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# Solving Singular Perturbation with One Boundary Layer Problem of Second Order ODE using The Method of Matched Asymptotic Expansion (MMAE)

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**Abstract**: Any physical problems when modeled into equations, may become linear or nonlinear equations with known and unknown boundaries. The exact solution for those equations is not easy to obtain. Hence, an analytical approximation solution in terms of asymptotic expansion is sought. This study seeks to solve a singular perturbation in second order ordinary differential equations. Solutions to several perturbed ordinary differential equations are obtained in terms of asymptotic expansion. The main focus of this study is to find an approximate analytical solution for perturbation problems (linear and nonlinear equations) that occur at one boundary layer using the classical method of matched asymptotic expansion (MMAE). Besides, the underlying concepts and principles of MMAE will also be clarified. Mathematica computer algebra system is used to perform the detail algebraic computations. The results of approximation analytical solution are illustrated by graphs using selected parameters which show the outer, inner and composite solutions separately. Hence, the exact solution and composite solution obtained using MMAE are compared by plotting the graph to show their accuracy. From the comparison, MMAE is one of the best methods to solve singular perturbation problems in second order ordinary differential equation since the results obtained are very close to the exact solution.

**Keywords**: MMAE, Outer Solution, Inner Solution, Composite Solution

## 1 Introduction

Many important phenomena in our real life can be modeled mathematically in terms of differential equations. Differential equation can take in many forms and could exhibit certain features such as variable/singularities coefficients, nonlinearities, complex boundary conditions or existence of small/large parameters which preclude exact analytical solutions. In order to obtain information about solutions to such equations, we are forced to resort to analytical approximations, numerical solutions or a combination of both.

First and foremost, the perturbation or asymptotic methods are among the approximation methods which have proven to be an invaluable analytical tool in applied mathematics. The term 'perturbation method' is referred to a method of obtaining an analytical approximation when some parameters of the problem are small. Both perturbation and numerical solutions complement each other. The analytic approximations can give an explicit dependence of the parameter involved rather than the isolated results at particular values from the numerical method. The main drawback of numerical solutions is they do not provide much insight into the physics of the problem. The checks between the two methods for satisfying agreement are essential in research. Another advantage of the perturbation solution is one can do the analysis that could reveal some deeper physical insight of the problem when taking the limit of the small parameters.

It has been discussed in many literatures that problems such as viscous fluid flow, solid mechanics, aerodynamics, celestial mechanics, wave propagation, quantum mechanics and nonlinear oscillations to name a few, give rise to the singular perturbation problems. With the great current interest in developing automated computer-algebra methods like Mathematica and the richness of behaviour of the governing equations have led to ever-increasing number of industrial applications of singular perturbation problems.

There is no single or best method to tackle singular perturbation problems, but one may exploit the small parameter given some experience and understanding of similar perturbation problems. In a nutshell, this work focuses on the technique of systematically constructing an analytical approximation of the solution to a perturbation problem. In this study, linear and nonlinear equations of singular perturbation that occur at one boundary is solved using the method of matched asymptotic expansion (MMAE).

## 2 Perturbation Theory

Perturbation method is the most popular method people used to find an approximate solution to the differential equations when the exact solution cannot be easily obtained. Perturbation theory is a large collection of analytical techniques for obtaining highly accurate solutions to mathematical problems that do not have a simple exact analytical solution.

Generally, numerical and perturbative methods are the two possible ways in solving a certain, difficult problem. The main advantages of perturbative methods are analytical analytical procedures, simple structures and quick formulations?. There are three fundamental steps in the perturbative solution of a hard problem. Firstly, introduce a small parameter $\varepsilon$, into the hard problem. Secondly, assume that the solution to the infinite class of hard problems has the form of a perturbation power series in $\varepsilon$. Lastly, set $\varepsilon = 0$ in the series to recover the solution to the original hard problem, Firdawati [1].

Regular and singular perturbations are two types of perturbation method. In the theory of singular perturbations however, a small perturbation cause a large impact, making it very intriguing and challenging to find the solution. Such singular perturbation models arise in many physics and engineering problems and are characterized by a rapid change of the solution which exhibits a non-uniform behavior in a thin region. The thin region is called boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in fluid and solid mechanics and transition points in quantum mechanics.

