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Capacity Bounds for Some Gaussian Interference Channels

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To my family

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Capacity Bounds for Some Gaussian Interference Channels

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In current wireless networks, co-channel interference is the major limiting factor in achieving high spectral efficiency. The effective interference at receivers can be minimized by using advanced interference management techniques. Given channel conditions, what is the fundamental limit on maximum spectral efficiency we can achieve, and which encoding and decoding techniques achieve this limi t? These research questions can be addressed as network information theory problems. In particular, the capacity of Gaussian interference channels is an important open problem dealing with these fundamental questions. Some special cases of the interference channels and their capacity regions are studied in this dissertation.

For a class of partially connected interference channels, approximate capacity regions are characterized. The impact of topology, interference alignment, and the interplay between interference and noise are discussed. The results show that for these channels, genie-aided outer bounds are tight to within a constant gap from capacity. Near-optimal achievable schemes, based on rate-splitting and lattice alignment, are presented.

The Gaussian X-channel is also an important Gaussian interference channel model. Lower and upper bounds on the sum-rate capacity are derived for this channel. The achievable schemes are based on layered lattice coding and compute-and-forward decoding. For different regimes of channel parameters, some combinations of encoding and decoding strategies are designed. For some range of channel parameters, the approximate sum-rate capacity is characterized to within a constant gap.

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Chapter 1

Introduction

1.1 Motivation

The capacity of the interference channel remains one of the most challenging open problems in network information theory. The capacity region is not known in general, except for a specific range of channel parameters. For the two-user scalar Gaussian interference channel, where the interference alignment is not required, the approximate capacity region to within one bit is known [1]. For the channels where interference alignment is required such as the K-user Gaussian interference channel [11, 2, 3, 4, 5, 7] and the Gaussian X-channel [11, 9, 10], a tight characterization of the capacity region is not known, even for symmetric channel cases.

A tractable approach to the capacity of interference channels is to consider partial connectivity of interference links and analyze the impact of topology on the capacity. Topological interference management [8] approach gives important insights on the degrees-of-freedom (DoF) of partially connected interference channels and their connection to index coding problems [19, 20, 21, 22, 23, 24, 25, 26]. It is shown that the symmetric DoF of a partially connected interference channel can be found by solving the corresponding index coding problem.

We consider a class of three-user partially connected interference channels and characterize approximate capacity regions at finite SNR. We focus on the impact of interference topology, interference alignment, and the interplay between interference and noise. We choose a few representative topologies where we can achieve clear interference alignment gain. For these topologies, Z-channel type outer bounds are tight to within a constant gap from the corresponding inner bound. For each topology, we present an achievable scheme based on rate-splitting, lattice alignment, and successive decoding.

The Gaussian X-channel is another challenging open problem, previously studied in [11, 9, 10]. The channel model has the same 2-by-2 physical links as those in the two-user Gaussian interference channel. But, there are four message sets, one for each transmitter-receiver pair. It was shown in [11] that the degrees-of-freedom (DOF) of $\frac{4}{3}$ is achievable for almost all channel realizations by using *real* interference alignment. In [9], the generalized degrees-of-freedom (GDOF) results are derived by using a deterministic channel approach and its application to the Gaussian case. In [10], a lower-triangular deterministic channel approach is developed to show a constant-gap capacity result for the channel realizations outside an explicit outage set.

1.2 Related Work

Lattice coding based on nested lattices is shown to achieve the capacity of the single user Gaussian channel in [12, 28]. The idea of lattice-based interference alignment by decoding the sum of lattice codewords appeared in the conference version of [4]. This lattice alignment technique is used to derive capacity bounds for three-user interference channel in [2, 3]. The idea of decoding the sum of lattice codewords is also used in [13, 14, 15] to derive the approximate capacity of the two-way relay channel. An extended approach, compute-and-forward [16, 17] enables to first decode some linear combinations of lattice codewords and then solve the lattice equation to recover the desired messages. This approach is also used in [7] to characterize approximate sum-rate capacity of the fully connected K-user interference channel.

The idea of sending multiple copies of the same sub-message at different signal levels, so-called Zigzag decoding, appeared in [5] where receivers collect side information and use them for interference cancellation.

The K-user cyclic Gaussian interference channel is considered in [6] where an approximate capacity for the weak interference regime ($\mathrm{SNR}_k \geq \mathrm{INR}_k$ for all k) and the exact capacity for the strong interference regime ($\mathrm{SNR}_k \leq \mathrm{INR}_k$ for all k) are derived. Our type 4 and 5 channels are $K = 3$ cases in *mixed* interference regimes, which were not considered in [6].

1.3 Organization

In Chapter 2, we explain some preliminaries on lattice coding. In Chapter 3, a class of partially connected interference channels are studied. The capacity outer bounds are derived in Section 3.2. Lattice coding-based achievable rate regions for each channel type and the corresponding gap analysis are given in Section 3.3–3.7, respectively. Random coding achievable regions are given in Section 3.8 and 3.9. In Chapter 4, the symmetric Gaussian X-channel is studied. In Section 4.4, the sum-rate capacity upper bound is proved. In Section 4.5, we present achievable schemes based on layered lattice coding, interference alignment, and layer-by-layer successive decoding, and we prove the achievability part. In Section 4.6, achievable schemes based on compute-and-forward decoding is explained. We discuss conclusions in Chapter 5.

1.4 Notation

Signal $\mathbf{x}_{ij}$ is a coded version of message M_{ij} with code rate R_{ij} unless otherwise stated. The single user capacity at receiver k is denoted by $C_k = \frac{1}{2}\log\left(1+\frac{P}{N_k}\right)$. Let $\mathcal{C}$ denote the capacity region of an interference channel. Also, let $\mathcal{R}_i$ and $\mathcal{R}_o$ denote the capacity inner bound and the capacity outer bound, respectively. Thus, $\mathcal{R}_i \subset \mathcal{C} \subset \mathcal{R}_o$. Let δ_k denote the gap on the rate R_k between $\mathcal{R}_i$ and $\mathcal{R}_o$. Let δ_{jk} denote the gap on the sum-rate $R_j + R_k$ between $\mathcal{R}_i$ and $\mathcal{R}_o$. For example, if

$$\mathcal{R}_i = \{(R_j, R_k) : R_k \leq L_k, R_j + R_k \leq L_{jk}\} \tag{1.1}$$

$$\mathcal{R}_o = \{(R_j, R_k) : R_k \leq U_k, R_j + R_k \leq U_{jk}\}, \tag{1.2}$$

then $\delta_k = U_k - L_k$ and $\delta_{jk} = U_{jk} - L_{jk}$. For side information graph, we use graph notation of [24]. For example, $\mathcal{G}_1 = \{(1|3), (2), (3|1)\}$ means that node 1 has an incoming edge from node 3, that node 2 has no incoming edge, and that node 3 has an incoming edge from node 1. $\log(\cdot)$ is base-2 logarithm. $a \simeq b$ means a and b are approximately equal up to a constant.

Chapter 2

Preliminaries

2.1 Lattice Coding

Lattice Λ is a discrete subgroup of $\mathbb{R}^n$, $\Lambda = \{\mathbf{t} = \mathbf{G}\mathbf{u} : \mathbf{u} \in \mathbb{Z}^n\}$ where $\mathbf{G} \in \mathbb{R}^{n\times n}$ is a real generator matrix. Quantization with respect to Λ is $Q_\Lambda(\mathbf{x}) = \arg\min_{\lambda\in\Lambda} \|\mathbf{x} - \lambda\|$. Modulo operation with respect to Λ is $M_\Lambda(\mathbf{x}) = [\mathbf{x}] \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x})$. For convenience, we use both notations $M_\Lambda(\cdot)$ and $[\cdot] \bmod \Lambda$ interchangeably. Fundamental Voronoi region of Λ is $\mathcal{V}(\Lambda) = \{\mathbf{x} : Q_\Lambda(\mathbf{x}) = \mathbf{0}\}$. Volume of the Voronoi region of Λ is $V(\Lambda) = \int_{\mathcal{V}(\Lambda)} d\mathbf{x}$. Normalized second moment of Λ is $G(\Lambda) = \frac{\sigma^2(\Lambda)}{V(\Lambda)^{2/n}}$ where $\sigma^2(\Lambda) = \frac{1}{nV(\Lambda)} \int_{\mathcal{V}(\Lambda)} \|\mathbf{x}\|^2 d\mathbf{x}$. Lattices Λ_1, Λ_2 and Λ are said to be nested if $\Lambda \subseteq \Lambda_2 \subseteq \Lambda_1$. For nested lattices $\Lambda_2 \subset \Lambda_1$, $\Lambda_1/\Lambda_2 = \Lambda_1 \cap \mathcal{V}(\Lambda_2)$.

We briefly review the lattice decoding procedure in [12]. We use nested lattices $\Lambda \subseteq \Lambda_t$ with $\sigma^2(\Lambda) = S$, $G(\Lambda) = \frac{1}{2\pi e}$, and $V(\Lambda) = (2\pi e S)^{\frac{n}{2}}$. The transmitter sends $\mathbf{x} = [\mathbf{t} + \mathbf{d}] \bmod \Lambda$ over the point-to-point Gaussian channel $\mathbf{y} = \mathbf{x} + \mathbf{z}$ where the codeword $\mathbf{t} \in \Lambda_t \cap \mathcal{V}(\Lambda)$, the dither signal $\mathbf{d} \sim \mathrm{Unif}(\mathcal{V}(\Lambda))$, the transmit power $\frac{1}{n}\|\mathbf{x}\|^2 = S$ and the noise $\mathbf{z} \sim \mathcal{N}(0, N\mathbf{I})$. The code rate is given by $R = \frac{1}{n}\log\left(\frac{V(\Lambda)}{V(\Lambda_t)}\right)$.

After linear scaling, dither removal, and mod-Λ operation, we get

$$\mathbf{y}' = [\beta\mathbf{y} - \mathbf{d}] \bmod \Lambda = [\mathbf{t} + \mathbf{z}_e] \bmod \Lambda \tag{2.1}$$

where the effective noise is $\mathbf{z}_e = (\beta - 1)\mathbf{x} + \beta\mathbf{z}_1$ and its variance $\sigma_e^2 =$

$\frac{1}{n}\mathbb{E}[\|\mathbf{z}_e\|^2] = (\beta - 1)^2 S + \beta^2 N$. With the MMSE scaling factor $\beta = \frac{S}{S+N}$ plugged in, we get $\sigma_e^2 = \beta N = \frac{SN}{S+N}$. The capacity of the mod-Λ channel [12] between $\mathbf{t}$ and $\mathbf{y}$ is

$$\begin{aligned}
\frac{1}{n} I(\mathbf{t};\mathbf{y}) &= \frac{1}{n} h(\mathbf{y}) - \frac{1}{n} h(\mathbf{y}|\mathbf{t}) \\
&= \frac{1}{n} h(\mathbf{y}) - \frac{1}{n} h(\mathbf{z} \bmod \Lambda) \\
&\geq \frac{1}{n} h(\mathbf{y}) - \frac{1}{n} h(\mathbf{z}) \\
&= \frac{1}{n} \log V(\Lambda) - \frac{1}{n} h(\mathbf{z}) \\
&= \frac{1}{2} \log\left(\frac{S}{\beta N}\right) \\
&= \frac{1}{2} \log\left(1 + \frac{S}{N}\right) \\
&= C
\end{aligned}$$

where $I(\cdot)$ and $h(\cdot)$ are mutual information and differential entropy, respectively. For reliable decoding of $\mathbf{t}$, we have the code rate constraint $R \leq C$. With the choice of lattice parameters, $\sigma^2(\Lambda_t) \geq \beta N$, $G(\Lambda_t) = \frac{1}{2\pi e}$ and $V(\Lambda_t)^{\frac{n}{2}} = \frac{\sigma^2(\Lambda_t)}{G(\Lambda_t)} \geq 2\pi e \beta N$,

$$\begin{aligned}
R &= \frac{1}{n} \log\left(\frac{V(\Lambda)}{V(\Lambda_t)}\right) \\
&\leq \frac{1}{n} \log\left(\frac{(2\pi e S)^{\frac{n}{2}}}{(2\pi e \beta N)^{\frac{n}{2}}}\right) \\
&= \frac{1}{2} \log\left(\frac{S}{\beta N}\right).
\end{aligned}$$

Thus, the constraint $R \leq C$ can be satisfied. By *lattice decoding* [12], we can recover $\mathbf{t}$, i.e.,

$$Q_{\Lambda_t}(\mathbf{y}') = \mathbf{t}, \tag{2.2}$$

with probability $1 - P_e$ where

$$P_e = \Pr[Q_{\Lambda_t}(\mathbf{y}') \neq \mathbf{t}] \tag{2.3}$$

is the probability of decoding error. If we choose Λ to be Poltyrev-good [28], then $P_e \to 0$ as $n \to \infty$.

