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Blow-up of the quantum potential for a free particle in one dimension

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Summary. — We derive a non-linear differential equation that must be satisfied by the quantum potential, in the context of the Madelung equations, in one dimension for a particular class of wave functions. In this case, we exhibit explicit conditions leading to the blow-up of the quantum potential of a free particle at the boundary of the compact support of the probability density.

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1. – Introduction

The hydrodynamical formulation of quantum mechanics is based on the work of David Bohm [1] and on the equations obtained by Madelung in his pioneering paper published in 1927 [2]. The lines of research related to this approach have grown during the last years and have found applications in several fields in the context of the Quantum Trajectories Method [3] (for instance, for the analysis of semiconductor devices [4]). One of the central concepts in this case is the so-called Quantum Potential, the term appearing in the quantum Hamilton-Jacobi equation (see sect. **2**) which depends on the structure of the probability density function associated to the wave function. Of particular interest is

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the problem related to the presence of singularities of the quantum potential. Typically, in the proximity of these points, due to the fact that the probability density goes to zero, the analysis of the problem is not easy and can require a particular care and proper strategies [5]. As a matter of fact, it would be interesting to find conditions in terms of the probability density that guarantees the presence of blow-up of the quantum potential.

In this paper we will follow a particular approach to this problem. More precisely, we will identify a class of wave functions whose squared modulus (in this case the probability density function of a single particle) can be directly expressed as a function of the quantum potential. We will confine our analysis to the case of one spatial dimension, then derive (at fixed time) a non-linear differential equation for the quantum potential. Finally, we will show that the solutions of this equation exhibit a blow-up behavior so that the probability density must have a compact support. We will see that this behavior is strictly connected to the assumed relation between quantum potential and probability density.

The paper is organized as follows. In sect. **2** we review the Madelung equations obtained by the Schrödinger equation and define the Quantum Potential. In sect. **3** we define the class of wave functions that we want to analyze; as an example we will show that the Gaussian wave packet belongs to this class. Moreover, we will obtain a differential equation to be satisfied by the quantum potential associated to the probability density. In sect. **4** we specialize this differential equation to the case of a simpler dependence and in sect. **5** we will show that the solution of the simplified differential equation (the quantum potential) must exhibit a blow-up for a finite value of the position. In sect. **6** we perform a qualitative analysis of the solutions in the proximity of the blow-up points. In sect. **7** we draw some conclusions.

2. – Quantum hydrodynamics equations and the quantum potential

In this section we briefly review the derivation of the Madelung equations [2] that are the basis of the hydrodynamical interpretation of quantum mechanics and of the Bohm theory [1].

Let us consider a single particle with mass $m > 0$ moving under the influence of a time-invariant potential described by the smooth function $V : \mathbb{R} \to \mathbb{R}$ (we restrict our analysis to the one-dimensional case). The associated time-dependent Schrödinger equation is

$$i\hbar \frac{\partial \Psi(q,t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} + V(q) \right) \Psi(q,t), \tag{1}$$

where $\hbar$ is the reduced Planck constant and $\Psi(q,t)$ is the normalized (square-integrable) complex-valued wave function that, in polar form, can be expressed by

$$\Psi(q,t) = \sqrt{\mu(q,t)}\, e^{\frac{i}{\hbar} S(q,t)}\,, \tag{2}$$

in terms of the probability density $\mu(q,t)$ and of the action function $S(q,t)$ (which are both real-valued).

Inserting eq. (2) into eq. (1) and rearranging the terms, we obtain the coupled partial differential equations

$$\frac{\partial\mu(q,t)}{\partial t}+\frac{\partial J(q,t)}{\partial q}=0, \tag{3}$$

$$\frac{\partial S(q,t)}{\partial t}+\frac{1}{2m}\left(\frac{\partial S(q,t)}{\partial q}\right)^2+V(q)+\Phi(q,t)=0, \tag{4}$$

where the terms $J(q,t)$ and $\Phi(q,t)$ are respectively a *probability current*

$$J(q,t)=\frac{1}{m}\mu(q,t)\frac{\partial S(q,t)}{\partial q}\,, \tag{5}$$

which depends on both $\mu(q,t)$ and $S(q,t)$ and the so-called *quantum potential*

