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The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston

Cultivated mind is the guardian genius of Democracy, and while guided and controlled by virtue, the noblest attribute of man. It is the only dictator that freemen acknowledge, and the only security which freemen desire.

Mirabeau B. Lamar

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Edited by

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FOREWORD

We are very much pleased to present in this number three papers read before the Mathematics Section of the Texas State Teachers Association meeting last November in Houston, Texas. These papers are "Philosophy and Mathematics," by Dr. J. A. Lynch, of Rice Institute; "Modern Tendencies and Aims of Secondary Mathematics," by Lilian Cash, of Bonham, Texas; and "Stimulating Interest in Mathematics by Creating a Mathematical Atmosphere," by Mary Ruth Cook, of Denton, Texas.

Attention is also called to the fact that the first chapter of Mrs. Della Houssels' paper, the second and third chapters of which are published in this number, will be found in the November 15, 1930, issue of the *Bulletin.*

Thanks are due Dean I. P. Hildebrand for bringing to our notice the reprint of the article by Pres. A. L. Lowell, which is quoted in this *Bulletin.*

PHILOSOPHY AND MATHEMATICS

By DR. J. ALVIS LYNCH
Rice Institute, Houston, Texas

The major premise of this paper is that man's thinking proceeds on the basis of analogies. He is always measuring and estimating that which lies in the distance by applying samples of the near at hand.

There are three ways, including thinking, by which man passes from the familiar and the close at hand to the unfamiliar and the remote. These three ways are, roughly, *apperception, intuition,* and *thinking.* What psychologists have termed *apperception* is a sort of a syncopated or telescoped thinking process between which and conscious thinking stands that which is commonly called intuition.

In apperception, we find the first tendency to anticipate the senses. This tendency must have appeared very early in the evolutionary process, because, without it, the senses would be sheer luxuries which could serve no purpose whatever in behavior. It was only after sensations became embedded in matrices of habits or behavior patterns that they assumed the anticipatory rôle. Man, and perhaps animals also, *perceives* immediately much more at any given time than is actually sensed. Mass, taste, and temperature are as certainly beheld in the distant object as are its colors and its odors. Objects look hot, heavy, bitter, or wet. Certain qualities act as significant cues and man, especially, has become adept at guessing the rest. When one of these cues penetrates the complex mechanisms of the mind, which, for the time being, lie in the subconscious, man guesses, and if successfully, he names the faculty *intuition.*

Now thinking literally lays out the processes of apperception and intuition and picks the best in stock and trade for the purpose at hand. This process is slower, because it requires time for the complex patterns of thought to expose themselves, but it is safer, and it increases the reach of the mind a thousand fold. Thinking extends further into space and time than apperception, and it is more open to correction, improvement, and criticism than intuition, which plies between so-called "hunches," on the one hand, and subconscious wishes, infinitely projected, on the other.

What might, for convenience, be called *objectivity* in thought is a function of the number of cues and the number of opportunities to try projected guesses. On the other hand, the fewer the cues, the less opportunity to try guesses, the more of the thinker's own organization enters in to fill up the wide gaps. As opposed to objectivity, the tendency in this direction may be called *subjectivity.*

Now it goes almost without saying that the nature of the philosopher's problems is such as to involve the largest and most fundamental structures of his own personality; because the farthest reaches of the mind, the boldest and most magnificent guesses about the world in which one lives, when made cautiously, constitutes one's philosophy. It is necessary to qualify these guesses as *cautious* to differentiate them from blind wishes and prejudices which are often projected behind experience without taking the trouble to balance the equation between that which is known at first hand and that which is set up as its transexperiential equivalent. The philosopher cannot afford to ignore any entity which lies on the side of the given. In other words, he must see that his solution comprehends adequately all the different elements of experience in so far as they are relevant to his dominant analogy. Therefore, while in one sense there is little, in another, there is much to restrain the philosopher. He is not required, as a philosopher, to give a complete account of the universe from every conceivable point of view. He does well if the pure light of his reason illumines one side, the side next to himself, of every large or small lump of reality. A few illustrations of what I mean will probably be welcome.

A savage, having his most significant experiences with living beings like himself, projects a spirit community back of the nature which touches him, to account, alike, for her order and her caprice.

The earliest known philosophers, those referred to in the history of philosophy as the school of Miletus, were looking for the all-pervading stuff of reality. They used as a base one or more of the familiar materials of common experience.

A Pythagoras, finding number significantly related to the pitch of musical sounds and other processes familiar to him, would read his discovery into the universe at large. Turning his big idea on the world, he found much of *order* expressed there which was not unlike that expressed in music. This was, indeed, a wonderful idea, and in so far as Pythagoras and his disciples were able to discover and relate the expressions of number as they had conceived it, they gave a true, a complete, as well as a significant account of the universe in which they lived.

A Socrates went forth to search the hearts of men for a few substantial ideas upon which a just and honest man could base his life. Plato, the faithful friend and pupil of Socrates, built out of the memories of the master a magnificent monument to him in the world's first great metaphysics of ideas. The discovery of ideas impressed Plato, so he set about to explain everything in terms of its idea. This gave a perspective of the universe, and who is here to say that it was not a true perspective?

Galileo, and those who finished drawing the conclusions from his premised discoveries, reduced the world to a system of falling bodies. If Galileo is known popularly as the discoverer of the telescope, it not because this constitutes his chief claim to greatness. The telescope was to him merely an instrument which aided him in applying his idea to the stellar universe. The isolation of the concepts of mass and motion was a fulcrum idea for future philosophies of nature. From this point of support, a universe was finally lifted out of the confusion of irrelevant qualities; and if the protest went up later that it did not contain anything which could touch the senses, the appetites, or the hearts of men, it was no less completely a universe from a particular point of view.

There is no point in multiplying instances of men who have seen the universe through the medium of one dominant idea. Those who have tried to "see life steadily and see it whole" have done this, not as a defect, but as a necessity. To reduce the *many* to the *one* and to find the *one* in the *many* is the perennial problem of the philosopher. And, as with the intellect, so it is with the emotions. Those who are obsessed with filial, conjugal, or religious love make out of their dominating affection a unifying principle for the whole of experience. If Jesus saw God, his ultimately real, as love, it is because the universe reverberated with the quality of his own heart throbs.

Such a conception of what constitutes knowing is the background for this short treatment of the relation of mathematics to philosophy. It will soon be sensed that I am not trying to produce a eulogy or an *apologia* for either philosophy or mathematics. Neither needs anything of the kind; and, if it did, these are to be found in current literature pertaining to both. However, those who make out of mathematics a sort of intellectual religion, believing that it expresses the essence of all respectable thought, will not be satisfied with what I have to say, because their view makes out of mathematics the only St. Peter who can open the gate to the heaven of ultimate truth. I am going to argue that there are other entrances, or, at least, that there are other heavens for those who seek a different variety of truth.

From the time of the Greeks, mathematics has stood out as one of two dominant habits of mind through which men have viewed their world. Two things, above all, have had a chance to impress their patterns upon the thoughts of men: one of these is the manipulatory process in which rigid bodies are lifted, handled, broken apart, spanned, traversed, counted, and compared; the other is the developing process as revealed in living things. Analogies taken from the former, when purified and cleared of all elements of particularity, that is, when they became habits of mind, supplied the rudimentary concepts of mathematics; while those derived from the latter are the basic patterns of developmental philosophies from Aristotle on down.

Now it is better known to no one than mathematicians that the application of an analogy, as an explanatory principle, gives not a whit more content to the object or situation explained than is contained in the analogy itself. It may give more than the situation seems to merit; but, if it does, the superfluity is due to the over-rich content of the analogy. For example, it might happen that a hylozoist would see more life and less of dead matter in the world about him than anybody else; while one who applies to the universe the patterns which are derived from the manipulation of rigid bodies sees it as a machine, perfect to the last detail perhaps, but dead.

The philosopher Protagoras said that "man is the measure of all things." This may be true, but it cannot be taken as implying too much subjectivity. Man cannot measure until he selects something as a standard, and it is the measuring rod which is reflected in the final results more vividly than the man. But he shows bias enough in his choice of a measuring rod. For when he decides what he wishes to do with the world, it becomes to him a medium, a potential field of operation, for his chosen type of activity. The patterns which have worked their way out of his habitual behavior become loaded responses, so to speak; and these responses, organized into a more or less harmonious whole, make up his personality. And the personality, so conceived, defines its world in terms of the stimuli which make up the complete occasion for its expression. It is in this sense that one's world is the *medium* in which he moves and has his being. It is, in a sense, one's environment, when one takes the trouble to define its outer boundaries, and this has been defined by John Dewey as the sum total of forces with which an organism varies. One's choice of attitudes, and the habits which are organically connected with them, lay out what we might call a perspective of the world.

Now the arguments which I am presenting through this approach are: (1) that the mathematical attitude of mind, the mathematical way of handling things, defines one of the perspectives; and (2) the point which I expect you to challenge, that there are other important perspectives which are not ultimately reducible to that of the mathematician.

If this makes you angry, I hope it will, at least, not leave you indifferent. For, I trust, it is as true as ever that "the hot" is closer to intellectual activity than "the cold." But I am sure that if you agree with me, so far, it is because you are not following me carefully. Abstractions require all the sympathy and love of brothers in the faith to make them convincing; therefore, one could not expect to get away with even a respectable defeat, within hostile territory, by relying wholly upon a battery of generalizations. Wherever generalizations gain in range they lose correspondingly in precision and accuracy. Therefore, since you are likely to press me, I shall do well to anticipate your demands for more concrete and up-to-date instances of these alleged non-mathematical, and yet true, perspectives of the world.

I have already given you the biological bias as an instance. But this is general; it is really based on a wrong conception of development; and, since it goes back to Aristotle, it might be said that some philosophers are merely falling into an ancient and well-respected error. Galileo was mentioned as a classical instance; but he had so much mathematics mixed up with the manipulation of his data that his case is not likely to impress you with its uniqueness.

Perhaps a better example than either of these is the philosophy which takes its principal patterns from organic evolution. The concepts of natural variation, however conceived, and natural selection can be manipulated in such a way as to give a world view which, if it involves mathematics at all, pushes it into the peripheral regions. And these concepts can be used to control the course of natural events, to a slight degree at least, in the artificial selection of plants and animals. The fact that they can be used successfully gives them the same claims to objectivity as any other approach. This view has, to be sure, fluctuated somewhat with the various explanations of the nature and cause of variations; but all the way through, it has maintained an identity of its own. Its fluctuations have been no more serious than those occurring in physics and the corresponding philosophy. The objection might be raised that this view explains life and not matter; but it might be worth while to point out that matter yields as well to biological as to physical patterns. The modern philosophical views of Bergson, Morgan, Alexander, Whitehead, and Eddington are projections of biological patterns more truly than of physical. The atom is as easily conceived as an organism as it is as a miniature solar system.

In a similar way, preoccupation with the psychology of habit, with its extension to the interpretation of customs and the more general data of anthropology, may become a generalized way of regarding the world. There is much in the rhythmic character of natural events to suggest itself to this type of explanation. In fact, this is but the biological explanation from a psychological angle. The habit conception simply places attention on the rhythmic, recurrent character of the behavior of organisms. Some of the dynamic psychologists, when they become philosophers, attach metaphysical significance to this. Sigmund Freud explains what he takes to be two opposing tendencies in nature, the tendency to self-renewal and the tendency to self-destruction, as expressions of the same universal tendency in nature to return to a former state or condition—in the one case recent, in the other remote.

A similar case might be made out for the geologist and the geological point of view. In a short stretch of events, the geologist finds patterns which he can apply to the interpretation of nature in all times and all places. It can become a philosophy when he applies these to ultimates.

It is certainly true that in almost every case which has been mentioned mathematics is used, might be used, or, at least, might conceivably be used in the manipulation of all or some data. It is because of the conspicuous nature of this relationship to mathematics that mathematical physics does not make a convincing case in support of the debatable assertion that some perspectives are irreducible to mathematical formulae or patterns. Without the application of mathematics, it is unlikely that Galileo and Newton would have been able to discover in all instances of motion only special cases of the falling-body principle. To universalize this pattern required a degree of precision which was possible only through the mathematical technique. The perspective in this case is still physical rather than mathematical; that is, the content of the world, from this point of view, is still physical material rather than mathematical formulae. The content still retains some of the properties of the original model or experimental objects, namely, mass, hardness or impenetrability, and the properties of motion. By adding to the precision of physical concepts, mathematics has undoubtedly exercised its greatest indirect influence on philosophy. The classical examples of this are found: (1) in the general upset of the geocentric point of view, and the ultimate reduction of the stellar universe to the same material as that of the earth, by Galileo and his successors; and (2) in the general overthrow of the Newtonian order by the movement which centered around Einstein. Each of these movements caused a corresponding upheaval in philosophy.

Mathematics enters into this relationship with practically all ways of thinking but in varying degrees. Sometimes only a small part of the data will yield to mathematical treatment, sometimes none, and sometimes they can be handled in no other way. Their relative independence of mathematics renders the psychological and biological approaches more impressive, superficially, as examples of nonmathematical perspectives. But the admission that the effective manipulation of certain physical analogies depends upon the use of mathematics is quite different from the assumption that the mathematical formula, as an empty form, describes the inner structure of reality toward which all other types of explanation, in so far as they are headed right, must focus.

What I am calling the mathematical habit of mind is reflected in the metaphysical views of Plato, of Descartes, and, recently, of Sir James Jeans. There are others, of course, but these are fair examples of men who have seen the world as a whole through mathematical glasses.

Plato is perfectly right in insisting, as he does, in Book VII of the Republic, that music, number theory, geometry, and astronomy constitute the stepping stones to philosophy; because Plato meant by philosophy *the philosophy of ideas,* and not *a philosophy.* After Plato had defined the final outcome of philosophy there was no further question about what the approach should be. The philosopher, contemplating the world of pure ideas, and regarding these as the true objects of which other things are shadows, is as much a product of foregoing conditions as any other event in nature. If the philosopher was to arrive, eventually, at the stage in which he could contemplate the world of pure ideas, pure forms after which the rational, and therefore the real, in the world of becoming was conceived to be imperfectly modeled, he would have to submit himself to the disciplines which were necessary to set the Platonic habit of mind. Plato was not interested in mathematics as an end in itself or as a preprofessional study; mathematics was considered merely a prelude. "For," says Socrates, "you surely would not regard the skilled mathematician as a dialectician?" "Assuredly not," replied Glaucon, "I have hardly ever known a mathematician who was capable of reasoning." "Then if geometry compels us to view being, it concerns us; if becoming only, it does not concern us?" To this Glaucon also gives his assent.

In dialectic, the science which is concerned with *being,* Plato reveals the mathematician's mind by making the subject matter of dialectic *pure form* or *pure pattern* without any of the contaminating, material content of the unreal world of becoming. Plato overlooked the fact that there is no absolute way of determining which is the *real* object and which is the shadow. Ideas might conceivably be present and be effective in a world of which they do not constitute the nucleus. We are to take the rational man's word for it, to be sure; because he has the world ordered and systematized, and we never prefer chaos to order. But Plato did not see that it is the *rational man*, rather than his pet conception of the real, which puts order into the world. He might very well start at the opposite end from the pure forms, as Bergson did, and construct a coherent world view. It is certainly a different world view, however. In the case of Bergson, the world of change is seething, living, and the ultimately real; whereas, both ideas and matter are the sloughed-off, metaphysical by-products.

Descartes is a second interesting example of one who found the patterns for the essence of reality in the forms of his thought. According to him, the primary qualities of the world are only shape, size, position, and motion. These are distinctly mathematical essences conceived in terms of analytical geometry. The qualities of an object which are capable of being sensed were conceived to be secondary and of a derivative nature. These were significant as signs of the underlying substratum of reality which was regarded as expressible in mathematical terms. Among perceptions was the same confusion which Plato found in the world of becoming. In making the knowable forms the core of his world, he was like Plato and somewhat different from Galileo and Newton, who conceived being in terms of solid, material substances, but substances which, nevertheless, obey mathematical laws.

