from the CAT the local character in time, in the sense that it is valid only for $t \in T$ and $\tau \in \mathcal{T}$. To extend the QAT beyond T, we can proceed as in the classical case for the harmonic oscillator, considering the different branches of the inverse function of $\tau(t)$, defining an *unfolded* QAT, $\hat{\hat{A}}$ [13]. It should be stressed that if in the different branches of the *unfolded* CAT proper solutions verifying (4) are not used, changes in signs can appear reflected as changes in phases in the different branches of the *unfolded* QAT. This phenomenon seems to be related to the Maslov correction (see for instance [14]).

4. – Applications of the QAT

From the commutative diagram (10) it is clear that the QAT is a unitary operator (see [15] for a proof), and this has interesting and far-reaching consequences. Among them we can mention the possibility of importing operators (symmetries) from one system to the other, importing wave functions, scalar product, computing the time evolution operator, etc. We shall only discuss some of them here, referring the reader to [1,12,15,16] for more details.

4·1. *Harmonic states for the free particle.* – Thanks to the commutativity of the diagram (10), and the unitarity of the operators appearing in it, we can map objects (wave functions, operators, expectation values, uncertainties) from one system to the other. In [1] we benefited from this fact for transporting the simplicity of the free particle to more involved systems, finding, for instance, an analytic expression for the evolution operator of the complicated systems (even with time-dependent Hamiltonians) in terms of the free particle evolution operator. Let us proceed the other way round (following [12]), transporting objects and properties from the harmonic oscillator to the free particle.

Let $\mathcal{H}$ be the Hilbert space of solutions of the time-dependent Schrödinger equation for the free particle, and $\mathcal{H}_{\rm HO}$ the one corresponding to the harmonic oscillator of a given frequency ω. We shall denote by $\psi(x,t) \in \mathcal{H}$ the free particle solutions and by $\psi'(x',t') \in \mathcal{H}_{\rm HO}$ the harmonic oscillator ones. Then, the inverse of the QAT is given by

$$\psi'(x',t') = \hat{A}^{-1}\psi(x,t) = \frac{1}{\sqrt{u_2(t')}} e^{\frac{i}{2}\frac{m}{\hbar}\frac{1}{W(t')}\frac{\dot{u}_2(t')}{u_2(t')}x'^2}\,\psi\left(\frac{x'}{u_2(t')}, \frac{u_1(t')}{u_2(t')}\right), \tag{11}$$

where expressions from subsect. **2**˙1 must be used.

Applying the QAT to the time-dependent eigenstates of the harmonic oscillator Hamiltonian $\hat{H}'_{\rm HO}$, with energy $E_n = \hbar\omega(n+\frac{1}{2})$,

$$\psi'_n(x',t') = \mathcal{N}_n e^{-i\omega(n+\frac{1}{2})t'} e^{-\frac{m\omega}{2\hbar}x'^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x'\right), \tag{12}$$

where $\mathcal{N}_n = (\frac{m\omega}{\hbar\pi})^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}$, we obtain the following set of states, solutions of the Schrödinger equation for the free particle:

$$\psi_n(x,t) = \mathcal{N}_n \frac{1}{\sqrt{|\delta|}} e^{-\frac{x^2\delta^*}{4L^2|\delta|^2}} \left(\frac{\delta^*}{|\delta|}\right)^{n+\frac{1}{2}} H_n\left(\frac{x}{\sqrt{2}L|\delta|}\right), \tag{13}$$

where we have used formulas of subsect. **2**˙1 for the CAT and, in order to obtain a more compact notation, we have introduced the quantities $L = \sqrt{\frac{\hbar}{2m\omega}}$, with the dimensions of length, and $\tau = \frac{2mL^2}{\hbar} = \omega^{-1}$, with the dimensions of time. We also denote by δ the complex, time-dependent, dimensionless expression $\delta = 1 + i\omega t = 1 + i\frac{\hbar t}{2mL^2} = 1 + i\frac{t}{\tau}$. We have also used the fact that $e^{-i\omega t'} = e^{-i\tan^{-1}(\omega t)} = \frac{\delta^*}{|\delta|}$. Note that with these definitions the normalization factor $\mathcal{N}_n$ can be written as $\mathcal{N}_n = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{2^n n! L}}$.