$$\begin{aligned}\Phi(q,t)&=-\frac{\hbar^2}{2m}\frac{1}{\sqrt{\mu(q,t)}}\frac{\partial^2\sqrt{\mu(q,t)}}{\partial q^2}\\&=-\frac{\hbar^2}{4m}\left[\frac{1}{\mu(q,t)}\frac{\partial^2\mu(q,t)}{\partial q^2}-\frac{1}{2\mu^2(q,t)}\left(\frac{\partial\mu(q,t)}{\partial q}\right)^2\right],\end{aligned} \tag{6}$$

which only depends on $\mu(q,t)$. So, while eq. (3) is a continuity equation for the quantum probability density $\mu(q,t)$ which, at time t flows at the point q with rate $J(q,t)$, eq. (4) is the quantum Hamilton-Jacobi equation.

In the following sections we analyze some particular cases of blow-up of the quantum potential for a particular class of wave functions.

3. – A non-linear equation for the quantum potential at fixed time

We consider the case of a free particle, *i.e.* we set $V(q)=0$, $\forall q\in\mathbb{R}$. Moreover, we fix a time $t=t_0$ and assume that the probability density exponentially depends on the quantum potential at time t_0, *i.e.* has the form

$$\rho(q):=\mu(q,t_0)=C_\alpha e^{-\alpha f(\phi(q))}\qquad\text{with}\qquad\phi(q)=\Phi(q,t_0), \tag{7}$$

where $\alpha\in\mathbb{R}$, C_α is a normalization constant and f is a regular real valued function. We notice that eq. (7) is a particular case of the general dependence

$$\rho(q)=g(\phi(q)), \tag{8}$$

where g is a regular real valued function.

3·1. *Example: Gaussian wave packet.* – We show that the probability density of a Gaussian wave packet, for instance, exhibits this behavior for a suitable $\alpha<0$, f being, in this case, the identity map. To this aim we set, at time $t_0=0$, the initial state of a Gaussian wave packet

$$\Psi(q,0) = \frac{1}{\left(2\pi\Delta q^2\right)^{1/4}} \exp\left[-\frac{q^2}{4\Delta q^2} + \frac{i}{\hbar} p_0 q\right], \tag{9}$$
$$\mu(q,0) = \frac{1}{\left(2\pi\Delta q^2\right)^{1/2}} \exp\left[-\frac{q^2}{2\Delta q^2}\right],$$

where p_0 is the average initial momentum and Δq^2 is the variance (uncertainty) of the position of the packet. Then, taking into account the analytic behavior obtained in [6] for the free evolution of a particle described by a Gaussian wave packet, we obtain

$$\mu(q,t) = \frac{1}{\left(2\pi\Delta_t q^2\right)^{1/2}} \exp\left[-\frac{(q-q_t)^2}{2\Delta_t q^2}\right], \tag{10}$$

where

$$\Delta_t q^2 = \Delta q^2 + \frac{\hbar^2 t^2}{4m^2\Delta q^2} \qquad \text{and} \qquad q_t = \frac{p_0}{m} t. \tag{11}$$

Rewriting eq. (6) with $\mu(q,t)$ given by eq. (10), we obtain the expression of the quantum potential for a free particle described by a Gaussian wave packet

$$\Phi(q,t) = \frac{\hbar^2}{4m\Delta_t q^2}\left(1 - \frac{(q-q_t)^2}{2\Delta_t q^2}\right), \tag{12}$$

from which we deduce

$$1 - \frac{(q-q_t)^2}{2\Delta_t q^2} = \frac{4m\Delta_t q^2}{\hbar^2}\Phi(q,t). \tag{13}$$

So, setting at some fixed $t > 0$, $\alpha = -(4m\Delta_t q^2)/\hbar^2 < 0$, we get from eq. (10) that the behavior assumed in eq. (7) is satisfied when f is the identity map and

$$C_\alpha = \frac{1}{\left(2\pi e^2 \Delta_t q^2\right)^{1/2}} = \sqrt{-\frac{2m}{\pi e^2 \hbar^2 \alpha}}. \tag{14}$$

Finally, from eq. (10) we get

$$\mu(q,t) = \sqrt{-\frac{2m}{\pi e^2\hbar^2\alpha}}\, e^{-\alpha\Phi(q,t)} = C_\alpha e^{-\alpha\Phi(q,t)}, \tag{15}$$

as claimed.