Let us consider the case where the transmitter sends a superposition of L lattice codewords

$$\mathbf{x} = \sum_{l=1}^{L} h^{l-1}\mathbf{x}_l$$

where $\mathbf{x}_l = [\mathbf{t}_l + \mathbf{d}_l] \bmod \Lambda$ with the transmit power $S = \frac{1}{n}\|\mathbf{x}\|^2 = \frac{1}{n}\sum_{l=1}^{L} h^{2(l-1)}\|\mathbf{x}_l\|^2 = \sum_{l=1}^{L} h^{2(l-1)}P$. If we use layer-by-layer successive decoding, it is straightforward to show that each layer can achieve

$$R_l = \frac{1}{2}\log\left(1 + \frac{h^{2(l-1)}P}{\sum_{m>l}^{L} h^{2(m-1)}P + N}\right),$$

and $\sum_{l=1}^{L} R_l = C$.

2.2 Dirichlet's Theorem and Farey sequence

Theorem 2.1 (Dirichlet (1842))**.** *Let h and Q be real numbers with $Q > 1$. Then their exist integers p and q such that $1 \leq q < Q$ and $|qh - p| \leq \frac{1}{Q}$.*

Proof. The proof can be found in Schmidt (1980). □

Definition 2.2 (Farey sequence)**.** *The Farey sequence $\mathcal{F}_n$ of order n with $n \geq 1$ is the sequence of rationals in their lowest terms between 0 and 1 with denominators less than or equal to n, written in ascending order.*

For example,

$$\mathcal{F}_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}.$$

Definition 2.3 (Farey decomposition)**.** *Given Q, a real number $h \in [0,1]$ can be decomposed into its quantized part*

$$h_Q = \operatorname*{argmin}_{\frac{p}{q} \in \mathcal{F}_{\lfloor Q \rfloor}} |qh - p|$$

and modulo part $\varepsilon = h - h_Q$.

Corollary 2.4. *For $h \in [0,1]$, $\varepsilon = h - h_Q$ is bounded by $|\varepsilon| \leq \frac{1}{q^\star Q}$ where $q^\star$ is the denominator of h_Q.*

Proof. We first show that

$$\frac{1}{Q} \geq \min_{(p,q):q\in[1,Q]} |qh - p| = \min_{q\in[1,Q]} |qh - \lfloor qh \rceil| = \min_{\frac{p}{q} \in \mathcal{F}_{\lfloor Q \rfloor}} |qh - p|.$$

The inequality holds due to Dirichlet's theorem. Now, note that

$$\mathcal{A} = \{(p,q) : \frac{p}{q} \in \mathcal{F}_{\lfloor Q \rfloor}\} \subseteq \mathcal{B} = \{(\lfloor qh \rceil, q) : q \in [1,Q]\} \subseteq \mathcal{C} = \{(p,q) : q \in [1,Q]\}.$$

We can see that $\mathcal{C} - \mathcal{B}$ includes only (p,q) with $p \neq \lfloor qh \rceil$, which is suboptimal. And, $\mathcal{B} - \mathcal{A}$ includes only rationals not in lowest terms, which are suboptimal solutions of the minimization. Thus, restriction to $\mathcal{A}$ is without loss of optimality. Let us define $q^\star = \operatorname{argmin}_{q\in[1,Q]} |qh - \lfloor qh \rceil|$. We can express h_Q equivalently by $h_Q = \frac{\lfloor q^\star h \rceil}{q^\star}$, and then

$$|\varepsilon| = \left|h - \frac{\lfloor q^\star h \rceil}{q^\star}\right| = \frac{1}{q^\star} |q^\star h - \lfloor q^\star h \rceil| \leq \frac{1}{q^\star Q}.$$

□

Definition 2.5 (Farey neighbors)**.** *If two numbers are successive terms in $\mathcal{F}_n$, they are said to be Farey neighbors in $\mathcal{F}_n$.*

Let us denote by $(\frac{p}{q}, \frac{a}{b})$ a pair of Farey neighbors with $\frac{p}{q} < \frac{a}{b}$. Note that $(\frac{p}{q}, \frac{a}{b})$ are Farey neighbors in $\mathcal{F}_{\max\{q,b\}}$ but not necessarily in $\mathcal{F}_n$ with $n > \max\{q, b\}$. For example, $(\frac{1}{2}, \frac{2}{3})$ are Farey neighbors in $\mathcal{F}_3$ but not in $\mathcal{F}_5$. The following is a well-known property of Farey neighbors.

Theorem 2.6 (Distance between Farey neighbors)**.** *If $(\frac{p}{q}, \frac{a}{b})$ are Farey neighbors in $\mathcal{F}_n$, then $qa - pb = 1$. Equivalently, the distance between them is $\frac{a}{b} - \frac{p}{q} = \frac{1}{qb}$.*

Proof. The proof can be found in Schmidt (1980). □

The converse is not true. Given two numbers $\frac{p}{q}$ and $\frac{a}{b}$ in $\mathcal{F}_n$, the equality $qa - pb = 1$ holds even when $(\frac{p}{q}, \frac{a}{b})$ are not Farey neighbors in $\mathcal{F}_n$ if they are Farey neighbors in some $\mathcal{F}_k$ with $k < n$.

Definition 2.7 (Farey umbrella)**.** *Given a number $\frac{p}{q}$ in $\mathcal{F}_Q$, the interval $\left[\frac{p}{q} - \frac{1}{qQ}, \frac{p}{q} + \frac{1}{qQ}\right]$ is said to be the Farey umbrella of $\frac{p}{q}$.*

Corollary 2.8. *The union of Farey umbrellas of $\mathcal{F}_Q$ covers the entire interval $[0, 1]$, i.e.,*

$$[0,1] \subset \bigcup_{\frac{p}{q} \in \mathcal{F}_Q} \left[\frac{p}{q} - \frac{1}{qQ}, \frac{p}{q} + \frac{1}{qQ}\right].$$

Proof. It is sufficient to show that $\frac{p}{q} + \frac{1}{qQ} \geq \frac{a}{b} - \frac{1}{bQ}$ for every pair $(\frac{p}{q}, \frac{a}{b})$ of Farey neighbors in $\mathcal{F}_Q$. Since $\frac{a}{b} = \frac{p}{q} + \frac{1}{qb}$ by the theorem above, it remains to show that $\frac{1}{qQ} + \frac{1}{bQ} = \frac{q+b}{qbQ} \geq \frac{1}{qb}$. Let us assume otherwise, i.e., $q + b < Q$, then $\frac{p+a}{q+b}$ must be a member of $\mathcal{F}_Q$. In this case, since $\frac{p}{q} < \frac{p+a}{q+b} < \frac{a}{b}$, both $(\frac{p}{q}, \frac{p+a}{q+b})$

and $(\frac{p+a}{q+b}, \frac{a}{b})$ are valid pairs of Farey neighbors in $\mathcal{F}_Q$, but $(\frac{p}{q}, \frac{a}{b})$ is not. This is contradictory. $\square$

Chapter 3

Partially Connected Interference Channels[1]

3.1 Channel Model and Main Results

We consider five channel types defined in Table 4.2 and described in Fig. 3.1 (a)–(e). Each channel type is a partially connected three-user Gaussian interference channel. Each transmitter is subject to power constraint $\mathbb{E}[X_k^2] \leq P_k = P$. Let us denote the noise variance by $N_k = \mathbb{E}[Z_k^2]$. Without loss of generality, we assume that $N_1 \leq N_2 \leq N_3$.

Definition 3.1 (side information graph)**.** *The side information graph representation of an interference channel satisfies the following.*

- *A node represents a transmitter-receiver pair, or equivalently, the message.*
- *There is a directed edge from node i to node j if transmitter i does not interfere at receiver j.*

The side information graphs for five channel types are described in Fig. 3.1 (f)–(j). We state the main results in the following two theorems, of which the proofs will be given in the main body of the paper.

[1]The result in this chapter was presented in part at the IEEE ISIT 2017 [29]. Muryong Kim as the first author performed the research and generated the main results in theorems.

Type	Channel model
1	$Y_1 = X_1 + X_2 + Z_1$ $Y_2 = X_1 + X_2 + X_3 + Z_2$ $Y_3 = X_2 + X_3 + Z_3$
2	$Y_1 = X_1 + X_2 + X_3 + Z_1$ $Y_2 = X_1 + X_2 + Z_2$ $Y_3 = X_1 + X_3 + Z_3$
3	$Y_1 = X_1 + X_3 + Z_1$ $Y_2 = X_2 + X_3 + Z_2$ $Y_3 = X_1 + X_2 + X_3 + Z_3$
4	$Y_1 = X_1 + X_3 + Z_1$ $Y_2 = X_1 + X_2 + Z_2$ $Y_3 = X_2 + X_3 + Z_3$
5	$Y_1 = X_1 + X_2 + Z_1$ $Y_2 = X_2 + X_3 + Z_2$ $Y_3 = X_1 + X_3 + Z_3$

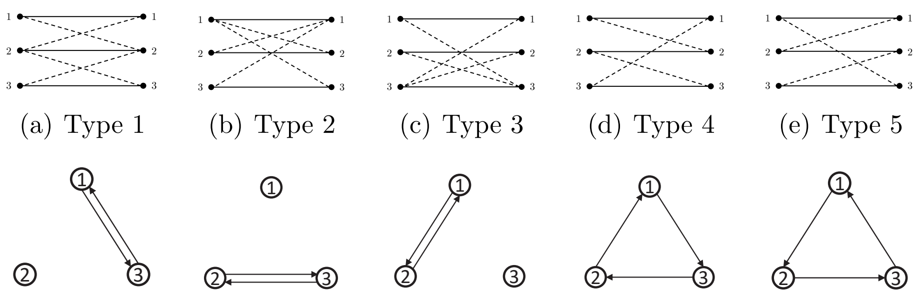
Table 3.1: Five channel types

Theorem 3.2 (Capacity region outer bound)**.** *For the five channel types, if* (R_1, R_2, R_3) *is achievable, it must satisfy*

$$\sum_{j \in \mathcal{K}} R_j \leq \frac{1}{2} \log \left(1 + \frac{|\mathcal{K}|P}{\min_{j \in \mathcal{K}}\{N_j\}} \right) \tag{3.1}$$

for every subset $\mathcal{K}$ *of the nodes* $\{1, 2, 3\}$ *that does not include a directed cycle in the side information graph over the subset.*

Theorem 3.3 (Capacity region to within one bit)**.**
For any rate triple (R_1, R_2, R_3) *on the boundary of the outer bound region, the point* $(R_1 - 1, R_2 - 1, R_3 - 1)$ *is achievable.*



(a) Type 1 (b) Type 2 (c) Type 3 (d) Type 4 (e) Type 5

(f) $\mathcal{G}_1$ (g) $\mathcal{G}_2$ (h) $\mathcal{G}_3$ (i) $\mathcal{G}_4$ (j) $\mathcal{G}_5$

Figure 3.1: Five channel types and their side information graphs.

