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Reduction of Lie-Jordan algebras: Classical

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Summary. — In this paper we present a unified algebraic framework to discuss the reduction of classical and quantum systems. The underlying algebraic structure is a Lie-Jordan algebra supplemented, in the quantum case, with a Banach structure. We discuss the reduction by symmetries, by constraints as well as the possible, non trivial, combinations of both. We finally introduce a new, general framework to perform the reduction of physical systems in an algebraic setup.

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1. – Introduction

This paper is the first part of two that jointly study the reduction procedure in classical and quantum mechanics. One of the difficulties when carrying out this program is to find the appropriate common language for both classical and quantum physics.

Quantum mechanics has been mainly formulated, since its foundations, in algebraic language [1]. The observables are elements of a C^* algebra and the states are functionals in this space [2]. Alternative, geometrical approaches to Quantum Mechanics have been formulated recently [4-11].

On the other hand the language of classical mechanics has been mainly geometrical [3] and in this framework different reduction procedures have been introduced like Marsden-Weinstein reduction, symplectic reduction, Poisson reduction,... However, it was soon realised that these procedures have their algebraic counterpart [12-14].

In this contribution we will focus mainly in classical mechanics, while the second part is devoted to the study of the quantum case. In both parts we will adopt a common algebraic language in terms of Lie-Jordan algebras, supplemented in the quantum case with a topological Banach space structure.

In the next section we start by a succinct description of Lie-Jordan algebras and its connection with Poisson algebras and C^* algebras. In the third section we discuss the reduction procedures for Poisson manifolds in the presence of symmetries or constraints. Section **4** contains the main results of the paper. In it we introduce a generalisation of the previous reductions and it is illustrated with an example. Finally sect. **5** is dedicated to the discussion of a possible extension of the reduction from the classical framework to the quantum one.

2. – Lie-Jordan algebras

A Lie-Jordan algebra $(\mathcal{L}, \circ, [\ ,\ ])$ is the combination of a real, abelian algebra (Jordan algebra) $(\mathcal{L}, \circ)$ and a Lie algebra $(\mathcal{L}, [\ ,\ ])$, *i.e.* $[\ ,\ ]$ is an antisymmetric, bilinear bracket that, for any $a, b, c \in \mathcal{L}$, fulfils the Jacobi identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

In addition we require two compatibility conditions between the two operations: the Leibniz rule

$$[a \circ b, c] = a \circ [b, c] + [a, c] \circ b,$$

and the associator identity

$$(a \circ b) \circ c - a \circ (b \circ c) = \hbar^2 [[a, c], b],$$

for some $\hbar \in \mathbb{R}$. Actually if $\hbar \neq 0$ we can take it one by an appropriate rescaling of any of the two operations that do not affect the rest of the properties of the algebra. The reason why we introduced the constant $\hbar$, apart from its obvious physical meaning, is because in the *classical* limit, $\hbar = 0$, the Jordan algebra becomes associative (we shall call it associative Lie-Jordan algebra) and $(\mathcal{A}, \circ, [\,,\,])$ is a Poisson algebra.

Notice that given a Lie-Jordan algebra $\mathcal{L}$, we can define on $\mathcal{L}^{\mathbb{C}}$ the following product

$$a \cdot b = a \circ b - i\hbar [a, b]$$

that makes $(\mathcal{L}^{\mathbb{C}}, \cdot)$ an associative algebra. Moreover we can introduce the following antilinear involution:

$$(a + ib)^* = a - ib,$$

that is an antihomomorphism of the algebra.

Conversely, given a complex associative algebra with an antihomomorphism $*$, antilinear and involutive $(\mathcal{A}, \cdot, *)$, the self-adjoint elements

$$\mathcal{A}_{\mathrm{sa}} = \{x \in \mathcal{A} | x^* = x\}$$

with the operations defined by

$$a \circ b = \frac{1}{2}(a \cdot b + b \cdot a), \qquad [a, b] = \frac{i}{2\hbar}(a \cdot b - b \cdot a).$$

form a Lie-Jordan algebra $(\mathcal{A}_{\mathrm{sa}}, \circ, [\ ,\ ])$ with $\hbar \neq 0$.