3·2. *Non-linear differential equation from eq.* (7). – In what follows we obtain a differential equation describing the behavior of the quantum potential at a fixed time for wave functions satisfying eq. (7). At time t_0 eq. (6) becomes

$$\phi(q) = -\frac{\hbar^2}{4m}\left[\frac{1}{\rho(q)}\frac{\mathrm{d}^2\rho(q)}{\mathrm{d}q^2} - \frac{1}{2\rho^2(q)}\left(\frac{\mathrm{d}\rho(q)}{\mathrm{d}q}\right)^2\right]. \tag{16}$$

We evaluate the first and second derivative of the function $\rho(q)$ getting

$$\frac{\mathrm{d}\rho(q)}{\mathrm{d}q} = \frac{\mathrm{d}g(\phi)}{\mathrm{d}\phi}\frac{\mathrm{d}\phi(q)}{\mathrm{d}q}, \qquad \frac{\mathrm{d}^2\rho(q)}{\mathrm{d}q^2} = \frac{\mathrm{d}g(\phi)}{\mathrm{d}\phi}\frac{\mathrm{d}^2\phi(q)}{\mathrm{d}q^2} + \frac{\mathrm{d}^2g(\phi)}{\mathrm{d}\phi^2}\left(\frac{\mathrm{d}\phi(q)}{\mathrm{d}q}\right)^2. \tag{17}$$

where we have abused the notation introduced in eq. (8) by writing

$$g(\phi) = g(\phi(q)). \tag{18}$$

Moreover, from eq. (8) we get

$$\frac{\mathrm{d}g(\phi)}{\mathrm{d}\phi} = -\alpha\rho(q)\frac{\mathrm{d}f(\phi)}{\mathrm{d}\phi}, \tag{19}$$

$$\begin{aligned} \frac{\mathrm{d}^2g(\phi)}{\mathrm{d}\phi^2} &= -\alpha\rho(q)\frac{\mathrm{d}^2f(\phi)}{\mathrm{d}\phi^2} + \alpha^2\rho(q)\left(\frac{\mathrm{d}f(\phi)}{\mathrm{d}\phi}\right)^2 \\ &= -\alpha\rho(q)\left[\frac{\mathrm{d}^2f(\phi)}{\mathrm{d}\phi^2} - \alpha\left(\frac{\mathrm{d}f(\phi)}{\mathrm{d}\phi}\right)^2\right], \end{aligned} \tag{20}$$

where, as above, $f(\phi)$ means $f(\phi(q))$.

Inserting eqs. (19)-(20) into eq. (17) and then into eq. (16) we find, rearranging the terms, that the quantum potential ϕ associated to a probability density $\mu(q,t)$ given by eq. (7) at time t_0 satisfies the following differential equation:

$$\frac{\mathrm{d}f(\phi)}{\mathrm{d}\phi}\frac{\mathrm{d}^2\phi(q)}{\mathrm{d}q^2} - \left[\frac{\alpha}{2}\left(\frac{\mathrm{d}f(\phi)}{\mathrm{d}\phi}\right)^2 - \frac{\mathrm{d}^2f(\phi)}{\mathrm{d}\phi^2}\right]\left(\frac{\mathrm{d}\phi(q)}{\mathrm{d}q}\right)^2 - \frac{4m}{\alpha\hbar^2}\phi(q) = 0. \tag{21}$$

4. – A simplified form of the non-linear equation

Motivated by the result obtained in the previous section for the Gaussian wave packet, we consider the case in which f is the identity map, so eq. (7) becomes

$$\rho(q) = C_\alpha e^{-\alpha\phi(q)}. \tag{22}$$

We also assume that the probability density $\rho(q)$ is symmetric around $q = 0$; this happens when ϕ is even (*i.e.* $\phi(-q) = \phi(q)$) as happens for the quantum potential relative to the Gaussian packet at time $t = 0$ (at any time if $p_0 = 0$), see eq. (12). The particular dependence of the probability density on the the quantum potential assumed in eq. (22) will be fundamental to obtain the main result of the paper. As a matter of fact, we will