According to Firdawati and Mohamad [2], singular perturbation problems occur in an ordinary differential equation when a small parameter $\varepsilon$ multiplies to the highest derivative. The obtained solution failed to satisfy certain boundary layers conditions when the small parameter, $\varepsilon$ approaches to zero. In numerical calculation, the perturbation parameter leads to difficulties when a classical technique is used to solve such problems and convergence will not be uniformed. Besides, the solution to the problem varies rapidly in some parts and slowly in some other parts. There are many techniques being studied to handle the singular perturbation problems. One example of the techniques is the method of matched asymptotic expansion and multiple scales. The latest technique in solving this kind of problem as discussed by Dogan *et al*. [3] is called a differential transformation method. A typical problem in singular perturbation theory is described by a second order differential equation as

$$\varepsilon y'' + a(x)y' + b(x)y = 0\,,\ x \in [x_1, x_2],\ x_1 < 0\,,\ x_2 > 0\,,$$

where $0 < \varepsilon < 1$ is a small positive parameter, $a(x)$ and $b(x)$ are sufficient by smooth real-valued functions. This problem has been well discussed by many researchers.

## 3 Method of Matched Asymptotic Expansion (MMAE)

MMAE is a fundamental and dependable method that requires new coordinates/stretched points.
This method has been successfully applied in a physical problem. The most physical problem that involves singular perturbation basically is solved by using this method. In this method, the solution is

composed by two or more asymptotic expansion which must be linked together by matching procedures.
In solving singular perturbation problems, Mohan and Reddy [4] solved the problem by using approximate method where the original second order problem was replaced by an asymptotically equivalent first order problem and solved as an initial value problem in the inner region. A terminal boundary condition was then obtained from the solution of the inner region problem.

Qassim and Silva Freire [5] focused on the application of the boundary value technique problems. They explained a view to evaluating some of the features of the boundary value technique, particularly in comparison with coefficient matching techniques as exemplified by the method asymptotic expansion. The classical boundary layer theory and the technique of matched asymptotic expansions can be served as an adequate test for the accuracy of the boundary value technique. The simplicity of the boundary value technique was shown to be valid for partial differential equations (PDE) as well.

Other researchers who successfully used this method are MacGillivray [6] and Kelley [7]. They discussed a singular perturbation problem of Carrier and Pearson. Kadalbajoo and Patidar [8] on the other hand, looked at the numerical solution of singular perturbed two-point boundary value problems by spline in tension. Moreover, Neufeld and Wettlaufer [9] studied the pattern formation in mushy layers and Kadalbajoo *et al*. [10] studied the collection method using artificial viscosity for solving stiff singularly perturbed turning point problem having two boundary layers. Argatov and Mishuris [11] computed the solution in oscillator with small mass by using the method of matched asymptotic expansion. The singular perturbation theory is a niche area of research and has attracted many applied and pure mathematicians, due to the fact that the singular perturbation problem occurs in some interesting behaviour such as in resonance phenomena. This is clearly illustrated by Mckelvey and Bohac [12] who revisited the Ackerberg – O'Malley resonance problem; and MacGillivray [6] and Wong and Yang [13] who constructed the solution on the Ackerberg – O'Malley resonance.

The main procedure of this method is to define at which point the boundary layer occurs. At the outer layer, it is called an outer solution while other layers are known as inner solutions. In such problem of the form $y''+a(x)y'+b(x)y=0$ with boundary layer are $y(0)=1$ and $y(1)=0$, the layer could occur at $x=0$ because $a(x)>0$. The boundary layer occurs the other way round if $a(x)<0$, Hunter [12]. The linear or nonlinear problems that occur in one boundary are always discussed using MMAE but most linear problems are discussed when they occur in two boundary layers.

### *A Outer Solution*

According to O'Malley [14] in any perturbation problem involving a small positive parameter $\varepsilon$, it is natural to seek an outer solution in the form of

$$y(x,y)\approx\sum_{j=0}^{\infty}a_j(x)\beta_j(\varepsilon) \quad \text{as} \quad \varepsilon\to 0,$$

where $x$ ranges over some or usually bounded domain $D$ and $\beta_j(\varepsilon)$ is an asymptotic sequence as $\varepsilon\to 0$.

