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Stochastic evolution of classical and quantum systems

D. Chruściński

Institute of Physics, Faculty of Physics, Astronomy and Informatics
Nicolaus Copernicus University - Grudziadzka 5, 87–100 Torun, Poland

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Summary. — We present a basic introduction to stochastic evolution of classical and quantum finite level systems. We discuss the properties of classical and quantum states and classical and quantum channels. Moreover, we provide the description of Markovian semigroups and discuss the structure of local in time master equations. A short discussion of non-Markovian dynamics is included as well.

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1. – Introduction

Stochastic systems play an important role both in classical and quantum physics [1,2]. The aim of this paper is to provide a basic introduction to the mathematical description of such systems. The main focus is on quantum systems, however, for pedagogical reasons we present a parallel discussion for both classical and quantum systems.

The dynamics of open quantum systems attracts nowadays increasing attention [3-7]. It is relevant not only for a better understanding of quantum theory but it is fundamental in various modern applications of quantum mechanics. Since the system-environment interaction causes dissipation, decay and decoherence it is clear that the dynamic of open systems is fundamental in modern quantum technologies, such as quantum communication, cryptography and computation [8].

We start with the discussion of classical and quantum states. These are convex sets of probability distributions and density operators, respectively. In the classical case a set of states shares an additional important property —it is a simplex. Quantum theory generalized a simplex to much more sophisticated convex sets. After introducing states we analyze an important concept of channels, *i.e.* linear maps mapping states into states. In the classical case they are represented by stochastic matrices and in the quantum one by the linear completely positive trace preserving maps. This is a powerful generalization which plays an important role in the modern formulation of quantum theory.

Equipped with these mathematical concepts we analyze the structure of classical and quantum dynamics described by families of classical and quantum channels —classical and quantum dynamical maps. We discuss Markovian semigroups and then general dynamics based on local in time master equations. We conclude with a short discussion of non-Markovian evolution.

2. – Classical states and classical channels

Let us consider n-states classical stochastic system. States of such system are represented by the probability vectors $\mathbf{p} = (p_1, \ldots, p_n)^{\mathrm{T}}$ and hence the corresponding space of states defines a simplex

$$\Sigma_n = \{\mathbf{p} \in \mathbb{R}^n_+ \mid p_1 + \ldots + p_n = 1\}. \tag{1}$$

Pure states correspond to vertices of Σ_n. Note that there are exactly n vertices and any point in Σ_n is uniquely represented as a convex combination of vertices

$$\mathbf{p} = p_1\mathbf{e}_1 + \ldots + p_n\mathbf{e}_n \ , \tag{2}$$

where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a set of vertices, that is, $\mathbf{e}_1 = (1, 0, \ldots, 0)^{\mathrm{T}}, \ldots, \mathbf{e}_n = (0, \ldots, 0, 1)^{\mathrm{T}}$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. It is called positive if $T(\mathbb{R}^n_+) \subset \mathbb{R}^n_+$. It is called a *classical channel* if $T(\Sigma_n) \subset \Sigma_n$, *i.e.* it maps classical states into classical states. It is clear that T is positive iff all matrix elements of T satisfy $T_{ij} \geq 0$. Moreover T defines a classical channel iff

$$T_{ij} \geq 0, \qquad \sum_{i=1}^{n} T_{ij} = 1. \tag{3}$$

A matrix satisfying (3) is called stochastic. It is clear that stochastic matrices form a convex set.

The vector space $\mathbb{R}^n$ is equipped with a family of p-norms

$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} . \tag{4}$$

One shows that if T is a stochastic matrix, then

$$\|T\mathbf{x}\|_1 \leq \|\mathbf{x}\|_1, \tag{5}$$

for all $\mathbf{x} \in \mathbb{R}^n$. Hence, classical channels (stochastic matrices) are contractions in 1-norm. Let us observe that $\|\mathbf{x}\|_1$ provides a natural distance measure in Σ_n: for any pair $\mathbf{p}, \mathbf{q} \in \Sigma_n$ one defines

$$D[\mathbf{p}, \mathbf{q}] = \frac{1}{2} \|\mathbf{p} - \mathbf{q}\|_1. \tag{6}$$

One calls $D[\mathbf{p}, \mathbf{q}]$ the distinguishability of $\mathbf{p}$ and $\mathbf{q}$. Note that $0 \leq D[\mathbf{p}, \mathbf{q}] \leq 1$ and $D[\mathbf{p}, \mathbf{q}] = 0$ if and only if $\mathbf{p} = \mathbf{q}$. Formula (5) implies that if T is a stochastic matrix, then

$$D[T\mathbf{p}, T\mathbf{q}] \leq D[\mathbf{p}, \mathbf{q}], \tag{7}$$

for all $\mathbf{p}, \mathbf{q} \in \Sigma_n$. Hence, the distance between any pair of states never increases under the action of a classical channel T. A similar property holds for a relative entropy

$$S(\mathbf{p}\|\mathbf{q}) = \sum_{k=1}^{n} p_k\,(\log p_k - \log q_k). \tag{8}$$

One shows that for any stochastic matrix T

$$S(T\mathbf{p}\|T\mathbf{q}) \leq S(\mathbf{p}\|\mathbf{q}), \tag{9}$$

for all $\mathbf{p}, \mathbf{q} \in \Sigma_n$.