A few days ago, the astronomer Sir James Jeans made a statement which is typical of any mathematician who has thought his way through to ultimates on strictly mathematical grounds. The *Christian Science Monitor* quotes him as saying:

> The universe does not appear to work, as was at one time thought, on animistic or anthropomorphic lines, nor as was recently thought, on mechanical lines; it rather works on purely mathematical lines. In brief, the universe appears to have been designed by a pure mathematician.

> The terrestrial pure mathematician does not concern himself with material substances but with pure thought. His creations are not only created by thought but consist of thought, just as the creations of the engineer consist of engines.
>
> This concept of the universe as a world of pure thought implies, of course, that the final truth about the phenomenon resides in the mathematical description of it: so long as there is no imperfection in this, our knowledge of the phenomenon is complete. We go beyond the mathematical formula at our own risk: we may find model or picture which helps us to understand it, but we have no right to expect this and our failure to find such a model or picture need not indicate that either our reasoning or our knowledge is at fault.

There is nothing new in this statement. It recalls Pythagoreanism and Kepler's divine arithmetic and geometry. It represents the mathematician's *idol of the theater,* if we may borrow a bit of Baconian terminology. Sir James would deny the premise with which this paper starts—that reasoning from the known to the unknown proceeds on the basis of analogies. His argument on this point would be perfectly clear: we have abandoned physical analogies, pictures, models, etc.; and by that act we have freed ourselves from analogies altogether. But does he not overlook the fact, which his own statements seem to obviate, that the mathematical formula, though perhaps not a physical one, is also, and none the less, an analogy? To say that it is not, is almost like abstracting our dancing habits from the dancing act and then denying their origin. Wherever the physical deviates from the mathematical description, according to Sir James, it is either wrong or irrelevant. One has but to point out to Sir James that his world view contains nothing qualitatively different from that which flits through the mind and muscles of mathematicians. Now where is his occasion for thanking God that in this he is not as other men? We have freed ourselves, he declares, from anthropomorphism; we are no longer satisfied with world views which are merely the projection of the personalities of men. These will not do for one who has been dubbed a knight of the mathematical round table. It requires something more exclusive. We must change the prefix in the word *anthropomorphism* so it will appertain, exclusively, to a particular class of men, namely, mathematicians.

One thinks that one would not look far to find anthropomorphism in this. But it is not in being anthropomorphic, but in denying his human kin, that Sir James has sinned. Anthropomorphics we are of necessity, because we are men; but if Sir James could change into a deity, the divine bias would still prevail, and he would have accomplished no more than to change the prefix again. We are probably safe in assuming that mathematicians are neither deities nor different from other men in this respect, except, perhaps, in their exclusiveness.

MODERN TENDENCIES AND AIMS OF SECONDARY MATHEMATICS

By LILIAN CASH
Bonham, Texas

As teachers of this most venerable of sciences, we very often think that mathematics is mathematics and needs no new ideas and methods. A short review of the history of this subject, however, proves to us that "the science of mathematics is a great stream rather than a static mass." All down through the ages, mathematics has changed as the minds and mentalities of the makers of civilization have changed. A study of the advancement of mathematics is parallel with a study of the progress of the world. Long ago mathematics outgrew its practical beginning and was developed into a logical science. Quetelet, the scientist, says, "The more advanced the other sciences have become, the more they have tended to enter the domain of mathematics, which is a sort of center toward which they converge. We can judge of the perfection to which a science has come by the facility, more or less great, with which it may be approached by calculation."

Most thinking persons will concede that the world of today is different in many important respects from the world of a few years ago. If we wish to prepare ourselves to meet future needs in any measure, we must realize that we are now living in a dynamic society. If we wish to select as essential elements of secondary education the subjects which promise the best to the future citizens, we must consider the past history of the civilized world and its probable future history during the next forty or fifty years. When certain fields of study have in the world's history shown their importance in contributing to the progress of knowledge and the advance of civilization, and when the changes that have taken place during recent years show a steadily increasing need for this same type of training, then we ought, without a doubt, to accept it as one of the necessities in our educational system. That seems to be the strategic point which mathematics has taken. Mathematics must "go modern." It is inevitable that it should not only follow civilization, but that it should lead the way in many instances. Realizing these things, we are not surprised that a cry has gone forth for reorganization in this important field.

About thirty years ago a few of our far-sighted educators saw that this need for change was upon us and began preparing the world for it. We can date our beginning of "Modern Tendencies in Mathematics" with an address made in 1901 by John Perry before the British Association which had far-reaching results both in England and in the United States. He was not in accord with the prevalent aim of teaching mathematics in order to pass examinations and to create mathematicians. His interest was in the education of the average citizen, and for this reason he claimed that usefulness should determine what subjects should be taught. He says, "The study began because it was useful and continues because it is useful, and it is valuable to the world because of the usefulness of its results." Other characteristic quotations from him are: "It is immensely important that if any one method of elementary teaching be generally adopted it should not be hurtful to the one boy in a thousand who is fond of abstract reasoning, but it is just as important that the average boy not be hurt"; and "Where would be the harm of letting a boy assume the truth of many propositions of the first four books of Euclid, letting him accept their truth partly by faith, partly by trial"; and again, "Perhaps the worst fault of our teaching is that the pupil is taught as if he were going to be a teacher himself." The details of Perry's address sound very familiar after the passing of a quarter of a century, but in 1901 his ideas were quite novel. Since that time much progress has been made. In the United States societies for the improvement of the teaching of mathematics were soon formed and have rapidly grown in strength and importance.

Then on December 29, 1902, E. H. Moore of the University of Chicago delivered an address before the American Mathematical Society on the "Foundations of Mathematics." It was an epoch-making address. Many of the ideas were similar to those advocated by Perry. He asked for diminished emphasis on the systematic and formal side, and increased emphasis on the practical side. He believed in the teaching of mathematics by the laboratory plan, so that students could get their mathematics in concrete form. "Pupils should study things, not words," was a high point in his address. He believed that the high-school student "might be brought into vital relation with the fundamental elements of trigonometry, analytic geometry, and the calculus on condition that the whole treatment in its origin is, and in its development remains, closely associated with thoroughly concrete phenomena."

Moore did not ask for a revolution to bring about these changes, but pleaded for close coöperation among all concerned to bring about an evolutionary change. Even though some of his ideas are impractical and debatable, he gave to educational mathematics much cause for analysis and study.

After these two addresses, teachers of mathematics began trying to improve their methods and results, and a spirit of coöperation invaded the countries. In 1908, at the suggestion of David Eugene Smith, an international committee on the teaching of mathematics met in Rome. Reports from most of the leading countries were made showing the work done in schools of all types throughout the world. By means of these, teachers obtained a broad view of what was being done in their country, and had the opportunity to compare their own curricula with those of foreign countries. It was found that after the fourth grade the schools of the United States lagged behind those of the leading countries of Europe. For instance the work in secondary-school mathematics was begun much earlier there. Plans have since been made where this difference has decreased somewhat.

It must be remembered, however, in any comparison between the work of our schools and those of European countries, that a much more select group attends the secondary schools of foreign countries.

In 1916 the Mathematical Association of America organized the National Committee on Mathematical Requirements for the purpose of studying the improvement of mathematical education in secondary schools and colleges. The first findings of the committee were published as tentative reports and discussed at various meetings of organized groups of mathematics teachers throughout the country, criticism being invited. In this way a great many of the progressive teachers of secondary mathematics in the United States had a part in the work of the committee. The improvement in the teaching of mathematics in the last decade has been due largely to the work of this committee, not only because of the value of its report, but also because of its preliminary work of organizing the coöperation of many teachers. The final report, covering more than six hundred pages, entitled "The Reorganization of Mathematics in Secondary Education," was published in 1923.

Another organization which has greatly benefited the teaching of mathematics and has sponsored at all times modern tendencies and improved ideas in this field is the National Council of Teachers of Mathematics. This is a "national organization of mathematics teachers whose purpose is to create and maintain interest in the teaching of mathematics, keep the values of mathematics before the educational world, help the young and inexperienced teacher to become a good teacher, improve teachers in service by making them better teachers, and raise in general the level of instruction in mathematics. All persons interested in mathematics and in the teaching of the subject are eligible to membership." The Council has two publications, "The Mathematics Teacher" and the "Yearbook."

For many years college entrance requirements and examinations have had an important effect upon the teaching of mathematics in secondary schools, often an ultraconservative and irksome influence. Teachers, in many cases, have had all their efforts concentrated on preparing their pupils to pass these extramural tests, for they knew they would be judged by the results of these efforts and these alone. Unfortunately, training pupils to pass examinations and teaching them mathematics are not always synonymous phrases. This situation has been much alleviated, if not entirely removed, by the present revision of requirements.

Before 1900 each college had its own entrance requirements, thinking little of the requirements of other colleges or the needs of the secondary schools. In 1900 the College Entrance Examination Board was organized, making possible a unification of examinations and consultations with secondary school administrators and teachers. In 1922 a representative committee was appointed by the board to revise the requirements. The report made in 1923 showed that the interests of both the colleges and secondary schools were considered. In algebra much useless material, such as long and elaborate manipulations with polynomials and much of the factoring, has been eliminated. The formula and the graph receive greater attention. The requirements in connection with fractions have been lightened. The work with equations has been limited to those which a pupil will use in subsequent study. The amount of time for the treatment of irrationals has been shortened. Numerical trigonometry and logarithms have been introduced. In geometry the number of book theorems has been greatly reduced in order to place a greater emphasis upon the original exercise. Plane and solid geometry are being taught within the space of one year.

Various states still have various requirements for graduation from their high schools. In a personal interview with Mr. Dunlevy, head of the mathematics department at San Diego Senior High School, I found that California requires only one year of high-school mathematics for graduation from their high schools. This is general mathematics, which is given in their ninth grade. In this senior high school, out of 3,200 students last year only 1,200 were taking any mathematics at all. Since mathematics was not required and not even strongly advised in many cases, many students did not take it. Thus it would seem that the modern tendency is to do away with Plato's requirement of "Let no man ignorant of mathematics enter here."

One modern movement which has been very beneficial to the educational world and especially to mathematics is the testing movement. At first the standardized tests were so crude and inaccurate that little information was obtained from them, but the later editions and forms are very good. There are many defects, of course, among which come the facts that not always are the most desirable objectives in the teaching of the subject used in compiling the problem material for the tests; that no thought is given to values; that they cannot easily be adapted or modified to meet the needs of different courses of study or conflicting ideas. The tests are often misused; this latter difficulty arises from ignorance on the part of the teacher. This to me seems to be the main hindrance to the tests. Very few teachers know how and when to use standardized tests. As long as they remain a means toward an end no harm is done and a great deal of benefit can be derived, but when a test is the end itself; when a teacher uses that as the objective of her teaching, then we need to open our eyes and see. Some of the benefits which testing has given us are as follows: (1) Individual differences have been shown more clearly. Many high schools are dividing their classes into ability groups using tests as the basis for such grouping. (2) These tests have shown us that a great deal of the traditional material is too difficult for most pupils, and therefore should not be taught. Likewise certain material has been shown easier and within the mastery of a large number of children. (3) Tests show the amount of time necessary for drill work, and they have aided us in developing certain standards for our personal satisfaction.

A few of the modern tendencies regarding methods are: (1) Supervised study, division of the period into half study and half teaching. (2) Project method and Socratic recitation, often calling for extra equipment and arrangement of the room. The laboratory plan for teaching geometry comes under this method; Oakland High School, Oakland, Calif., is trying this out. (3) Contract plan, invented by Morrison, is good for teaching certain subjects. (4) Unit system. Houston is carrying out this idea. (5) Mathematics clubs, as a class device. Meetings of these clubs should be held once or twice each month for the discussion of more or less recreational mathematics.

One subject which has aroused much interest during recent years is the subject of graphing. This never fails to be of interest to both pupils, visitors, and teachers. A study of graphing commercially and mathematically will motivate itself if properly introduced. In teaching commercial graphs in the first-year algebra, all the different types of graphs may be introduced and as many practical illustrations as possible shown. Machine-made graphs showing daily and hourly gas or water pressure, power in electrical plants, nightly visitations of the night watchman; any and all of these never fail to arouse interest and these can be easily obtained. Then teaching graphs in equations as equations are taught is very interesting. Begin as soon as two variables are introduced and do not stop until various equations of the conic sections have been memorized and graphed. Many of our difficult theories, such as imaginaries, infinities, and inconsistent equations, can easily be explained and will be accepted in terms of the graph.

In the report of the National Committee on Mathematics Requirements, aims for mathematics are divided into three groups: (1) practical, (2) disciplinary, and (3) cultural. By a practical or utilitarian aim they meant the immediate or direct usefulness in life of a fact, method, or process in mathematics. Under this come (1) the fundamental processes in arithmetic, (2) understanding of the language of algebra, (3) development of ability to use fundamental laws of algebra and the formula (the formula is given special recognition and importance), (4) ability to understand graphic representation of facts, and (5) familiarity with geometric forms, including their mensuration, and development of space perception and spatial imagination.

Under disciplinary aims the old controversial idea of "transfer of training" arises. Rather than go into both sides of the question, I will give some of the points upon which the majority of psychologists agree concerning transfer of training in mathematics:

(1) In itself the transfer value of mathematics (or of any other subject) is insufficient to justify its being required of all pupils.

(2) The amount of transfer from mathematics depends upon the method of teaching the subject.

(3) Meanings, methods of attack, and attitudes are more transferable than skills and information.

Other disciplinary aims revert to this idea of transfer and yet they are usually spoken of as "training in thinking" along this line and that. One high-school girl told me the other day that she was trying hard to learn mathematics because Bacon said that mathematics makes one subtle. I do hope she accomplishes her desire.

Under cultural aims we may mention the development or acquisition of: (1) appreciation of beauty in the geometrical forms of nature, art, and industry; (2) ideals of perfection as to logical structure, precision of statement and thought, logical reasoning, discrimination between the true and the false, etc.; and (3) an appreciation of the power of mathematics. Byron expressively called mathematical reasoning "the power of thought, the magic of the mind." Let us have an ever-deepening appreciation of the power that factual knowledge gives us. Inspire our pupils with the rôle that mathematics and abstract thinking, in general, have played in the development of civilization; in science, industry, and philosophy. We, as teachers, can give our pupils a vision of not only arithmetic, algebra, and geometry, but also something of the mathematics which follows. A teacher of college mathematics had just returned from abroad to his native town after three years study of mathematics. His brother was accosted on Main Street by a friend of the family who had heard of the return of Robert, the student and scholar.

"They tell me Bob has just got back from Germany," said the kindly friend.

"Yes, he returned yesterday," replied the brother.

"What has he been doing over there all these years?"

"Studying mathematics."

"Nothing but mathematics?"

"That's all."

The man gazed meditatively across the street at the blank wall of a brick building. He tried to grasp the immensity of the thing. Finally he blurted out, "I'll bet he could tell in a minute how many bricks there are in that wall."

Couldn't we as teachers of this mathematical science teach something of the magnitude of mathematics comparable to this very universe itself? Couldn't we in teaching make the student realize what a wealth of knowledge, satisfaction, and pleasure awaits him who loves mathematics for mathematics' sake?

STIMULATING INTEREST IN MATHEMATICS BY CREATING A MATHEMATICAL ATMOSPHERE

By MARY RUTH COOK
Denton, Texas

Mathematics teachers are forced to admit that the attitude of many high-school students toward mathematics has been the feeling that the subject is dull, uninteresting, hard, something forced upon them and to be endured until enough credits have been earned to meet the requirements for graduation. Far too many students have never realized the goal of graduation because of their dislike for the subject and consequent inefficiency in the work. This feeling is largely due to the way in which mathematics has been taught. The zeal for teaching the subject matter of mathematics has been so keen that teachers have failed to keep in mind the necessity for arousing interest on the part of their students in that which is being taught. Dr. Rankin of Duke University very fittingly says:

> Our textbooks and those of us who teach mathematics are so intent that our students shall acquire a certain amount of technique in manipulating the symbols used in mathematics that we have bounded mathematics on the north by X, on the east by sin A, on the south by log X, and on the west by by $\sqrt{-1}$ in about the same way we learned to bound Ohio when we studied geography. The beauty and power of mathematics to set forth truth is lost sight of.