3.2 Capacity Outer Bounds

We prove the capacity outer bound in *Theorem 1* for each channel type. The result is summarized in Table 3.2. The shape of the outer bound region is illustrated in Fig. 3.2. For all channel types, we assume $P_1 = P_2 = P_3 = P$ and $N_1 \leq N_2 \leq N_3$.

3.2.1 Channel Type 1

In this section, we present an outer bound on the capacity region of Type 1 channel defined by

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}.$$

We state the outer bound in the following theorem.

Theorem 3.4. *The capacity region of Type 1 channel is contained in the* *following outer bound region:*

$$\begin{aligned}
R_k &\leq C_k, \;\; k = 1, 2, 3 \\
R_1 + R_2 &\leq \frac{1}{2}\log\left(1 + \frac{P}{N_1}\right) + \frac{1}{2}\log\left(\frac{2P + N_2}{P + N_2}\right) \\
R_2 + R_3 &\leq \frac{1}{2}\log\left(1 + \frac{P}{N_2}\right) + \frac{1}{2}\log\left(\frac{2P + N_3}{P + N_3}\right).
\end{aligned}$$

Proof. The individual rate bounds are obvious. We proceed to sum-rate bounds.

$$\begin{aligned}
& n(R_1 + R_2 - \epsilon) \\
& \quad \leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) \\
& \quad \leq I(X_1^n; Y_1^n | X_2^n) + I(X_2^n; Y_2^n | X_3^n) \\
& \quad = h(Y_1^n | X_2^n) - h(Y_1^n | X_1^n, X_2^n) + h(Y_2^n | X_3^n) - h(Y_2^n | X_2^n, X_3^n) \\
& \quad = h(X_1^n + Z_1^n) - h(Z_1^n) + h(X_1^n + X_2^n + Z_2^n) - h(X_1^n + Z_2^n) \\
& \quad \leq \frac{n}{2}\log\left(\frac{P + N_1}{N_1}\right) + \frac{n}{2}\log\left(\frac{2P + N_2}{P + N_2}\right)
\end{aligned}$$

where the first inequality is by Fano's inequality, the second inequality due to the independence of X_1, X_2, X_3. The third inequality holds from the fact that Gaussian distribution maximizes differential entropy and that $h(X_1^n + Z_1^n) - h(X_1^n + Z_2^n)$ is also maximized by Gaussian distribution. Similarly,

$$\begin{aligned}
& n(R_2 + R_3 - \epsilon) \\
& \quad \leq I(X_2^n; Y_2^n) + I(X_3^n; Y_3^n) \\
& \quad \leq I(X_2^n; Y_2^n | X_1^n, X_3^n) + I(X_3^n; Y_3^n) \\
& \quad = h(Y_2^n | X_1^n, X_3^n) - h(Y_2^n | X_1^n, X_2^n, X_3^n) + h(Y_3^n) - h(Y_3^n | X_3^n) \\
& \quad = h(X_2^n + Z_2^n) - h(Z_2^n) + h(X_2^n + X_3^n + Z_3^n) - h(X_2^n + Z_3^n) \\
& \quad \leq \frac{n}{2}\log\left(\frac{P + N_2}{N_2}\right) + \frac{n}{2}\log\left(\frac{2P + N_3}{P + N_3}\right).
\end{aligned}$$

□

3.2.2 Channel Type 2

In this section, we present an outer bound on the capacity region of Type 2 channel defined by

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}.$$

We state the outer bound in the following theorem.

Theorem 3.5. *The capacity region of Type 2 channel is contained in the following outer bound region:*

$$\begin{aligned} R_k &\le C_k, \;\; k = 1, 2, 3 \\ R_1 + R_2 &\le \frac{1}{2} \log\left(1 + \frac{P}{N_1}\right) + \frac{1}{2} \log\left(\frac{2P + N_2}{P + N_2}\right) \\ R_1 + R_3 &\le \frac{1}{2} \log\left(1 + \frac{P}{N_1}\right) + \frac{1}{2} \log\left(\frac{2P + N_3}{P + N_3}\right). \end{aligned}$$

Proof.

$$\begin{aligned} & n(R_1 + R_2 - \epsilon) \\ & \le I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) \\ & \le I(X_1^n; Y_1^n | X_2^n, X_3^n) + I(X_2^n; Y_2^n) \\ & = h(Y_1^n | X_2^n, X_3^n) - h(Y_1^n | X_1^n, X_2^n, X_3^n) + h(Y_2^n) - h(Y_2^n | X_2^n) \\ & = h(X_1^n + Z_1^n) - h(Z_1^n) + h(X_1^n + X_2^n + Z_2^n) - h(X_1^n + Z_2^n) \\ & \le \frac{n}{2} \log\left(\frac{P + N_1}{N_1}\right) + \frac{n}{2} \log\left(\frac{2P + N_2}{P + N_2}\right). \end{aligned}$$

$$
\begin{aligned}
&n(R_1 + R_3 - \epsilon) \\
&\quad \le I(X_1^n; Y_1^n) + I(X_3^n; Y_3^n) \\
&\quad \le I(X_1^n; Y_1^n | X_2^n, X_3^n) + I(X_3^n; Y_3^n) \\
&\quad = h(Y_1^n | X_2^n, X_3^n) - h(Y_1^n | X_1^n, X_2^n, X_3^n) + h(Y_3^n) - h(Y_3^n | X_3^n) \\
&\quad = h(X_1^n + Z_1^n) - h(Z_1^n) + h(X_1^n + X_3^n + Z_3^n) - h(X_1^n + Z_3^n) \\
&\quad \le \frac{n}{2} \log\left(\frac{P + N_1}{N_1}\right) + \frac{n}{2} \log\left(\frac{2P + N_3}{P + N_3}\right).
\end{aligned}
$$

□

3.2.3 Channel Type 3

In this section, we present an outer bound on the capacity region of Type 3 channel defined by

$$
\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}.
$$

We state the outer bound in the following theorem.

Theorem 3.6. *The capacity region of Type 3 channel is contained in the following outer bound region:*

$$
\begin{aligned}
R_k &\le C_k, \;\; k = 1, 2, 3 \\
R_1 + R_3 &\le \frac{1}{2} \log\left(1 + \frac{P}{N_1}\right) + \frac{1}{2} \log\left(\frac{2P + N_3}{P + N_3}\right) \\
R_2 + R_3 &\le \frac{1}{2} \log\left(1 + \frac{P}{N_2}\right) + \frac{1}{2} \log\left(\frac{2P + N_3}{P + N_3}\right).
\end{aligned}
$$

Proof.

$$
\begin{aligned}
& n(R_1 + R_3 - \epsilon) \\
& \le I(X_1^n; Y_1^n) + I(X_3^n; Y_3^n) \\
& \le I(X_1^n; Y_1^n | X_3^n) + I(X_3^n; Y_3^n | X_2^n) \\
& = h(Y_1^n | X_3^n) - h(Y_1^n | X_1^n, X_3^n) + h(Y_3^n | X_2^n) - h(Y_3^n | X_2^n, X_3^n) \\
& = h(X_1^n + Z_1^n) - h(Z_1^n) + h(X_1^n + X_3^n + Z_3^n) - h(X_1^n + Z_3^n) \\
& \le \frac{n}{2} \log\left(\frac{P + N_1}{N_1}\right) + \frac{n}{2} \log\left(\frac{2P + N_3}{P + N_3}\right).
\end{aligned}
$$

$$
\begin{aligned}
& n(R_2 + R_3 - \epsilon) \\
& \le I(X_2^n; Y_2^n) + I(X_3^n; Y_3^n) \\
& \le I(X_2^n; Y_2^n | X_3^n) + I(X_3^n; Y_3^n | X_1^n) \\
& = h(Y_2^n | X_3^n) - h(Y_2^n | X_2^n, X_3^n) + h(Y_3^n | X_1^n) - h(Y_3^n | X_1^n, X_3^n) \\
& = h(X_2^n + Z_2^n) - h(Z_2^n) + h(X_2^n + X_3^n + Z_3^n) - h(X_2^n + Z_3^n) \\
& \le \frac{n}{2} \log\left(\frac{P + N_2}{N_2}\right) + \frac{n}{2} \log\left(\frac{2P + N_3}{P + N_3}\right).
\end{aligned}
$$

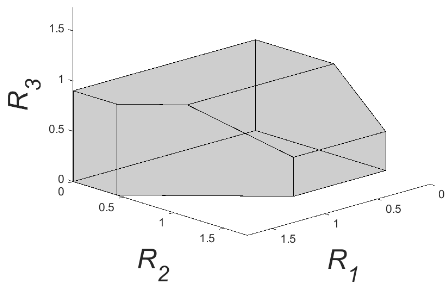
□

3.2.4 Channel Type 4

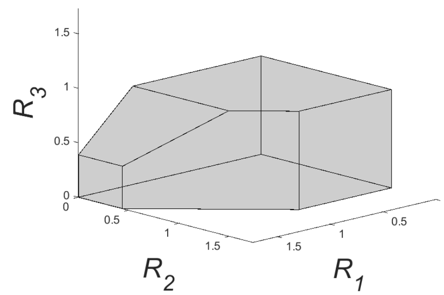
In this section, we present an outer bound on the capacity region of Type 4 channel defined by

$$
\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}.
$$

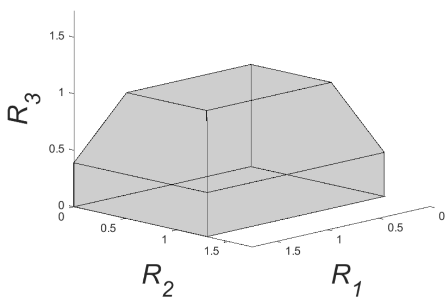
This is a cyclic Gaussian interference channel [6]. We first show that channel type 4 is in the mixed interference regime. By normalizing the noise variances,



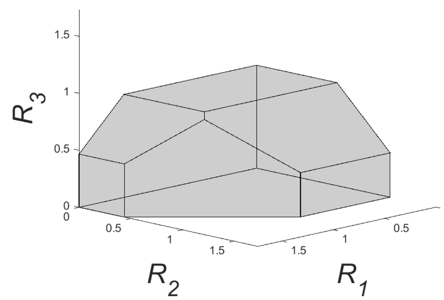
(a) Channel type 1



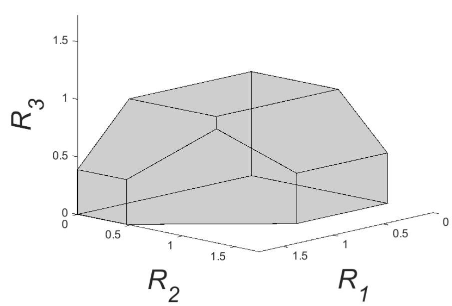
(b) Channel type 2



(c) Channel type 3



(d) Channel type 4



(e) Channel type 5

Figure 3.2: The shape of the outer bound region.

we get the equivalent channel given by

$$\begin{bmatrix} Y_1' \\ Y_2' \\ Y_3' \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1' \\ Z_2' \\ Z_3' \end{bmatrix}$$

where $Y_k' = \frac{1}{\sqrt{N_k}} Y_k$, $Z_k' = \frac{1}{\sqrt{N_k}} Z_k$, $N_0 = \mathbb{E}[Z_k'^2] = 1$, $\mathbb{E}[X_k^2] \leq P_k = P$ and

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{N_1}} & 0 & \frac{1}{\sqrt{N_1}} \\ \frac{1}{\sqrt{N_2}} & \frac{1}{\sqrt{N_2}} & 0 \\ 0 & \frac{1}{\sqrt{N_3}} & \frac{1}{\sqrt{N_3}} \end{bmatrix}.$$

With the usual definitions of $\mathrm{SNR}_k = \frac{h_{kk}^2 P_k}{N_0}$ and
$\mathrm{INR}_k = \frac{h_{jk}^2 P_k}{N_0}$ for $j \neq k$ as in [1, 6],

$$\mathrm{SNR}_1 = \frac{P}{N_1} \geq \mathrm{INR}_1 = \frac{P}{N_2} \tag{3.2}$$

$$\mathrm{SNR}_2 = \frac{P}{N_2} \geq \mathrm{INR}_2 = \frac{P}{N_3} \tag{3.3}$$

$$\mathrm{SNR}_3 = \frac{P}{N_3} \leq \mathrm{INR}_3 = \frac{P}{N_1}. \tag{3.4}$$

We state the outer bound in the following theorem.