Now, let us pass to the dual picture, *i.e.* a space of classical observables. It is a real unital commutative algebra $\mathcal{A}_n = (\mathbb{R}^n, \circ)$ such that $\mathbf{a} \circ \mathbf{b} = \mathbf{c}$, with $c_k = a_k b_k$. The unit element $\mathbf{e}$ is defined by $\mathbf{e} = (1, \ldots, 1)^{\mathrm{t}}$. Note that if T is a stochastic a matrix then the dual map T^{t} is unital, that is, $T^{\mathrm{t}}\mathbf{e} = \mathbf{e}$. The algebra $\mathcal{A}_n$ is equipped with the max-norm $\|\mathbf{a}\|_\infty := \max_k |a|_k$. The analog of (5) reads

$$\|T^{\mathrm{t}}\mathbf{a}\|_\infty \leq \|\mathbf{a}\|_\infty, \tag{10}$$

for all $\mathbf{a} \in \mathbb{R}^n$. Hence the dual channel T^{t} is a contraction in the max-norm.

3. – Quantum states and quantum channels

In the algebraic approach to quantum theory one considers a unital $\mathbb{C}^*$-algebra [9-11] $\mathfrak{U}$ and quantum states correspond to positive unital functionals $\omega : \mathfrak{U} \to \mathbb{C}$, that is, $\omega(aa^*) \geq 0$ for all $a \in \mathfrak{U}$ and $\omega(e) = 1$, where e denotes a unit element in $\mathfrak{U}$. Self-adjoint part of $\mathfrak{U}$ serves as an algebra of observables and standard Gelfand-Naimark-Segal (GNS) construction enables one to reconstruct a Hilbert space given a state ω [12, 13, 11].

Consider now a linear map $\Phi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$, where $\mathcal{H}$ is an arbitrary (in general infinite dimensional) Hilbert space and as usual $\mathfrak{B}(\mathcal{H})$ denotes a linear space of bounded linear operators in $\mathcal{H}$. A map Φ is called positive [9, 10] if $\Phi(aa^*) \geq 0$ for all $a \in \mathfrak{U}$. Φ is unital if $\Phi(e) = \mathbb{I}_{\mathcal{H}}$. It is clear that unital positive map Φ provide a natural generalization of a state. Let $M_k(\mathbb{C})$ be an algebra of $k \times k$ complex matrices and let id_k denote an identity map in $M_k(\mathbb{C})$, that is, $\mathrm{id}_k(X) = X$ for any $X \in M_k(\mathbb{C})$. Finally, let us introduce a linear map

$$\Phi_k := \mathrm{id}_k \otimes \Phi : M_k(\mathbb{C}) \otimes \mathfrak{U} \ \to\ M_k(\mathbb{C}) \otimes \mathfrak{U}\ , \quad k = 1, 2, \ldots\ . \tag{11}$$

A map Φ is k-positive iff Φ_k is positive and Φ is completely positive (CP) iff it is k-positive for all $k = 1, 2, \ldots$. A celebrated GNS construction is generalized by the following:

Theorem 1 (Stinespring dilation theorem [14]*). A map Φ is CP if and only if there exist a Hilbert space $\mathcal{K}$ and the $*$-homomorpsim*

$$\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{K}), \tag{12}$$

such that for each $a \in \mathfrak{A}$ *one has*

$$\Phi(a) = V\pi(a)V^\dagger, \tag{13}$$

with V *being a bounded linear operator* $V : \mathcal{K} \to \mathcal{H}$ *satisfying* $\|\Phi(\mathbb{I}_\mathcal{K})\| = \|V\|^2$.

The triple $(\pi, V, \mathcal{K})$ is usually called a Stinespring representation of Φ.

In this paper we restrict ourselves to finite-dimensional situation where both $\mathfrak{U}$ and $\mathcal{H}$ are finite dimensional and $\mathfrak{U} = \mathfrak{B}(\mathcal{H})$. In this case one has $\mathfrak{B}(\mathcal{H}) = M_n(\mathbb{C})$, where $n = \dim \mathcal{H}$. Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. Interestingly, one has the following characterization:

Proposition 1 (Choi [15]*). If* $\dim \mathcal{H} = n$*, then* Φ *is CP if and only if* Φ *is* n*-positive.*

Denoting by $\mathcal{P}_k$ a convex set of k-positive maps one has the following chain of inclusions:

$$\mathrm{CP} \equiv \mathcal{P}_n \subset \ldots \subset \mathcal{P}_2 \subset \mathcal{P}_1 \equiv \text{Positive maps}. \tag{14}$$

If $\Phi_1 \in \mathcal{P}_k$ and $\Phi_2 \in \mathcal{P}_l$ then both compositions $\Phi_1 \circ \Phi_2$ and $\Phi_2 \circ \Phi_1$ belong to $\mathcal{P}_{k\wedge l}$, where $k \wedge l = \min\{k, l\}$. In particular if Λ is CP and $\Phi \in \mathcal{P}_k$ with $k < n$, then $\Lambda \circ \Phi$ is in general only k-positive and hence not CP. Note, however, that if both Φ_1 and Φ_2 are CP then $\Phi_1 \circ \Phi_2$ and $\Phi_2 \circ \Phi_1$ are CP as well. Hence CP maps define a subalgebra in the algebra of linear maps in $M_n(\mathbb{C})$. In the finite-dimensional case the Stinespring theorem reduces to the following:

Proposition 2 (see [9, 10, 15, 16]*). A map* Φ *is CP if and only if*

$$\Phi(X) = \sum_\alpha K_\alpha X\, K_\alpha^\dagger, \tag{15}$$

for $X \in M_n(\mathbb{C})$.

Formula (15) is usually called a Kraus representation of Φ and K_α are called Kraus operators. Actually, the above formula appeared already in the Sudarshan *et al.* paper [17]. It should be stressed that a Kraus representation is highly non-unique.

Let $\{e_1, \ldots, e_n\}$ be a fixed orthonormal basis in $\mathcal{H}$ and let $|\psi_n^+\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^n e_k \otimes e_k$ denote maximally entangled state in $\mathcal{H} \otimes \mathcal{H}$. Moreover, let $P_n^+ = |\psi_n^+\rangle\langle\psi_n^+|$ denote the corresponding rank-1 projector.