Mathematics should mean more to the pupil than mere problem solving. If the proper attitudes are being acquired by the pupils, the teacher need have no fear that her pupils will not acquire the necessary "technique in manipulating the symbols used in mathematics."

What better way have teachers of mathematics of portraying this "beauty and power of mathematics to set forth truth" at the same time that they are developing the necessary skills and knowledge than by helping their students to develop in themselves proper emotional attitudes by becoming conscious of the presence of mathematics in things all about them?

Why Should We Create a Mathematical Atmosphere?

1. To help make the child mathematics conscious, i. e., to help him to recognize the presence of mathematics in all things about him.
2. To serve as an incentive for growth and development in mathematics.
3. To challenge respect and appreciation on the part of the child for the beauty, truth, and power of mathematics.
4. To make the child a more intelligent user of mathematics and thus increase his efficiency both to himself and to society.
5. To awaken the casual public to some of the uses of mathematics. (This would be accomplished indirectly through the children and directly through the contact of the public with exhibits, etc., portraying the uses of mathematics.)

How May We Create a Mathematical Atmosphere?

1. One of the first means of creating a mathematical atmosphere is by the physical equipment of the mathematics class room.

   Does the appearance of your mathematics class room, because of its equipment, bespeak the fact that it is a room in which mathematics is taught, just as the appearance of the science class room tells one that it is a place where science is taught? What is some of the equipment that will lend an atmosphere of mathematics to the class room?

   (1) One rather essential piece of equipment for the mathematics class room is a graph board. This can easily be made by having one section of the blackboard marked off in inch squares, preferably with green paint.

   (2) A second thing that is quite an asset to a mathematics class room is a bulletin board on which the pupils and the teacher may post interesting articles, pictures, graphs, etc., which show the uses of mathematics.

   (3) A third means of facilitating the work of students in the mathematics class room is through having such blackboard equipment as yard sticks, blackboard compasses, blackboard protractors, T-squares, triangles, etc.

(4) A fourth bit of equipment for the mathematics class room consists of models, a carpenter's level, if possible a transit, an angle mirror, a good slide rule, a decimalized tape, etc.

(5) A fifth thing that lends atmosphere to the class room is the use of pictures of mathematical interest. One inexpensive set of pictures that almost any school can afford for the rooms in which mathematics is taught is the set of prints of mural paintings on the walls of the mathematics rooms of the Lincoln School in New York City. These lovely colored prints portray the history of mathematics. There are three of these pictures, together with a short story of them written by Dr. David Eugene Smith, which may be purchased from the Bureau of Publications, Columbia University. Prints of the men who have been prominent in the development of mathematics may be had from various sources, and these contribute materially to the mathematical atmosphere in the class room. Recently I visited a mathematics class in the Sherman High School. I was delighted to see on the walls of this room pencil and pen sketches of Thales, Plato, Euclid, the Tower of Knowledge, and others of the prints which are coming out each month in the "Mathematics Teacher." The teacher in this room told me that pupils in his classes had made these sketches. What was the value of this? These pupils were helping to make their mathematics room distinctive; they were becoming acquainted at least with the names of some of the men who have contributed to mathematics, and probably they were gaining some knowledge of what each contributed; they in turn may have been awakened to some lasting interest in mathematics. Another type of picture which is appropriate for a mathematics room is a lovely scene or picture of some beautiful building which shows mathematical forms or examples of symmetry.

(6) Slogans, proverbs, mottoes, and the like bearing upon mathematics are a sixth means of calling attention to the fact that the room is one in which mathematics is taught. In this mathematics class which I visited in Sherman, in addition to the sketches of famous mathematicians there were several such proverbs on the walls. Some of these which I recall are "God Eternally Geometrizes,"

"Mathematics Is the Corner Stone of Successful Business," "The Laws of Nature Are the Mathematical Thoughts of God," " 'Ye Shall Know the Truth and the Truth Shall Make You Free.' Mathematics Is Truth." These mottoes had also been made by the pupils in the mathematics classes. If these serve no other purpose than to set the child wondering about such things they have served a worth while purpose.

(7) A reading table in the room with some of the supplementary reading materials that the children can use in connection with the problems they are working on or for leisure reading is another means of creating a mathematical atmosphere. Some of the books which should be on that table are Smith's *Number Stories of Long Ago;* Week's *Boy's Own Arithmetic;* Abbott's *Flatland;* Dudeney's *Amusements in Mathematics,* also his *The Canterbury Puzzles and Other Curious Problems;* Lick's *Recreations in Mathematics;* Smith's *History of Mathematics,* in two volumes, or Sanford's *A Brief History of Mathematics.*

2. A second means of creating a mathematical atmosphere is by making the child mathematics-conscious.

H. G. Wells in *Mankind in the Making* says, "The new mathematics is a sort of supplement to language, affording a means of thought about form and quantity and a means of expression, more exact, compact, and ready than ordinary language. The great body of physical science, a great deal of the facts of financial science and endless social and political problems are only thinkable to those who have had sound training in mathematical analysis. . . ." It is just such an appreciation of the necessity for mathematical efficiency together with an emotionalized attitude toward the universal presence of mathematics that we hope to awaken in our students. One of the best ways of accomplishing this goal is by doing interesting problems or projects which will develop the necessary manipulative skill, and which at the same time will be related to things within the interest and daily experience of the children. Some of the ways of doing this are:

(1) First, by the use of booklets. My own seventh grade mathematics class in their study of general mathematics have decided to keep a sort of class scrap book or diary of the work they do this year. They have finished their first unit, which in their text is called "Measurement—Computation." Before starting into the actual work of this unit, they took a series of standard tests on the four fundamentals in integers and fractions. These they called "Inventory Tests," placing the taking of these tests on the same basis as the merchant taking inventory to see how much he had accomplished and where he needed to restock. In order to get a picture of their standing with reference to the possible score, each child made a bar graph of the class scores and the possible score. In connection with this preliminary work in graphs they brought in bar graphs clipped from the current issues of newspapers and discussed their significance. In discussing a graph showing the growth of a certain insurance company during the past ten years, one of the children asked, "Do they have a bar predicting what the insurance during this year would be?" Upon being told no, he said, "Well I'll bet if we got a graph that would show this year that bar would be less than the last one because this is such hard times." Was this child getting at the real significance of these graphs? The section in their booklet devoted to these tests they have called " 'Check and Double Check' on the Four Fundamentals." Along with the mounted graphs showing their standing, two members of the group have written an essay setting forth the necessity of checking results and checking up on what one has accomplished, giving examples from everyday business and banking.

These children have decided to dedicate their book to "The Makers of Mathematics." The group who are working on the dedication will contribute to the class a little of the history of mathematics by looking up and reporting to the class on some of the "makers of mathematics" and what their contributions have been.

Another group in this class are working up the division of their book which they have chosen to call "Geometric Forms Daily Seen." In this division the pupils have drawn the figures they have studied about, giving the characteristics of each. These drawings are followed by pictures showing examples of these forms as seen in daily use, for example, the triangle in the bridge and in the gable end of the house, the circles in design, etc.

"Measuring Instruments in Daily Use" is the title chosen for another division. This group has mounted a page of pictures of the most common measuring instruments such as scales, measuring cups, teaspoons, tablespoons, clocks, thermometers, rulers, compasses, etc. This is accompanied by an essay on the constant uses made of measuring instruments.

As one part of "Mathematics in Architecture," two members have written an essay on scale drawings, their use in house building, and have illustrated it by the picture of the plan of a house accompanied by the picture of the house.

Another division that has been suggested by members of the group is "Newspaper Clippings That Require an Understanding of Mathematics for Intelligent Reading."

This is only the beginning of the many things that this group of children will do in working up their book. I have not attempted to point out the drill in manipulative skills that the children are getting through measuring to get the size for their book, measuring in drawing their figures and in arranging their work on the pages, etc., for the purpose of this paper is to point out the possibilities of stimulating interest in mathematics by creating a mathematical atmosphere. A few quotations from the essays of the children and remarks made by the children while doing their work will answer the question as to whether these children are interested in the work which they are doing and whether they are becoming more mathematics-conscious:

"Why, I'd never thought of how much we use measurements."

"People don't stop to think of all the measuring instruments they use daily. Just think, the cook in the kitchen uses measuring instruments morning, noon, and night."

"The most important part of house building is the plan. The architect had to know mathematics to know how to let one inch represent so many feet, and how to get the right shape for the various parts of his house."

"We were doing many interesting things in mathematics; so we decided to make a book which will be a record of our year's work. We had an interesting time and everybody enjoyed it."

"I did not think that people used mathematics so much until we began talking about its many uses in our class."

"If you just think about it, there's mathematics in 'most everything you see."

(2) A second way of helping to make children mathematics-conscious is by the use of posters.

In a similar way to that described in the use made of the booklet, posters may be used to create a mathematical atmosphere that will stimulate interest in mathematics. Some interesting subjects which have been suggested for posters are:

"Mathematics in Business."
"What Can Man Do with Mathematics?"
"Mathematics, the Key to the Universe."
"Mathematics in the Home."
"Mathematics in Nature."
"Mathematics in Costumes."
"Mathematics in Art."
"Does the Architect Use Mathematics?"

The description of an experiment carried on in the uses and making of such posters is given by Olive A. Kee in the *Third Year Book* of the National Council of Mathematics Teachers.

(3) A third way of helping to make the child mathematics-conscious is by mathematics exhibits. Posters and booklets together with slogans and the like may be used in the working up of mathematics exhibits. This is also one of the best means of awakening the casual public to some appreciation of mathematics and its many uses.

The stimulation of interest through the development of a proper interpretation of the ever-present mathematical atmosphere is dependent upon the teacher who has an everlasting belief in the importance of his subject, a thorough understanding of the subject matter, an ever-growing stock of interesting bits of mathematical history, an increasing supply of general information upon which to draw for enrichment materials, together with contagious enthusiasm. If teachers of mathematics will resolve to connect mathematics in every way possible with practical life experiences and will strive to open the eyes of each individual to the great possibilities of mathematics, it is not too much to hope that a different notion of mathematics will be developed; so that its real wealth will be recognized and "instead of seeming to be a wilderness of nightmares and terrors it will be a fairyland of flowers and murmuring brooks." Mathematics will be considered "fascinating and easy to understand, a joy to study, a satisfaction when learned."

THEORY VERSUS PRACTICE IN ALGEBRA

By E. E. HEIMANN

New Braunfels High School

One of the biggest problems that confronts a teacher of algebra in teaching the fundamental operations is that of teaching his pupils both to perform these operations and at the same time to understand the principles involved. The question arises as to just how much theory it is possible to teach beginners in algebra. Teachers are apt to go to one of two extremes in the matter, one group believing that it makes little difference whether the student knows why the rules involved are true so long as he can perform the operations quickly and accurately; the other group maintaining that it does one little good to be able to perform the various operations unless he knows the theory involved. Neither one is entirely wrong nor entirely right. It is possible to teach the operations without teaching the theory involved, and, on the other hand, it is possible to teach the theory without developing a great amount of speed and accuracy in performing the various operations. The modern tendency in education seems to favor the former idea rather than the latter. We want people who can do things; who can actually put into practice the things they learn. Theories never put into practice are of little value.

Any teacher of algebra who has kept a close check on the results of his teaching knows that both theory and practice are necessary if his students are to learn algebra so that they will become efficient in performing the operations and in remembering how to perform them later on. Efficiency, which involves accuracy and speed, can be accomplished only through a great amount of drill. But a thorough understanding of the subject comes from a knowledge of the principles involved. Every teacher will find some students who, regardless of his method, learn to perform all operations in a purely mechanical way and who, nevertheless, become quite expert. Such students may be the pride of the teacher at the time, but when they are tested on the same material a few years later, one usually finds that they have forgotten nearly everything they ever knew about the subject. Such was found to be the case with senior students in high school who were given one of the tests given by the Mathematics Department of The University of Texas to their freshman classes. Some of the students who had been "A" students in algebra made very low grades on this test. On the other hand, some students who had been only average students had remembered a great deal more. This test dealt only with algebraic operations, not with theory.

Although the above results show that both theory and practice are necessary in order to give the student a thorough and lasting understanding of the simple operations of algebra, one has to admit that the aim should be to teach these operations until they become almost automatic. That is, we should teach both the actual operations and the theory involved at first, but after awhile the student should learn to perform these operations without having to stop and recall the rules involved. This is possible in algebra just as well as it is possible to add, subtract, multiply, and divide arithmetic numbers automatically. As was pointed out above, this is brought about by a great amount of drill. No student can acquire any speed in performing algebraic operations as long as he has to stop and recall the rule involved for everything he does. In other words, rules have served their usefulness once the student has mastered their practical application. The same is true in any other subject. We learn the parts of speech and the rules of grammar, for instance, in order to learn to use correct English, but the aim should be to make the use of correct English a habit so that rules need no longer be recalled. Not many people remember a great deal about grammar later on in life but they continue to use correct English all the same because it has become a habit. This may not be exactly the same in the use of algebra, for people don't use algebra as constantly as they do English, but it serves to illustrate the principle; namely, that rules and theory are of value only as a means to an end but are not an end in themselves.

Having decided that both theory and its application should be taught to beginners in algebra, it still remains to be decided just how much theory can and should be taught and how this can be accomplished. In the first place, we should realize that a great deal of theory cannot be taught to freshmen. That is, it is possible for the teacher to explain thoroughly the "why" of all of the operations, but one must not expect too thorough an explanation of these from the student. It is possible to see why certain things are true without being able to explain to some one else why they are true. Every teacher of algebra knows that some of the most simple operations are the hardest to explain. Even the teacher finds it hard at times to give a very clear explanation of some of them. That students have more difficulty in giving explanations than they do with the actual operations was found to be the case in a test given to freshman classes. This test included both theory and practice and covered only the fundamental operations. Not only were rules asked for but explanations and examples were required. The group as a whole were found to be able to perform the various operations much better than they were able to answer the questions concerning theory, this in spite of the fact that special efforts had been taken by the teacher to explain all operations thoroughly. There were a few individuals who could state a few rules but could neither explain nor apply them, and there were more who could work all problems but who could not answer any of the questions on theory.

Teaching theory, of course, does not mean merely assigning rules to be memorized. It requires that the student be able to formulate the rules in his own words and to be able at least to apply them, if not to explain them. It does little good to assign pages of rules and explanatory matter to beginners; in fact, if it were not for the drill exercises which expedite the teacher's work, we would be better off without textbooks. All definitions and explanations of new terms and rules should be made by the teacher. The application of rules to problems should be demonstrated first, and then the student should be allowed to formulate his own rules. Any rule stated in the student's own words, however crudely it may be done, is of much more value to him than a parrot-like repetition of a rule in the textbook.

We conclude, then, that the teaching of both theory and practice is necessary in order to give the student a thorough understanding of the subject, but that the theory should be explained by the teacher to the student and not necessarily by the student to the teacher. In other words, teach theory but be satisfied when the student has learned to apply it instead of requiring of the student a detailed explanation of it. It is not necessary to test on everything we teach. If the student has learned to perform the fundamental operations quickly and accurately and the teacher has done his part in explaining the theory underlying these operations, then one may conclude that the student has made a beginning in the subject that will enable him to carry it on successfully.