### *B Inner Solution*

In order to determine the inner solution, we have to know the stretching point where it is defined as $\tau=\frac{x-x_0}{\delta(\varepsilon)}$, while $x_0$ is an inner point and $\delta(\varepsilon)$ is the dominant balance. In terms of a stretched point, we seek an asymptotic solution of the form

$$y(x,\tau)\approx\sum_{j=0}^{\infty}b_j(\tau)\alpha_j(\varepsilon) \qquad \text{as} \quad \varepsilon\to 0,$$

where the sequence $\alpha_j(\varepsilon)$ is asymptotic as $\varepsilon \to 0$ is valid for values of $\tau$ in some inner region.

### *C Overlap*

Overlap means the term that satisfies both outer and inner solution for each order of terms. In the solution, we may or may not have overlap region. For example, if we have the inner solution for a boundary layer problem as in $Y(x,\tau) = e - e^{1-2\tau}$. Therefore, the overlap is $e$. If the inner solution is $Y(x,\tau) = e^{-\tau}$, no overlap region is obtained.

### *D Composite Solution*

Analytical approximation solutions obtained in solving a problem using MMAE is the composite solution. The composite solution is determined by adding the outer solution with the inner and subtracting with the overlap. Basically it is written as $y_c(x) = y_o(x) + y_i(x) - overlap$ where $y_c(x),\ y_o(x)$ and $y_i(x)$ are composite, outer and inner solutions respectively, Firdawati [1].

## 4 Analytical Approximation Solution using MMAE

### *A Linear Equation*

Consider the equation of $\varepsilon y'' + y' + y = 0$ with boundary conditions of $y(0) = \alpha$ and $y(1) = \beta$, as in our study. Assume that the boundary layer occurs at $x = 0$. From that we know the outer solution is at $x = 1$ and inner solution at $x = 0$ .Write the linear equation in straightforward expansion as

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + ...$$

(1)

Substitute the linear equation into equation (1) as

$$\varepsilon\left[y''_0(x) + \varepsilon y''_1(x) + O(\varepsilon^2)\right]$$
$$+\left[y'_0(x) + \varepsilon y'_1(x) + O(\varepsilon^2)\right]$$
$$+\left[y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)\right] = 0.$$

Therefore, from the above expansion we carried out

$$O(1):\quad y'_0(x) + y_0(x) = 0,$$

(2)

is subject to boundary condition $y_0(1) = \beta$ and

$$O(\varepsilon):\ y''_0(x) + y'_1(x) + y_1(x) = 0. \quad (3)$$

Solving equation (2) gives the general solution as

$$y_0(x) = Ae^{-x},$$

(4)

where $A$ is a constant. Substitute the boundary condition, $y(1) = \beta,$ into equation (4) gives $A = \beta e$. Now,

$$y_0(x) = \beta e^{1-x}$$

(5)

Equation (5) is the first order for outer solution. Repeat the same process as above which gives the general solution for second order as

$$y_1(x) = \beta e^{1-x} - \beta x e^{1-x}$$

(6)

Write equation (5) and (6) together give the first two order outer solution as

$$y(x) = \beta e^{1-x} + \varepsilon\left[\beta e^{1-x} - x\beta e^{1-x}\right] + O\left(\varepsilon^2\right).$$

(7)

To determine inner solution, we used the principle of stretched variable where it is given by $\tau = \frac{x - x_0}{\varepsilon}$ and $t_0 = 0$. So, $\tau = \frac{x}{\varepsilon}$ . Rewriting our equation in terms of stretched variable gives

$$\varepsilon \frac{1}{\delta(\varepsilon)^2} Y'' + \frac{1}{\delta(\varepsilon)} Y' + Y = 0 \tag{8}$$

Now we must compare the entire coefficient to choose $\delta(\varepsilon)$ to determine the dominant balance. There are three possibilities in equating to zero of three coefficients which are $\frac{\varepsilon}{\delta(\varepsilon)^2}, \frac{1}{\delta(\varepsilon)}$ and 1.