Theorem 3.7. *The capacity region of Type 4 channel is contained in the following outer bound region:*

$$\begin{aligned} R_k &\leq C_k, \;\; k = 1, 2, 3 \\ R_1 + R_2 &\leq \frac{1}{2}\log\left(1 + \frac{P}{N_1}\right) + \frac{1}{2}\log\left(\frac{2P + N_2}{P + N_2}\right) \\ R_1 + R_3 &\leq \frac{1}{2}\log\left(1 + \frac{2P}{N_1}\right) \\ R_2 + R_3 &\leq \frac{1}{2}\log\left(1 + \frac{P}{N_2}\right) + \frac{1}{2}\log\left(\frac{2P + N_3}{P + N_3}\right). \end{aligned}$$

Proof.

$$\begin{aligned}
&n(R_1 + R_2 - \epsilon) \\
&\quad \le I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) \\
&\quad \le I(X_1^n; Y_1^n | X_3^n) + I(X_2^n; Y_2^n) \\
&\quad = h(Y_1^n | X_3^n) - h(Y_1^n | X_1^n, X_3^n) + h(Y_2^n) - h(Y_2^n | X_2^n) \\
&\quad = h(X_1^n + Z_1^n) - h(Z_1^n) + h(X_1^n + X_2^n + Z_2^n) - h(X_1^n + Z_2^n) \\
&\quad \le \frac{n}{2} \log\left(\frac{P + N_1}{N_1}\right) + \frac{n}{2} \log\left(\frac{2P + N_2}{P + N_2}\right).
\end{aligned}$$

$$\begin{aligned}
&n(R_2 + R_3 - \epsilon) \\
&\quad \le I(X_2^n; Y_2^n) + I(X_3^n; Y_3^n) \\
&\quad \le I(X_2^n; Y_2^n | X_1^n) + I(X_3^n; Y_3^n) \\
&\quad = h(Y_2^n | X_1^n) - h(Y_2^n | X_1^n, X_2^n) + h(Y_3^n) - h(Y_3^n | X_3^n) \\
&\quad = h(X_2^n + Z_2^n) - h(Z_2^n) + h(X_2^n + X_3^n + Z_3^n) - h(X_2^n + Z_3^n) \\
&\quad \le \frac{n}{2} \log\left(\frac{P + N_2}{N_2}\right) + \frac{n}{2} \log\left(\frac{2P + N_3}{P + N_3}\right).
\end{aligned}$$

$$\begin{aligned}
&n(R_1 + R_3 - \epsilon) \\
&\quad \le I(X_1^n; Y_1^n) + I(X_3^n; Y_3^n) \\
&\quad \le I(X_1^n; Y_1^n) + I(X_3^n; Y_3^n | X_2^n) \\
&\quad \le I(X_1^n; Y_1^n) + I(X_3^n; Y_1^n | X_1^n) \\
&\quad \le I(X_1^n, X_3^n; Y_1^n) \\
&\quad = h(Y_1^n) - h(Y_1^n | X_1^n, X_3^n) \\
&\quad = h(X_1^n + X_3^n + Z_1^n) - h(Z_1^n) \\
&\quad \le \frac{n}{2} \log\left(\frac{2P + N_1}{N_1}\right)
\end{aligned}$$

where we used the fact that $I(X_3^n; Y_3^n | X_2^n) = I(X_3^n; X_3^n + Z_3^n) \leq I(X_3^n; X_3^n + Z_1^n) = I(X_3^n; Y_1^n | X_1^n)$. $\square$

3.2.5 Channel Type 5

In this section, we present an outer bound on the capacity region of Type 5 channel defined by

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}.$$

This is a cyclic Gaussian interference channel [6]. We first show that channel type 5 is in the mixed interference regime. By normalizing the noise variances, we get the equivalent channel given by

$$\begin{bmatrix} Y_1' \\ Y_2' \\ Y_3' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{N_1}} & \frac{1}{\sqrt{N_1}} & 0 \\ 0 & \frac{1}{\sqrt{N_2}} & \frac{1}{\sqrt{N_2}} \\ \frac{1}{\sqrt{N_3}} & 0 & \frac{1}{\sqrt{N_3}} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1' \\ Z_2' \\ Z_3' \end{bmatrix}.$$

We can see that

$$\mathrm{SNR}_1 = \frac{P}{N_1} \geq \mathrm{INR}_1 = \frac{P}{N_3} \tag{3.5}$$
$$\mathrm{SNR}_2 = \frac{P}{N_2} \leq \mathrm{INR}_2 = \frac{P}{N_1} \tag{3.6}$$
$$\mathrm{SNR}_3 = \frac{P}{N_3} \leq \mathrm{INR}_3 = \frac{P}{N_2}. \tag{3.7}$$

We state the outer bound in the following theorem.

Theorem 3.8. *The capacity region of Type 5 channel is contained in the*

Type	Outer bound region $\mathcal{R}_o$
1	$R_k \leq C_k,\ k = 1, 2, 3$ $R_1 + R_2 \leq \frac{1}{2}\log\left(\frac{P+N_1}{N_1} \cdot \frac{2P+N_2}{P+N_2}\right)$ $R_2 + R_3 \leq \frac{1}{2}\log\left(\frac{P+N_2}{N_2} \cdot \frac{2P+N_3}{P+N_3}\right)$
2	$R_k \leq C_k,\ k = 1, 2, 3$ $R_1 + R_2 \leq \frac{1}{2}\log\left(\frac{P+N_1}{N_1} \cdot \frac{2P+N_2}{P+N_2}\right)$ $R_1 + R_3 \leq \frac{1}{2}\log\left(\frac{P+N_1}{N_1} \cdot \frac{2P+N_3}{P+N_3}\right)$
3	$R_k \leq C_k,\ k = 1, 2, 3$ $R_1 + R_3 \leq \frac{1}{2}\log\left(\frac{P+N_1}{N_1} \cdot \frac{2P+N_3}{P+N_3}\right)$ $R_2 + R_3 \leq \frac{1}{2}\log\left(\frac{P+N_2}{N_2} \cdot \frac{2P+N_3}{P+N_3}\right)$
4	$R_k \leq C_k,\ k = 1, 2, 3$ $R_1 + R_2 \leq \frac{1}{2}\log\left(\frac{P+N_1}{N_1} \cdot \frac{2P+N_2}{P+N_2}\right)$ $R_1 + R_3 \leq \frac{1}{2}\log\left(\frac{2P+N_1}{N_1}\right)$ $R_2 + R_3 \leq \frac{1}{2}\log\left(\frac{P+N_2}{N_2} \cdot \frac{2P+N_3}{P+N_3}\right)$
5	$R_k \leq C_k,\ k = 1, 2, 3$ $R_1 + R_2 \leq \frac{1}{2}\log\left(\frac{2P+N_1}{N_1}\right)$ $R_2 + R_3 \leq \frac{1}{2}\log\left(\frac{2P+N_2}{N_2}\right)$ $R_1 + R_3 \leq \frac{1}{2}\log\left(\frac{P+N_1}{N_1} \cdot \frac{2P+N_3}{P+N_3}\right)$

Table 3.2: Capacity outer bounds

Type	Relaxed outer bound region $\mathcal{R}'_o$	Two-dimensional cross-section of $\mathcal{R}'_o$
1	$R_k \leq \frac{1}{2}\log\left(\frac{P}{N_k}\cdot\frac{4}{3}\right)$ $R_1 + R_2 \leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right)$ $R_2 + R_3 \leq \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right)$	At some $R_2 \in [0, C_2]$, $R_1 \leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - R_2, \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{4}{3}\right)\right\}$ $R_3 \leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right) - R_2, \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right)\right\}$
2	$R_k \leq \frac{1}{2}\log\left(\frac{P}{N_k}\cdot\frac{4}{3}\right)$ $R_1 + R_2 \leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right)$ $R_1 + R_3 \leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right)$	At some $R_1 \in [0, C_1]$, $R_2 \leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - R_1, \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{4}{3}\right)\right\}$ $R_3 \leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - R_1, \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right)\right\}$
3	$R_k \leq \frac{1}{2}\log\left(\frac{P}{N_k}\cdot\frac{4}{3}\right)$ $R_1 + R_3 \leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right)$ $R_2 + R_3 \leq \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right)$	At some $R_3 \in [0, C_3]$, $R_1 \leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - R_3, \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{4}{3}\right)\right\}$ $R_2 \leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right) - R_3, \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{4}{3}\right)\right\}$
4	$R_k \leq \frac{1}{2}\log\left(\frac{P}{N_k}\cdot\frac{4}{3}\right)$ $R_1 + R_2 \leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right)$ $R_1 + R_3 \leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right)$ $R_2 + R_3 \leq \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right)$	At some $R_1 \in [0, C_1]$, $R_2 \leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - R_1, \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{4}{3}\right)\right\}$ $R_3 \leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - R_1, \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right)\right\}$ $R_2 + R_3 \leq \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right)$
5	$R_k \leq \frac{1}{2}\log\left(\frac{P}{N_k}\cdot\frac{4}{3}\right)$ $R_1 + R_2 \leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right)$ $R_2 + R_3 \leq \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right)$ $R_1 + R_3 \leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right)$	At some $R_2 \in [0, C_2]$, $R_1 \leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - R_2, \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{4}{3}\right)\right\}$ $R_3 \leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right) - R_2, \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right)\right\}$ $R_1 + R_3 \leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right)$

Table 3.3: Relaxed outer bounds

following outer bound region:

$$\begin{aligned} R_k &\leq C_k, \;\; k = 1,2,3 \\ R_1 + R_2 &\leq \frac{1}{2}\log\left(1 + \frac{2P}{N_1}\right) \\ R_2 + R_3 &\leq \frac{1}{2}\log\left(1 + \frac{2P}{N_2}\right) \\ R_1 + R_3 &\leq \frac{1}{2}\log\left(1 + \frac{P}{N_1}\right) + \frac{1}{2}\log\left(\frac{2P + N_3}{P + N_3}\right). \end{aligned}$$

Proof.

$$\begin{aligned} & n(R_1 + R_2 - \epsilon) \\ & \leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) \\ & \leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n | X_3^n) \\ & \leq I(X_1^n; Y_1^n) + I(X_2^n; Y_1^n | X_1^n) \\ & \leq I(X_1^n, X_2^n; Y_1^n) \\ & = h(Y_1^n) - h(Y_1^n | X_1^n, X_2^n) \\ & = h(X_1^n + X_2^n + Z_1^n) - h(Z_1^n) \\ & \leq \frac{n}{2}\log\left(\frac{2P + N_1}{N_1}\right) \end{aligned}$$

where we used the fact that $I(X_2^n; Y_2^n | X_3^n) = I(X_2^n; X_2^n + Z_2^n) \leq I(X_2^n; X_2^n +$