The wave functions (13) are known in the literature as Hermite-Gauss wave packets [17], and they have been widely used, in their two-dimensional version (see [12]), in paraxial wave optics [18]. However, this kind of states are better understood in the framework of the QAT. Note that, making use of the classical solutions only, and through the QAT, we have been able to import the time evolution from the stationary states of the harmonic oscillator, $\psi'_n(x',t')$, to the non-stationary ones, $\psi_n(x,t)$, without solving the time-dependent Schrödinger equation.

The QAT also allows to map invariant operators from one Hilbert space ($\mathcal{H}_{\rm HO}$) to the other ($\mathcal{H}$). These invariant operators are also known as constant or integral of motion operators, in the sense that their matrix elements are constant with respect to their corresponding time evolution, and preserve their respective Hilbert spaces. Particularizing the general expression given in [1] or by direct computation, the conserved position and momentum operator for the harmonic oscillator can be written as

$$
\begin{aligned}
\hat{X}' &= \frac{\dot{u}_1(t')}{W(t')}x' + \frac{i\hbar}{m}u_1(t')\frac{\partial}{\partial x'} = \cos\omega t'\, x' + \frac{i\hbar}{m\omega}\sin\omega t' \frac{\partial}{\partial x'}\,,\\
\hat{P}' &= -i\hbar\, u_2(t')\frac{\partial}{\partial x'} - m\frac{\dot{u}_2(t')}{W(t')}x' = -i\hbar\cos\omega t' \frac{\partial}{\partial x'} + m\omega\, \sin\omega t'\, x',
\end{aligned}
$$

and therefore, the conserved creation and annihilation operators are

$$
\begin{aligned}
\hat{a}' &= \frac{1}{2L}\hat{X}' + i\frac{L}{\hbar}\hat{P}' = e^{i\omega t'}\left(\frac{1}{2L}x' + L\frac{\partial}{\partial x'}\right),\\
\hat{a}'^{\dagger} &= \frac{1}{2L}\hat{X}' - i\frac{L}{\hbar}\hat{P}' = e^{-i\omega t'}\left(\frac{1}{2L}x' - L\frac{\partial}{\partial x'}\right).
\end{aligned}
$$

Note that these are the operators acting on solutions of the time-dependent Schrödinger equation for the harmonic oscillator $\psi'_n(x', t')$ as ladder operators.

Position and momentum operators $\hat{X}'$ and $\hat{P}'$ are mapped into operators representing conserved position and momentum operators for the free particle through the QAT:

$$
\begin{aligned}
\hat{X} &= x + \frac{i\hbar}{m}t\frac{\partial}{\partial x}\,,\\
\hat{P} &= -i\hbar\frac{\partial}{\partial x}\,.
\end{aligned}
$$

As a consequence, ladder operators for the harmonic oscillator can be mapped into ladder operators for the free particle that act as creation and annihilation operators for the (time-dependent) Hermite-Gauss states:

$$
\begin{aligned}
\hat{a} &= L\delta\frac{\partial}{\partial x} + \frac{x}{2L}\,,\\
\hat{a}^{\dagger} &= -L\delta^{*}\frac{\partial}{\partial x} + \frac{x}{2L}\,.
\end{aligned} \tag{14}
$$

The action of $\hat{a}$ and $\hat{a}^{\dagger}$ on the Hermite-Gauss wave functions is the usual one:

$$
\hat{a}\psi_n(x,t) = \sqrt{n}\,\psi_{n-1}(x,t), \qquad \hat{a}^{\dagger}\psi_n(x,t) = \sqrt{n+1}\,\psi_{n+1}(x,t). \tag{15}
$$

It is possible to introduce this discrete basis without resorting to the QAT in a very intuitive manner. The key point is that the operator $\hat{a}$ annihilates the Gaussian wave packet, and this fact characterizes it. The whole family of states (13) can be generated by acting with the adjoint operator $\hat{a}^{\dagger}$ of $\hat{a}$. The rest of the construction, *i.e.* uncertainties, coherent and squeezed states, etc. would proceed without the need of resorting to the QAT. However, the QAT is very useful when performing involved computations in a very easy way [12].