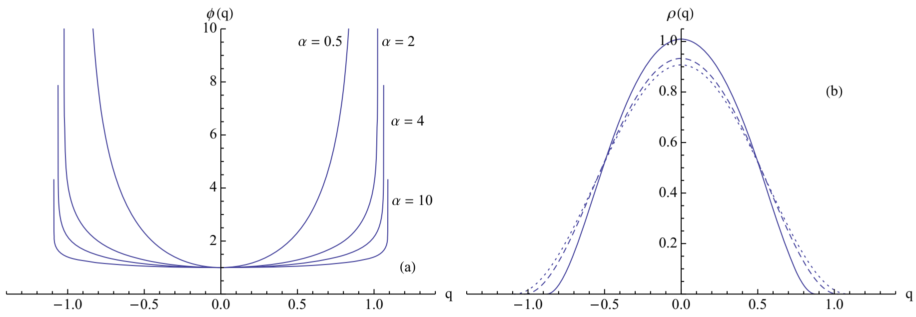

Fig. 1. – (Color online) (a) Solution of eq. (23) for $\alpha = 0.5$, $\alpha = 2$, $\alpha = 4$ and $\alpha = 10$. (b) Plot of the normalized probability density $\rho(q)$ corresponding to the solution of eq. (23) for $\alpha = 0.5$ (solid line), $\alpha = 2$ (dashed line), and $\alpha = 10$ (dotted line). We have set $m = 1$, $\hbar = 1$, $\phi_0 = 1$ and $\phi'(0) = 0$.

show that this dependence, under certain assumptions, implies a blow-up of the quantum potential at finite position.

Using eq. (22) we see that eq. (21) reduces to

$$\phi''(q) - \frac{\alpha}{2}(\phi'(q))^2 - \frac{4m}{\alpha\hbar^2}\phi(q) = 0, \tag{23}$$

where we have used the standard notation

$$\phi'(q) = \frac{\mathrm{d}\phi(q)}{\mathrm{d}q}, \qquad \phi''(q) = \frac{\mathrm{d}^2\phi(q)}{\mathrm{d}q^2}. \tag{24}$$

Then we assume $\phi(0) = \phi_0 > 0$ and, being ϕ even, ϕ' must be odd (*i.e.* $\phi'(-q) = -\phi'(q)$) and, as a consequence, $\phi'(0) = 0$. By substituting these conditions into eq. (23) we immediately obtain that the solution has

$$\phi''(0) = \frac{4m}{\alpha\hbar^2}\phi_0 > 0, \qquad \text{with} \qquad \alpha > 0, \tag{25}$$

differently from the case of the Gaussian wave packet where $\alpha < 0$. Therefore, for $q \simeq 0$, the quantum potential behaves as a confining potential. In fig. 1(a) we plot the (numerical) solution of eq. (23) for different values of the parameter $\alpha > 0$; fig. 1(b) shows some examples of probability densities corresponding to the quantum potentials obtained from eq. (23); in both cases we have set $m = 1$, $\hbar = 1$ with the initial conditions $\phi_0 = 1$, $\phi'(0)$=0. The numerical integration indicates that the quantum potential exhibits a blow-up for finite values of q and, therefore, the corresponding $\rho(q)$'s have a compact support and vanishes on the boundary.

In order to analyze this property, it is useful to manipulate eq. (23). Let us define the function

$$w(\phi) = (\phi')^2 + B\phi, \tag{26}$$

where $A = \alpha/2$ and $B = 4m/(\alpha\hbar^2)$. Thus, eq. (23) takes the form

$$\phi'' = Aw(\phi) + B(1 - A)\phi. \tag{27}$$

By taking the derivative of eq. (26) with respect to q, we deduce

$$B\phi' = \frac{\mathrm{d}w(\phi)}{\mathrm{d}q} - 2\phi'\phi'' = \frac{\mathrm{d}w(\phi)}{\mathrm{d}\phi}\phi' - 2\phi'\phi'', \tag{28}$$

and therefore

$$\phi'\left(B - \frac{\mathrm{d}w}{\mathrm{d}\phi} + 2\phi''\right) = 0. \tag{29}$$

Inserting eq. (27) into eq. (29) we obtain

$$\phi'\left(B - \frac{\mathrm{d}w}{\mathrm{d}\phi} + 2Aw + 2B(1-A)\phi\right) = 0. \tag{30}$$

Taking into account that $\phi \equiv 0$ is the only constant solution of eq. (23) which does not satisfy the initial condition $\phi(0) > 0$, we will focus on the equation

$$\frac{\mathrm{d}w(\phi)}{\mathrm{d}\phi} = 2Aw(\phi) + 2B(1-A)\phi + B. \tag{31}$$

Equation (31) is a first order normal differential equation, so we get, by standard calculations, that, for any $C \in \mathbb{R}$,

$$w(\phi) = Ce^{2A\phi} + \frac{B}{A}(A-1)\phi - \frac{B}{2A^2}\,, \tag{32}$$

is a solution to eq. (31) (as it is possible to check by direct substitution).