$Z_1^n) = I(X_2^n; Y_1^n | X_1^n)$.

$$
\begin{aligned}
&n(R_2 + R_3 - \epsilon) \\
&\quad \le I(X_2^n; Y_2^n) + I(X_3^n; Y_3^n) \\
&\quad \le I(X_2^n; Y_2^n) + I(X_3^n; Y_3^n | X_1^n) \\
&\quad \le I(X_2^n; Y_2^n) + I(X_3^n; Y_2^n | X_2^n) \\
&\quad \le I(X_2^n, X_3^n; Y_2^n) \\
&\quad = h(Y_2^n) - h(Y_2^n | X_2^n, X_3^n) \\
&\quad = h(X_2^n + X_3^n + Z_2^n) - h(Z_2^n) \\
&\quad \le \frac{n}{2} \log\left(\frac{2P + N_2}{N_2}\right)
\end{aligned}
$$

where we used the fact that $I(X_3^n; Y_3^n | X_1^n) = I(X_3^n; X_3^n + Z_3^n) \le I(X_3^n; X_3^n + Z_2^n) = I(X_3^n; Y_2^n | X_2^n)$.

$$
\begin{aligned}
&n(R_1 + R_3 - \epsilon) \\
&\quad \le I(X_1^n; Y_1^n) + I(X_3^n; Y_3^n) \\
&\quad \le I(X_1^n; Y_1^n | X_2^n) + I(X_3^n; Y_3^n) \\
&\quad = h(Y_1^n | X_2^n) - h(Y_1^n | X_1^n, X_2^n) + h(Y_3^n) - h(Y_3^n | X_3^n) \\
&\quad = h(X_1^n + Z_1^n) - h(Z_1^n) + h(X_1^n + X_3^n + Z_3^n) - h(X_1^n + Z_3^n) \\
&\quad \le \frac{n}{2} \log\left(\frac{P + N_1}{N_1}\right) + \frac{n}{2} \log\left(\frac{2P + N_3}{P + N_3}\right).
\end{aligned}
$$

□

3.2.6 Relaxed Outer Bounds

For ease of gap calculation, we also derive relaxed outer bounds. First, we can see that for $N_j \le N_k$,

$$
\frac{1}{2} \log\left(1 + \frac{P}{N_j}\right) + \frac{1}{2} \log\left(\frac{2P + N_k}{P + N_k}\right) \le \frac{1}{2} \log\left(1 + \frac{2P}{N_j}\right).
$$

Five outer bound theorems in this section, together with this inequality, give the sum-rate bound expression in Theorem 1.

Next, we can assume that $P \geq 3N_j$ for $j = 1, 2, 3$. Otherwise, showing one-bit gap capacity is trivial as the capacity region is included in the unit hypercube, i.e., $R_j \leq \frac{1}{2}\log\left(1 + \frac{P}{N_j}\right) < 1$. For $P \geq 3N_j$,

$$\begin{aligned}
\frac{1}{2}\log\left(1 + \frac{2P}{N_j}\right) &= \frac{1}{2}\log\left(\frac{P}{N_j}\right) + \frac{1}{2}\log\left(\frac{N_j}{P} + 2\right) \\
&\leq \frac{1}{2}\log\left(\frac{P}{N_j}\right) + \frac{1}{2}\log\left(\frac{7}{3}\right) \\
\frac{1}{2}\log\left(1 + \frac{P}{N_j}\right) &\leq \frac{1}{2}\log\left(\frac{P}{N_j}\right) + \frac{1}{2}\log\left(\frac{4}{3}\right).
\end{aligned}$$

The resulting relaxed outer bounds $\mathcal{R}'_o$ are summarized in Table 3.2.

3.3 Inner Bound: Channel Type 1

Theorem 3.9. *Given $\alpha = (\alpha_0, \alpha_2) \in [0,1]^2$, the rate region $\mathcal{R}_\alpha$ is defined by*

$$\begin{aligned}
R_1 &\leq \frac{1}{2}\log^+\left(\frac{1-\alpha_0}{2-\alpha_0} + \frac{(1-\alpha_0)P}{(\alpha_0+\alpha_2)P + N_2}\right) \\
&\qquad\qquad + \frac{1}{2}\log\left(1 + \frac{\alpha_0 P}{N_1}\right) \\
R_2 &\leq \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2}\right) \\
R_3 &\leq \frac{1}{2}\log^+\left(\frac{1}{2-\alpha_0} + \frac{P}{(\alpha_0+\alpha_2)P + N_3}\right)
\end{aligned}$$

where $\log^+(\cdot) = \max\{0, \log(\cdot)\}$. *And,*

$$\mathcal{R} = \textsc{conv}\left(\bigcup_\alpha \mathcal{R}_\alpha\right)$$

is achievable where $\textsc{conv}(\cdot)$ *is convex hull operator.*

3.3.1 Achievable Scheme

We present an achievable scheme for the proof of *Theorem 8*. The achievable scheme is based on rate-splitting, lattice coding, and interference alignment. Message $M_1 \in \{1, 2, \ldots, 2^{nR_1}\}$ is split into two parts: $M_{11} \in \{1, 2, \ldots, 2^{nR_{11}}\}$ and $M_{10} \in \{1, 2, \ldots, 2^{nR_{10}}\}$, so $R_1 = R_{11} + R_{10}$. Transmitter 1 sends $\mathbf{x}_1 = \mathbf{x}_{11} + \mathbf{x}_{10}$ where $\mathbf{x}_{11}$ and $\mathbf{x}_{10}$ are coded signals of M_{11} and M_{10}, respectively. Transmitters 2 and 3 send $\mathbf{x}_2$ and $\mathbf{x}_3$, coded signals of $M_2 \in \{1, 2, \ldots, 2^{nR_2}\}$ and $M_3 \in \{1, 2, \ldots, 2^{nR_3}\}$. In particular, $\mathbf{x}_{11}$ and $\mathbf{x}_3$ are lattice-coded signals.

We use the lattice construction of [14, 15] with the lattice partition chain $\Lambda_c/\Lambda_1/\Lambda_3$, so $\Lambda_3 \subset \Lambda_1 \subset \Lambda_c$ are nested lattices. Λ_c is the coding lattice for both $\mathbf{x}_{11}$ and $\mathbf{x}_3$. Λ_1 and Λ_3 are shaping lattices for $\mathbf{x}_{11}$ and $\mathbf{x}_3$, respectively. The lattice signals are formed by

$$\mathbf{x}_{11} = [\mathbf{t}_{11} + \mathbf{d}_{11}] \bmod \Lambda_1 \tag{3.8}$$

$$\mathbf{x}_3 = [\mathbf{t}_3 + \mathbf{d}_3] \bmod \Lambda_3 \tag{3.9}$$

where $\mathbf{t}_{11} \in \Lambda_c \cap \mathcal{V}(\Lambda_1)$ and $\mathbf{t}_3 \in \Lambda_c \cap \mathcal{V}(\Lambda_3)$ are lattice codewords. The dither signals $\mathbf{d}_{11}$ and $\mathbf{d}_3$ are uniformly distributed over $\mathcal{V}(\Lambda_1)$ and $\mathcal{V}(\Lambda_3)$, respectively. To satisfy power constraints, we choose $\mathbb{E}[\|\mathbf{x}_{11}\|^2] = n\sigma^2(\Lambda_1) = (1-\alpha_1)nP$, $\mathbb{E}[\|\mathbf{x}_{10}\|^2] = \alpha_1 nP$, $\mathbb{E}[\|\mathbf{x}_2\|^2] = \alpha_2 nP$, $\mathbb{E}[\|\mathbf{x}_3\|^2] = n\sigma^2(\Lambda_3) = nP$.

With the choice of transmit signals, the received signals are given by

$$\begin{aligned}
\mathbf{y}_1 &= \mathbf{x}_{11} + \mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_1 \\
\mathbf{y}_2 &= [\mathbf{x}_{11} + \mathbf{x}_3] + \mathbf{x}_2 + \mathbf{z}_2' \\
\mathbf{y}_3 &= \mathbf{x}_3 + \mathbf{z}_3'.
\end{aligned}$$

where $\mathbf{x}_f = [\mathbf{x}_{11} + \mathbf{x}_3]$ is the sum of interference, and $\mathbf{z}'_2 = \mathbf{x}_{10} + \mathbf{z}_2$ and $\mathbf{z}'_3 = \mathbf{x}_2 + \mathbf{z}_3$ are the effective Gaussian noise. The signal scale diagram at each receiver is shown in Fig. 3.3 (a).

At the receivers, successive decoding is performed in the following order: $\mathbf{x}_{11} \to \mathbf{x}_2 \to \mathbf{x}_{10}$ at receiver 1, $\mathbf{x}_f \to \mathbf{x}_2$ at receiver 2, and receiver 3 only decodes $\mathbf{x}_3$.

Note that the aligned lattice codewords $\mathbf{t}_{11} + \mathbf{t}_3 \in \Lambda_c$, and $\mathbf{t}_f = [\mathbf{t}_{11} + \mathbf{t}_3] \bmod \Lambda_1 \in \Lambda_c \cap \mathcal{V}(\Lambda_1)$. We state the relationship between $\mathbf{x}_f$ and $\mathbf{t}_f$ in the following lemmas.

Lemma 3.10. *The following holds.*

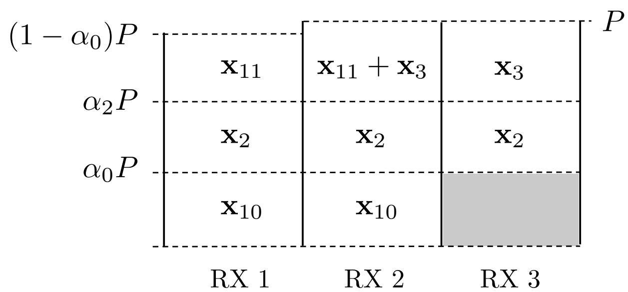
$$[\mathbf{x}_f - \mathbf{d}_f] \ mod\ \Lambda_1 = \mathbf{t}_f$$

where $\mathbf{d}_f = \mathbf{d}_{11} + \mathbf{d}_3$.

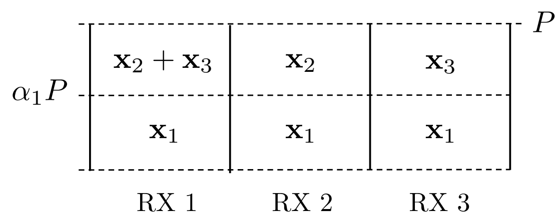
Proof.

$$\begin{aligned}
&[\mathbf{x}_f - \mathbf{d}_f] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{t}_{11} + \mathbf{d}_{11}) + M_{\Lambda_3}(\mathbf{t}_3 + \mathbf{d}_3) - \mathbf{d}_f] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{t}_{11} + \mathbf{d}_{11}) + M_{\Lambda_1}(\mathbf{t}_3 + \mathbf{d}_3) - \mathbf{d}_f] \bmod \Lambda_1 \\
&= [\mathbf{t}_{11} + \mathbf{d}_{11} + \mathbf{t}_3 + \mathbf{d}_3 - \mathbf{d}_f] \bmod \Lambda_1 \\
&= [\mathbf{t}_{11} + \mathbf{t}_3] \bmod \Lambda_1 \\
&= \mathbf{t}_f.
\end{aligned}$$

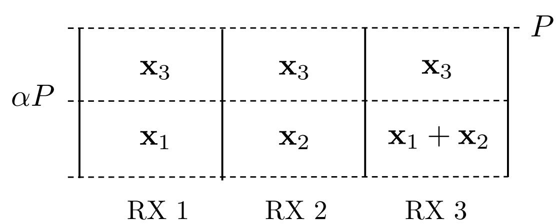
The second and third equalities are due to distributive law and the identity in the following lemma. $\square$



(a) Channel type 1



(b) Channel type 2



(c) Channel type 3

Figure 3.3: Signal scale diagram.