In the next section we will discuss at length the case of associative Lie-Jordan algebras and its reduction in the context of Poisson manifolds.

3. – Reduction of Poisson algebras

The first example of Poisson algebra that we will consider is the set of smooth functions in a Poisson manifold M, $(C^\infty(M), \circ, \{,\})$, where $\circ$ is the pointwise product of functions and, for $f, g \in C^\infty(M)$,

$$\{f, g\} = \Pi(df, dg)$$

with $\Pi \in \Gamma(\bigwedge^2 TM)$ such that $[\Pi, \Pi] = 0$. Here the square brackets represent the Schouten-Nijenhuis bracket for multivector fields. The latter property guarantees that the Poisson bracket satisfies the Jacobi identity and all the properties of an associative Lie-Jordan algebra ($\hbar = 0$) are fulfilled.

The data to construct the Poisson algebra have been give in terms of geometrical objects, this will be also the case when we discuss the reduction procedures. One of the goals of the paper is to translate the geometric data to the algebraic language, in order to compare with the quantum case, where the discussion is carried out in purely algebraic terms.

3˙1. *Reduction by symmetries.* – Suppose that we have a Lie group acting on M and we want to restrict our Poisson algebra to functions that are invariant under the action of the group.

The infinitesimal action of the group induces a family of vector fields $E \subset \mathfrak{X}(M)$ that we assume to be an integrable distribution. With these geometric data we introduce the subspace

$$\mathcal{E} = \{f \in C^\infty(M) \text{ s.t. } Xf = 0,\ \forall X \in \Gamma(E)\}$$

that is a Jordan subalgebra ($\mathcal{E} \circ \mathcal{E} \subset \mathcal{E}$), but not necessarily a Lie subalgebra. When this is the case, *i.e.* if

$$\{\mathcal{E}, \mathcal{E}\} \subset \mathcal{E},$$

the restrictions of the operations to $\mathcal{E}$ endows it with the structure of a Poisson subalgebra.

From the algebraic point of view the action of vector fields on functions is a derivation of the Jordan algebra:

$$X(f \circ g) = Xf \circ g + f \circ Xg,$$

and if this derivation is also a Lie derivation:

$$X\{f, g\} = \{Xf, g\} + \{f, Xg\},$$

then one easily sees that $\mathcal{E}$ is a Lie subalgebra.

An example of the previous situation is when E is a family of Hamiltonian vector fields, *i.e.* there exists a Lie subalgebra $\mathcal{G} \subset \mathcal{C}^\infty(\mathcal{M})$ such that $X \in E$ if and only if there is a $g \in \mathcal{G}$ with $Xf = \{g, f\}$ for any $f \in C^\infty(M)$. This kind of derivations, defined through the Lie product, are called inner derivations, they are always Lie derivations and therefore they define a Lie-Jordan subalgebra with the procedure described above.

3˙2. *Reduction by constraints*. – In this case the geometric input is a submanifold $N \subset M$ and the goal is to define a Poisson algebra in the set of smooth functions on N or, at least, in a subset of it.

In order to carry out the algebraic reduction we introduce the Jordan ideal of functions that vanish on N,

$$\mathcal{I} = \{f \in C^\infty(M) \text{ s.t. } f|_N = 0\}.$$

Then, the Lie normaliser of $\mathcal{I}$,

$$\mathcal{N} = \{g \in C^\infty(M) \text{ s.t. } \{g, \mathcal{I}\} \subset \mathcal{I}\},$$

is a Lie-Jordan subalgebra, as a straightforward computation shows, and $\mathcal{N} \cap \mathcal{I}$ is its Lie-Jordan ideal. Therefore $\mathcal{N}/(\mathcal{N} \cap \mathcal{I})$ inherits the structure of a Lie-Jordan algebra.