So, by eq. (26), we obtain a new first order equation for the quantum potential at time t_0 relative to a probability density $\mu(q,t)$ which satisfies eq. (22) at fixed time t_0

$$|\phi'(q)| = \sqrt{Ce^{2A\phi(q)} - \frac{B}{A}\phi(q) - \frac{B}{2A^2}} = \sqrt{Ce^{\alpha\phi(q)} - \frac{8m}{\alpha^2\hbar^2}\phi(q) - \frac{8m}{\alpha^3\hbar^2}}\,. \tag{33}$$

In the following section we will analyze some properties of the solutions to eq. (33).

5. – Blow-up at finite positions

We will focus on the case $\alpha > 0$ which is not covered by the quantum potential relative to a Gaussian wave packet. In particular, we shall deal with

1) $\phi(0) = \phi_0 > 0$;

2) ϕ even to have, by eq. (22), a probability density $\rho(q)$ symmetric around $q = 0$; under such assumption ϕ' must be odd (*i.e.* $\phi'(-q) = -\phi'(q)$) and, as a consequence, $\phi'(0) = 0$;

3) ϕ increasing on $[0, +\infty[$ (*i.e.* $\phi'(q) = |\phi'(q)|$ if $q \geq 0$) and, as a consequence, decreasing on $]-\infty, 0]$ (*i.e.* $\phi'(q) = -|\phi'(q)|$ if $q \leq 0$).

Therefore, we can find an even solution to eq. (23) with initial conditions $\phi(0) = \phi_0$, $\phi'(0) = 0$ by an even extension of the unique solution to the equation

$$\phi'(q) = \sqrt{Ce^{\alpha\phi(q)} - \frac{8m}{\alpha^2\hbar^2}\phi(q) - \frac{8m}{\alpha^3\hbar^2}}\,. \tag{34}$$

with initial condition $\phi(0) = \phi_0$.

We are now ready to show the existence of finite blow-up of the quantum potential, which is solution of eq. (34) and, after even extension, of eq. (23). This property is characterized by the following statement.

Proposition 1. *For every $\phi_0 > 0$ the solution to eq.* (34) *satisfying $\phi(0) = \phi_0$ exhibits a blow-up at the finite value $q_* < +\infty$ satisfying the following inequality:*

$$q_* \leq \frac{\sqrt{\alpha}\hbar}{\sqrt{2m}} \max\left(\frac{\sqrt{\alpha^3} + e^{-\alpha\phi_0/2}}{\sqrt{F_\alpha(\phi_0) - F_\alpha(2\phi_0)}} e^{-\alpha\phi_0/2}, \sqrt{\alpha} + \frac{e^{-\alpha\phi_0}}{\sqrt{F_\alpha(\phi_0) - F_\alpha(2\phi_0)}}\right), \tag{35}$$

where, for all $x \in \mathbb{R}$, $F_\alpha(x) = (\alpha x + 1)e^{-\alpha x}$.

Proof. By imposing the condition $\phi'(0) = 0$, or equivalently the "pairing" condition $w(\phi_0) = -B\phi_0$, we get from eq. (34)

$$C = \frac{8m}{\alpha^2\hbar^2}\left(\phi_0 + \frac{1}{\alpha}\right)e^{-\alpha\phi_0} \tag{36}$$

and therefore

$$\phi'(q) = \sqrt{\frac{8m}{\alpha^3\hbar^2}}e^{\alpha\phi/2}\sqrt{(\alpha\phi_0 + 1)e^{-\alpha\phi_0} - (\alpha\phi + 1)e^{-\alpha\phi}}. \tag{37}$$