Lemma 3.11. *For any nested lattices $\Lambda_3 \subset \Lambda_1$ and any $\mathbf{x} \in \mathbb{R}^n$, it holds that*

$$[M_{\Lambda_3}(\mathbf{x})] \text{ mod } \Lambda_1 = [\mathbf{x}] \text{ mod } \Lambda_1.$$

Proof.

$$\begin{aligned}
&[M_{\Lambda_3}(\mathbf{x})] \bmod \Lambda_1 \\
&= [\mathbf{x} - \lambda_3] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{x}) - M_{\Lambda_1}(\lambda_3)] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{x}) - \lambda_3 + Q_{\Lambda_1}(\lambda_3)] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{x})] \bmod \Lambda_1 \\
&= [\mathbf{x}] \bmod \Lambda_1.
\end{aligned}$$

where $\lambda_3 = Q_{\Lambda_3}(\mathbf{x}) \in \Lambda_1$, thus $Q_{\Lambda_1}(\lambda_3) = \lambda_3$. □

Lemma 3.12. *The following holds.*

$$[\mathbf{t}_f + \mathbf{d}_f] \text{ mod } \Lambda_1 = [\mathbf{x}_f] \text{ mod } \Lambda_1.$$

Proof.

$$\begin{aligned}
&[\mathbf{t}_f + \mathbf{d}_f] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{t}_{11} + \mathbf{t}_3) + \mathbf{d}_f] \bmod \Lambda_1 \\
&= [\mathbf{t}_{11} + \mathbf{t}_3 + \mathbf{d}_f] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{t}_{11} + \mathbf{d}_{11}) + M_{\Lambda_1}(\mathbf{t}_3 + \mathbf{d}_3)] \bmod \Lambda_1 \\
&= [M_{\Lambda_1}(\mathbf{t}_{11} + \mathbf{d}_{11}) + M_{\Lambda_3}(\mathbf{t}_3 + \mathbf{d}_3)] \bmod \Lambda_1 \\
&= [\mathbf{x}_{11} + \mathbf{x}_3] \bmod \Lambda_1 \\
&= [\mathbf{x}_f] \bmod \Lambda_1.
\end{aligned}$$

□

Receiver 2 does not need to recover the codewords $\mathbf{t}_{11}$ and $\mathbf{t}_3$ but the real sum $\mathbf{x}_f$ to remove the interference from $\mathbf{y}_2$. Since $\mathbf{x}_f = M_{\Lambda_1}(\mathbf{x}_f)+Q_{\Lambda_1}(\mathbf{x}_f)$, we first recover the modulo part and then the quantized part to cancel out $\mathbf{x}_f$. This idea appeared in [17] as an achievable scheme for the many-to-one interference channel.

The mod-Λ_1 channel between $\mathbf{t}_f$ and $\mathbf{y}'_2$ is given by

$$\begin{aligned}
\mathbf{y}'_2 &= [\beta_2 \mathbf{y}_2 - \mathbf{d}_f] \bmod \Lambda_1 &(3.10)\\
&= [\mathbf{x}_f - \mathbf{d}_f + \mathbf{z}_{e2}] \bmod \Lambda_1 &(3.11)\\
&= [\mathbf{t}_f + \mathbf{z}_{e2}] \bmod \Lambda_1 &(3.12)
\end{aligned}$$

where the effective noise $\mathbf{z}_{e2} = (\beta_2 - 1)\mathbf{x}_f + \beta_2(\mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_2)$. Note that $\mathbb{E}[\|\mathbf{x}_f\|^2] = (\bar{\alpha}_0 + 1)nP$, and the effective noise variance $\sigma_{e2}^2 = \frac{1}{n}\mathbb{E}[\|\mathbf{z}_{e2}\|^2] = (\beta_2 - 1)^2(\bar{\alpha}_0 + 1)P + \beta_2^2 N_{e2}$ where $N_{e2} = (\alpha_0 + \alpha_2)P + N_2$. With the MMSE scaling factor $\beta_2 = \frac{(\bar{\alpha}_0+1)P}{(\bar{\alpha}_0+1)P+N_{e2}}$ plugged in, we get $\sigma_{e2}^2 = \beta_2 N_{e2} = \frac{(\bar{\alpha}_0+1)PN_{e2}}{(\bar{\alpha}_0+1)P+N_{e2}}$. The capacity of the mod-Λ_1 channel between $\mathbf{t}_f$ and $\mathbf{y}'_2$ is

$$\begin{aligned}
&\frac{1}{n} I\left(\mathbf{t}_f; \mathbf{y}'_2\right)\\
&\geq \frac{1}{n}\log\left(\frac{V(\Lambda_1)}{2^{h(\mathbf{z}_{e2})}}\right)\\
&= \frac{1}{2}\log\left(\frac{\bar{\alpha}_0 P}{\beta_2 N_{e2}}\right)\\
&= \frac{1}{2}\log\left(\frac{\bar{\alpha}_0(\bar{\alpha}_0+1)P + \bar{\alpha}_0 N_{e2}}{(\bar{\alpha}_0+1)N_{e2}}\right)\\
&= \frac{1}{2}\log\left(\frac{\bar{\alpha}_0}{\bar{\alpha}_0+1} + \frac{\bar{\alpha}_0 P}{N_{e2}}\right)\\
&= \frac{1}{2}\log\left(\frac{\bar{\alpha}_0}{\bar{\alpha}_0+1} + \frac{\bar{\alpha}_0 P}{(\alpha_0+\alpha_2)P + N_2}\right)\\
&= C_f.
\end{aligned}$$

For reliable decoding of $\mathbf{t}_f$ at receiver 2, we have the code rate constraint $R_{11} = \frac{1}{n}\log\left(\frac{V(\Lambda_1)}{V(\Lambda_c)}\right) \leq C_f$. This also implies that $R_3 = \frac{1}{n}\log\left(\frac{V(\Lambda_2)}{V(\Lambda_c)}\right) \leq C_f + \frac{1}{n}\log\left(\frac{V(\Lambda_2)}{V(\Lambda_1)}\right) = \frac{1}{2}\log\left(\frac{P}{\beta_2 N_{e2}}\right) = \frac{1}{2}\log\left(\frac{1}{\bar{\alpha}_0+1} + \frac{P}{(\alpha_0+\alpha_2)P+N_2}\right)$. By lattice decoding, we can recover the modulo sum of interference codewords $\mathbf{t}_f$ from $\mathbf{y}_2'$. Then, we can recover the real sum $\mathbf{x}_f$ in the following way.

- Recover $M_{\Lambda_1}(\mathbf{x}_f)$ by calculating $[\mathbf{t}_f + \mathbf{d}_f] \bmod \Lambda_1$ (lemma 3).
- Subtract it from the received signal,

$$\mathbf{y}_2 - M_{\Lambda_1}(\mathbf{x}_f) = Q_{\Lambda_1}(\mathbf{x}_f) + \mathbf{z}_2'' \tag{3.13}$$

  where $\mathbf{z}_2'' = \mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_2$.

- Quantize it to recover $Q_{\Lambda_1}(\mathbf{x}_f)$,

$$Q_{\Lambda_1}\left(Q_{\Lambda_1}(\mathbf{x}_f) + \mathbf{z}_2''\right) = Q_{\Lambda_1}(\mathbf{x}_f) \tag{3.14}$$

  with probability $1 - P_e$ where

$$P_e = \Pr[Q_{\Lambda_1}\left(Q_{\Lambda_1}(\mathbf{x}_f) + \mathbf{z}_2''\right) \neq Q_{\Lambda_1}(\mathbf{x}_f)] \tag{3.15}$$

  is the probability of decoding error. If we choose Λ_1 to be simultaneously Rogers-good and Poltyrev-good [28] with $V(\Lambda_1) \geq V(\Lambda_c)$, then $P_e \to 0$ as $n \to \infty$.

- Recover $\mathbf{x}_f$ by adding two vectors,

$$M_{\Lambda_1}(\mathbf{x}_f) + Q_{\Lambda_1}(\mathbf{x}_f) = \mathbf{x}_f. \tag{3.16}$$

We now proceed to decoding $\mathbf{x}_2$ from $\mathbf{y}_2 - \mathbf{x}_f = \mathbf{x}_2 + \mathbf{z}_2'$. Since $\mathbf{x}_2$ is a codeword from an i.i.d. random code for point-to-point channel, we can achieve rate up to

$$R_2 \leq \frac{1}{2}\log\left(\frac{\alpha_2 P}{\alpha_0 P + N_2}\right). \tag{3.17}$$

At receiver 1, we first decode $\mathbf{x}_{11}$ while treating other signals $\mathbf{x}_2+\mathbf{x}_{10}+\mathbf{z}_1$ as noise. The effective noise in the mod-Λ_1 channel is $\mathbf{z}_{e1} = (\beta_1 - 1)^2\mathbf{x}_{11} + \beta_1(\mathbf{x}_2+\mathbf{x}_{10}+\mathbf{z}_1)$ with variance $\sigma_{e1}^2 = \frac{1}{n}\mathbb{E}[\|\mathbf{z}_{e1}\|^2] = (\beta_1-1)^2\bar{\alpha}_0 P+\beta_1^2 N_{e1}$ where $N_{e1} = (\alpha_0+\alpha_2)P + N_1$. For reliable decoding, the rate R_{11} must satisfy

$$R_{11} \leq \frac{1}{2}\log\left(\frac{\sigma^2(\Lambda_1)}{\beta_1\sigma_{e1}^2}\right) = \frac{1}{2}\log\left(1+\frac{\bar{\alpha}_0 P}{(\alpha_0+\alpha_2)P+N_1}\right)$$

where the MMSE scaling parameter $\beta_1 = \frac{\bar{\alpha}_0 P}{\bar{\alpha}_0 P + N_{e1}}$. Similarly, we have the other rate constraints at receiver 1:

$$R_2 \leq \frac{1}{2}\log\left(1+\frac{\alpha_2 P}{\alpha_0 P+N_1}\right) \tag{3.18}$$

$$R_{10} \leq \frac{1}{2}\log\left(1+\frac{\alpha_0 P}{N_1}\right). \tag{3.19}$$

At receiver 3, the signal $\mathbf{x}_3$ is decoded with the effective noise $\mathbf{x}_2+\mathbf{z}_3$. For reliable decoding, R_3 must satisfy

$$R_3 \leq \frac{1}{2}\log\left(1+\frac{P}{\alpha_2 P+N_3}\right). \tag{3.20}$$

In summary,

- $\mathbf{x}_{11}$ decoded at receivers 1 and 2

$$R_{11} \leq T'_{11} = \frac{1}{2}\log\left(1+\frac{(1-\alpha_0)P}{(\alpha_0+\alpha_2)P+N_1}\right)$$
$$R_{11} \leq T''_{11} = \frac{1}{2}\log\left(c_{11}+\frac{(1-\alpha_0)P}{(\alpha_0+\alpha_2)P+N_2}\right)$$

  where $c_{11} = \frac{(1-\alpha_0)P}{(1-\alpha_0)P+P} = \frac{1-\alpha_0}{2-\alpha_0}$.