A NEW ARRANGEMENT OF THE HIGH SCHOOL MATHEMATICS COURSE

By SUPT. L. L. WILKES
Hubbard, Texas

High-school and university instructors are alike deploring the poor work the freshmen students are doing in college mathematics. Much has been written on the subject and some progress has been made in correcting the very apparent deficiencies. The transition from high-school to college mathematics is too abrupt, and the fault lies largely in the construction of the high-school course of study—chiefly the absence of algebra in the tenth and eleventh grades.

In 1927 the Hubbard High School began the reorganization of its mathematics courses in an effort to correlate them better with the college courses. The results as evidenced by favorable reports from last year's graduates, who are now enrolled in twelve different universities and colleges, have justified the experiment and encouraged us to continue and develop it.

None of these pupils has failed in freshman mathematics. Furthermore they have stated that the work was materially lightened by reason of the change.

Briefly stated, this is the plan. Algebra is taught during the eighth and the first half of the ninth grades. Advanced arithmetic is then optional with the pupils for the last semester, if their grades show they are capable of taking this course, otherwise they may make a half credit in some other branch of work. Solid geometry is offered the first semester of the senior year. This course is optional, but we encourage the entire class to enroll because of its importance in preparing them for college mathematics. The possibility of advanced credit and the necessity of taking this work in most colleges without receiving any degree credit are also used to get them to enroll. Algebra is offered during the last semester of the fourth year and is required of all students.

For pupils who are fond of mathematics and expect to major in this subject in the university an additional course is offered the senior class, consisting of advanced arithmetic and trigonometry. This is not a required course. If pupils are pursuing a commercial course they are encouraged to enroll in the arithmetic class. If they expect to enter college the trigonometry is suggested.

However, the entire course is planned in such a manner as to give all the seniors a course in algebra during the last half of the eleventh grade. Thus, more or less algebra is taught all through high school.

The algebra in the eleventh grade begins with quadratics and includes logarithms, the theory of exponents, progressions, permutations and combinations, graphs, the binomial theorem, and as many supplementary topics as time and the progress of the class permit. The point at which the class begins is determined by the progress made in the eighth and ninth grades.

The plan now being followed is by no means complete or fixed, but in general it is more satisfactory than the old order, where algebra was dropped at the end of the ninth year. Some might suggest that plane geometry be introduced in the last semester of the ninth year, and this has been done successfully in many schools; but so many students in this school have had difficulty with plane geometry, that it has seemed wise to defer it until the tenth year, when their minds are more mature. Advanced arithmetic might seem too difficult for ninth-grade pupils, but it is taken only by students who have excellent grades or wish to take up bookkeeping.

Thus far the plan has met with success. It will take another year for the four classes completely to adjust themselves to the program. In the meantime, as we compare results, we may change the arrangement of the optional groups, but this will not affect the half year of algebra in the senior year. The use of this subject in college mathematics creates the necessity for sending pupils from high school with a review of the fundamentals and an introduction to some of the terms and processes which they will use in their university courses. It is to be hoped it will remedy to some extent the present unsatisfactory conditions.

SOME MATHEMATICAL WHIMSICALITIES BY LEWIS CARROLL

(Contributed)

Few out of the thousands of English-speaking children and grown-ups who each year read *Alice in Wonderland* know anything of the author who hid his identity behind the familiar pseudonym Lewis Carroll. They might be surprised to learn that its comical inventions and fanciful fun emanated from a churchman and a teacher, but they would be still more astonished to find that its author was also an erudite mathematician. But such he was—Charles Lutwidge Dodgson, an ordained deacon, for twenty-five years lecturer in mathematics in Christ Church College, Oxford.

There is a story that Queen Victoria was so pleased with *Alice* that she requested that all the author's books be sent to her. Her surprise at receiving a huge package containing such titles as *An Elementary Treatise on Determinants, Formulae of Plane Geometry, Syllabus of Plane Algebraical Geometry* was hardly greater than that of visitors to the Stark Library at The University of Texas, when they find these same volumes side by side with *Alice's Adventures in Wonderland, Through the Looking-Glass,* and *The Hunting of the Snark.*

The gentle humor and exuberant imagination of the creative mind which produced these fanciful works seem as far as possible removed from the severe, undeviating logic necessary to an understanding of the laws of mathematics, fixed and immutable from eternity to eternity. Yet it has often happened that delicate, subtle, whimsical fancy has gone hand in hand with unusual ability in the most austere, exacting, and unforgiving of all sciences. So it was with Professor Charles Dodgson.

This paper is not concerned, however, with his attainments and powers as a mathematician, but rather with a few of his whimsical applications of the vocabulary of his science to matters of current interest.

In 1855, the election of Benjamin Jowett as Regius Professor of Greek in Balliol College, Oxford, raised a great protest from those who suspected him of heresy. The chair to which he was elected was one of four endowed by Henry VIII with 40 pounds per year, and it was the only one of the four that had not been increased by subsequent endowment. Former encumbents, however, had received additional emoluments from other sources; the usual salary at that time was 400 pounds. The enemies of Jowett succeeded in depriving him of this supplement for ten years, until 1865, when the income of the position was raised to 500 pounds per year. Apropos of this long drawn out dispute, Dodgson wrote in 1865 a pamphlet which he called "The New Method of Evaluation as Applied to π." The author's humorous perversion of the symbol π was indicated by the Mother Goose Rhyme on the title page:

Little Jack Horner
Sat in a Corner,
Eating a Christmas Pie.

The problem of evaluating π, he says, has been from the earliest ages down to the present time considered as purely arithmetical. But this generation has discovered that "it is really a dynamical problem," and the true value of π has at last been "obtained under pressure." As the main data of the problem, he submits:

> Let U = the University, G = Greek, and P = Professor. Then GP = Greek Professor; let this be reduced to its lowest terms, and call the result J [Jowett].
>
> Also let W = the Work done, T = the Times, p = the given payment, π = the payment according to T, and S = the sum required, so that π = S.

The problem is to obtain a value for π which shall be commensurate with W.

> In the early treatises on this subject, the mean value assigned to π will be found to be 40.000000. Later writers suspected that the decimal point had been accidentally shifted, and that the proper value was 400.-000000. . . .

No other progress was made in the subject till our own time, though several most ingenious methods were tried for solving the problem.

One of these methods he calls the process of *Rationalization.* After letting H = High Church, L = Low Church, and B = Broad Church, he derives the equation

$$HL = B$$

Also let x and y represent unknown qualities. The process now requires the breaking up of U into its partial factions, and the introduction of certain combinations.

Of the two principal factions thus formed, one represented no further difficulty, but it proved hopeless to rationalize the other, and after "several ingenious substitutions and transformations" had been resorted to it was found that all the y's [wise] were on one side. As repeated trials only "produced the same irrational results, the process was finally abandoned."

The next method tried was that of *Indifferences,* in which v is used to represent novelty and (E + R) [*Essays and Reviews,* the volume in which Jowett's supposedly heretical article had appeared] is assumed as a function of v. This method also disappointed expectations; the opponents of this theorem actually succeeded in demonstrating that the v-element did not even enter into the function.

The attempt to solve the problem by the *Elimination* of *J* [Jowett] having also failed, *Evaluation Under Pressure* was resorted to. By this the value

$$\pi = S = 500.000$$

was reached.

This result differed considerably from the anticipated value, namely 400.000. Still there can be no doubt that the process has been correctly performed, and that the learned world may be congratulated on the final settlement of this most difficult problem.

Another tiny pamphlet in which Dodgson lends his mathematical learning to pure fun is *The Dynamics of a Particle,* less serious in purpose than *The Evaluation of* π, and a bit more evident in its humor. It is introduced thus:

It was a lovely autumn evening, and the glorious effects of chromatic aberration were beginning to show them-themselves in the atmosphere as the earth revolved away from the great western luminary, when two lines might have been observed wending their weary way across a plane superficies. The elder of the two had by long practice acquired the art, so painful to young and impulsive loci, of lying evenly between his extreme points; but the younger, in her girlish impetuosity, was ever longing to diverge and become an hyperbola or some such romantic and boundless curve. They had lived and loved: fate and the intervening superficies had hitherto kept them asunder, but this was no longer to be: *a line had intersected them, making the two interior angles together less than two right angles.* It was a moment never to be forgotten, and, as they journeyed on, a whisper thrilled the superficies in isochronous waves of sound, "Yes. We shall at length meet if continually produced!" (Jacobi's Course of Mathematics, Chap. 1.)

We have commenced with the above quotation as a striking illustration of the advantage of introducing the human element into the hitherto barren region of Mathematics. Who shall say what germs of romance, hitherto unobserved, may not underlie the subject? Who can tell whether the parallelogram, which in our ignorance we have defined and drawn, and the whole of whose properties we profess to know, may not be all the while panting for exterior angles, sympathetic with the interior, or sullenly repining at the fact that it cannot be inscribed in a circle? What mathematician has ever pondered over an hyperbola, mangling the unfortunate curve with lines of intersection here and there, in his effort to prove some property that perhaps after all is a mere calumny, who has not fancied at last that the ill-used locus was spreading out its asymptotes as a silent rebuke, or winking one focus at him in contemptuous pity?

In some such spirit as this we have compiled the following pages. Crude and hasty as they are, they yet exhibit some of the phenomena of light or 'enlightenment,' considered as a force, more fully than has hitherto been attempted by other writers.

The contents of the main body of the treatise are indicated by the following extracts:

Plain Superficiality is the character of speech, in which any two points being taken, the speaker is found to lie wholly with regard to these two points.

Thucydides . . . tells us that the favorite cry of encouragement during a trireme race was that touching allusion to Polar Co-ordinates which is still heard during the races of our own time, 'ρ_5, ρ_6, $\cos\phi$, they're gaining.'

A Surd is a radical whose meaning cannot be exactly ascertained. This class comprises a very large number of particles.

A takes in 10 books in the Final Examination, and gets a 3d Class: B takes in the Examiners, and gets a 2nd. Find the value of the Examiners in terms of books. Find also their value in terms in which no Examination is held.

Serious in its purpose, if humorous in its style, is Dodgson's *Euclid and His Modern Rivals,* the great cause which the author has at heart being the vindication of Euclid's great masterpiece as the best manual for teaching geometry. His arguments are presented, he explains, in the dramatic form, "partly because it seemed a better way of exhibiting in alternation the arguments on both sides of the question"; partly that he might find himself "at liberty to treat it in a rather lighter style than would have suited an essay." Pitying friends warned him, he adds, that in adopting a lighter tone he would alienate the sympathies of all true scientific readers. He answered, "If there is a Scylla before me, there is also a Charybdis. . . . In my fear of being read as a jest, I may incur the darker destiny of not being read at all. . . . I am content to run some risk; thinking it better that the purchaser of this little book should *read* it, though it be with a smile, than that, with the deepest conviction of its seriousness of purpose, he should still leave it unopened on the shelf."

The scene is a college study. Time midnight. Minos is reading examination papers in geometry. The answers have caused him to run his fingers through his hair until it radiates in all directions like the second Corollary of Euclid. Rhadamanthus appears and asks, "Are we bound to mark an answer that's a clear logical fallacy?"

Minos answers, "Of course you are—with that peculiar mark which cricketers call 'a duck's egg,' and thermometers 'zero'."

By and by Minos falls asleep, and the ghost of Euclid enters.

"Now what is it you really require in a Manual of Geometry?" he asks.

Minos protests that the question is an abrupt opening of a conversation between two beings "two thousand years apart in history," and suggests that they pass a few preliminary remarks by way of becoming acquainted. His visitor answers, "Centuries are long, my good sir, but *my* time to-night is short: and I never was a man of many words."

The long dialogue ends with Euclid's speech:

> 'The cock doth craw, the day doth daw,' and all respectable ghosts ought to be going home. Let me carry with me the hope that I have convinced you of the importance, if not the necessity, of retaining my order and numbering, and my method of treating straight lines, angles, right angles, and (most especially) Parallels. Leave me these untouched, and I shall look on with great contentment while other changes are made.

To the sound of slow music Euclid and the other visiting ghosts "heavily vanish" according to Shakespeare's approved stage directions. Minos wakes with a start, and betakes himself to bed, a "sadder and wiser man."

The prophecy of Dodgson's "pitying friends" has been fulfilled in that the world has refused to accept this work seriously, but in encountering Scylla, he did not avoid Charybdis, for neither does it enjoy at the present, if indeed it ever did, any great degree of popularity. Yet it is an able defense of the unique position of Euclid's *Elements* as a first textbook in geometry and a serious contribution to Euclidean geometry—and it is good fun as idle reading.

USE OF MATHEMATICS IN THE STUDY OF CERTAIN PROBLEMS IN EDUCATIONAL PSYCHOLOGY

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The effort to deal with certain problems in educational psychology in a scientific manner has led to the use of a considerable amount of mathematical theory. In using such material, psychology is following the lead of other sciences, like physics, chemistry, and engineering. It is true that much of this type of work in psychology has been done by those not concerned with education, but the relation of such work to educational psychology is often so clear-cut that there can be no excuse on the part of the psychologist for not being familiar with the field.

The first type of work to be mentioned comes in the field of probabilities. The specific thing which I have in mind is the curve of distribution. The equation of this curve in a simplified form is as follows:

$$y = Ke^{-Cx^2}$$

In this equation, y and x are variables and K, e, and C are constants.

It has been found by experimentation that many psychological data distribute themselves approximately according to this curve. As a result of this, the curve is often accepted as a kind of norm for determining the reliability of tests. That is, if a test is given to a large group of unselected children and the results distribute themselves according to this curve, this is taken as one type of evidence that the test has been well constructed.

This curve also serves as a basis for the study of central tendencies and variability. By this is meant that the probable error or the standard deviation can be stated in terms of chance. That is, if a considerable number of test scores are tabulated and found to distribute themselves according to this curve, it can be determined that the chances are so many out of one hundred or so many out of a thousand that the true average will vary only so much from the calculated average.

Other uses might be given for the curve if space permitted; so that the two cited above should be considered only as representative.

Another field in which mathematics has been applied is that of Individual Differences. In 1884, Sir Francis Galton began work which has given a definite slant to much of American psychology. One of the methods which he employed is correlation.

It is important from many standpoints to know something of the relation that exists between two abilities, such as that in English and that in mathematics. Correlation is a tool by which such a relation can be obtained. Relations of this type are often expressed by the following formula:

$$x = r \frac{\sigma_x}{\sigma_y} y$$

where x and y are variables, r is the coefficient of correlation, and σ_x and σ_y are the standard deviations of the two variables.

It is clear that often there will be more than two variables concerned in such problems. When this is true, the situation is dealt with by means of partial and multiple correlation. It is also true that there are several other forms of correlation, so that data in a variety of forms can be used.

At present correlation is accepted as one of the most powerful tools at the command of psychology, and is being used in what might be spoken of as both extensive and intensive ways.

Another type of work of interest at this time is the famous law of Weber in the field of psychophysics. Weber's problem had to do with determining how accurately small differences between different stimuli can be perceived. As a result of his experimental work, he derived the law

$dR/R = C$, in which C is a constant, R is a standard stimulus, and dR is the increment which must be added to R so that a distinction can be made by the subject between R and $R + dR$.

Later, Fechner took up this work and restated Weber's law in the following manner: $S = C \log R$, where S is the sensation, R the stimulus, and C a constant. A recent modification of this work has resulted in certain educational scales for the measurement of educational products, as handwriting and drawing. Thorndike assumed that differences in quality of specimens of handwriting which are equally often noticed are equal. On this basis, he devised a scale of handwriting made up of specimens of which the differences in quality had been determined. Such scales have proved their importance in educational work.

The next type of work to be mentioned has to do with the learning curve. It has been shown that progress in learning often obeys the following law:

$$Y = \frac{LX}{X + R},$$

in which Y is attainment in terms of number of successful acts per unit of time; X is formal practice in terms of total number of practice acts after the beginning of formal practice; L is the limit of practice in terms of attainment units; and R is the rate of learning and indicates the relative rapidity with which the limit of practice is being approached. While this equation has been criticized by some, and while it probably does not apply to all types of learning, yet it has been of considerable help in understanding some phases of learning.