4·2. *The Arnold-Ermakov-Pinney transformation.* – Although the CAT and the QAT relate any LSODE system with the free particle, the dynamics of both systems can be very different([1]). Thus, it could be interesting to relate directly two arbitrary LSODE systems with similar behavior, and this can be achieved by composing a CAT and an inverse CAT (or a QAT and an inverse QAT). This idea has been presented in [15], and we reproduce it in the following.

Let A_1 and A_2 denote the CATs relating the LSODE-system 1 and LSODE-system 2 to the free particle, respectively. Then, $E = A_1^{-1}A_2$ relates LSODE-system 2 to LSODE-system 1. E can be written as

$$\begin{aligned} E : \mathbb{R} \times T_2 &\to \mathbb{R} \times T_1 \\ (x_2, t_2) &\mapsto (x_1, t_1) = E(x_2, t_2). \end{aligned} \tag{16}$$

The explicit form of the transformation can be easily computed by composing the two CATs, resulting in

$$x_1 = \frac{x_2}{b(t_2)} \qquad W_1(t_1)\mathrm{d}t_1 = \frac{W_2(t_2)}{b(t_2)^2}\mathrm{d}t_2, \tag{17}$$

where $b(t_2) = \frac{u_2^{(2)}(t_2)}{u_2^{(1)}(t_1)}$ satisfies the non-linear SODE:

$$\ddot{b} + \dot{f}_2\dot{b} + \omega_2 b = \frac{W_2^2}{W_1^2}\frac{1}{b^3}\left[\omega_1^2 + \dot{f}_1\frac{\dot{u}_2^{(1)}}{u_2^{(1)}}\left(1 - b^2\frac{W_1}{W_2}\right)\right], \tag{18}$$

and where $u_i^{(j)}$ refers to the i-th particular solution for system j; W_j, $\dot{f}_j$ and ω_j stand for the Wronskian and the LSODE coefficients for system j; and the dot means derivation with respect to the corresponding time function.

For the particular case where LSODE-system 1 is a harmonic oscillator ($\omega_1(t_1) \equiv \omega_0$ and $\dot{f}_1 = 0$), this expression simplifies to

$$\ddot{b} + \dot{f}_2\dot{b} + \omega_2 b = \frac{W_2^2}{b^3}\omega_0^2, \tag{19}$$

resulting in a generalization of the Ermakov-Pinney equation. For $\dot{f}_2 = 0$ the Ermakov-Pinney equation (also known as Milne-Pinney) is recovered [19-21]:

$$\ddot{b} + \omega_2 b = \frac{\omega_0^2}{b^3}, \tag{20}$$

representing a harmonic oscillator of frequency ω_2 with an extra inverse squared potential $\frac{\omega_0^2}{x^2}$. For $\omega_0 = 0$, the Arnold-Ermakov-Pinney transformation reduces to the ordinary CAT, *i.e.* $E = A$.

([1]) Consider, for example, the harmonic oscillator with a bounded, periodic motion as compared with the free particle, with an unbounded non-periodic motion.

The Ermakov-Pinney equation is related to the Ermakov invariant [19, 4] and appears in many branches of physics, such as Cosmology [22], BEC [23], etc. Its generalization to higher dimensions, known as Ermakov pairs (or system), appears in BEC [10] and in what is known as Kepler-Ermakov systems [24].