The reduced algebra as written above does not seem to have a direct connection with the functions on N. In order to uncover this connection we use the second isomorphism theorem for vector spaces

$$\mathcal{N}/(\mathcal{N} \cap \mathcal{I}) \simeq (\mathcal{N} + \mathcal{I})/\mathcal{I}$$

and taking into account that the quotient by $\mathcal{I}$ can be identified with the restriction to N the right hand side can be described as the restriction to N of the functions in $\mathcal{N} + \mathcal{I}$. If $\mathcal{N} + \mathcal{I} = C^\infty(M)$ (the constraints are second class in Dirac's terminology [15]) we obtain a Poisson algebra structure in $C^\infty(N)$. The Poisson bracket, in this case, is restriction to N of the Dirac bracket [15] in M determined by the second class constraints.

4. – More general Poisson reductions

One attempt to combine the previous two reductions to define a more general one is contained in [16]. We shall rephrase in algebraic terms the original construction that was presented in geometric language.

The data are an embedded submanifold $\iota : N \to M$ of a Poisson manifold and a subbundle $B \subset T_N M := \iota^*(TM)$. With these data we define the Jordan ideal $\mathcal{I} = \{f \in C^\infty(M) \text{ s.t. } f|_N = 0\}$, as before and the Jordan subalgebra $\mathcal{B} = \{f \in C^\infty(M) \text{ s.t. } Xf = 0\ \forall X \in \Gamma(B)\}$. The goal is to define an associative Lie-Jordan structure in $\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$.

Following [16] we assume that $\mathcal{B}$ is also a Lie subalgebra, then if $\mathcal{B} \cap \mathcal{I}$ is a Lie ideal of $\mathcal{B}$ the sought reduction is possible.

However, the condition that $\mathcal{B}$ is a subalgebra is a rather strong one [17] and, consequently, the reduction procedure is much less general than initially expected. Actually, as we will show, it consists on a succesive application of the reductions introduced in the previous section. One can prove the following result.

Theorem 1. *With the previous definitions, if $\mathcal{B}$ is not the whole algebra, i.e. $B \neq 0$, and in addition it is a Lie subalgebra, then the following hold:*

a) $\mathcal{B} \subset \mathcal{N} := \{g \in C^\infty(M) \text{ s.t. } \{\mathcal{I}, g\} \subset \mathcal{I}\}$.

b) *$\mathcal{B} \cap \mathcal{I}$ is Poisson ideal of $\mathcal{B}$.*

c) *$\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$ always inherits a Poisson bracket.*

d) *Take another $0 \neq B' \subset T_N(M)$ and define $\mathcal{B}'$ accordingly. If $B \cap TN = B' \cap TN$ $\Leftrightarrow \mathcal{B} + \mathcal{I} = \mathcal{B}' + \mathcal{I}$ by the second isomorphism theorem we have*

$$\mathcal{B}/(\mathcal{B} \cap \mathcal{I}) \simeq (\mathcal{B} + \mathcal{I})/\mathcal{I} \simeq \mathcal{B}'/(\mathcal{B}' \cap \mathcal{I})$$

and the two Poisson brackets induced on $(\mathcal{B} + \mathcal{I})/\mathcal{I}$ coincide.

Proof. We prove a) by contradiction. Assume that $\mathcal{B} \not\subset \mathcal{N}$ then there exist functions $f \in \mathcal{B}$, $g \in \mathcal{I}$ and an open set $U \subset N$, such that

$$\{g, f\}(p) \neq 0, \quad \text{for any } p \in U.$$

But certainly $g^2 \in \mathcal{B}$ as a simple consequence of the Leibniz rule for the action of vector fields. Therefore, using that $\mathcal{B}$ is a Lie subalgebra we have

$$\{g^2, f\} = 2g\{g, f\} \in \mathcal{B}$$

and due to the fact that $g \in \mathcal{I}$ and $\{g, f\}(p) \neq 0$ this implies $g \in \mathcal{B}_U$, where $\mathcal{B}_U$ is the set of functions whose restriction to U coincide with the restriction of someone in $\mathcal{B}$.