The last type of work to be mentioned is concerned with the problem of mental organization. Spearman, an English psychologist, has been largely responsible for this, although a number of psychologists have made contributions to it. The problem in which Spearman is interested is concerned with whether mental ability may be resolved into two types of factors, one of them to be denoted by g, which is general in nature, and the other designated by s, which is specific in nature.

In dealing with a simple form of this problem, Professor Spearman sets down what he speaks of as a factor pattern by writing:

$$x_1 = m_1 g + n_1 s_1$$
$$x_2 = m_2 g + n_2 s_2$$
$$x_3 = m_3 g + n_3 s_3$$
$$x_4 = m_4 g + n_4 s_4$$

In dealing with these equations, the basic theorem for the statistical evidence is often given as follows: four variables may be considered as due to one general factor (g) plus four uncorrelated specific factors (s) when $r_{12} r_{34} = r_{13} r_{24} = r_{14} r_{23}$, where r_{12} is the correlation between x_1 and x_2 and r_{34} is the correlation between x_3 and x_4 and so forth. Attention should also be called to the fact that the letters x, g, and s denote variables measured from their respective means, while the m's and n's are constant. Further, an assumption is made that the four variables designated by the s's are uncorrelated with g.

The converse theorem concerning these equations is also of interest. This is, if the relations $r_{12} r_{34} = r_{13} r_{24} = r_{14} r_{23}$ hold, then the factor pattern given by the above equations may be obtained with zero correlation between the s's and g. This mathematical theory has been hailed by a number of psychologists as being one of the powerful tools at the hand of psychology.

In many of the investigations where these conditions have been set up and carried through, it has been found that there is a general factor and specific factors, although the g factor is apparently often small. It is clear that such a theory has a direct bearing on the problem of transfer of training, if not in every phase of learning.

In giving courses pertaining to the foregoing type of work, the matter of prerequisites becomes important. While it may not be advisable to lay down definite courses, yet it is often necessary for the student to know something of determinants, conic sections, and curve fitting, along with the calculus. There is also some theory in which differential equations have value.

In a recent number of *Science* there appeared a note concerning some work in which a certain law or principle of modern physics has been employed to explain the conditional reflex. This may mean that the psychologist of the future will be called upon to know some phases of modern physics, along with its background of mathematics.

REASONS ASSIGNED FOR THE POOR PREPARATION OF UNIVERSITY FRESHMEN

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For many years it has been a source of wonder to me that students from our best high schools are frequently unable to perform the most usual arithmetic and algebraic processes and lack familiarity with the most widely known and generally used geometric theorems. Why should such be the case? Being unable to give a satisfactory answer to this question myself, I have asked others many times for an answer. I am now giving the replies that I have received from various classes of people, among whom are University students, high-school students, parents of both of the preceding classes, teachers, and school-board members. The teachers whose various opinions I record are employed in military schools, in high schools—both junior and senior, in towns and cities of different sizes, in elementary schools, and in colleges and universities. I am in no wise passing judgment on these reasons assigned. I do not say that they are right or wrong, true or false. I merely state them as an effort towards the solution of the problem of college preparation, and as a reply to the many queries that have been addressed to me personally asking for an explanation of the situation in which we find ourselves.

The first teacher to whom I propounded my query concerning the lack of preparation is an old student friend of mine, now an instructor of physics in a small town high school. I well remember her reply, characteristically brief. She said, "It's the fault of the superintendents and principals, I believe." "Recently," she continued, "monthly report cards were sent out. A teachers' meeting was held a few days later and the principal informed me that a large proportion of the failing grades given on the cards was in my physics. He began to criticize, whereupon I begged immediately that he desist while I proceeded to explain the necessity for the grades I had given." She then added, that while the criticism of the principal at this point meant little to her and that she continued to give the grades that were made, such an attitude did seriously affect many younger and less firmly established teachers.

Only this spring an earnest teacher in a junior high school in a large city of Texas told me how delighted she was when her principal allowed her to fail a large number of her pupils whom to pass would seriously interfere with their future progress. She added, "You know most principals object to students not being passed."

The second person interrogated was also a teacher of science, but in a high school in one of our large cities. Strange to say, she again put the blame on principals and superintendents, but from a slightly different angle. She cited as evidence the fact that when their head teacher of English reported adversely in regard to graduating one of the seniors, the superintendent required the teacher to give the girl several English examinations, resulting ultimately in the girl's being graduated. One occurrence of this kind serves to intimidate teachers, and when such incidents are almost frequent, the teacher soon becomes powerless.

On the other hand, a principal recently left the matter of graduating a student entirely in the teacher's hands, the teacher saw no reason for giving another examination, and the student failed to graduate.

Other teachers have given as a reason for poor preparation on the part of the students the statement that the teacher is "too near the parents." This is intended to imply that some parents do not support the teachers in their dealings with the pupil. Perhaps this incident will make clear the attitude of some parents. A teacher had sent a U-note to the mother and father of one of the pupils; the note said that the pupil's work was poor and asked the parents to see to it that the boy studied his mathematics at home. The reply from the mother was to the effect that if the teacher would explain the examples, the boy would work at home, but that because the teacher did not explain sufficiently the boy could not study by himself. The teacher happened to be of the type who explained fully both in class and in conference; so the parent had not even investigated the justice of her comment. I have heard another teacher—admittedly an excellent one—tell through the years the remarks made to her by some of the parents whose children she failed or gave bad marks, until I sometimes have felt that indeed she did live too near some parents.

Another teacher assigns as one reason the fact that parents, if not openly criticizing teachers, fail to coöperate with them. Such parents do not see to it that their children study their daily lessons carefully and regularly. In fact, a surprisingly large number of teachers have said that parents are not at home enough to superintend the children's study.

Another phase of this same complaint is that parents allow their children to attend social functions until late hours on school nights, the next day finding them at school with unlearned lessons and depressed vitality. Again, week-end visits make the pupil and teacher join the printer in having a blue Monday, I am told.

The most incredible explanation that I was ever accorded was given by a teacher who had been principal of a school in a small town. He assured me that in visiting a class in arithmetic in his school he actually heard the teacher, then a junior in a Texas college, drill the pupils on adding fractions, teaching them that $1/2 + 1/3 = 2/5$, adding numerators for a new numerator and denominators for a new denominator. Recently a teacher told me that she thought that a fertile source of poor teaching, particularly in mathematics, is the fact that many athletic coaches are hired for their athletic ability and then given mathematics to teach. A somewhat similar reason was that many teachers who have been assigned mathematics had specialized in the universities in English, Spanish, or other nonmathematical subjects, the further statement being made that the reason for students' not specializing in mathematics is that mathematics requires more intense application and often longer hours of labor.

Now comes a lady who has lived in at least a dozen states in the Union assuring me that in many schools the course in arithmetic is not given according to directions, but that frequently considerable portions of it are omitted for the simple reason that the teachers do not understand it and cannot teach it.

A board member told me that he attributed poor preparation on the part of freshman students to the fact that teachers are judged by the number of students that are passed. The same statement was made in a junior college in South Carolina, and the president and teachers of this college devoted quite a bit of time to explaining to the college regents why such a criterion is a false one. It is a frequently heard statement that what the public schools of the state need to make them more efficient is a liberal number of failures for several years. Teachers in all grades of schools tell me that the board member's opinion is correct and that teachers are expected to pass entirely too large a proportion of their students.

It was also a member of a school board who said that teachers are not only required to teach, but that they are expected to "learn" the children their lessons, not being able to demand sufficient home study on the part of the children. This same idea was reiterated only yesterday, a teacher from another state saying that children are not even expected to put forth enough effort to learn any subject thoroughly.

Nor have I failed to propound my query to many parents. One reason assigned by the mother of a high-school girl was that the students are graded too high. English themes with numerous errors and lacking in neatness were given "A." This she thought resulted in low standards on the part of the child. Another reason assigned by a parent was that students' papers are returned with mistakes uncorrected or not returned at all. She felt that not enough individual attention was given each student. Another mother told me that her children had to take part in so many activities in the school other than lessons that they actually had no time to get lessons. In regard to this same criticism, teachers say that they have little or no time to give individual attention to pupils and fail to do their work in a way really satisfactory to themselves because they have to devote much time to matters other than giving instruction.

Parents also speak of lack of coöperation on the part of teachers with them and lack of interest on the part of teachers, but in general their criticism seems to be of failure on the part of schools to hold up high standards of performance in academic subjects.

One parent objected to lessons assigned on sheets instead of from books, urging that children in the public schools could not keep such sheets intact and were hence left without adequate subject matter for their courses.

Reasons assigned by students in the University for their lack of preparation are numerous. I have often been told by boys interested in athletics that their athletic excellence won them immunity from academic failure. Often they tell me that they had a spell of illness when some particular subject was being studied in the high school and that they never did understand the principles involved. Again, the excuse is that a certain teacher spent his time selling real estate, or that the teacher gave them a "model" example always instead of insisting on independence on the student's part, or that cheating was customary and hence pupils did not learn their work thoroughly, that nobody studied in the high schools, and so on ad infinitum.

One of the most interesting comments on this subject was made by a professor of education from another state. He expressed great chagrin because he found his students unable to take a variety of viewpoints in any field of knowledge which he had carefully expounded for them. They seemed able to see only the method of approach which he himself had used. They wanted him to work examples for them which they might use as models. And then he added, "It is too late to develop in them a broad viewpoint now;

they should have been trained by such a method in the primary grades."

Of course the statement is often made that some who enter the University are not "university material" or "can't learn mathematics." On the other hand, it has been said that "I do not want to exonerate the schools by saying that some folks can't learn mathematics, for I do not believe that." Some rather discerning teachers are urging that the trouble lies considerably in the methods used in teaching mathematics. They say that the subject is taught as a mass of facts and performances and not as a logical development; that pupils are allowed to reproduce processes without understanding underlying principles; and that intellectual mastery is not set as a standard to be attained by the pupil. It might be put thus, that the true meaning of education (= *e-*, out of, and *ducere,* to lead) has passed from the center of the stage, and that the main object on the part of the teacher is not to incite the pupil to intellectual interest and activity, with the result that the pupil comes to set for his aim the gentle art of "getting by." Or again, it might be said that the greatest failure is not that pupils learn few facts and processes—these are by-products—but that they fail to acquire a method of study and an abiding interest and pleasure in intellectual activity.

THE THREE FAMOUS PROBLEMS OF GREEK GEOMETRY

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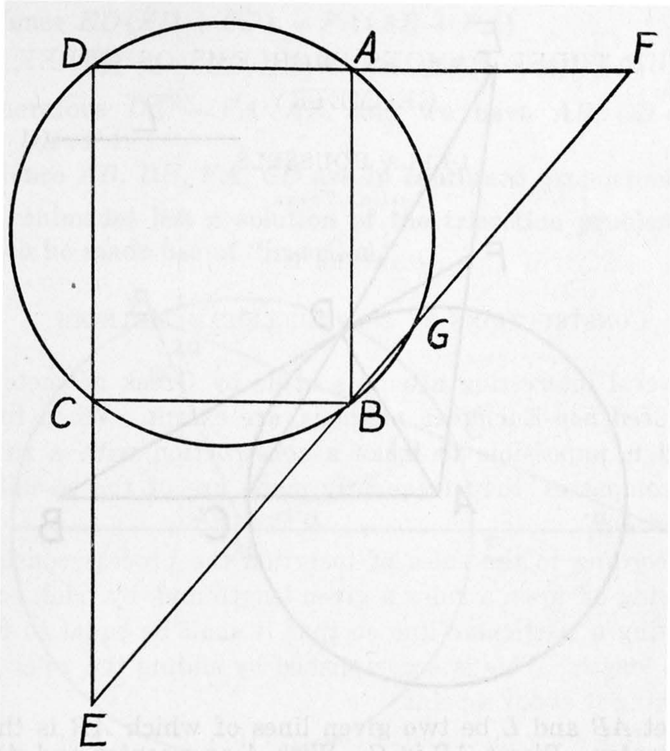
CHAPTER II

Constructions by Non-Euclidean Methods

Several interesting attempts made by Greek geometers, who used non-Euclidean methods, are extant. When they found it impossible to make a construction with a ruler and compasses, they frequently made use of the so-called "insertion."

According to the rules of insertion the process consists of laying off upon a ruler a given length and, by trial, constructing a particular line so that it shall be equal to the given length. This is accomplished by sliding the ruler or revolving it about a point.

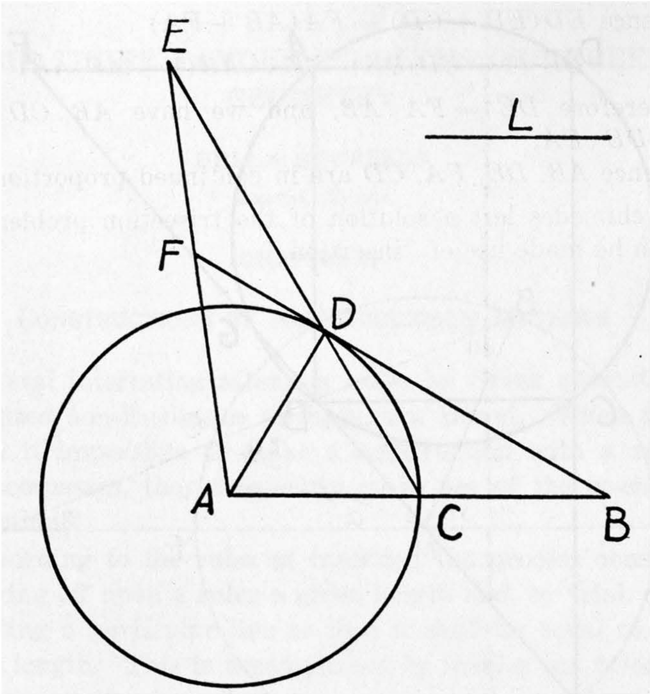
The following mechanical construction of two mean proportionals between two given straight lines is due to Philo of Byzantium:



Let the extremes *AB, BC* be placed at right angles to each other; complete the rectangle *ABCD*, and describe a circle about it. Produce *DA, DC*, and let a graduated ruler be made to revolve around the point *B*, and so adjusted that *BE* shall be equal to *GF;* then *AF, CE* are two mean proportionals between *AB, BC*. Since *BE* is equal to *GF*, the rectangle $BE \cdot GE = BF \cdot GF$. Therefore $DE \cdot CE = DF \cdot AF$; hence $DE : DF :: AF : CE$; and by similar triangles, $AB : AF :: DE : DF$, and $CE : CB :: DE : DF$. Hence $AB : AF :: AF : CE$, and $AF : CE :: CE : CB$. Therefore *AB, AF, CE, CB* are continual proportionals. Hence *AF, CE* are two mean proportionals between *AB* and *BC*.[2]

Newton also gave a mechanical solution to the same problem.

[2]Casey, John: *Elements of Euclid*, p. 304.