Proposition 3 (Choi [15]*).* Φ *is CP if and only if* $(\mathrm{id}_n \otimes \Phi)(P_n^+) \geq 0$.

This simple characterization gives rise to the following Kraus representation of Φ: assuming that Φ is CP one has $(\mathrm{id}_n \otimes \Phi)(P_n^+) \geq 0$ and hence the corresponding spectral representation reads

$$(\mathrm{id}_n \otimes \Phi)(P_n^+) = \sum_{\alpha=1}^{n^2} x_\alpha P_\alpha, \tag{16}$$

with $x_\alpha \geq 0$, and $P_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$. Note that $|\psi_\alpha\rangle \in \mathcal{H}\otimes\mathcal{H}$ and hence

$$|\psi_\alpha\rangle = \sum_{k,l=1}^{n} \Psi_{kl}^{(\alpha)}\, e_k \otimes e_l,$$

where $\Psi_{kl}^{(\alpha)}$ are complex coefficients. Let us introduce $F_\alpha \in M_n(\mathbb{C})$ defined as follows: $F_\alpha e_k = \sum_{l=1}^n \Psi_{kl}^{(\alpha)}\, e_l$. One arrives at the following representation:

$$P_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha| = \sum_{k,l=1}^{n} e_{kl} \otimes F_\alpha e_{kl} F_\alpha^\dagger, \tag{17}$$

where $e_{kl} := |e_k\rangle\langle e_l|$ denote the matrix units in $M_n(\mathbb{C})$. Finally, one obtains

$$(\mathrm{id}_n \otimes \Phi)(P_n^+) = \sum_{k,l=1}^{n} e_{kl} \otimes \sum_{\alpha=1}^{n^2} x_\alpha F_\alpha e_{kl} F_\alpha^\dagger, \tag{18}$$

and recalling that $P_n^+ = \frac{1}{n}\sum_{k,l=1}^n e_{kl}\otimes e_{kl}$, one has $\Phi(e_{kl}) = \sum_\alpha K_\alpha e_{kl}\, K_\alpha^\dagger$, where we introduced $K_\alpha = \sqrt{n x_\alpha} F_\alpha$. Taking into account that e_{kl} provide an orthonormal basis one ends up with formula (15).

Let $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a linear map. It is called trace preserving iff $\mathrm{tr}[\Phi(X)] = \mathrm{tr}\, X$ for all $X \in M_n(\mathbb{C})$. A completely positive trace preserving map (CPTP) is called a quantum channel. Note, that if Φ is CPTP then its Kraus representation (15) satisfies

$$\sum_\alpha K_\alpha^\dagger\, K_\alpha = \mathbb{I}_{\mathcal{H}}. \tag{19}$$

Remark 1. Note that fixing an orthonormal basis $\{e_k\}$ in $\mathcal{H}$ and defining $P_k = |e_k\rangle\langle e_k|$ one easily shows that if Φ is a quantum channel, then the following $n \times n$ matrix

$$T_{ij} = \mathrm{Tr}(P_i \Phi(P_j)) \tag{20}$$

is stochastic, i.e. it defines a classical channel.

Theorem 2. Any quantum channel Φ may be constructed as follows:

$$\Phi(\rho) = \mathrm{tr}_E\big[U(\rho\otimes\omega_E)U^\dagger\big], \tag{21}$$

where U is a unitary operator in $\mathcal{H}\otimes\mathcal{H}_E$ and ω_E is a density operator in $\mathcal{H}_E$.

One usually interprets $\mathcal{H}_E$ as a Hilbert space of an environment and ω_E as a fixed state on an environment. Let $\omega_E e_k = \lambda_k e_k$, with $\lambda_k \geq 0$. Moreover, let $U = \sum_{k,l} U_{kl} \otimes e_{kl}$. Formula (21) implies

$$\begin{aligned}\Phi(\rho) &= \sum_{m,n}\sum_{i,j}\sum_k \lambda_k\, \mathrm{tr}_E\big[(U_{ij}\otimes e_{ij})(\rho\otimes e_{kk})(U_{mn}^\dagger\otimes e_{nm})\big]\\ &= \sum_{m,n}\sum_{i,j}\sum_k \lambda_k\, \mathrm{tr}[e_{ij}e_{kk}e_{nm}]\, U_{ij}\rho U_{mn}^\dagger.\end{aligned}$$

Using $\mathrm{tr}[e_{ij}e_{kk}e_{nm}] = \delta_{im}\delta_{jk}\delta_{kn}$ and introducing $K_\alpha := K_{mn} = \sqrt{\lambda_n}\,U_{mn}$ one arrives at the Kraus representation $\Phi(\rho) = \sum_\alpha K_\alpha \rho K_\alpha^\dagger$ which proves that Φ defined *via* formula (21) is completely positive. One easily proves that Φ is also trace preserving and hence defines a quantum channel. To show that any quantum channel may be represented *via* formula (21) one uses the Stinespring dilation theorem (see *e.g.* [5]).

For any linear map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ one introduces a dual map $\Phi^* : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by

$$\mathrm{tr}[A\Phi(X)] = \mathrm{tr}[\Phi^*(A)X], \tag{22}$$

for all $A, X \in M_n(\mathbb{C})$. Note that Φ is trace preserving if and only if Φ^* is unital, that is $\Phi^*(\mathbb{I}) = \mathbb{I}$. One may consider Φ and Φ^* as Schrödinger and Heisenberg representation, respectively. The natural arena in the Schrödinger picture is a Banach space $M_n(\mathbb{C})$ equipped with the trace norm $\|X\|_1$. In the Heisenberg picture one deals with a $\mathbb{C}^*$-algebra $M_n(\mathbb{C})$ equipped with the operator norm $\|A\|$. The basic property of a quantum channel Φ and its dual Φ^* is summarized in

Proposition 4 (see [9, 10]*). Φ and Φ^* are contractions, that is,*

$$\|\Phi(X)\|_1 \leq \|X\|_1, \qquad \|\Phi^*(X)\| \leq \|X\|, \tag{23}$$

for any $X \in M_n(\mathbb{C})$.