The Ermakov-Pinney equation entails a kind of nonlinear superposition principle, in the sense that its solutions can be written in terms of the solutions $y_1(t), y_2(t)$ of the corresponding linear equation (with $\omega_0 = 0$):

$$b(t)^2 = c_1 y_1(t)^2 + c_2 y_2(t)^2 + 2c_3 y_1(t) y_2(t), \quad c_1 c_2 - c_3^2 = \omega_0^2. \tag{21}$$

The other way round, the general solution $y(t)$ of the linear equation can be written in terms of a particular solution $\rho(t)$ of the Ermakov-Pinney equation (20) as

$$y(t) = c_1 \rho(t) \cos(\omega_0\, \theta(t) + c_2), \tag{22}$$

where c_1, c_2 are arbitrary constants and $\theta(t) = \int^t \rho^{-2} \mathrm{d}t'$. Note that this equation is just (17) for $W_1 = W_2 = 1$, $t_1 = \theta(t_2)$, $\rho = b$, $x_2 = y(t_2)$ and $x_1 = y(t_2)/b(t_2) = c_1 \cos(\omega_0 t_1 + c_2)$. As a result, the general solution of (20) can be determined from a particular solution $\rho(t)$ using (22) and (21).

The quantum version of the Arnold-Ermakov-Pinney transformation, $\hat{E}$, can be obtained computing the composition of a QAT and an inverse QAT, to give

$$\begin{aligned} \hat{E} : \mathcal{H}^{(2)}_{t_2} &\longrightarrow \mathcal{H}^{(1)}_{t_1}, \\ \phi(x_2, t_2) &\longmapsto \varphi(x_1, t_1) = \hat{E}\,(\phi(x_2, t_2)) \\ &\qquad = E^* \left(\sqrt{b(t_2)}\, e^{-\frac{i}{2}\frac{m}{\hbar}\frac{1}{W_2(t_2)}\frac{\dot{b}(t_2)}{b(t_2)} x_2^2} \phi(x_2, t_2) \right). \end{aligned} \tag{23}$$

The Quantum Arnold-Ermakov-Pinney transformation is a unitary map importing solutions of a GCK Schrödinger equation from solutions of a different, auxiliary GCK Schrödinger equation which, in particular, can be the one corresponding to a harmonic oscillator. In that case the transformation is very similar to the one used in Bose-Einstein Condensates, known as scaling transformation to transform the time-dependent potential (oscillator traps with time-dependent frequencies) into a time-independent harmonic oscillator potential [10]. Also, in that case (*i.e.* for $\dot{f}_2 = 0$, $W_2 = 1$, $\dot{f}_1 = 0$, $W_1 = 1$) equation (23) reduces to the transformation given by Hartley and Ray [25] (this was already given by Lewis and Riesenfeld in [4]). However, the Quantum Arnold-Ermakov-Pinney transformation allows to choose in a suitable way the auxiliary system from which the solutions may be imported.

4·3. *Release & Recapture*. – Let us describe the situation of a particle in a harmonic potential which is switched off (Release) at a given time T_0. After that, the particle will evolve freely and then, at time T_1, is captured again (Recapture) by another (in particular, the same) harmonic potential. Time evolution will then be harmonic again until an arbitrary time T_2. The complete process is know as Release & Recapture (R&R) and it is usually used to measure the effective temperature of an ensemble of atoms in a trap [26], or even of a single atom [27]. When the effective temperature of the ensemble or single atom is low enough, the trap can be considered harmonic. At even lower temperatures, the thermal effects can be neglected and the state of the atom can be described by a wave function([2]).

The process of Release is of practical relevance([3]), for instance in the preparation of the 1-dimensional Hermite-Gauss free states (13). This idea was proposed in [28], and was named "Quamtum Sling". The vacuum state of the harmonic oscillator, after the harmonic potential is switched off, will provide the "vacuum" Gaussian wave packet with width $L = \sqrt{\frac{\hbar}{2m\omega}}$, where m is the mass of the particle and ω the frequency of the oscillator. Note that the dispersion time τ coincides with the inverse of the frequency of the oscillator.