Let AB and L be two given lines of which AB is the greater. Bisect AB in C. With A as a center and AC as a radius, describe a circle, and in it place the chord CD equal to the second line L. Join B, D, and draw by trial through A a line meeting BD, CD produced in the points E, F, so that the intercept EF shall be equal to the radius of the circle. DE and FA are the mean proportionals required. Join AD. Since BF cuts the sides of the triangle ACE, we have $AB \, . \, CD \, . \, EF = CB \, . \, DE \, . \, FA$; but $EF = CB$; therefore $AB \, . \, CD = DE \, . \, FA$ or $CD/DE = FA/AB$. Again, since the triangle ACD is isosceles, we have

$$\begin{aligned} ED \, . \, EC &= \overline{AE}^2 - \overline{AC}^2 \\ &= (FA + AC)^2 - \overline{AC}^2 \\ &= 2FA \, . \, AC + \overline{FA}^2 \\ &= FA \, . \, AB + \overline{FA}^2 \end{aligned}$$

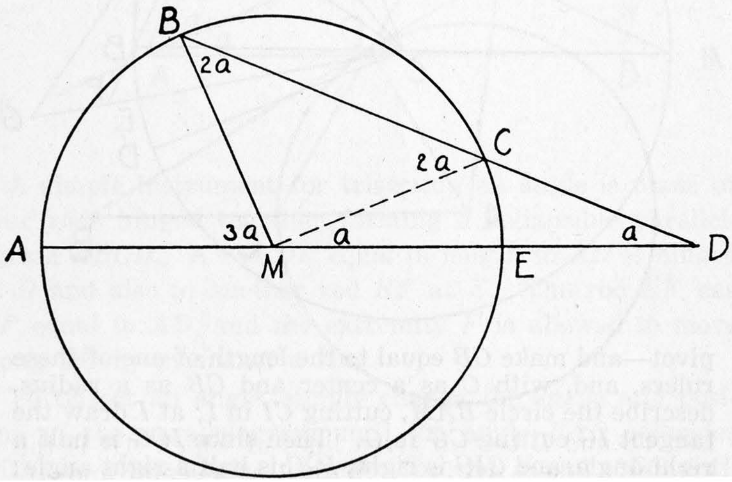
Hence $ED(ED + CD) = FA(AB + FA)$

$$\overline{DE}^2(1 + CD/DE) = FA \, . \, AB(1 + AF/AB)$$

therefore $\overline{DE}^2 = FA \, . \, AB$, and we have $AB \, . \, CD = DE \, . \, FA$.

Hence *AB, DE, FA, CD* are in continued proportion.[3]

Archimedes left a solution of the trisection problem in which he made use of "insertion."

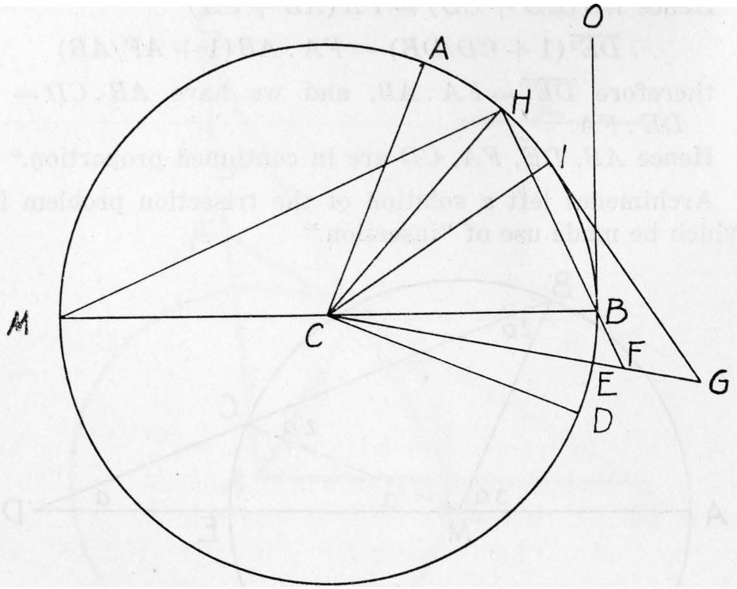


Required: to divide the arc *AB* of the circle with center *M* into three equal parts.
Draw the diameter *AE*, and through *B* draw a secant cutting the circumference in *C* and the diameter *AE* in *D*, so that *CD* equals the radius *r* of the circle. Then arc $CE = 1/3$ arc *AB*.[4]

The following mechanical method of trisecting an angle was worked out by Dr. John Casey of the Royal University of Ireland:

To trisect a given angle *ACB*. Erect *CD* perpendicular to *AC;* bisect angle *BCD* by *CG* and make the angle *ECI* equal to half a right angle; it is evident that *CI* will fall between *CA* and *CB*. Then, if we use a jointed ruler—that is two equal rulers connected by a

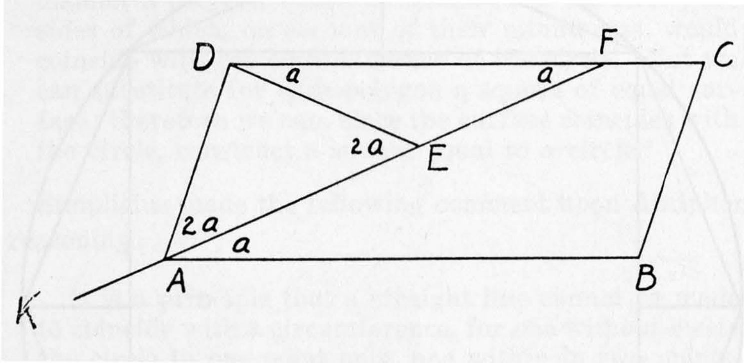
[3]Casey, John: *Elements of Euclid,* pp. 305–306.
[4]Fink, Karl: *A Brief History of Mathematics,* p. 208. (Translated by W. W. Beman and D. E. Smith.)



pivot—and make CB equal to the length of one of these rulers, and, with C as a center and CB as a radius, describe the circle BAM, cutting CI in I; at I draw the tangent IG cutting CG in G. Then since ICG is half a right angle, and CIG is right, IGC is half a right angle; therefore IC is equal to IG; but IC equals CB; therefore $IG = CB$ — equal length of one of the two equal rulers. Hence if the rulers are opened at right angles, and placed so that the pivot will be at I, and one extremity at C, the other extremity at G; it is evident that the point B will be between the two rulers; then, while the extremity at C remains fixed, let the other be made to traverse the line GF, until the edge of the second ruler passes through B; it is plain that the pivot moves along the circumference of the circle. Let CH, HF, be the position of the rulers when this happens: draw CH; the angle ACH is one-third of ACB.

Produce BC to M. Join HM. Erect BO at right angles to BM. Then, because $CH = HE$, the angle $HCF = HFC$, and the angle $DCE = ECB$ (construction). Hence the angle $HCD = HBC$ and the right angles ACD, CBO are equal; therefore the angle ACH

is equal to *HBO;* that is equal to *HMB;* or half the angle *HCB*. Hence *ACH* is one-third of *ACB*.[5]



A simple instrument for trisecting an angle is made of four rods hinged together forming a collapsible parallelogram *ABCD*. A rod *DE* equal in length to *AD* is hinged at *D* and also to another rod *KF* at *E*. The rod *KF* has *EF* equal to *AD*, and the extremity *F* is allowed to move along *DC* by means of a ring.

To trisect an angle, simply change the shape and position of the parallelogram until the angle *BAD* coincides with the given angle and move the end *F* along *DC* until *FK* passes through *A*. $\triangle EFD$ and $\triangle ADE$ are isosceles; $\angle DEA = \angle EAD = 2\angle EFD = 2\angle BAF$.

$$\therefore \angle BAF = 1/2 \angle FAD, \text{ or } \angle BAF = 1/3 \angle BAD.$$

The following method of squaring the circle is the one given by Eudemus as that used by Antiphon:

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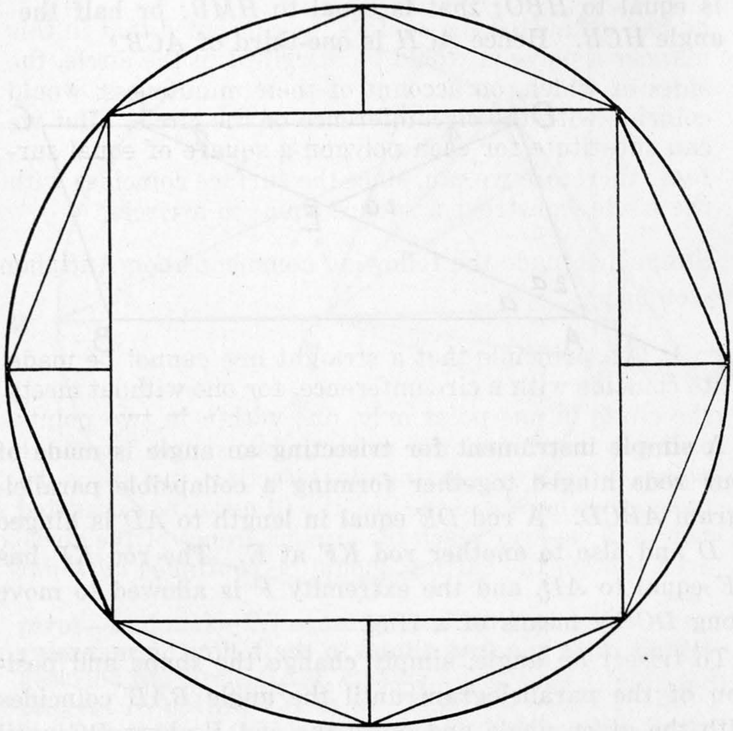
[5]Casey, John: *Elements of Euclid*, pp. 307–308.



Antiphon, having drawn a circle, inscribed in it one of those regular polygons that can be inscribed—let it be a square. Then he bisected each side of the square, and through the points of section drew straight lines at right angles to them, producing them to meet the circumference; these lines evidently bisect the corresponding segments of the circle. He then joined the new points of section to the ends of the sides of the square, so that four triangles were formed, and the whole inscribed figure became an octagon. And again, in the same way, he bisected the sides of the octagon, and drew from the points of bisection perpendiculars; he then joined the points where these perpendiculars met the circumference with the extremities of the octagon, and thus formed an inscribed figure of sixteen sides. Again, in the same manner, bisecting the sides of the inscribed polygon of sixteen sides, and drawing straight lines, he formed a polygon of twice as many sides; and doing the same thing again and again, until he had exhausted the surface, he concluded that in this manner a polygon would be inscribed in the circle, the sides of which, on account of their minuteness, would coincide with the circumference of the circle. But we can substitute for each polygon a square of equal surface; therefore we can, since the surface coincides with the circle, construct a square equal to a circle.[6]

Simplicius made the following comment upon Antiphon's reasoning:

> It is a principle that a straight line cannot be made to coincide with a circumference, for one without meets the circle in one point only, one within in two points, and not more, and the meeting takes place in single points. Yet, by continually bisecting the space between the chord and the arc, it will never be exhausted, nor shall we ever reach the circumference of the circle, even though the cutting should be continued *ad infinitum*.[7]

Hippocrates squared a lune in the following manner and attempted to square a circle in the same manner.

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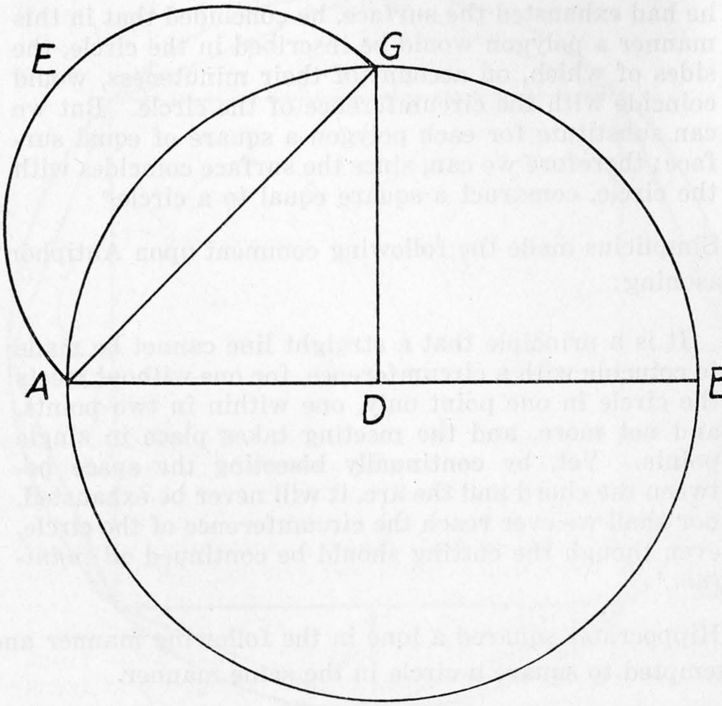
[6]Allman, George J.: *Greek Geometry from Thales to Euclid*, pp. 65–66.

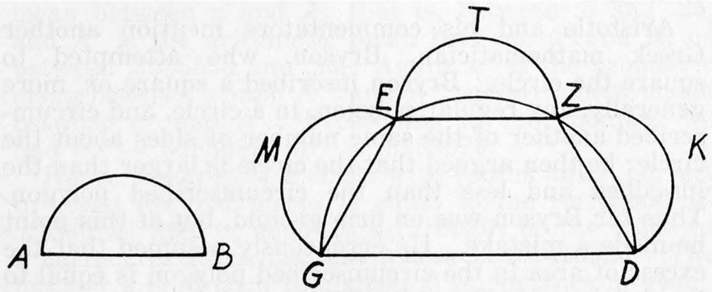
[7]Rupert, W. W.: *Famous Geometrical Theorems and Problems with Their History*, p. 44.



Let a semicircle *ABG* be described on the straight line *AB;* bisect *AB* in *D;* from the point *D* draw a perpendicular *DG* to *AB,* and join *AG;* this will be the side of a square inscribed in the circle of which *ABG* is the semicircle. On *AG* describe a semicircle *AEG.* Now, since the square on *AB* is equal to double the square on *AG* (and since the squares on the diameters are to each other as the respective circles or semicircles), the semicircle *AGB* is double the semicircle *AEG.* The quadrant *AGD* is equal to the semicircle *AEG.* Take away the common segment lying between the circumference *AG* and the side of the square; then the remaining lune *AEG* will be equal to the $\triangle$ *AGD;* but this triangle is equal to a square.

Having thus shown that the lune can be squared, Hippocrates next tries by means of the preceding demonstration to square the circle thus:—

Let there be a straight line as *AB,* and let a semicircle be described upon it; take *GD,* double of *AB* and on it also describe a semicircle; and let the sides of a hexagon *GE, EZ, ZD,* be inscribed in it.



On these sides describe semicircles, *GME, ETZ, ZKD.* Then each of these semicircles described on the sides of the hexagon is equal to the semicircle on *AB,* for *AB* is equal to the side of the hexagon. The four semicircles are equal to each other, and together are then equal to four times the semicircle on *AB:* but the semicircle on *GD* is also four times that on *AB.* The semicircle on *GD* is therefore equal to the four semicircles on the sides of the hexagon. Take away from the semicircles on the sides of the hexagon and from that on *GD* the common segments contained by the sides of the hexagon and the periphery of the semicircle *GD;* the remaining lunes *GME, ETZ, ZKD* together with the semicircle on *AB* will be equal to the trapezium *GEZD.* If we now take away from the trapezium the excess, that is, a surface equal to the lunes (for it has been shown that there exists a rectilineal figure equal to a lune), we shall obtain a remainder equal to the semicircle *AB;* we double this rectilineal figure which remains and construct a square equal to it. That square will be equal to the circle of which *AB* is the diameter, and thus the circle has been squared.[8]

Eudemus gave the following criticism of this effort to square the circle:

This treatment of the problem is indeed ingenious; but the wrong conclusion arises from assuming that

[8]Allman, George J.: *Greek Geometry from Thales to Euclid,* pp. 67–68.

as demonstrated generally which is not so; for not every lune has been shown to be squared; but only that which stands over the side of a square inscribed in a circle; but the lunes in question stand over the side of a hexagon.[9]

Aristotle and his commentators mention another Greek mathematician, Bryson, who attempted to square the circle. Bryson inscribed a square or, more generally, any regular polygon, in a circle, and circumscribed another of the same number of sides about the circle; he then argued that the circle is larger than the inscribed and less than the circumscribed polygon. Thus far Bryson was on firm ground, but at this point he made a mistake. He erroneously assumed that the excess of area in the circumscribed polygon is equal to the deficiency in the case of the inscribed polygon, and concluded that the circle is the mean between the two.[10]

APPROXIMATIONS OF π

Several expeditious methods, depending on higher mathematics, are known for calculating the value of π. The following is an outline of a very simple elementary method for approximating this important constant. It depends upon a theorem which is at once inferred from Ex. 87, Book VI of Euclid, namely, If a, A denote the reciprocals of the areas of any two polygons of the same number of sides inscribed and circumscribed to a circle; a', A' the corresponding quantities for polygons of twice the number; a' is the geometric mean between a and A, and A' the arithmetic mean between a' and A'. Hence, if a and A are known, we can, by the process of finding arithmetic and geometric means, find a' and A'. In like manner, from a', A' we can find a'', A'' related to a', A'; as a', A' are to a, A. Therefore proceeding in this manner until we arrive at values $a^{(n)}$, $A^{(n)}$ that will agree in as many decimal places as there are in the degree of accuracy we wish to obtain; and since the area of a circle is intermediate between the reciprocals of $a^{(n)}$ and $A^{(n)}$, the area of the circle can be found to any required degree of approximation. If for simplicity we take the radius of the circle to be unity, and commence with the inscribed and circumscribed squares, we have

[9]Rupert, W. W.: *Famous Geometrical Theorems and Problems with Their History*, p. 46.
[10]*Ibid.:* p. 47.