The dominant balance, $\delta(\varepsilon) = \varepsilon$ was identified in our study. Putting this to equation (8) so that

$$\frac{1}{\varepsilon} Y'' + \frac{1}{\varepsilon} Y' + Y = 0. \tag{9}$$

Simplify equation (9) as

$$Y'' + Y' + \varepsilon Y = 0. \tag{10}$$

Then substitute equation (1) into (10) and we obtain

$$\begin{aligned} &\left[Y_0''(\tau) + \varepsilon Y_1''(\tau) + o\left(\varepsilon^2\right)\right] + \\ &\left[Y_0'(\tau) + \varepsilon Y_1'(\tau) + o\left(\varepsilon^2\right)\right] + \\ &\varepsilon\left[Y_0(\tau) + \varepsilon Y_1(\tau) + o\left(\varepsilon^2\right)\right] \end{aligned} \tag{11}$$

Therefore from equation (11), we get

$$O(1):\ \ Y''_0(\tau) + Y'_0(\tau) = 0,$$

(12)

$$O(\varepsilon):\ Y''_1(\tau) + Y'_1(\tau) + Y_0(\tau) = 0. \tag{13}$$

The general solution to the equation (12) is

$$Y_0(\tau) = A\left(1 - e^{-\tau}\right), \tag{14}$$

and equation (14) is subject to boundary condition $Y_0(0) = \alpha$ . Hence, the first term of inner solution is

$$Y_0(\tau) = \alpha + A\left(1 - e^{-\tau}\right). \tag{15}$$

In order to determine the value of constant $A$ appears in equation (15) we match it to the outer solution as

$$\lim_{\tau\to\infty} Y_0(\tau) = \lim_{x\to 0} y_0(x), \tag{16}$$

$$\lim_{\tau\to\infty} \alpha + A\left(1 - e^{-\tau}\right) = \lim_{x\to 0} \beta e^{1-x}. \tag{17}$$

Comparing both side of equation (17) therefore,

$$A = \beta e - \alpha. \tag{18}$$

Putting equation (18) into equation (15) we get the first order of inner solution as

$$Y_0(\tau) = \beta e + \alpha e^{-\tau} - \beta e^{1-\tau} \tag{19}$$

Now, determine the general solution for equation (13) we get

$$Y_1(\tau) = B\left(1 - e^{\tau}\right) - \beta e\tau - \alpha\tau e^{-\tau} + \beta\tau e^{1-\tau},$$

(20)

where the constant $B$ appears in equation (20) and is determined by repeating the matching process procedure as before. Therefore,

$$\lim_{\tau\to\infty}\left[B - Be^{\tau} - \beta e\tau - \alpha\tau e^{-\tau} + \beta\tau e^{1-\tau}\right]$$

$$= \lim_{x\to 0}\left[\beta e^{1-x} - x\beta e^{1-x}\right].$$

Hence,

$$B = \beta e. \tag{21}$$

Substitute equation (21) into (20) we get inner solution of order 2 as

$$Y_1(\tau) = \beta e - \beta e^{1-\tau} - \beta e\tau - \alpha\tau e^{-\tau} + \beta\tau e^{1-\tau}.$$

(22)

Combine and write equations (19) and (22) as inner solution

$$Y(\tau) = \beta e + \alpha e^{-\tau} - \beta e^{1-\tau} + \varepsilon\left[\beta e - \beta e^{1-\tau} - \beta e\tau - \alpha\tau e^{-\tau} + \beta\tau e^{1-\tau}\right] \tag{23}$$

Next we continue by determine the Composite solution by

$$y_c(x) = y_i(x) + y_o(x) - overlap.$$

And notice that the $overlap = \beta e + \varepsilon\left[\beta e - x\beta e\right]$. Hence,

$$y_c(x) = \beta e^{1-x} + \varepsilon\left[\beta e^{1-x} - x\beta e^{1-x}\right] + \left\{\begin{array}{l}\beta e + \alpha e^{-\tau} - e^{1-\tau} + \\ \varepsilon\left[\beta e - \beta e^{1-\tau} - \beta e\tau - \alpha\tau e^{-\tau} + \beta\tau e^{1-\tau}\right]\end{array}\right\} - \left\{\beta e + \varepsilon\left[\beta e - x\beta e\right]\right\}.$$

(24)

Substitute the stretched variable $\tau = \frac{x}{\varepsilon}$ into equation (24) and simplify it gives

$$y_c(x) = \left[\beta e^{1-x} + (\alpha - \beta e)(1+x)e^{-\frac{x}{\varepsilon}}\right] + \varepsilon\left[(1-x)\beta e^{1-x} - \beta e^{1-\frac{x}{\varepsilon}}\right]. \tag{25}$$

Equation (25) is called the analytical approximation solution using MMAE.