So far we know that $g \in \mathcal{B}_U \cap \mathcal{I}$ and therefore $hg \in \mathcal{B}_U \cap \mathcal{I}$ for any $h \in C^\infty(M)$. But using that $\mathcal{B}_U$ is a Lie subalgebra as it is $\mathcal{B}$ (due to the local character of the Poisson bracket) we have

$$\{hg, f\} = h\{g, f\} + g\{h, f\} \in \mathcal{B}_U \Rightarrow h\{g, f\} \in \mathcal{B}_U \Rightarrow h \in \mathcal{B}_U.$$

But h is any function, then $\mathcal{B}_U = C^\infty(M)$ and $B|_U = 0$ which implies $B = 0$ as we assumed that it is a subbundle. This contradicts the hypothesis of the theorem and a) is proved.

b) follows immediately from a). Actually if $\mathcal{B} \subset \mathcal{N}$ we have $\{\mathcal{I}, \mathcal{B}\} \subset \mathcal{I}$ and moreover $\{\mathcal{B}, \mathcal{B}\} \subset \mathcal{B}$. Then $\{\mathcal{I} \cap \mathcal{B}, \mathcal{B}\} \subset \mathcal{I} \cap \mathcal{B}$.

c) is a simple consequence of the fact that $\mathcal{B}$ is a Lie-Jordan subalgebra and $\mathcal{B} \cap \mathcal{I}$ its Lie-Jordan ideal.

To prove d) take $f_i \in \mathcal{B}$ and $f_i' \in \mathcal{B}'$, $i = 1, 2$, such that $f_i + \mathcal{I} = f_i' + \mathcal{I}$. The Poisson bracket in $(\mathcal{B} + \mathcal{I})/\mathcal{I}$ is given by

$$\{f_1 + \mathcal{I}, f_2 + \mathcal{I}\} = \{f_1, f_2\} + \mathcal{I} \in (\mathcal{B} + \mathcal{I})/\mathcal{I},$$

where for simplicity we use the same notation for the Poisson bracket in the different spaces, which should not lead to confusion. We compute now the alternative expression $\{f_i' + \mathcal{I}, f_2' + \mathcal{I}\} = \{f_1', +f_2'\} + \mathcal{I}$. We assumed $f_i' = f_i + g_i$ with $g_i \in \mathcal{I} \cap (\mathcal{B} + \mathcal{B}')$ and therefore, as a consequence of a), we have $\{f_1, g_2\}, \{g_1, f_2\}, \{g_1, g_2\} \in \mathcal{I}$, which implies

$$\{f_1', +f_2'\} + \mathcal{I} = \{f_1, f_2\} + \mathcal{I}$$

and the proof is completed. □

Last property implies that the reduction process does not depend effectively on B but only on $B \cap TN$. Actually one can show that this procedure is simply a successive application of the two previous reductions presented before: first we reduce the Poisson bracket by constraints to N and then by symmetries with $E = B \cap TN$.

For completeness we would like to comment on the situation when $B = 0$. In this case $\mathcal{B} = C^\infty(M)$ and, of course, it is always a Lie subalgebra. Under these premises the reduction is not possible unless $\mathcal{I}$ is a Lie ideal which is not the case in general. Anyhow, if the conditions to perform the reduction are met and we consider some $B' \neq 0$ such that $B' \cap TN = 0$ and $\mathcal{B}'$ is a Lie subalgebra, then we obtain again property d) of the theorem: the Poisson brackets induced by $B = 0$ and B' on $\mathcal{B}/\mathcal{I}$ are the same.