The trace norm defines a natural distance between quantum states represented by density operators: given two density operators ρ and σ one defines

$$D[\rho, \sigma] = \frac{1}{2}\|\rho - \sigma\|_1. \tag{24}$$

The quantity $D[\rho, \sigma]$ is usually interpreted as a measure of distinguishability of the quantum states ρ and σ. One has

$$0 \leq D[\rho, \sigma] \leq 1, \tag{25}$$

and $D[\rho, \sigma] = 0$ iff ρ and σ are perfectly distinguishable, that is, they are orthogonally supported $\mathrm{tr}(\rho\sigma) = 0$, and $D[\rho, \sigma] = 0$ iff $\rho = \sigma$. Proposition 4 implies the following:

Corollary 1. If Φ is a quantum channel, then

$$D[\Phi(\rho), \Phi(\sigma)] \leq D[\rho, \sigma], \tag{26}$$

that is, under the action of Φ the distinguishability never increases.

Given two density operators ρ and σ one defines Uhlmann fidelity [18]

$$F[\rho, \sigma] = \left(\mathrm{tr}\left[\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}\right]\right)^2. \tag{27}$$

Equivalently, one has $F[\rho, \sigma] = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$ which shows that $F[\rho, \sigma] = F[\sigma, \rho]$. One proves the following relation between these characteristics:

$$1 - F[\rho, \sigma] \leq D[\rho, \sigma] \leq \sqrt{1 - F[\rho, \sigma]^2}. \tag{28}$$

Proposition 5. If Φ *is a quantum channel, then*

$$F[\Phi(\rho),\Phi(\sigma)] \geq F[\rho,\sigma], \tag{29}$$

that is, under the action of Φ *the fidelity never decreases.*

Finally, let us recall the definition of relative entropy

$$S(\rho\,\|\,\sigma) = \mathrm{Tr}(\rho[\log\rho - \log\sigma]), \tag{30}$$

(one assumes that $S(\rho\,\|\,\sigma) = \infty$ when supports of ρ and σ do not satisfy $\mathrm{supp}\,\rho \subset \mathrm{supp}\,\sigma$).

Proposition 6. If Φ *is a quantum channel, then*

$$S(\Phi(\rho)\,\|\,\Phi(\sigma)) \leq S(\rho\,\|\,\sigma), \tag{31}$$

that is, under the action of Φ *the relative entropy never increases.*

It should be stressed that contrary to the "common wisdom" it is not always true that $S(\Phi(\rho)) \geq S(\rho)$. However, if Φ is a unital channel, then

$$S(\Phi(\rho)\,\|\,\Phi(\mathbb{I}/n)) = S(\Phi(\rho)\,\|\,\mathbb{I}/n) = \log n - S(\Phi(\rho)),$$

and hence one arrives at the following:

Corollary 2. If Φ *is a unital channel, then* $S(\Phi(\rho)) \geq S(\rho)$.

Finally, let us recall that if Φ is a quantum channel then its dual Φ^* being CP unital map satisfies celebrated Kadison inequality

$$\Phi^*(AA^\dagger) \geq \Phi^*(A)\Phi^*(A^\dagger), \tag{32}$$

for any $A \in M_n(\mathbb{C})$.

4. – Classical Markovian semigroup

Having defined a space of classical states and legitimate classical operations (classical channels) mapping states into states, let us consider a classical stochastic evolution. Such evolution is uniqely described by a family of channels T_t with $t \geq 0$ such that $T_0 = \mathbb{I}_n$. One calls T_t a *classical dynamical map*. If $\mathbf{p} \in \Sigma_n$ is an initial state then $\mathbf{p}_t := T_t\mathbf{p}$ defines a trajectory in Σ_n starting at $\mathbf{p}$. Suppose that T_t satisfies a linear equation (so called classical master equation [1])

$$\frac{\mathrm{d}}{\mathrm{d}t}T_t = MT_t, \qquad T_0 = \mathbb{I}_n, \tag{33}$$

where the $n \times n$ matrix M is called a generator of classical evolution. The formal solution $T_t = e^{Mt}$ guarantees that T_t satisfies the following semigroup property:

$$T_t \cdot T_u = T_{t+u}, \tag{34}$$

for all $t, u \geq 0$. A natural question is what are the properties of M such that e^{Mt} provides a classical dynamical map. The answer is given by the following:

Theorem 3 (see [1]*). A classical master equation* (33) *provides a legitimate solution* T_t *iff* M *satisfies the following conditions:*

– $M_{ij} \geq 0$ *for* $i \neq j$,

– $\sum_{i=1}^{n} M_{ij} = 0$ *for all* $j = 1, \ldots, n$.

The above conditions are usually called *Kolmogorov conditions*. Originally the master equation (33) was written in terms of $\mathbf{p}_t$ as the following Pauli rate equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_i(t) = \sum_{j=1}^{n} \Big[\pi_{ij} p_j(t) - \pi_{ji} p_i(t) \Big], \tag{35}$$

where $\pi_{ij} \geq 0$ $(i \neq j)$ describes probability rates for the transition from "j" to "i" (note that a term with $i = j$ does not appear in the summation). One rewrites the above equation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_i(t) = \sum_{j=1}^{n} M_{ij} p_j(t), \tag{36}$$

with

$$M_{ij} = \pi_{ij} - \delta_{ij} \sum_{k=1}^{n} \pi_{kj}. \tag{37}$$

It is clear that M_{ij} satisfies Kolmogorov conditions iff M_{ij} satisfies (37) with $\pi_{ij} \geq 0$ $(i \neq j)$.