## *B NonLinear Equation*

Now we look at a nonlinear equation of $\varepsilon y'' + 2y' + e^{y} = 0$ in the interval of $0 \le x \le 1$ with the boundary conditions are $y(0) = 0$ and $y(1) = 0$. In this case we will look into the analytical solution for order 1 which has the outer at boundary $x = 1$. Substitute equation (1) into nonlinear equation as

$$\varepsilon\left[y"_0(x)+\varepsilon y"_1(x)+O(x)\right]+2\left[y'_0(x)+\varepsilon y'_1(x)+O(x)\right]$$
$$+e^{\left[y_0(x)+\varepsilon y_1(x)+O(x)\right]}=0.$$

From the above equation, we can write order 1 as

$$O(1):\ \ 2y'_0(x)+e^{y_0(x)}=0. \tag{26}$$

Equation (26) is subject to boundary condition $y_0(1)=0$. Therefore the general solution for equation (26) given as

$$y_0(x)=\log\left(\frac{2}{x-2A}\right) \tag{27}$$

where $A$ is a constant. Constant $A$ is determined by imposing the boundary condition $y_0(1)=0$, therefore $A=\frac{-1}{2}$ and substitute into equation (27) we get

$$y_0(x)=\log\left(\frac{2}{x+1}\right) \tag{28}$$

The inner is at $x=0$ since our outer is at $x=1$. By the principle of stretched variable our inner layer is $\tau=\frac{x}{\varepsilon}$ and the dominant balance is $\delta=\varepsilon$. Now, we write the inner expansion of nonlinear equation as

$$\frac{\varepsilon Y"}{\varepsilon^2}+\frac{2Y'}{\varepsilon}+e^Y=0. \tag{29}$$

Simplify equation (29) gives

$$Y"+2Y'+\varepsilon e^Y=0. \tag{30}$$

Substitute equation (1) into (29) we get

$$\left[Y"_0(\tau)+\varepsilon Y"_1(\tau)+O(\tau)\right]+$$
$$2\left[Y'_0(\tau)+\varepsilon Y'_1(\tau)+O(\tau)\right] \tag{31}$$
$$+\varepsilon e^{\left[Y_0(\tau)+\varepsilon Y_1(\tau)+O(\tau)\right]}=0.$$

Carry out the first order of the expansion in equation (31) as

$$O(1):\ \ Y"_0(\tau)+2Y'_0(\tau)=0, \tag{32}$$

Equation (32) is subject to boundary condition $Y_0(0)=0$.

So, the general solution for equation (32) is

$$Y_0(\tau)=B\left(1-e^{-2\tau}\right), \tag{33}$$

where constant $B$ appear in equation (33) is determine by matching process. In the matching procedure, matched the outer solution as

$$\lim_{\tau\to\infty}Y_0(\tau)=\lim_{x\to 0}y_0(x), \tag{34}$$

$$\lim_{\tau\to\infty}A\left(1-e^{-2\tau}\right)=\lim_{x\to 0}\left(\log\left(\frac{2}{x+1}\right)\right). \tag{35}$$

Solving equation (35) gives $A=\log 2$. Therefore, our inner expansion is

$$Y_0(\tau)=\log 2\left(1-e^{-2\tau}\right). \tag{36}$$

Notice that the overlap for this nonlinear equation is $\log 2$. Therefore the composite solution is

$$y_c(x)=\log\left(\frac{2}{x+1}\right)+\log 2\left(1-e^{-2\tau}\right)-\log 2. \tag{37}$$

Simplify the equation (37) as

$$y_c(x) = \log\left(\frac{2}{x+1}\right) - \log 2e^{-2\tau} \qquad (38)$$

Rewrite equation (38) by substituting $\tau = \dfrac{x}{\varepsilon}$, so our composite expansion of order 1 becomes

$$y_c(x) = \log\left(\frac{2}{x+1}\right) - \log 2e^{\frac{-2x}{\varepsilon}} \qquad (39)$$

## 5 Result and Discussion

This paper discusses linear and nonlinear equations that occur at one boundary layer. The detailed procedures and techniques in solving this problem by using MMAE are successfully shown. For linear cases, we indicate up to two iterations while for nonlinear cases, just one iteration in order to understand the concept of MMAE. In order to discuss other iterations we need help from computer tools to programme the steps in MMAE, as Kaufmann solved an oscilation equation using MMAE up to a desired order through the use of Mathematica tool and programming, Firdawati [1].