The question then is if given N and B there is a more general way to obtain the desired associative Lie-Jordan structure in $\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$ where $\mathcal{B}$ and $\mathcal{I}$ are defined as before.

To answer this question we will rephrase the problem in purely algebraic terms. We shall assume that together with an associative Lie-Jordan algebra we are given a Jordan ideal $\mathcal{I}$ and a Jordan subalgebra $\mathcal{B}$. Of course, a particular example of this is the geometric scenario discussed before. Under these premises $\mathcal{B} \cap \mathcal{I}$ is a Jordan ideal of $\mathcal{B}$ and $\mathcal{B} + \mathcal{I}$ is a Jordan subalgebra, then it is immediate to define Jordan structures on $\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$ and on $(\mathcal{B} + \mathcal{I})/\mathcal{I}$ such that the corresponding projections π_B and π are Jordan homomorphisms. Moreover, the natural isomorphism between both spaces is also a Jordan isomorphism. The problem is whether we can also induce a Poisson bracket in the quotient spaces. One first step to carry out this program is contained in the following theorem.

Theorem 2. *Given an associative Lie-Jordan algebra,* $(\mathcal{L}, \circ, \{\ ,\ \})$*, a Jordan ideal* $\mathcal{I}$ *and a Jordan subalgebra* $\mathcal{B}$*, assume*

$$\text{a) } \{\mathcal{B}, \mathcal{B}\} \subset \mathcal{B} + \mathcal{I}, \qquad \text{b) } \{\mathcal{B}, \mathcal{B} \cap \mathcal{I}\} \subset \mathcal{I}, \tag{1}$$

then the following commutative diagram

$$\begin{tikzcd}
\mathcal{B} \times \mathcal{B} \arrow[rr, "\{\ ,\ \}"] \arrow[d, "\pi_B \times \pi_B"] & & \mathcal{B} + \mathcal{I} \arrow[d, "\pi"'] \\
\mathcal{B}/(\mathcal{B} \cap \mathcal{I}) \times \mathcal{B}/(\mathcal{B} \cap \mathcal{I}) \arrow[r] & \mathcal{B}/(\mathcal{B} \cap \mathcal{I}) \arrow[r, leftrightarrow, "\simeq"] & (\mathcal{B} + \mathcal{I})/\mathcal{I}
\end{tikzcd} \tag{2}$$

defines a unique bilinear, antisymmetric operation in $\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$ *that satisfies the Leibniz rule.*

Proof. In order to show that we define uniquely an operation we have to check that π_B is onto and that $\ker(\pi_B) \times \mathcal{B}$ and $\mathcal{B} \times \ker(\pi_B)$ are mapped into $\ker(\pi) = \mathcal{I}$. But first property holds because π_B is a projection and the second one is a consequence of (1b). The bilinearity of the induced operation follows form the linearity or bilinearity of all the maps involved in the diagram and its antisymmetry derives form that of $\{\ ,\ \}$. Finally Leibniz rule is a consequence of the same property for the original Poisson bracket and the fact that π and π_B are Jordan homomorphisms. $\square$

The problem with this construction is that, in general, the bilinear operation does not satisfy the Jacobi identity as shown in the following example.

Example 1. *Consider $M = \mathbb{R}^3 \times \mathbb{R}^3$, with coordinates $(\mathbf{x}, \mathbf{y})$ and Poisson bracket given by the bivector $\Pi = \sum_{i=1}^3 \frac{\partial}{\partial_{x_i}} \wedge \frac{\partial}{\partial_{y_i}}$. take $N = \{(0, 0, x_3, \mathbf{y})\}$ and for a given $\lambda \in C^\infty(N)$ define $B = \operatorname{span}\{\partial_{x_1}, \partial_{x_2} - \lambda\partial_{y_1}\} \subset T_N M$ and*

$$\mathcal{B} = \{f \in C^\infty(M), \text{ s.t. } Xf|_N = 0, \forall\ X \in \Gamma(B)\}.$$