$a = .5,$ $A = .25$

$a' = .3535533,$ $A' = .3017766$

$a'' = .3264853,$ $A'' = .3141315$

These numbers are found thus: a' is the geometric mean between a and A: that is, between .5 and .25, and A' is the arithmetic mean between a' and A, or between .3535533 and .25. Again, a'' is the geometric mean between a' and A'; and A'' the arithmetic mean between a'' and A'. Proceeding in this manner, we find $a^{(13)} = .3183099$; $A^{(13)} = .3183099$.

Hence the area of a circle with radius unity, correct to seven decimal places is equal to the reciprocal of .3183099; that is, equal to 3.1415926; or the value of π correct to seven decimal places is 3.1415926. The number π is of such fundamental importance in geometry, that mathematicians have devoted great attention to its calculation. Mr. Shanks, an English computer, carried the calculation to 707 decimal places. . . . The result is here carried far beyond all requirements of mathematics. Ten decimals are sufficient to give the circumference of the earth to the fraction of an inch, and thirty decimal places would give the circumference of the whole visible universe to a quantity imperceptible with the most powerful microscope.[11]

CHAPTER III

A Proof of the Impossibility of These Constructions with Ruler and Compasses

Before attempting to show that the constructions under consideration cannot be made with ruler and compasses, it is necessary to determine just what constructions are possible with these instruments.

The first step in considering a problem for construction is to formulate it analytically. In some instances it is possible to set up simple algebraic equations. For instance, $x^3 = 2a^3$ represents the duplication problem, where a is the edge of a given cube.

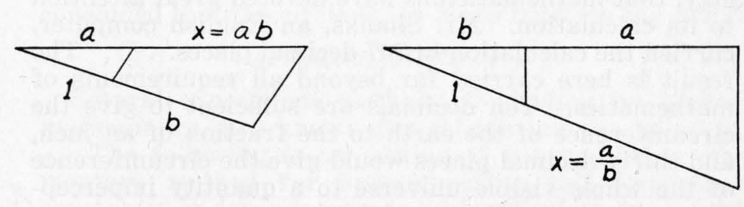
Usually it is possible to employ analytical geometry. A point is determined by its coördinates x and y with reference to fixed axes, a straight line by an equation of the first

[11]Casey, John: *Elements of Euclid,* pp. 309–310.

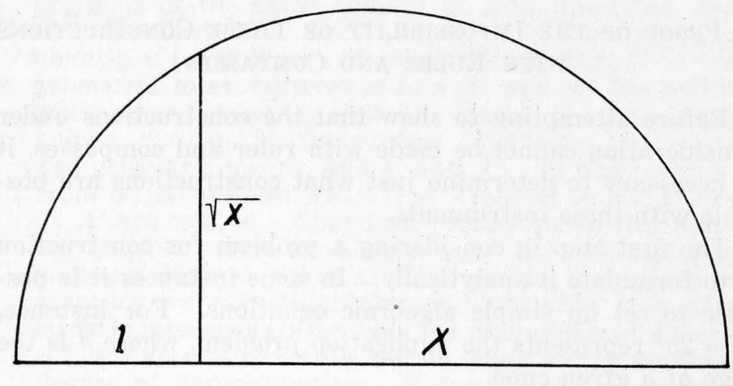
degree, and a circle by an equation of the second degree. We are therefore concerned with numbers, some of which are the coördinates of points, some the coefficients of equations, while others represent lengths, areas, or volumes. These numbers define analytically the different elements involved in the construction.

The construction of the sum or difference of line segments is equivalent to transferring a length, which is possible with ruler and compasses.

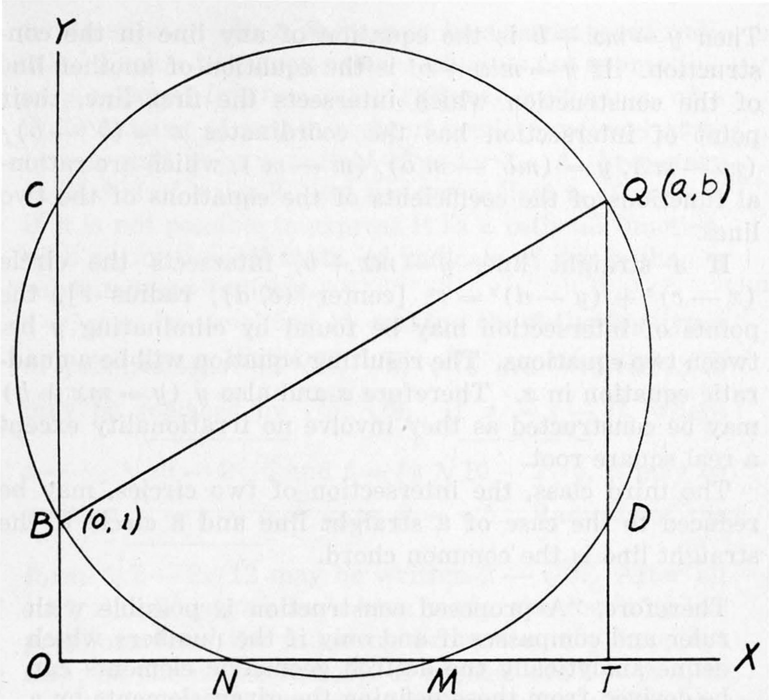
As parallel lines intercept proportional segments on all transversals, the product or quotient of line segments may be constructed also.



The square root of a given number may be constructed:



The quadratic equation $x^2 - ax + b = 0$ may be solved if the lengths a and b are known.



Take a pair of axes and mark the points $B(0,1)$ and $Q(a,b)$. With BQ as a diameter construct a circle. The abscissas of the points of intersection of the circle and the x-axis are the roots of the equation. The coördinates of the center of the circle are $(a/2, (b+1)/2)$, and $\overline{BQ}^2 = a^2 + (b-1)^2$. $\therefore$ the equation of the circle is $(x - a/2)^2 + [y - (b+1)/2]^2 = [a^2 + (b-1)^2]/4$, which reduces to $x^2 - ax + b = 0$ when $y = 0$. The two roots are real and unequal, real and equal, or imaginary, as the circle cuts the x-axis, is tangent to it, or fails to intersect it.

If a construction is possible with ruler and compasses, the straight lines and circles drawn in making the construction are determined by points initially given or obtained by the intersection of two straight lines, a straight line and a circle, or two circles.

We may take the axes of coördinates so that no line of the proposed construction will be parallel to the y-axis.

Then $y = mx + b$ is the equation of any line in the construction. If $y = m'x + b'$ is the equation of another line of the construction which intersects the first line, their point of intersection has the coördinates $x = (b' - b)/(m - m')$, $y = (mb' - m'b)/(m - m')$, which are rational functions of the coefficients of the equations of the two lines.

If a straight line, $y = mx + b$, intersects the circle $(x - c)^2 + (y - d)^2 = r^2$ [center (c, d), radius r], the points of intersection may be found by eliminating y between two equations. The resulting equation will be a quadratic equation in x. Therefore x and also y $(y = mx + b)$ may be constructed as they involve no irrationality except a real square root.

The third class, the intersection of two circles, may be reduced to the case of a straight line and a circle if the straight line is the common chord.

> Therefore, "A proposed construction is possible with ruler and compasses if and only if the numbers which define analytically the desired geometric elements can be derived from those defining the given elements by a finite number of rational operations and extractions of real square roots."[12]

The duplication of the cube and the trisection of an arbitrary angle may each be represented by cubic equations with rational coefficients but having no rational roots.

> Let the equation $x^3 + mx^2 + nx + p = 0$ have rational coefficients m, n, and p and let x_1 be a root of this equation such that $\pm x_1$ may be constructed with ruler and compasses. If x_1 is rational we have already shown that it can be constructed so we will now consider the case when x_1 is irrational. As x_1 is irrational and can be constructed with a ruler and compasses, it can be obtained by a finite number of rational operations (addition, subtraction, multiplication and division) and the extraction of real square roots. Therefore x_1 involves at least one square root but no other

[12]Dickson, L. E.: *First Course in the Theory of Equations*, p. 30.

irrationalities. In x_1 there may be superimposed radicals. Such a two-story radical which is not expressible as a rational function, with rational coefficients, of a finite number of square roots of positive rational numbers is said to be a radical of order 2. A three-story radical is of order 3. An n-story radical is of order n if it is not possible to express it as a rational function, with rational coefficients, of radicals of fewer than n superimposed radicals.

x_1 may be simplified by making the following types of substitutions: If $\sqrt{5}$, $\sqrt{3}$, $\sqrt{15}$ are involved $\sqrt{15}$ may be replaced by $\sqrt{5}$, $\sqrt{3}$. If $x_1 = s - 7t$, where $s = \frac{1}{2}\sqrt{10 - 2\sqrt{5}}$ and $t = \frac{1}{2}\sqrt{10 + 2\sqrt{5}}$, it may be written $x_1 = s - 7\sqrt{5}/s$, as $st = \sqrt{5}$. Radicals of the form $\sqrt{7 - 2\sqrt{12}}$ may be written $2 - \sqrt{3}$. After all such simplifications have been made, the resulting expressions have the following properties (by hypothesis); no one of the radicals of highest order n in x_1 is equal to a rational function, with rational coefficients, of the other radicals of the same or lower order, and no radical of $(n - 1)$ order is equal to a rational function of the other radicals of the same or of lower order, and so forth.

Let $\sqrt{K}$ be a radical of highest order which appears in x_1. Then $x_1 = (a + b\sqrt{K})/(c + d\sqrt{K})$ where a, b, c, d do not involve $\sqrt{K}$ but may involve other radicals. If $d = 0$, then $c \neq 0$ and $x_1 = (a + b\sqrt{K})/c$. If $d \neq 0$ we may multiply both numerator and denominator by $(c - d\sqrt{K})$ and write $[(a + b\sqrt{K})(c - d\sqrt{K})]/[c^2 - d^2K]$. In either case $\sqrt{K}$ would appear only in the numerator. By hypothesis $x_1 = (a + b\sqrt{K})/c$ is a root of $x^3 + mx^2 + nx + p = 0$, therefore we may write: $[(a + b\sqrt{K})/c]^3 + m[(a + b\sqrt{K})/c]^2 +$

$n[(a + b\sqrt{K}) / c] + p = A + B\sqrt{K}$, where A and B are polynomials in a, b, c, K and the rational numbers m, n, and p. Since $(a + b\sqrt{K}) / c$ is a root of $x^3 + mx^2 + nx + p = 0$, $A + B\sqrt{K} = 0$. If $B \neq 0$, $\sqrt{K} = -A/B$ and $\sqrt{K}$ can be expressed as a rational function, with rational coefficients, of the radicals of lower order than $\sqrt{K}$ which appear in x_1. As this is contrary to the hypothesis, $B = 0$ and hence $A = 0$.

If $(a - b\sqrt{K}) / c$ is substituted for x in $x^3 + mx^2 + nx + p = 0$, the result is $A - B\sqrt{K} = 0$. But $B = A = 0$, therefore, $x_2 = (a - b\sqrt{K}) / c$ is another root of our equation. The sum of the three roots of our equation must equal $-m$, so the third root is $x_3 = -m - (2a/c)$. By hypothesis m is rational and if $2a/c$ is also rational we have found a rational root to our equation.

Suppose $2a/c$ is irrational. As $2a/c$ is part of a root which can be constructed with ruler and compasses, its only irrationalities are square roots. Let $\sqrt{s}$ be one of the radicals of the highest order in $2a/c$. Then by the same argument used earlier in this discussion, $x_3 = g + h\sqrt{s}$, where neither g nor h involves $\sqrt{s}$. But if $g + h\sqrt{s}$ is a root of the equation, $g - h\sqrt{s}$ is also a root different from x_3 and must be equal to x_1 or x_2 as there are only three roots to our equation.

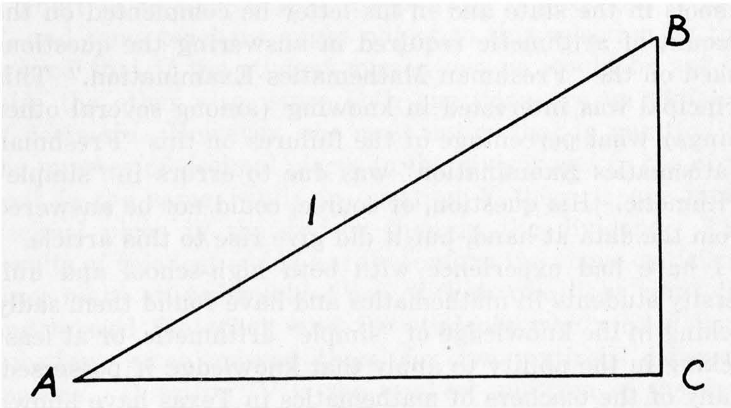
$\therefore g - h\sqrt{s} = (a \pm b\sqrt{K})/c$. $\sqrt{s}$ was defined as one of the radicals occurring in a/c and every radical occurring in g or h occurs in x_3 and hence in $a/c = \frac{1}{2}(-m - x_3)$, m being rational. Therefore all radicals occurring in a/c occur also in $x_3 = (a - b\sqrt{K})/c$. Hence $\sqrt{K}$ may be expressed rationally in terms of the radicals occurring in a, b, c, and hence in x_1, which is contrary to our hypothesis. Hence a/c is rational and the equation $x^3 + mx^2 +$

$nx + p = 0$ must have a rational root if it can be constructed with ruler and compasses.

Therefore it is not possible to construct, with ruler and compasses, a line whose length is a root or the negative of a root of a cubic equation with rational coefficients but having no rational roots.[13]

The equation $x^3 = 2a^3$, where a is the edge of the cube, represents the duplication of the cube and has no rational root, therefore it is not possible to construct the roots of this equation with ruler and compasses.

The trisection problem would be possible if we could construct the cosine of one-third of the given angle. Given an angle A:



If $AB = 1$, $\cos A = AC$. By using the trigonometric identity $\cos A = 4\cos^3(A/3) - 3\cos(A/3)$ and multiplying each member by 2, then substituting $x = 2\cos(A/3)$, we obtain the equation $x^3 - 3x = 2\cos A$, which represents the trisection problem.

For certain values of A, as 90°, 180°, this equation has a rational root. But for $A = 120°$, it becomes $x^3 - 3x + 1 = 0$, which has no rational root. Therefore not all angles can be trisected by the use of ruler and compasses.

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[13]Dickson, L. E.: *First Course in the Theory of Equations*, p. 32.