Then, we define the function

$$F_\alpha(x) = (\alpha x + 1)e^{-\alpha x}, \tag{38}$$

which results to be strictly decreasing $\forall x \geq 0$ so that eq. (37) takes the form

$$\phi'(q) = \sqrt{\frac{8m}{\alpha^3\hbar^2}}e^{\alpha\phi/2}\sqrt{F_\alpha(\phi_0) - F_\alpha(\phi)} =: \theta(\phi), \tag{39}$$

where $\theta(\phi)$ is well defined for $\phi \geq \phi_0$. Notice that $\phi \mapsto \theta(\phi)$ is a positive and continuous function. To prove the thesis we are going to show that the Osgood condition [7] is satisfied, namely that

$$q_* = \int_{\phi_0}^{+\infty} \frac{1}{\theta(\phi)}\mathrm{d}\phi < +\infty. \tag{40}$$

To this aim, since the funcion $1/\theta(\phi)$ is unbounded near ϕ_0, we additively split the integral in eq. (40) into two terms and proceed to estimate them. We set

$$q_1^* = \int_{\phi_0}^{2\phi_0} \frac{1}{\theta(\phi)}\mathrm{d}\phi \qquad \text{and} \qquad q_2^* = \int_{2\phi_0}^{+\infty} \frac{1}{\theta(\phi)}\mathrm{d}\phi. \tag{41}$$

In order to evaluate q_1^* we have to estimate

$$\lim_{\omega\to\phi_0^+}\int_\omega^{2\phi_0}\frac{1}{\theta(\phi)}\mathrm{d}\phi. \tag{42}$$

We firstly observe that,

$$\exists\lim_{\phi\to\phi_0}\frac{\sqrt{F_\alpha(\phi_0)-F_\alpha(\phi)}}{\sqrt{\phi-\phi_0}}=\alpha\sqrt{\phi_0}\,e^{-\alpha\phi_0/2}=:l_0, \tag{43}$$

so that, not only

$$\sqrt{F_\alpha(\phi_0)-F_\alpha(\phi)}\simeq l_0\sqrt{\phi-\phi_0}=\alpha\sqrt{\phi_0}\,e^{-\alpha\phi_0/2}\sqrt{\phi-\phi_0}\qquad\text{as}\qquad\phi\to\phi_0, \tag{44}$$

but, *a posteriori*, also

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\sqrt{F_\alpha(\phi_0)-F_\alpha(\phi)}\simeq l_0\frac{\mathrm{d}}{\mathrm{d}\phi}\sqrt{\phi-\phi_0}\qquad\text{as}\qquad\phi\to\phi_0. \tag{45}$$

Therefore, setting

$$\eta_0=\max\left(\frac{l_0\sqrt{2\phi_0-\phi_0}}{\sqrt{F_\alpha(\phi_0)-F_\alpha(2\phi_0)}},1\right)=\max\left(\frac{\alpha\phi_0e^{-\alpha\phi_0/2}}{\sqrt{F_\alpha(\phi_0)-F_\alpha(2\phi_0)}},1\right) \tag{46}$$

we easily obtain the following bound

$$\begin{aligned} q_1^* &= \sqrt{\frac{\alpha^3\hbar^2}{8m}}\int_{\phi_0}^{2\phi_0}\frac{e^{-\alpha\phi/2}}{\sqrt{F_\alpha(\phi_0)-F_\alpha(\phi)}}\mathrm{d}\phi\\ &\le \sqrt{\frac{\alpha^3\hbar^2}{8m}}\int_{\phi_0}^{2\phi_0}e^{-\alpha\phi_0/2}\frac{\eta_0}{\alpha\sqrt{\phi_0}e^{-\alpha\phi_0/2}\sqrt{\phi-\phi_0}}\mathrm{d}\phi\\ &\le \sqrt{\frac{\alpha^2\hbar^2}{8m}}\frac{\eta_0}{\sqrt{\phi_0}}\int_{\phi_0}^{2\phi_0}\frac{1}{\sqrt{\phi-\phi_0}}\mathrm{d}\phi=\sqrt{\frac{\alpha^2\hbar^2}{2m}}\eta_0<+\infty. \end{aligned} \tag{47}$$