If we "recapture" one of these traveling states at time T_1 switching on a harmonic oscillator potential with a frequency ω_1, it would "freeze" in a harmonic-oscillator state (without dispersion). The frequency ω_1 used to capture the dispersed wave packet might be fine tuned in such a way that the wave packet, at the time T_1, matches an appropriate eigenstate (with the same n) of the harmonic oscillator with frequency ω_1.

We can wonder which is the resulting wave function $\psi'(T_2)$ after the whole R&R process, given a state prepared for the initial harmonic oscillator $\psi'(T_0)$? Assuming the sudden approximation (instead of the adiabatic one), the formal solution will be a product of two evolution operators, describing free evolution from T_0 to T_1, and harmonic one with the new frequency ω_1 from T_1 to T_2:

$$\psi'(T_2) = \hat{U}'_{\omega_1}(T_2, T_1)\hat{U}(T_1, T_0)\psi'(T_0). \tag{24}$$

Before proceeding, let us discuss a little bit of notation. In subsect. **4˙1**, we have used lowercase, unprimed letters for quantities referring to the free particle and lowercase, primed ones for those of the harmonic oscillator. Now we are indicating *physical* or *true* time with capital T's. As the Arnold transformation includes a diffeomorphism in time, if we are going to use it as a tool to perform calculations, we have to be very careful of not confusing the *physical* time with that used in the Arnold transformations. Recovering the notation of lowercase letters, fix $T_0 = t_0 = t'_0$. Then $T_1 = t_1 = \frac{u_1(t'_1)}{u_2(t'_1)}$, with u_1 and u_2 satisfying (4) at t'_0, and denote by $t' = T_2$.

Denote by $\hat{A}_{\omega,t_0}(t)$ the QAT from a harmonic oscillator of frequency ω performed at time t, and by $\hat{U}'_\omega$ the unitary time evolution operator for that harmonic oscillator. With this notation and the previous choice of classical solutions u_1 and u_2 (subsect. **2˙1**), for $t = t_0$ $\hat{A}_{\omega,t_0}(t_0)$ is the identity.

Now, the product (24) can be decomposed by a sequence of a QAT and harmonic oscillator evolution operators splitting the free evolution operator (see figure in [16], where a more general case is considered):

$$\psi'(t') = \hat{U}'_{\omega_1}(t', t_1)\hat{A}_{\omega,t_0}(t_1)\hat{U}'_\omega(t'_1, t_0)\psi'(t_0). \tag{25}$$

An obvious generalization of the proposed method consists in increasing the number of captures and releases (in [16] a R&R&R process is considered). On the other hand,

([2]) If this is not the case, the following treatment is still valid, but density matrices should be used, see sect. 6 in [12].

([3]) It is also used in BECs, there called ballistic expansion, where it is used to measure the distribution of velocities of the condensate.

depending on the particular initial state, it may be a better choice a path of calculation along purely free evolutions and inverse Arnold transformations.

Note that the harmonic evolution will be particularly simple if the frequency ω_1 is such that the width of the wave packet fits a natural width of this second-harmonic potential. Intuitively, one might say that, in this situation, the particle would be captured in an eigenstate of the oscillator Hamiltonian with frequency ω_1. However, this is not the case: it can be checked that an extra, position-dependent phase $\exp(i\frac{\omega^2 t_1^2}{4L^2|\delta_1|^2}x^2)$ appears, where $\delta_1 = 1 + i\omega t_1$. This fact and the relationship with the "quasistationary" or "pseudostationary" states appearing in [5] will be analyzed elsewhere.

When the same frequency ω is used to capture the state in the harmonic-oscillator trap, the resulting state will be a radially squeezed state with squeezing parameter r given by $r = -\log(|\delta_1|)$, which is negative. This can be seen as a feasible way of producing squeezing in trapped states, simply by a R&R process on the trap for a lapse of time t, resulting in a squeezing parameter $r = -\frac{1}{2}\log(1 + \omega^2 t^2)$. In fact, a similar way of producing squeezed states in Bose-Einstein Condensates was reported in [29].

* * *

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