### A *Linear Equation*

The linear equation has been solved analytically using MMAE (refer equation (25)). According to Firdawati [1], the exact analytical solution for linear equation is

$$\frac{e^{x\frac{\left(-1-\sqrt{1-4\varepsilon}\right)}{2\varepsilon}+\frac{\sqrt{1-4\varepsilon}}{\varepsilon}}\alpha - e^{\frac{x\left(-1+\sqrt{1-4\varepsilon}\right)}{2\varepsilon}}\alpha - e^{\frac{1}{2\varepsilon}+\frac{x\left(-1-\sqrt{1-4\varepsilon}\right)}{2\varepsilon}+\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}}\beta + e^{\frac{1}{2\varepsilon}+\frac{x\left(-1+\sqrt{1-4\varepsilon}\right)}{2\varepsilon}+\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}}\beta}{-1+e^{\frac{\sqrt{1-4\varepsilon}}{\varepsilon}}}.$$

The exact analytical solutions above will be compared graphically to the approximate analytical solution obtained by MMAE.

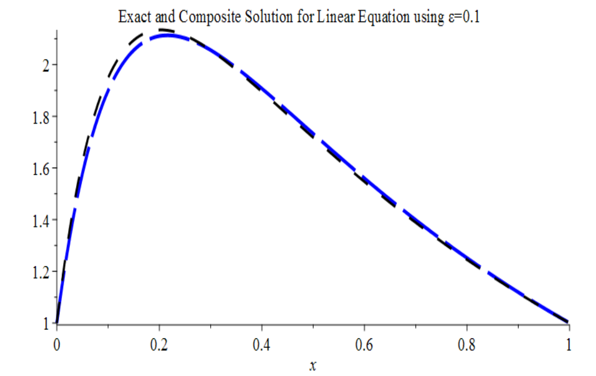


Figure 1

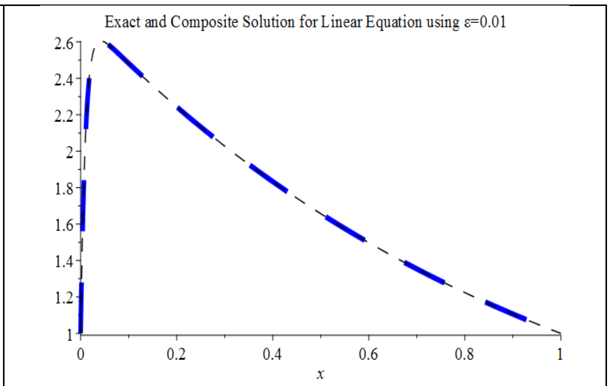


Figure 2

The plotted graphs in Figure 1 – 2 use selected value of small parameter, $\varepsilon = 0.1$ and $\varepsilon = 0.01$ and the boundary condition according to Padmaja and Reddy [15] are $\alpha = 1$ and $\beta = 1$. Figure 1 and 2 show that the analytical approximation obtained using MMAE (long datch line) is very close to exact solution (datch line) as $\varepsilon$ has become smaller.

### *B NonLinear Equation*

Analytical approximation of nonlinear equation using MMAE and exact solution plotted using a different value of small parameter, $\varepsilon$. According to Reddy and Chakravarthy [16], the exact solution of nonlinear equation is

$$y(x) = \log_e\left(\frac{2}{1+x}\right) - (\log_e 2)e^{-2\left(\frac{x}{\varepsilon}\right)}.$$

The result using MMAE (equation (39)) and the exact solution look similar. Figure 5 and 6 below use $\varepsilon = 0.01$ and $\varepsilon = 0.001$ respectively. From the graphs, we can see that the analytical solution using MMAE (datch line) and the exact solution (line) is very close since it is on the same line.