- $\mathbf{x}_{10}$ decoded at receiver 1

$$R_{10} \leq T_{10} = \frac{1}{2}\log\left(1+\frac{\alpha_0 P}{N_1}\right) \tag{3.21}$$

- $\mathbf{x}_2$ decoded at receivers 1 and 2

$$R_2 \leq T_2' = \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_1}\right) \tag{3.22}$$

$$R_2 \leq T_2'' = \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2}\right) \tag{3.23}$$

- $\mathbf{x}_3$ decoded at receivers 2 and 3

$$R_3 \leq T_3' = \frac{1}{2}\log\left(c_3 + \frac{P}{(\alpha_0 + \alpha_2)P + N_2}\right)$$

$$R_3 \leq T_3'' = \frac{1}{2}\log\left(1 + \frac{P}{\alpha_2 P + N_3}\right) \tag{3.24}$$

  where $c_3 = \frac{P}{(1-\alpha_0)P+P} = \frac{1}{2-\alpha_0}$.

Note that $0 \leq c_{11} \leq \frac{1}{2}$, $c_{11} + c_3 = 1$, and $\frac{1}{2} \leq c_3 \leq 1$. Putting together, we can see that the following rate region is achievable.

$$R_1 \leq T_1 = \min\{T_{11}', T_{11}''\} + T_{10} = T_{11}'' + T_{10}$$

$$R_2 \leq T_2 = \min\{T_2', T_2''\} = T_2''$$

$$R_3 \leq T_3 = \min\{T_3', T_3''\}$$

where

$$T_1 = \frac{1}{2}\log\left(c_{11} + \frac{(1-\alpha_0)P}{(\alpha_0 + \alpha_2)P + N_2}\right) + \frac{1}{2}\log\left(1 + \frac{\alpha_0 P}{N_1}\right) \tag{3.25}$$

$$T_2 = \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2}\right) \tag{3.26}$$

$$T_3 \geq \frac{1}{2}\log\left(c_3 + \frac{P}{(\alpha_0 + \alpha_2)P + N_3}\right). \tag{3.27}$$

Thus, *Theorem 8* is proved.

3.3.2 The Gap

We choose the parameter $\alpha_0 = \frac{N_2}{P}$, which is suboptimal but good enough to achieve a constant gap. This choice of parameter, inspired by [1], ensures making efficient use of signal scale difference between N_1 and N_2 at receiver 1, while keeping the interference of $\mathbf{x}_{10}$ at the noise level N_2 at receiver 2. By substitution, we get

$$T_1 = \frac{1}{2}\log\left(c_{11} + \frac{P - N_2}{\alpha_2 P + 2N_2}\right) + \frac{1}{2}\log\left(1 + \frac{N_2}{N_1}\right) \tag{3.28}$$

$$T_2 = \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{2N_2}\right) \tag{3.29}$$

$$T_3 \geq \frac{1}{2}\log\left(c_3 + \frac{P}{\alpha_2 P + N_2 + N_3}\right). \tag{3.30}$$

Since $\alpha_0 = \frac{N_2}{P} \in \left[0, \frac{1}{3}\right]$, it follows that $c_{11} = \frac{1-N_2/P}{2-N_2/P} \geq \frac{2}{5}$, and $c_3 = \frac{1}{2-N_2/P} \geq \frac{1}{2}$.

Starting from $\mathcal{R}_o$ from Table 3.2, we can express the two-dimensional outer bound region at R_2 as

$$\begin{aligned}
R_1 &\leq \min\left\{\frac{1}{2}\log\left(1 + \frac{2P}{N_1}\right) - R_2, C_1\right\} \\
&\leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - R_2, \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{4}{3}\right)\right\} \\
R_3 &\leq \min\left\{\frac{1}{2}\log\left(1 + \frac{2P}{N_2}\right) - R_2, C_3\right\} \\
&\leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right) - R_2, \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right)\right\}.
\end{aligned}$$

Depending on the bottleneck of $\min\{\cdot,\cdot\}$ expressions, there are three cases:

- $R_2 \leq \frac{1}{2}\log\left(\frac{7}{4}\right)$

- $\frac{1}{2}\log\left(\frac{7}{4}\right) \le R_2 \le \frac{1}{2}\log\left(\frac{N_3}{N_2}\cdot\frac{7}{4}\right)$

- $R_2 \ge \frac{1}{2}\log\left(\frac{N_3}{N_2}\cdot\frac{7}{4}\right)$.

At $R_2 = \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\cdot\frac{7}{4}\right)$, the outer bound region is

$$R_1 \le \min\left\{\frac{1}{2}\log\left(\frac{P}{\alpha_2 P}\cdot\frac{N_2}{N_1}\cdot\frac{4}{3}\right), \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{4}{3}\right)\right\}$$
$$R_3 \le \min\left\{\frac{1}{2}\log\left(\frac{P}{\alpha_2 P}\cdot\frac{4}{3}\right), \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right)\right\}.$$

Depending on the bottleneck of $\min\{\cdot,\cdot\}$ expressions, we consider the following three cases:

- $\alpha_2 P \ge N_3$

- $N_2 \le \alpha_2 P \le N_3$

- $\alpha_2 P \le N_2$.

Case i) $\alpha_2 P \ge N_3$: The outer bound region at $R_2 = \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\cdot\frac{7}{4}\right)$ is

$$R_1 \le \frac{1}{2}\log\left(\frac{P}{\alpha_2 P}\cdot\frac{N_2}{N_1}\cdot\frac{4}{3}\right), R_3 \le \frac{1}{2}\log\left(\frac{P}{\alpha_2 P}\cdot\frac{4}{3}\right). \tag{3.31}$$

For comparison, let us take a look at the achievable rate region. The first term of T_1 is lower bounded by

$$T_{11}'' = \frac{1}{2}\log\left(c_{11} + \frac{P - N_2}{\alpha_2 P + 2N_2}\right) \tag{3.32}$$
$$\ge \frac{1}{2}\log\left(\frac{2}{5} + \frac{P - \alpha_2 P}{3\alpha_2 P}\right) \tag{3.33}$$
$$> \frac{1}{2}\log\left(\frac{P}{3\alpha_2 P}\right). \tag{3.34}$$

We get the lower bounds:

$$
\begin{aligned}
T_1 &= T''_{11} + T_{10} && (3.35)\\
&> \frac{1}{2}\log\left(\frac{P}{3\alpha_2 P}\right) + \frac{1}{2}\log\left(1+\frac{N_2}{N_1}\right) && (3.36)\\
&> \frac{1}{2}\log\left(\frac{P}{3\alpha_2 P}\cdot\frac{N_2}{N_1}\right) && (3.37)\\
T_3 &\geq \frac{1}{2}\log\left(\frac{1}{2}+\frac{P}{\alpha_2 P + N_2 + N_3}\right) && (3.38)\\
&> \frac{1}{2}\log\left(\frac{P}{3\alpha_2 P}\right). && (3.39)
\end{aligned}
$$

For fixed α_2 and $R_2 = \frac{1}{2}\log\left(\frac{\alpha_2 P}{2N_2}\right)$, the two-dimensional achievable rate region is given by

$$
R_1 \leq \frac{1}{2}\log\left(\frac{P}{3\alpha_2 P}\cdot\frac{N_2}{N_1}\right),\ R_3 \leq \frac{1}{2}\log\left(\frac{P}{3\alpha_2 P}\right). \tag{3.40}
$$

Case ii) $N_2 \leq \alpha_2 P \leq N_3$: The outer bound region at $R_2 = \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\cdot\frac{7}{4}\right)$ is

$$
R_1 \leq \frac{1}{2}\log\left(\frac{P}{\alpha_2 P}\cdot\frac{N_2}{N_1}\cdot\frac{4}{3}\right),\ R_3 \leq \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right). \tag{3.41}
$$

Now, let us take a look at the achievable rate region. We have the lower bounds:

$$
\begin{aligned}
T_1 &> \frac{1}{2}\log\left(\frac{P}{3\alpha_2 P}\cdot\frac{N_2}{N_1}\right) && (3.42)\\
T_3 &\geq \frac{1}{2}\log\left(\frac{1}{2}+\frac{P}{\alpha_2 P + N_2 + N_3}\right) && (3.43)\\
&> \frac{1}{2}\log\left(\frac{P}{3N_3}\right). && (3.44)
\end{aligned}
$$

For fixed α_2 and $R_2 = \frac{1}{2}\log\left(\frac{\alpha_2 P}{2N_2}\right)$, the two-dimensional achievable rate region is given by

$$R_1 \leq \frac{1}{2}\log\left(\frac{P}{3\alpha_2 P}\cdot\frac{N_2}{N_1}\right),\ R_3 \leq \frac{1}{2}\log\left(\frac{P}{3N_3}\right). \tag{3.45}$$

Case iii) $\alpha_2 P \leq N_2$: The outer bound region at $R_2 = \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\cdot\frac{7}{4}\right)$ is

$$R_1 \leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{4}{3}\right),\ R_3 \leq \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right). \tag{3.46}$$

For this range of α_2, the rate R_2 is small, i.e., $R_2 = \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\cdot\frac{7}{4}\right) \leq \frac{1}{2}\log\left(\frac{7}{4}\right) < \frac{1}{2}$, and R_1 and R_3 are close to single user capacities C_1 and C_3, respectively.

Let us take a look at the achievable rate region. The first term of T_1 is lower bounded by

$$\begin{aligned}
T''_{11} &= \frac{1}{2}\log\left(c_{11} + \frac{P - N_2}{\alpha_2 P + 2N_2}\right) && (3.47)\\
&\geq \frac{1}{2}\log\left(\frac{2}{5} + \frac{P - N_2}{3N_2}\right) && (3.48)\\
&> \frac{1}{2}\log\left(\frac{P}{3N_2}\right). && (3.49)
\end{aligned}$$

We get the lower bounds:

$$\begin{aligned}
T_1 &= T''_{11} + T_{10} && (3.50)\\
&> \frac{1}{2}\log\left(\frac{P}{3N_2}\right) + \frac{1}{2}\log\left(1 + \frac{N_2}{N_1}\right) && (3.51)\\
&> \frac{1}{2}\log\left(\frac{P}{3N_1}\right) && (3.52)\\
T_3 &\geq \frac{1}{2}\log\left(\frac{1}{2} + \frac{P}{\alpha_2 P + N_2 + N_3}\right) && (3.53)\\
&> \frac{1}{2}\log\left(\frac{P}{3N_3}\right). && (3.54)
\end{aligned}$$

For fixed α_2 and $R_2 = \frac{1}{2}\log\left(\frac{\alpha_2 P}{2N_2}\right)$, the following two-dimensional rate region is achievable.

$$R_1 \leq \frac{1}{2}\log\left(\frac{P}{3N_1}\right), \; R_3 \leq \frac{1}{2}\log\left(\frac{P}{3N_3}\right). \tag{3.55}$$

In all three cases above, by comparing the inner and outer bound regions, we can see that $\delta_1 \leq \frac{1}{2}\log\left(3\cdot\frac{4}{3}\right) = 1$, $\delta_2 \leq \frac{1}{2}\log\left(2\cdot\frac{7}{4}\right) = 0.91$ and $\delta_3 \leq \frac{1}{2}\log\left(3\cdot\frac{4}{3}\right) = 1$. Therefore, we can conclude that the gap is to within one bit per message.

3.4 Inner Bound: Channel Type 2

Theorem 3.13. *Given $\alpha_1 \in [0,1]$, the region $\mathcal{R}_\alpha$ is defined by*

$$\begin{aligned} R_1 &\leq \frac{1}{2}\log\left(1 + \frac{\alpha_1 P}{N_1}\right) \\ R_2 &\leq \frac{1}{2}\log^+\left(\frac{1}{2} + \frac{P}{\alpha_1 P + N_2}\right) \\ R_3 &\leq \frac{1}{2}\log^+\left(\frac{1}{2} + \frac{P}{\alpha_1 P + N_3}\right), \end{aligned}$$

and $\mathcal{R} = \textsc{conv}\left(\bigcup_{\alpha_1}\mathcal{R}_\alpha\right)$ is achievable.