Notice that $T_N M$ is a direct sum of B and TN, therefore we immediately get

$$\{\mathcal{B}, \mathcal{B}\} \subset \mathcal{B} + \mathcal{I} = C^\infty(M) \quad \text{and} \quad \{\mathcal{B}, \mathcal{B} \cap \mathcal{I}\} \subset \mathcal{I},$$

and we meet all the requirements to define a bilinear, antisymmetric operation on $\mathcal{B}/(\mathcal{B} \cap \mathcal{I}) \simeq C^\infty(N)$.

Using coordinates $(x^3, \mathbf{y})$ for N the bivector field is

$$\Pi_N = \frac{\partial}{\partial_{x_3}} \wedge \frac{\partial}{\partial_{y_3}} + \lambda \frac{\partial}{\partial_{y_1}} \wedge \frac{\partial}{\partial_{y_2}}$$

that does not satisfy the Jacobi identity unless $\partial_{x_3}\lambda = \partial_{y_3}\lambda = 0$.

Now the problem is to supplement (1) with more conditions to guarantee that the induced operation satisfies all the requirements for a Poisson bracket. We do not know a simple description of the minimal necessary assumption but a rather general scenario is the following proposition:

Proposition 3. *Suppose that in addition to the conditions of theorem 2 we have two Jordan subalgebras $\mathcal{B}_+$, $\mathcal{B}_-$*

$$\mathcal{B}_- \subset \mathcal{B} \subset \mathcal{B}_+ \quad \textit{and} \quad \mathcal{B}_\pm + \mathcal{I} = \mathcal{B} + \mathcal{I},$$

such that

$$\text{a) } \{\mathcal{B}_-, \mathcal{B}_-\} \subset \mathcal{B}_+, \quad \text{b) } \{\mathcal{B}_-, \mathcal{B}_+ \cap \mathcal{I}\} \subset \mathcal{I}. \tag{3}$$

Then the antisymmetric, bilinear operation induced by (2) *is a Poisson bracket, i.e. it fulfils the Jacobi identity.*

Proof. To prove this statement consider any two functions $f_1, f_2 \in \mathcal{B}$ and, for $i = 1, 2$, denote by $f_{i,-}$ a function in $\mathcal{B}_-$ such that $f_i + \mathcal{I} = f_{i,-} + \mathcal{I} \subset \mathcal{B} + \mathcal{I}$. Due to (1) we know that

$$\{f_{1,-}, f_{2,-}\} + \mathcal{I} = \{f_1, f_2\} + \mathcal{I},$$

but if (3a) also holds,

$$\{f_{1,-}, f_{2,-}\} \in \mathcal{B}_+,$$

in addition we have that

$$\{f_{1,-}, f_{2,-}\}_- - \{f_{1,-}, f_{2,-}\} \in \mathcal{B}_+ \cap \mathcal{I},$$

and using (3,b)

$$\{\{f_{1,-}, f_{2,-}\}_-, f_{3,-}\} + \mathcal{I} = \{\{f_{1,-}, f_{2,-}\}, f_{3,-}\} + \mathcal{I}.$$

Therefore the Jacobi identity for the reduced antisymmetric product derives from that of the original Poisson bracket. □

Notice that the whole construction has been made in algebraic terms and therefore it will have an immediate translation to the quantum realm. But before going to that scenario we re-examine the example to show how it fits into the general result.