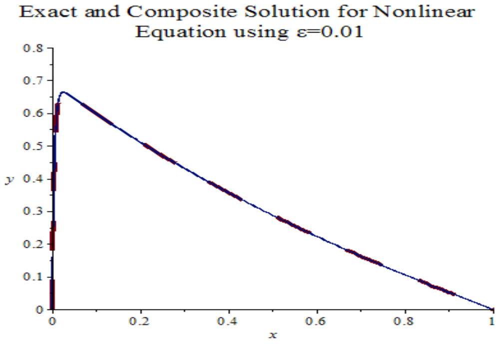


Figure 5

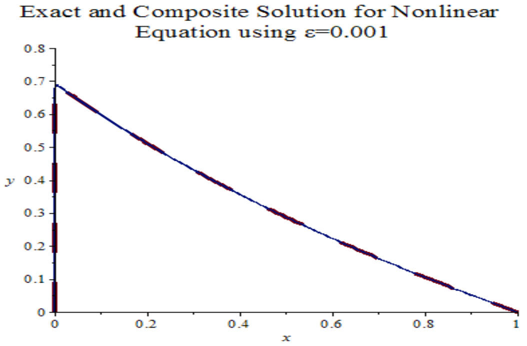


Figure 6

In conclusion, there is no single, best method to tackle singular perturbation problems. Even in MMAE, we face problems to determine the constant appear. But, MMAE is one of the best methods to solve singular perturbation problem in second order ordinary differential equation since the results obtained are very close to the exact solution (refer to Figure 3 and 4). Further research is to look for the other methods to see their comparisons. Besides, we are interested to look for the problem of linear and nonlinear ordinary differential equation rather than order 2, which has been examined in many areas.

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**References**

[1] Firdawati, M., Solving Singular Perturbation Problem of an Ordinary Differential Equation using The Method of Matched Asymptotic Expansion (MMAE), Master Dissertation, University Science of Malaysia, 2012.
[2] Firdawati, M, & Mohamad, F. A. K., Solving Singular Perturbation Problem of an Ordinary Differential Equation using The Method of Matched Asymptotic Expansion (MMAE), AIP Conferences Proceeding 1682, 020014 (2015).
[3] Dogan, N., Erturk, V. S. & Akin, O., Numerical Treatment of Singularly Perturbed Two-Point Boundary Value Problems by using Differential Transformation Method, pp1-10, 2012.
[4] Mohan, K., Reddy, Y. N., An Approximate Method for Solving a Class of Singular Pertubation Problems. Journal of Mathematical Analysis and Applications 133, 3066323, 1988.
[5] Qassim, R. Y., Silva Freire, A. P., Application of the Boundary-Value Technique to Singular Perturbation Problems at High Reynolds Numbers. Journal of Mathematical Analysis and Applications J22. 70-87, 1987.
[6] MacGillivray, A. D., A method for Incorporating Transcendentally Small terms into the Method of Matched Asymptotic Expansions., Journal of Studies in Applied Mathematics. 99, 285-310, 1997.
[7] Kelley, W. G., A singular perturbation Problem of Carrier and Pearson. Journal of Mathematical Analysis and Applications. 255, 678-696, 1999.

[8] Kadalbajoo, M. K. & Patidar, K. C., Numerical Solution of Singularly Perturbed Two-Point Boundary Value Problems by Spline in Tension. Applied Mathematics and Computation. 131, 299-320, 2002.
[9] Neufeld, J. A. and Wettlaufer, J. S., Shear flow, Phase Change and Matched Asymptotic Expansion: Pattern Formation in Mushy Layers. Journal of Physical D. 240, 140-149, 2010.
[10] Kadalbajoo, M., Arora, P. & Gupta, V. , Collacation method using Artificial Viscosity for Solving Stiff Singularly Perturbed Turning Points Problem having twin Boundary Layers. Computers and Mathematics with Applications. 61, 1595-1607, 2011.
[11] Argatov, I. & Mishuris, G,. Thesis,. Aberystwyth University, 2011.
[12] McKelvey, R. &Bohac, R., Ackerberg-O'Malley Resonance Revisited. Journal of Mathematics. 6, 637- 649, 1976.
[13] Wong, R. & Yang, H., On the Ackerberg-O'Malley Resonance. Studies in Applied Mathematics. 110, 157-179, 2003.
[14] O'Malley, R. E. , Introduction to Singular Perturbations. New York & London: Academic Press Inc, 1974.
[15] Padmaja, P., & Reddy, Y. N. A Numerical Patching Method for Solving Singular Perturbation Problems Via Pade` Approximations. International Journal of Applied Science and Engineering, 11(1), 51 – 67, 2013.
[16] Reddy, Y. N., & Chakravarthy, P. P. Numerical Patching method for Singularly Perturbed Two-Point Boundary Value Problems using Cubic Splines. Applied Mathematics and Computation, 149(2), 441 – 468, 2004.