In order to compare classical and quantum dynamics, let us reformulate the structure of M as follows: since a term with $i = j$ is irrelevant we may assume that $\pi_{ij} \geq 0$ for all $i, j = 1, \ldots, n$. In this case $\pi : \mathbb{R}^n \to \mathbb{R}^n$ represents a positive map and hence the formula for M may be rewritten as follows:

$$M\mathbf{p} = \pi \mathbf{p} - (\pi^{\mathrm{t}} \mathbf{e}) \circ \mathbf{p}, \tag{38}$$

where $\mathbf{a} \circ \mathbf{b}$ is a commutative product $(\mathbf{a} \circ \mathbf{b})_k = a_k b_k$. Introducing $\{\mathbf{a}, \mathbf{b}\} = \mathbf{a} \circ \mathbf{b} + \mathbf{b} \circ \mathbf{a}$ one finds

$$M\mathbf{p} = \pi \mathbf{p} - \frac{1}{2} \{\pi^{\mathrm{t}} \mathbf{e}, \mathbf{p}\}. \tag{39}$$

Definition 1. A linear operator $X : \mathbb{R}^n \to \mathbb{R}^n$ *is dissipative if*

$$X(\mathbf{a} \circ \mathbf{a}) \geq 2\, \mathbf{a} \circ X\mathbf{a} = \{\mathbf{a}, X\mathbf{a}\}, \tag{40}$$

for all $\mathbf{a} \in \mathbb{R}^n$.

One proves the following:

Proposition 7. M satisfies Kolmogorov conditions iff its dual $M^{\rm t}$ is dissipative.

It follows from the commutative version of Kadison inequality [10]: if T is stochastic matrix then

$$T^{\rm t}(\mathbf{a} \circ \mathbf{a}) \geq T^{\rm t}\mathbf{a} \circ T^{\rm t}\mathbf{a}, \tag{41}$$

for all $\mathbf{a} \in \mathbb{R}^n$.

Example 1. Let us consider 2-level system with

$$M = \begin{pmatrix} -\gamma_2 & \gamma_1 \\ \gamma_2 & -\gamma_1 \end{pmatrix}, \tag{42}$$

with $\gamma_1, \gamma_2 \geq 0$. Evidently M satisfies Kolmogorov conditions and it is the most general form of M for 2-level system. One easily finds the following equations for the probability vector $\mathbf{p}_t = (p_1(t), p_2(t))^{\rm t}$:

$$\dot{p}_1(t) = -\gamma_2 p_1(t) + \gamma_1 p_1(t), \tag{43}$$

$$\dot{p}_2(t) = \gamma_2 p_1(t) - \gamma_1 p_1(t), \tag{44}$$

and the corresponding solution reads

$$p_1(t) = p_1(0)\, e^{-(\gamma_1+\gamma_2)t} + p_1^*\Big[1 - e^{-(\gamma_1+\gamma_1)t}\Big], \tag{45}$$

$$p_2(t) = p_2(0)\, e^{-(\gamma_1+\gamma_2)t} + p_2^*\Big[1 - e^{-(\gamma_2+\gamma_2)t}\Big], \tag{46}$$

where we introduced

$$p_1^* = \frac{\gamma_1}{\gamma_1+\gamma_2}, \qquad p_2^* = \frac{\gamma_2}{\gamma_1+\gamma_2}. \tag{47}$$

It is clear that $\mathbf{p}^* = (p_1^*, p_2^*)^{\rm t}$ defines an equilibrium state which becomes maximally mixed if $\gamma_1 = \gamma_2$.

Remark 2. Note that $T_t = e^{Mt}$ is an invertible map. One obviously has $T_t^{-1} = e^{-Mt} = T_{-t}$. However, the inverse is not a stochastic matrix which means that the dynamics is not reversible. Consider for example M given by (42) with $\gamma_1 = \gamma_2 = \gamma > 0$. One easily finds

$$T_t = e^{-\gamma t}\begin{pmatrix} \cosh\gamma t & \sinh\gamma t \\ \sinh\gamma t & \cosh\gamma t \end{pmatrix}, \tag{48}$$

which is evidently stochastic for all $t \geq 0$. However it fails to be a stochastic for $t < 0$.

5. – Quantum Markovian semigroup

The quantum analog of classical master equation (33) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_t = L\Lambda_t, \quad \Lambda_0 = \mathrm{id}, \tag{49}$$

with time-independent generator $L : M_n(\mathbb{C}) \to M_n(\mathbb{C})$. The formal solution $\Lambda_t = e^{Lt}$ guaranties that Λ_t satisfies the following semigroup property:

$$\Lambda_t \, \Lambda_u = \Lambda_{t+u}, \tag{50}$$

for all $t, u \geq 0$. A family Λ_t of quantum channels with $\Lambda_0 = \mathrm{id}_n$ is called a *(quantum) dynamical map* [17]. A natural question is what are the properties of a generator L such that e^{Lt} provides a quantum dynamical map. The answer is given by the following:

Theorem 4 (see [19, 20]*). The quantum master equation* (49) *provides a legitimate dynamical map* Λ_t *if and only if* L *has the following form*

$$L(\rho) = -i[H,\rho] + \frac{1}{2}\sum_\alpha \big([V_\alpha \rho, V_\alpha^\dagger] + [V_\alpha, \rho V_\alpha^\dagger]\big), \tag{51}$$

where $H, V_\alpha \in M_n(\mathbb{C})$ *with* $H^\dagger = H$.