Example 2. *We take definitions and notations from example 1. Now let $\tilde{\lambda}$ be an arbitrary smooth extension of λ to M, i.e. $\tilde{\lambda} \in C^\infty(M)$ such that $\tilde{\lambda}|_N = \lambda$, we define $E = \mathrm{span}\{\partial_{x_1}, \partial_{x_2} - \tilde{\lambda}\partial_{y_1}\} \subset TM$ and $\mathcal{B}_- = \{f \in C^\infty(M) \text{ s.t. } Xf = 0,\ \forall X \in \Gamma(E)\}$.*

If we define $\mathcal{B}_+ = \mathcal{B}$, it is clear that $\mathcal{B}_- \subset \mathcal{B} \subset \mathcal{B}_+$, $\mathcal{B}_\pm + \mathcal{I} = \mathcal{B} + \mathcal{I}$ and $\{\mathcal{B}_-, \mathcal{B}_+ \cap \mathcal{I}\} \subset \mathcal{I}$. But $\{\mathcal{B}_-, \mathcal{B}_-\} \subset \mathcal{B}_+$ if and only if $\partial_{x_3}\lambda = \partial_{y_3}\lambda = 0$.

Therefore, in our construction we can accommodate the most general situation in which the example provides a Poisson bracket. We believe that this is not always the case, but we do not have any counterexamples.

5. – Final comments

We want to end this contribution with a comment on the possible application of the reduction described in the previous section to quantum systems. In this case the Lie-Jordan algebra is non-associative and due to the associator identity there is a deeper connection between the Jordan and Lie products. As a result the different treatment between the Jordan and the Lie part, that we considered in the case of associative algebras, is not useful any more and the natural thing to do is to consider a more *symmetric* prescription.

We propose a generalisation of the standard reduction procedure (the quotient of subalgebras by ideals) along similar lines to those followed in the associative case.

The statement of the problem is the following: given a Lie-Jordan algebra $\mathcal{L}$ and two subspaces $\mathcal{B}$, $\mathcal{S}$ the goal is to induce a Lie-Jordan structure in the quotient $\mathcal{B}/(\mathcal{B} \cap \mathcal{S})$.

If we assume

$$\begin{aligned} &\mathcal{B} \circ \mathcal{B} \subset \mathcal{B} + \mathcal{S}, \quad &&\mathcal{B} \circ (\mathcal{B} \cap \mathcal{S}) \subset \mathcal{S}, \\ &[\mathcal{B}, \mathcal{B}] \subset \mathcal{B} + \mathcal{S}, \quad &&[\mathcal{B}, \mathcal{B} \cap \mathcal{S}] \subset \mathcal{S}, \end{aligned}$$

then a diagram similar to (2) allows to induce commutative and anticommutative, bilinear operations in the quotient. Now, in order to fulfil the ternary properties (Jacobi, Leibniz and associator identity) we need more conditions. We can show that, again, it is enough to have two more subspaces $\mathcal{B}_- \subset \mathcal{B} \subset \mathcal{B}_+$ such that $\mathcal{B}_\pm + \mathcal{S} = \mathcal{B} + \mathcal{S}$ and moreover

$$\begin{aligned} &\mathcal{B}_- \circ \mathcal{B}_- \subset \mathcal{B}_+, \quad &&\mathcal{B}_- \circ (\mathcal{B}_+ \cap \mathcal{S}) \subset \mathcal{S}, \\ &[\mathcal{B}_-, \mathcal{B}_-] \subset \mathcal{B}_+, \quad &&[\mathcal{B}_-, (\mathcal{B}_+ \cap \mathcal{S})] \subset \mathcal{S}. \end{aligned}$$

Then, under these conditions, one can correctly induce a Lie-Jordan structure in the quotient.

There are at least two aspects of this construction that need more work. The first one is to find examples in which this reduction procedure is relevant, similarly to what we did for the classical case in the previous section. The second problem is of topological nature: given a Banach space structure in the big algebra $\mathcal{L}$, compatible with its operations, we can correctly induce a norm in the quotient provided $\mathcal{B}$ and $\mathcal{S}$ are closed subspaces. However, the induced operations need not to be continuous in general; though they are, if $\mathcal{B}$ is a subalgebra and $\mathcal{S}$ an ideal [18]. The study of more general conditions for continuity and compatibility of the norm will be the subject of further research.

* * *

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