In order to estimate q_2^* we argue as follows. Since $x\mapsto F_\alpha(x)$ is strictly decreasing on $\mathbb{R}_+$, for every $\phi>2\phi_0$ we have

$$\frac{1}{\sqrt{F_\alpha(\phi_0)-F_\alpha(\phi)}}<\frac{1}{\sqrt{F_\alpha(\phi_0)-F_\alpha(2\phi_0)}}. \tag{48}$$

Therefore we find

$$\begin{aligned} q_2^* &= \sqrt{\frac{\alpha^3\hbar^2}{8m}}\int_{2\phi_0}^{+\infty}\frac{e^{-\alpha\phi/2}}{\sqrt{F_\alpha(\phi_0)-F_\alpha(\phi)}}\mathrm{d}\phi\\ &\le \sqrt{\frac{\alpha^3\hbar^2}{8m}}\frac{1}{\sqrt{F_\alpha(\phi_0)-F_\alpha(2\phi_0)}}\int_{2\phi_0}^{+\infty}e^{-\alpha\phi/2}\mathrm{d}\phi\\ &= \sqrt{\frac{\alpha\hbar^2}{2m}}\frac{1}{\sqrt{F_\alpha(\phi_0)-F_\alpha(2\phi_0)}}e^{-\alpha\phi_0}<+\infty. \end{aligned} \tag{49}$$

Then, by setting $q_*=q_1^*+q_2^*$ and employing eq. (47) and eq. (49) we get the thesis. □

6. – Qualitative analysis of the quantum potential in the proximity of the blow-up

We can now perform a qualitative analysis of the features of the solutions of eq. (34) in proximity of the blow-up points.

Using the function $F_\alpha(x)$ introduced in eq. (38) by which $C = 8mF_\alpha(\phi_0)/(\alpha^3\hbar^2)$, eq. (34) becomes

$$\phi'(q) = \sqrt{C}e^{\alpha\phi(q)/2}\sqrt{1 - \frac{F_\alpha(\phi)}{F_\alpha(\phi_0)}}\,. \tag{50}$$

Taking into account that $F_\alpha(\phi)/F_\alpha(\phi_0)$ tends to zero as $\phi \to +\infty$, eq. (50) takes the form

$$\phi'(q) \simeq \sqrt{C}e^{\alpha\phi(q)/2} = \sqrt{\frac{8m(\alpha\phi_0 + 1)}{\alpha^3\hbar^2}}\exp\left[-\frac{\alpha(\phi - \phi_0)}{2}\right]. \tag{51}$$

for ϕ large enough. Equation (51) admits a solution of the form

$$\phi(q) \simeq -\frac{2}{\alpha}\ln\left(-\frac{\alpha}{2}\sqrt{C}q + D\right), \tag{52}$$

where D is an integration constant. In this approximation the quantum potential exhibits a blow-up when $q_+ = 2D/(\alpha\sqrt{C})$, with $D, C < +\infty$. The points q_+ and $q_- = -q_+$ represent, in the above approximation, the boundary of the support of the even solution to eq. (51).

7. – Conclusions

In this paper we have defined a class of wave functions whose probability density (assumed symmetric with respect to $q = 0$) can be directly expressed in terms of the associated quantum potential. Moreover, we have found that it is possible to derive a differential equation for the quantum potential whose solution exhibits a blow-up behavior forcing the probability density to have a compact support. This kind of analysis is of interest to predict the existence of singularities of the quantum potential and, therefore, the identification of probability densities with compact support. It would be interesting to extend these results to a more general functional dependence of the probability density on the quantum potential going beyond the identity map. Moreover, this analysis have been performed fixing a certain value of time t_0. One could also extend this work in order to study the evolution of the quantum potential during the propagation of the particle.

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REFERENCES

[1] Bohm D., *Phys. Rev.*, **85** (1952) 166; 180.
[2] Madelung E., *Z. Phys.*, **40** (1927) 322.
[3] Wyatt R. E., *Quantum Dynamics with Trajectories: Introduction to Quantum Hydrodynamics* (Springer, New York) 2005.
[4] Gardner C. L., *SIAM J. Appl. Math.*, **54** (1994) 409.
[5] Wyatt R. E., *J. Chem. Phys.*, **121** (2004) 9230.
[6] Shankar R., *Principles of Quantum Mechanics*, 2nd edition (Plenum) 1994.
[7] Osgood W. F., *Monatsh. Math. Phys.*, **9** (1898) 331.