Remark 3. The above Theorem may be generalized for infinite-dimensional Hilbert space $\mathcal{H}$ [20]*. In this case one assumes that* L *is a bounded operator and* $H, V_\alpha \in \mathfrak{B}(\mathcal{H})$.

In what follows we shall call such L a Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) generator. Defining a CP map

$$\Phi(\rho) = \sum_\alpha V_\alpha \rho V_\alpha^\dagger, \tag{52}$$

one may rewrite (51) as follows:

$$L(\rho) = -i[H,\rho] + \Phi(\rho) - \frac{1}{2}\{\Phi^*(\mathbb{I}), \rho\}, \tag{53}$$

where $\{a, b\} = ab + ba$ denotes anticommutator. It is, therefore, clear that formula (53) provides a quantum (non-commutative) generalization of (39).

Remark 4. Note that fixing an orthonormal basis $\{e_k\}$ *in* $\mathcal{H}$ *one easily shows that if* L *is a GKSL generator, then the following* $n \times n$ *matrix*

$$M_{ij} = \mathrm{Tr}(P_i L(P_j)), \tag{54}$$

satisfies Kolmogorov conditions.

Interestingly, one proves

Proposition 8 (see [19]*). L is a GKSL generator if and only if the following $n^2 \times n^2$ matrix*

$$\mathbb{M}_{\alpha\beta} = \mathrm{Tr}\big[\mathbb{P}_\alpha(\mathrm{id}\otimes L)(\mathbb{P}_\beta)\big], \tag{55}$$

satisfies Kolmogorov conditions for each set of orthonormal projectors $\mathbb{P}_\alpha$ in $\mathcal{H}\otimes\mathcal{H}$, i.e. $\mathbb{P}_\alpha\mathbb{P}_\beta = \delta_{\alpha\beta}\mathbb{P}_\alpha$ and $\sum_\alpha \mathbb{P}_\alpha = \mathbb{I}\otimes\mathbb{I}$.

The evolution in the Heisenberg picture is described by the dual map Λ_t^* which satisfies

$$\frac{d}{dt}\Lambda_t^* = L^*\Lambda_t^*, \quad \Lambda_0^* = \mathrm{id}. \tag{56}$$

Formula (53) implies

$$L^*(A) = i[H,A] + \Phi^*(A) - \frac{1}{2}\{\Phi^*(\mathbb{I}), A\}, \tag{57}$$

for $A \in M_n(\mathbb{C})$.

Definition 1. A linear map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called dissipative if

$$\phi(AA^\dagger) \geq \phi(A)\,A^\dagger + A\,\phi(A^\dagger), \tag{58}$$

for all $A \in M_n(\mathbb{C})$. It is called completely dissipative if $\mathrm{id}\otimes\phi$ *is dissipative.*

Proposition 9 (see [20]*). L is a GKSL generator if and only if its dual L^* is completely dissipative.*

The proof [20] easily follows from the Kadison inequality [10]: if Φ is 2-positive and trace-preserving then

$$\Phi^*(AA^\dagger) \geq \Phi^*(A)\,\Phi^*(A^\dagger), \tag{59}$$

for all $A \in M_n(\mathbb{C})$.

Example 2. Let us consider a qubit generator defined by $H = \frac{\omega}{2}\sigma_z$ and the following CP map

$$\Phi(\rho) = \gamma_1\sigma_+\rho\,\sigma_+^\dagger + \gamma_2\sigma_-\rho\,\sigma_-^\dagger + \gamma\sigma_z\rho\,\sigma_z, \tag{60}$$

where $\sigma_+ = |2\rangle\langle 1|$ and $\sigma_- = |1\rangle\langle 2| = \sigma_+^\dagger$ are standard qubit raising and lowering operators. The corresponding generator reads $L(\rho) = -i[H,\rho] + L_D(\rho)$ with the dissipative part

$$L_D(\rho) = \frac{\gamma_1}{2}\left([\sigma_+,\rho\sigma_-] + [\sigma_+\rho,\sigma_-]\right) + \frac{\gamma_2}{2}\left([\sigma_-,\rho\sigma_+] + [\sigma_-\rho,\sigma_+]\right) + \frac{\gamma}{2}\left(\sigma_z\rho\sigma_z - \rho\right). \tag{61}$$

To solve the master equation $\dot{\rho}_t = L\rho_t$ let us parameterize ρ_t as follows:

$$\rho_t = p_1(t)P_1 + p_2(t)P_2 + \alpha(t)\sigma_+ + \overline{\alpha(t)}\sigma_-, \tag{62}$$

with $P_k = |k\rangle\langle k|$. *Using the following relations*

$$
\begin{aligned}
L(P_1) &= \gamma_1(P_2 - P_1) = -\gamma_1\,\sigma_3,\\
L(P_2) &= \gamma_2(P_1 - P_2) = \gamma_2\,\sigma_3,\\
L(\sigma_+) &= (i\omega - \Gamma)\,\sigma_+\ ,\\
L(\sigma_-) &= (-i\omega - \Gamma)\,\sigma_-\ ,
\end{aligned}
$$

where

$$\Gamma = \frac{\gamma_1+\gamma_2}{2} + \gamma,$$

one finds the following Pauli master equations equations for the probability distribution $(p_1(t), p_2(t))$*:*

$$\dot{p}_1(t) = -\gamma_1\,p_1(t) + \gamma_2\,p_2(t), \tag{63}$$

$$\dot{p}_2(t) = \ \gamma_1\,p_1(t) - \gamma_2\,p_2(t), \tag{64}$$

together with $\alpha(t) = e^{(i\omega-\Gamma)t}\alpha(0)$. *Interestingly, equations for* $(p_1(t), p_2(t))$ *are the same as in Example 1. Hence, we have purely classical evolution of probability vector* $(p_1(t), p_2(t))$ *on the diagonal of* ρ_t *and very simple evolution of the off-diagonal element* $\alpha(t)$. *Note, that asymptotically one obtains completely decohered density operator*

$$\rho_t \quad \longrightarrow \quad \begin{pmatrix} p_1^* & 0 \\ 0 & p_2^* \end{pmatrix},$$

where p_1^* *and* p_2^* *are defined in* (47).