HIGH-SCHOOL STUDENTS AND SIMPLE ARITHMETIC

By BEE GRISSOM
Austin High School

A few weeks ago I had occasion to visit Miss Mary E. Decherd, Adjunct Professor of Pure Mathematics at The University of Texas. While in her office, she showed me one of the letters which she had received pertaining to the report of the examination on high-school algebra and plane geometry given to all students taking freshman mathematics in The University of Texas in the fall of 1930. This communication was from the principal of one of our high schools in the state and in his letter he commented on the amount of arithmetic required in answering the questions asked on the "Freshman Mathematics Examination." This principal was interested in knowing (among several other things) what percentage of the failures on this "Freshman Mathematics Examination" was due to errors in "simple" arithmetic. His question, of course, could not be answered from the data at hand, but it did give rise to this article.

I have had experience with both high-school and university students in mathematics and have found them sadly lacking in the knowledge of "simple" arithmetic, or at least lacking in the ability to apply that knowledge if possessed. Many of the teachers of mathematics in Texas have known of this weakness in our students for several years, and some of them have tried in a small way to remedy the situation.

One of the larger high schools in the state very graciously consented to give an arithmetic test to its students in mathematics, and the following examination was given to 986 students:

Perform the indicated operations:

I. $1/2 + 1/5 - 1/6 =$
II. $62.8 \div 2.6 =$ (to two decimal places)
III. $46.29 \times 23.4 =$
IV. Add: 9,864, 603, 7,425, and 30
V. $895.3 \div 3.75 =$ (to two decimal places)

The students were given no notice of this examination and each teacher gave it as if it were one of her own "pop quizzes." Some of the students, after seeing the nature of the examination, asked if the grades were going to be recorded and in practically every case the teacher's answer was yes. The students were very much pleased with the nature of the test and they thought that it was extremely easy. Each teacher was told to conduct the test as if it were her own and to take up the papers when, in her opinion, a sufficient number of students had finished. A ten to fifteen minute time limit was suggested, but each teacher was to use her own judgment, and the result was an average time limit of fifteen minutes.

There is, to some extent, a repetition in questions II and V, and some teachers might object to this selection on the ground that if the student missed one he would be apt to miss the other also, since both questions are on division of decimals. However, one must notice that in question II the number of decimal places in the divisor and in the divident is the same, while in question V there is one more decimal place in the divisor than in the dividend. The results of this test would hardly confirm the above criticism, since many students solved one of these questions correctly and missed the other, and the students who missed both questions often missed them for two entirely different reasons. Question IV is the weakest question of the five. I probably let my curiosity get the better of my judgment in making out this question when I omitted all decimals. Since 88 students missed this "simple" question, I hesitate to think of the number who would have missed it if decimals had been included.

We gave this examination with the intention of following it up with another test on the same subject matter. We thus expected to obtain some idea of the value of a review of these fundamental operations of arithmetic, which the student had learned as more or less separate operations and had been endeavoring to apply to more or less meaningless problems. The first test, however, was all that time would permit our giving.

For convenience in tabulation and comparison the students taking the first half of second-year algebra shall be referred to as 9A; those in the last half of the second year of algebra, as 9B; the first half of plane geometry, as 10A; the second half of plane geometry, as 10B; the students taking the elective courses, namely, solid geometry, trigonometry, and a specially designed algebra course for seniors, shall be referred to as 11A-B students. The questions were marked either right or wrong, although a slight margin was allowed in marking questions II and V which will be explained later. Hence a student missing two or more problems made a grade of 60 or less and failed the examination.

Classification	Number Students Passing	Number Students Failing	Per Cent Passing
9A	80	198	28.7
9B	57	119	32.3
10A	101	152	39.9
10B	47	78	37.6
	285	547	34.2
11A-B	115	39	74.6
Total	400	586	40.5

The following table will be of interest to those who wish to go further than general totals. Failures in questions II, III, and V were marked as to one of two causes, namely, "arithmetic" and "decimal." By "arithmetic" I mean that the problem was missed because of some error in one of the four fundamental operations of arithmetic, i. e., addition, subtraction, multiplication, or division of numbers, and by "decimal" I mean a misplaced decimal point. When the student was in error as to both arithmetic and decimal I tabulated the cause as arithmetic. For example, the answer to problem II is 24.15. Should the student have 0.24 as his answer he is in error as to the decimal point, but if his answer were 24.07, or 240.7 his error is in arithmetic.

Students who have taken the "Freshman Mathematics Examination" tell me that the statement, "exact to two decimal places," is very confusing. They ask if the statement means to carry the result to two decimal places and stop, or if it means to carry the result to three decimal places and drop the number in the third decimal place if that number is less than five and add one to the number in the second decimal place if the number in the third decimal place is five or greater than five. The confusion arises, of course, from the lack of distinction between the two statements: find the result to two decimal places and find the result exact to two decimal places. In grading these test papers it was decided not to make this distinction, since some teachers had emphasized the difference in meaning of these two statements more than others had. Hence problem V, for example, was marked correct if the student found any of these results: 238.74, 238.746, or 238.75.

With this explanation I present the second table:

Class	Total Taking Exam.	I	II Arith.	II Dec.	III Arith.	III Dec.	IV	V Arith.	V Dec.
9A	278	56	60	94	78	9	34	92	107
9B	176	34	47	52	64	3	11	67	65
10A	253	37	50	63	79	9	22	105	77
10B	125	31	26	25	48	6	11	52	41
11A-B	154	14	17	11	31	3	10	43	14
Total	986	172	200	245	300	30	88	359	304

The tables are self-explanatory and hence there is little need for comment. Certainly no one can be very proficient in mathematics if he is not a master of the fundamentals of "simple" arithmetic. Something is surely radically wrong when only 40.5 per cent of 986 high-school students pass a "simple" arithmetic examination. I have no doubt that a similar condition may be found in most if not all of our high schools in Texas.

The gradual increase in the percentage of students passing as one progresses from the 9A group to the 10B group might be due to several causes, but I think that it is probably due to differences in conditioning and experience of the students. The fact that the 10A's have a slightly higher percentage of passing than the 10B's is probably not significant. The 10A's had but recently completed their algebra course, in which they had more arithmetic than the

10B's had in their 10A geometry course, and this may have had some effect. The 11A-B group has a decidedly higher rating than any of the other groups, and this is due primarily to the type of students in it. This group is composed entirely of students who chose, for some reason of their own, to take more than the required number of courses in mathematics; hence they form a better group of students.

A glance at the second table discloses a very interesting but very disagreeable fact; namely, our students cannot multiply, divide, and subtract. I mention subtraction because it is involved in division and was the cause of many an error. It is no wonder that our high-school students when they become university freshmen have so much difficulty with freshman mathematics.

Is there any legitimate reason why some of the students missed all five of the above problems? Is there any reason why the answer to problem IV should be 9,872,058 or that the answer to problem III should be 107,918.616? Why should students in arranging problem IV for addition arrange the left side of the column in a straight line? Some students added decimals to some of the numbers in this problem but had no decimals in the result, and other students placed no decimals in the items to be added, yet a decimal appeared in the result. The number of students who missed problem III is probably very surprising also.

What shall we do about this "ignorance" and "carelessness"? We need both to broaden and narrow our courses of study. We should broaden our courses to enable the reteaching of some of the essentials taught in lower grades, and narrow or rather shorten our wanderings into the world of mere words. We ought to spend more time on word study, to teach our students to "read" mathematics, to teach them to "translate" the English into the language of mathematics and the mathematics into English, and spend less time on "puzzle hunting" for the answer. We should add a taste of the fundamentals to each of our courses even at the expense of a "prize puzzling" word problem or theorem.

A football team must first learn the fundamentals of football if it hopes to rank high in the percentage column. These fundamentals once learned, cannot be "laid on the shelf" as mastered for all time, but must be practiced over and over. Each football season must begin with a drill on fundamentals. How like this phase of football is mathematics. Without constant review and drill even the best of us lose much of our efficiency. A special effort should be made to keep our students more fit by adding even a little "simple" arithmetic to our course of study in each grade.

COLLEGE STUDIES AND PROFESSIONAL TRAINING

A Statistical Study in Harvard University

By PRESIDENT A. LAWRENCE LOWELL
Harvard University

Educators are proverbially conservative, and it is only in comparatively recent times they have turned to the experimental methods of science as a substitute for deductive reasoning based upon hypotheses. The experiments have dealt in great part with the most pregnant of all educational problems—the question how far an aptitude acquired in one field can be transferred to another. The upshot of many experiments made of late years by a number of independent observers would appear to be: that, in so far as the acquired aptitude depends upon familiarity with the subject matter, it can be transferred little, if at all; that, in so far as it rests upon methods of thought, in other words, upon mental processses, it can be transferred to a considerable extent wherever the same methods are applicable; and finally that, in so far as it is based upon general principles of work, or what is perhaps the same thing, upon a general moral attitude, upon such qualities as diligence, persistence, and intensity of effort, it can be transferred almost indefinitely.

Now we do not know to what extent each of these three elements enters as a factor in any continuous course of study, or in what way their relative importance may vary at different stages of maturity. The experiments have been made, for the most part, with school children, and it may be that young men and women are more capable than children of applying in a new field methods of thought acquired in an old one. The experiments have been made also—and very properly—with a view of isolating particular aptitudes, and it may be that in the more complex pursuits of higher education, where many methods of thought are employed, and perhaps a number of aptitudes developed, the transferability becomes greater. This is, indeed, what we call resourcefulness, and has been supposed to be one of the chief benefits of a broad education.

Moreover, the experiments have covered very brief periods, often not more than a couple of weeks, and it may well be that in so short a time methods of thought are acquired little, and a general moral attitude still less, as compared with a familiarity with the subject matter. If so, the last of those elements would be given an exaggerated prominence. If experiment should show that a boy who played baseball or tennis for a fortnight gained nothing in ability to row a boat, it would not follow that the general dexterity and the physical and nervous vigor he would acquire by playing those games for several years would not give him a vast advantage in rowing over a boy who, during all that time, has taken no strenuous exercise at all. Nor does it follow that, although when first placed in a boat he might be very inferior to a trained oarsman, he might not with a little practice improve rapidly until the difference had almost, if not quite, disappeared. Education can never be made fully an experimental science, for highly instructive as the experiments are, they cannot be continued long enough to test thoroughly the results of different curriculums. We cannot, merely in order to see what the consequences will be, subject boys to a course of study covering several years, unless we believe that it is the best for them. Except for experiments, therefore, too brief to have a lasting effect upon the pupil, we are confined to observing the results of various types of education adopted, not for experimental objects, but because they are supposed to be good for the student.

Fortunately, in the case of our colleges we have in America a great variety of curriculums chosen by the students themselves, and it would seem that some light may be thrown upon these problems by a careful examination of the different courses pursued by undergraduates. A difficulty presents itself, however, in measuring the results. Archimedes declared that if he had a lever long enough, and a fixed point to rest it on, he could move the world. In this case it is not easy to find the fixed point. Statistics have been compiled on several occasions to show the relation between rank in college, and success in life, the success in life being usually measured either by the appearance of the names of the graduates in *Who's Who in America,* or by the estimates of classmates and others. But the first of these tests gives too much weight to purely literary or scholastic qualities, the latter to a prominence often due in great part to wealth acquired by inheritance or marriage. Neither of them is, therefore, an accurate criterion of personal achievement. For this purpose the rank attained in a professional school after graduating from college has great value. It is not, indeed, a measure of ultimate success in life. It is by no means remote enough from college for that; but it has, at least, the merit of being a fairly accurate test of capacity so far as it goes.

In applying, however, any method of determining the relation between elective college studies and subsequent achievement, we must beware of assuming that the results are necessarily a consequence of the college work, for there are two factors in the problem, two independent variables in the equation; one of them the value of the college work in training the mind, the other the selective process whereby men of different tastes and abilities are drawn to elect the subjects for which they have a natural aptitude. If, therefore, the men who pursue certain subjects, or who attain high marks in them, have a better rank in the professional school than their comrades, it may be because the study of those subjects, or the effort put forth to obtain the marks, has fitted them in a peculiar degree for the professional work; or it may be that the men are impelled to choose those subjects, or to work hard, by the very qualities which would insure proficiency in the professional school whatever their college training had been; or again these two factors may be combined in unknown proportions. Now it is clearly important to separate the effects of the two factors, if possible, so that one of them can be measured by itself; and this is in fact done where the result is negative. If the men, for example, who have pursued quite different subjects in college do equally well in the professional school, it proves that the training given by one of those subjects is not distinctly superior to that given by another, for there is obviously no selective factor at work which would counterbalance any such superiority. This is one of the cases where a negative result is more decisive than a positive one. If, on the other hand, the result is positive, and college work in any subject, or of any grade, is followed by a peculiarly high average rank in the professional school, we must have the presence of these two factors constantly in mind, and try to contrive some other test which will eliminate or reduce the effect of one of them; we must seek another equation between the two independent variables.

Harvard University is singularly rich in material for determining the relation of college studies to the work of professional schools, because nowhere in the world have so large a body of undergraduates been so free, for so long a period, as in Harvard College to study whatever they chose, and to make any combination of courses they pleased. With the exception of one required course in English, and sometimes one in another modern language, the election of courses has been almost wholly free for a quarter of a century, and in fact the variety of combinations made has been almost limitless. Moreover, the Law and Medical Schools have contained a large number of graduates of Harvard College, and this is essential for a fair comparison of the results, because there is little use in comparing men who have studied, let us say, history in our college with those who have studied science in another, unless we can assume, what is rarely true, that the quality of students and the standards of scholarship in the two colleges are the same.

The statistics here presented cover, therefore, only bachelors of arts of Harvard College who graduated afterwards from the Harvard Law and Medical Schools, and they comprise only men who took twelve courses, or nearly three years' work, in the college.

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We are now in a position to consider the one apparently positive result of the pursuit of different subjects in college, that is the superiority in the Law School of men who took six or more courses in the group of philosophy and mathematics. This, as already pointed out, is in fact due to those who elected mathematics; but the number of men who took six courses in the subject is so small that it seemed well to enlarge the class by including all those who took four or more courses. There were only sixteen such men in all during the twenty years; of whom ten won a degree *cum laude* in the Law School, whereas at the average rate only three would be expected to do so. But if one examine the grades of these men in college, we find that they were far above the ordinary rank, three of them having graduated *cum laude,* eight *magna cum laude,* and two *summa cum laude.* Calculating the chances for men of that rank, it appears that 5.28 of them would be expected to attain a *cum laude* in the Law School. This is nearer, but still far below, the actual result.

It is interesting to compare with the mathematicians the men who took four or more courses in the classics. They numbered one hundred and fifteen, being decidedly more frequent in the earlier years. At the average rate for the whole class, twenty-one of them would have obtained a *cum laude* in the Law School, whereas in fact thirty-one achieved it. But the men who elect classics are also better scholars than the average and calculating their chances according to their rank in college, we should expect to find twenty-seven of them in the *cum laude* list of the Law School, a number not very far from the actual result. If, therefore, one can draw any inference from figures so small, the case of mathematics is singular. Unless some other element enters into the problem, such as an unusually high standard in the department, or an unusually vigorous intellectual appetite on the part of students who elect the subject, the result may be supposed to indicate, so far as it goes, that mathematics, although rarely selected for the purpose, is a particularly good preparation for the study of law; perhaps because the methods of thought in the two subjects are more nearly akin than is commonly supposed.

Leaving aside this possibly exceptional case, the conclusions to be derived from the facts presented in this paper would seem to be that, as a preparation for the study of law or medicine, it makes comparatively little difference what subject is mainly pursued in college, but that it makes a great difference with what intensity the subject is pursued—or, to put the same proposition in a more technical form, familiarity with the subject matter, which can be transferred little if at all, is of small importance in a college education, as compared with mental processes that are capable of being transferred widely, or with the moral qualities of diligence, perseverance, and intensity of application which can be transferred indefinitely. The practical deduction is that in the administration of our colleges, and, indeed, in all our general education, as distinguished from direct vocational or professional training, we have laid too much stress on the subject, too little on the excellence of the work and on the rank attained.

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