6. – Local master equations

Consider now a master equation with time-dependent generator

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_t = L_t\Lambda_t, \qquad \Lambda_0 = \mathrm{id}. \tag{65}$$

The formal solution has the following form:

$$\Lambda_t = \mathrm{T}\exp\left(\int_0^t L_u \mathrm{d}u\right), \tag{66}$$

where T denotes the chronological product. The above formula has rather a formal meaning. The T-product exponential is defined by the following Dyson series:

$$\mathrm{T}\exp\left(\int_0^t L_u \mathrm{d}u\right) = \mathrm{id}_n + \int_0^t \mathrm{d}t_1 L_{t_1} + \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1\, L_{t_2}\, L_{t_1} + \dots, \tag{67}$$

which is in general untractable. One of the mathematical problems in this approach is to formulate necessary and sufficient conditions for a local generator L_t which lead to legitimate dynamical map *via* formula (66). This problem is still open. Interestingly, one meets a similar problem for classical stochastic systems described by

$$\frac{\mathrm{d}}{\mathrm{d}t}T_t = M_t T_t, \qquad T_0 = \mathbb{I}, \tag{68}$$

with formal solution given by

$$T_t = \mathrm{T}\exp\left(\int_0^t M_u \mathrm{d}u\right). \tag{69}$$

Again we do not know conditions for M_t that lead to legitimate stochastic dynamics T_t. Surprisingly a classical problem is as hard as the quantum one.

In what follows we analyze two important classes of local generators which provide legitimate dynamical maps T_t and Λ_t in the classical and quantum case, respectively.

We call a dynamical map Λ_t commutative if $[\Lambda_t, \Lambda_u] = 0$ for all $t, u \geq 0$. It is easy to show that commutativity of Λ_t is equivalent to commutativity of the local generator

$$[L_t, L_u] = 0, \tag{70}$$

for any $t, u \geq 0$. Note that in this case the formula (66) considerably simplifies: the "T" product drops out and the solution is fully controlled by the integral $\int_0^t L_u \mathrm{d}u$. One has, therefore, the following:

Theorem 5. If L_t satisfies (70)*, then L_t is a legitimate generator if and only if $\int_0^t L_\tau \mathrm{d}\tau$ is a GKSL generator for all $t \geq 0$.*

A typical example of commutative dynamics is provided by

$$L_t = \omega(t)L_0 + \alpha_1(t)L_1 + \ldots \alpha_N(t)L_N, \tag{71}$$

where $[L_i, L_j] = 0$ with $L_0(\rho) = -i[H, \rho]$, and for $i > 0$ the generators L_i are purely dissipative, that is, $L_i(\rho) = \Phi_i(\rho) - \frac{1}{2}\{\Phi_i^*(\mathbb{I}), \rho\}$. One has for the corresponding dynamical map

$$\Lambda_t = e^{\Omega(t)L_0} \cdot e^{A_1(t)L_1} \cdot \ldots \cdot e^{A_N(t)L_N}, \tag{72}$$

with $\Omega(t) = \int_0^t \omega(u)\mathrm{d}u$ and $A_i(t) = \int_0^t \alpha_i(u)\mathrm{d}u$. It is clear that Λ_t is CP iff $A_i(t) \geq 0$ for all $i = 1, \ldots, N$.

We call a dynamical map Λ_t *divisible* if for any $t \geq s \geq 0$ one has the following decomposition:

$$\Lambda_t = V_{t,s}\,\Lambda_s, \tag{73}$$

with completely positive propagator $V_{t,s}$. Note, that $V_{t,s}$ satisfies the inhomogeneous composition law

$$V_{t,s}V_{s,u} = V_{t,u}, \tag{74}$$

for any $t \geq s \geq u$. In this paper, following [21], we accept the following definition of Markovian evolution: *a dynamical map* Λ_t *corresponds to Markovian evolution if and only if it is divisible*. Interestingly, the property of being Markovian (or divisible) is fully characterized in terms of the local generator L_t. Note, that if Λ_t satisfies (65) then $V_{t,s}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{t,s} = L_t V_{t,s}, \quad V_{s,s} = \mathbb{1}, \tag{75}$$

and the corresponding solution reads $V_{t,s} = \mathrm{T}\ \exp(\int_s^t L_u \mathrm{d}u)$. One proves [22] the following:

Theorem 6. The map Λ_t *is divisible if and only if* L_t *is defined by* (51) *for all* t.

Example 3. Consider a qubit dynamics governed by

$$L_t(\rho) = \frac{1}{2}\gamma(t)(\sigma_z \rho \sigma_z - \rho), \tag{76}$$

and let

$$\Gamma(t) = \int_0^t \gamma(\tau)\mathrm{d}\tau.$$

It is clear that L_t *belongs to a commutative class. One finds*

1. L_t *is a legitimate generator iff* $\Gamma(t) \geq 0$,
2. L_t *generates Markovian evolution iff* $\gamma(t) \geq 0$,
3. L_t *generates Markovian semigroup iff* $\gamma(t) = \mathrm{const} > 0$.

7. – Markovian vs. non-Markovian dynamics

Consider a quantum evolution represented by a dynamical map Λ_t. We call it Markovian if Λ_t is a divisible map, that is, the corresponding local in time generator L_t is GKSL for all $t \geq 0$. It is, therefore, clear that divisible maps provide direct generalization of Markovian semigroups. Using general properties of quantum channels (see sect. **3**) we can easily formulate several simple necessary conditions for Markovian evolution.

Corollary 1 implies that

$$D[\Lambda_t(\rho), \Lambda_t(\sigma)] \leq D[\rho, \sigma], \tag{77}$$

for any pair of initial states ρ and σ.

Proposition 10. If Λ_t *is a divisible map, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} D[\Lambda_t(\rho), \Lambda_t(\sigma)] \leq 0, \tag{78}$$

for any pair of initial states ρ *and* σ.

Interestingly, authors of [23] consider the above inequality as a definition of Markovian evolution.

Example 4. Consider the dynamics governed by the local in time generator

$$L_t\rho = \gamma(t)\,(\omega_t\,\mathrm{tr}\,\rho - \rho)\,, \tag{79}$$

where ω_t *is a family of Hermitian operators satisfying* $\mathrm{tr}\,\omega_t = 1$*. The above generator gives rise to Markovian evolution iff* L_t *has GKLS form for all* $t \geq 0$*, that is, iff* $\gamma(t) \geq 0$ *and* ω_t *defines a legitimate state, i.e.* $\omega_t \geq 0$*. The corresponding solution of the Master equation* $\dot{\rho}_t = L_t\rho_t$ *with an initial condition* ρ *reads as follows:*

$$\rho_t = e^{-\Gamma(t)}\rho + [1 - e^{-\Gamma(t)}]\Omega_t\,\mathrm{tr}\rho, \tag{80}$$

where

$$\Omega_t = \frac{1}{e^{\Gamma(t)} - 1}\int_0^t \gamma(\tau)e^{\Gamma(\tau)}\,\omega_\tau \mathrm{d}\tau\ . \tag{81}$$

It is therefore clear that L_t *generates a legitimate quantum evolution iff* $\Gamma(t) \geq 0$ *and* $\Omega_t \geq 0$*, that is,* Ω_t *defines a legitimate state. In particular, if* $\omega_t = \omega$ *is time independent, then* $\Omega_t = \omega$ *and the solution simplifies to a convex combination of the initial state* ρ *and the asymptotic invariant state* ω

$$\rho_t = e^{-\Gamma(t)}\rho + [1 - e^{-\Gamma(t)}]\,\omega.$$

One easily shows that the evolution is Markovian iff $\gamma(t) \geq 0$ *and* ω_t *is a legitimate density operator (that is,* $\omega_t \geq 0$*). Consider now the condition* (78)*. One has* $\rho_t - \sigma_t = e^{-\Gamma(t)}(\rho - \sigma)$ *and hence*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|\rho_t - \sigma_t\|_1 = -\gamma(t)\,e^{-\Gamma(t)}\,\|\rho - \sigma\|_1 \leq 0,$$

implies only $\gamma(t) \geq 0$ *but says nothing about positivity of* ω_t*. It shows that any* ω_t *which gives rise to* $\Omega_t \geq 0$ *leads to the evolution satisfying condition* (78) *but only* $\omega_t \geq 0$ *gives rise to Markovian dynamics. Hence, we may have non-Markovian dynamics (governed by nondivisible dynamical map) which satisfies condition* (78) *for all* $t \geq 0$*.*

One derives similar monotonicity conditions for fidelity and relative entropy using Propositions 5 and 6.

Proposition 11. If Λ_t *is a divisible map, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,F(\Lambda_t(\rho), \Lambda_t(\sigma)) \geq 0, \tag{82}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\,S(\Lambda_t(\rho)\|\Lambda_t(\sigma)) \leq 0, \tag{83}$$

for any pair of initial states ρ *and* σ*.*

Moreover

Proposition 12. If Λ_t *is a unital divisible map, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} S(\Lambda_t(\rho)) \geq 0, \tag{84}$$

for any initial state ρ.

It proves that for unital Markovian evolution the entropy monotonically increases.

Example 5. Consider once more pure decoherence of a qubit from Example 3. Note, that $L_t(\mathbb{I}) = 0$ *and hence the maximally mixed state is invariant. Therefore, Markovianity implies*

$$\frac{\mathrm{d}}{\mathrm{d}t} S(\rho_t) = -\dot{\lambda}_t^+ \log \frac{\lambda_t^+}{\lambda_t^-}, \tag{85}$$

where $\lambda_t^+ \geq \lambda_t^-$ *are eigenvalues of* ρ_t. *Hence* $\frac{\mathrm{d}}{\mathrm{d}t} S(\rho_t) \geq 0$ *if* $\dot{\lambda}_t^+ \leq 0$. *One easily finds*

$$\lambda_t^{\pm} = \frac{1}{2}\left(1 \pm \sqrt{(\rho_{11} - \rho_{22})^2 + |\rho_{12}|^2 e^{-2\Gamma(t)}}\right).$$

It is therefore clear that $S(\rho_t)$ *monotonically increases if and only if* $\dot{\Gamma}(t) = \gamma(t) \geq 0$.

For more information about quantum non-Markovian evolution the reader is referred to recent papers [24-30]. This topic is currently intensively studied with potential applications in various modern quantum technologies.

8. – Conclusions

We provided a basic introduction to the mathematical description of classical and quantum systems. The presentation includes classical and quantum stochastic states and classical and quantum channels. These concepts are used to describe classical and quantum stochastic evolution. We discuss both Markovian semigroups and go beyond the semigroup case. The presentation is concluded by a short analysis of Markovian and non-Markovian behavior.

* * *

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