3.4.1 Achievable Scheme

For this channel type, rate splitting is not necessary. Transmit signal $\mathbf{x}_k$ is a coded signal of $M_k \in \{1, 2, \ldots, 2^{nR_k}\}, k = 1, 2, 3$. In particular, $\mathbf{x}_2$ and $\mathbf{x}_3$ are lattice-coded signals using the same pair of coding and shaping lattices. As a result, the sum $\mathbf{x}_2 + \mathbf{x}_3$ is a dithered lattice codeword. The power allocation satisfies $\mathbb{E}[\|\mathbf{x}_1\|^2] = \alpha_1 nP$, $\mathbb{E}[\|\mathbf{x}_2\|^2] = nP$, and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$. The received signals are

$$\mathbf{y}_1 = [\mathbf{x}_2 + \mathbf{x}_3] + \mathbf{x}_1 + \mathbf{z}_1$$
$$\mathbf{y}_2 = \mathbf{x}_2 + \mathbf{x}_1 + \mathbf{z}_2$$
$$\mathbf{y}_3 = \mathbf{x}_3 + \mathbf{x}_1 + \mathbf{z}_3.$$

The signal scale diagram at each receiver is shown in Fig. 3.3 (b). Decoding is performed in the following way.

- At receiver 1, $[\mathbf{x}_2+\mathbf{x}_3]$ is first decoded while treating $\mathbf{x}_1+\mathbf{z}_1$ as noise. Next, $\mathbf{x}_1$ is decoded from $\mathbf{y}_1 - [\mathbf{x}_2+\mathbf{x}_3] = \mathbf{x}_1+\mathbf{z}_1$. For reliable decoding, the code rates should satisfy
$$R_2 \leq T_2' = \frac{1}{2}\log\left(\frac{1}{2} + \frac{P}{\alpha_1 P + N_1}\right) \tag{3.56}$$
$$R_3 \leq T_3' = \frac{1}{2}\log\left(\frac{1}{2} + \frac{P}{\alpha_1 P + N_1}\right) \tag{3.57}$$
$$R_1 \leq T_1 = \frac{1}{2}\log\left(1 + \frac{\alpha_1 P}{N_1}\right). \tag{3.58}$$

- At receiver 2, $\mathbf{x}_2$ is decoded while treating $\mathbf{x}_1+\mathbf{z}_2$ as noise. Similarly at receiver 3, $\mathbf{x}_3$ is decoded while treating $\mathbf{x}_1+\mathbf{z}_3$ as noise. For reliable decoding, the code rates should satisfy
$$R_2 \leq T_2'' = \frac{1}{2}\log\left(1 + \frac{P}{\alpha_1 P + N_2}\right) \tag{3.59}$$
$$R_3 \leq T_3'' = \frac{1}{2}\log\left(1 + \frac{P}{\alpha_1 P + N_3}\right). \tag{3.60}$$

Putting together, we get

$$R_1 \leq T_1$$
$$R_2 \leq T_2 = \min\{T_2', T_2''\}$$
$$R_3 \leq T_3 = \min\{T_3', T_3''\}$$

where

$$T_1 = \frac{1}{2}\log\left(1+\frac{\alpha_1 P}{N_1}\right) \tag{3.61}$$

$$T_2 \geq \frac{1}{2}\log\left(\frac{1}{2}+\frac{P}{\alpha_1 P+N_2}\right) \tag{3.62}$$

$$\geq \frac{1}{2}\log\left(\frac{1}{2}+\frac{P}{2\cdot\max\{\alpha_1 P, N_2\}}\right) \tag{3.63}$$

$$T_3 \geq \frac{1}{2}\log\left(\frac{1}{2}+\frac{P}{\alpha_1 P+N_3}\right) \tag{3.64}$$

$$\geq \frac{1}{2}\log\left(\frac{1}{2}+\frac{P}{2\cdot\max\{\alpha_1 P, N_3\}}\right). \tag{3.65}$$

3.4.2 The Gap

Starting from $\mathcal{R}_o$ from Table 3.2, we can express the two-dimensional outer bound region at R_1 as

$$\begin{aligned}
R_2 &\leq \min\left\{\frac{1}{2}\log\left(1+\frac{2P}{N_1}\right)-R_1, C_2\right\}\\
&\leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right)-R_1, \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{4}{3}\right)\right\}\\
R_3 &\leq \min\left\{\frac{1}{2}\log\left(1+\frac{2P}{N_1}\right)-R_1, C_3\right\}\\
&\leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right)-R_1, \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right)\right\}.
\end{aligned}$$

Depending on the bottleneck of $\min\{\cdot,\cdot\}$ expressions, there are three cases:

- $R_1 \leq \frac{1}{2}\log\left(\frac{N_2}{N_1}\cdot\frac{7}{4}\right)$
- $\frac{1}{2}\log\left(\frac{N_2}{N_1}\cdot\frac{7}{4}\right) \leq R_1 \leq \frac{1}{2}\log\left(\frac{N_3}{N_1}\cdot\frac{7}{4}\right)$
- $R_1 \geq \frac{1}{2}\log\left(\frac{N_3}{N_1}\cdot\frac{7}{4}\right)$.

At $R_1 = \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{7}{4}\right)$, the region can be expressed as

$$R_2 \le \min\left\{\frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\cdot\frac{4}{3}\right), \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{4}{3}\right)\right\}$$
$$R_3 \le \min\left\{\frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\cdot\frac{4}{3}\right), \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right)\right\}.$$

Depending on the bottleneck of $\min\{\cdot,\cdot\}$ expressions, we consider the following three cases.

Case i) $\alpha_1 P \ge N_3$: The two-dimensional outer bound region at $R_1 = \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{7}{4}\right)$ is

$$R_2 \le \frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\cdot\frac{4}{3}\right),\ R_3 \le \frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\cdot\frac{4}{3}\right). \tag{3.66}$$

For fixed α_1 and $R_1 = \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\right)$, the following two-dimensional region is achievable.

$$R_2 \le \frac{1}{2}\log\left(\frac{P}{2\alpha_1 P}\right),\ R_3 \le \frac{1}{2}\log\left(\frac{P}{2\alpha_1 P}\right). \tag{3.67}$$

Case ii) $N_2 \le \alpha_1 P \le N_3$: The two-dimensional outer bound region at $R_1 = \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{7}{4}\right)$ is

$$R_2 \le \frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\cdot\frac{4}{3}\right),\ R_3 \le \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right). \tag{3.68}$$

For fixed α_1 and $R_1 = \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\right)$, the following two-dimensional region is achievable.

$$R_2 \le \frac{1}{2}\log\left(\frac{P}{2\alpha_1 P}\right),\ R_3 \le \frac{1}{2}\log\left(\frac{P}{2N_3}\right). \tag{3.69}$$

Case iii) $\alpha_1 P \le N_2$: The two-dimensional outer bound region at $R_1 = \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{7}{4}\right)$ is

$$R_2 \le \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{4}{3}\right),\ R_3 \le \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right). \tag{3.70}$$

For fixed α_1 and $R_1 = \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\right)$, the following two-dimensional region is achievable.

$$R_2 \leq \frac{1}{2}\log\left(\frac{P}{2N_2}\right), \ R_3 \leq \frac{1}{2}\log\left(\frac{P}{2N_3}\right). \tag{3.71}$$

In all three cases above, by comparing the inner and outer bounds, we can see that $\delta_1 \leq \frac{1}{2}\log\left(\frac{7}{4}\right) < 0.41$, $\delta_2 \leq \frac{1}{2}\log\left(2\cdot\frac{4}{3}\right) < 0.71$, and $\delta_3 \leq \frac{1}{2}\log\left(2\cdot\frac{4}{3}\right) < 0.71$. We can conclude that the inner and outer bounds are to within one bit.

3.5 Inner Bound: Channel Type 3

Theorem 3.14. *Given $\alpha \in [0,1]$, the region $\mathcal{R}_\alpha$ is defined by*

$$\begin{aligned} R_1 &\leq \frac{1}{2}\log\left(1+\frac{\alpha P}{N_1}\right) \\ R_2 &\leq \frac{1}{2}\log\left(1+\frac{\alpha P}{N_2}\right) \\ R_3 &\leq \frac{1}{2}\log\left(1+\frac{P}{2\alpha P + N_3}\right), \end{aligned}$$

and $\mathcal{R} = \textsc{conv}\left(\bigcup_\alpha \mathcal{R}_\alpha\right)$ is achievable.

3.5.1 Achievable Scheme

For this channel type, neither rate splitting nor aligned interference decoding is necessary. Transmit signal $\mathbf{x}_k$ is a coded signal of $M_k \in \{1, 2, \ldots, 2^{nR_k}\}, k = 1, 2, 3$. The power allocation satisfies $\mathbb{E}[\|\mathbf{x}_1\|^2] = \alpha n P$, $\mathbb{E}[\|\mathbf{x}_2\|^2] = \alpha n P$, and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$. The received signals are

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}_3 + \mathbf{x}_1 + \mathbf{z}_1 \\ \mathbf{y}_2 &= \mathbf{x}_3 + \mathbf{x}_2 + \mathbf{z}_2 \\ \mathbf{y}_3 &= \mathbf{x}_3 + \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}_3. \end{aligned}$$

The signal scale diagram at each receiver is shown in Fig. 3.3 (c). Decoding is performed in the following way.

- At receiver 1, $\mathbf{x}_3$ is first decoded while treating $\mathbf{x}_1 + \mathbf{z}_1$ as noise. Next, $\mathbf{x}_1$ is decoded from $\mathbf{y}_1 - \mathbf{x}_3 = \mathbf{x}_1 + \mathbf{z}_1$. For reliable decoding, the code rates should satisfy

$$R_3 \leq T_3' = \frac{1}{2} \log \left(1 + \frac{P}{\alpha P + N_1}\right) \tag{3.72}$$

$$R_1 \leq T_1 = \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_1}\right). \tag{3.73}$$

- At receiver 2, $\mathbf{x}_3$ is first decoded while treating $\mathbf{x}_2 + \mathbf{z}_2$ as noise. Next, $\mathbf{x}_2$ is decoded from $\mathbf{y}_2 - \mathbf{x}_3 = \mathbf{x}_2 + \mathbf{z}_2$. For reliable decoding, the code rates should satisfy

$$R_3 \leq T_3'' = \frac{1}{2} \log \left(1 + \frac{P}{\alpha P + N_2}\right) \tag{3.74}$$

$$R_2 \leq T_2 = \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_2}\right). \tag{3.75}$$

- At receiver 3, $\mathbf{x}_3$ is decoded while treating $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}_3$ as noise. For reliable decoding, the code rates should satisfy

$$R_3 \leq T_3''' = \frac{1}{2} \log \left(1 + \frac{P}{2\alpha P + N_3}\right). \tag{3.76}$$

Putting together, we get

$$R_1 \leq T_1$$
$$R_2 \leq T_2$$
$$R_3 \leq T_3 = \min\{T_3', T_3'', T_3'''\}$$

where

$$T_1 = \frac{1}{2}\log\left(1+\frac{\alpha P}{N_1}\right) \tag{3.77}$$

$$T_2 = \frac{1}{2}\log\left(1+\frac{\alpha P}{N_2}\right) \tag{3.78}$$

$$T_3 = \frac{1}{2}\log\left(1+\frac{P}{2\alpha P+N_3}\right) \tag{3.79}$$

$$\geq \frac{1}{2}\log\left(1+\frac{P}{3\cdot\max\{\alpha P, N_3\}}\right). \tag{3.80}$$

3.5.2 The Gap

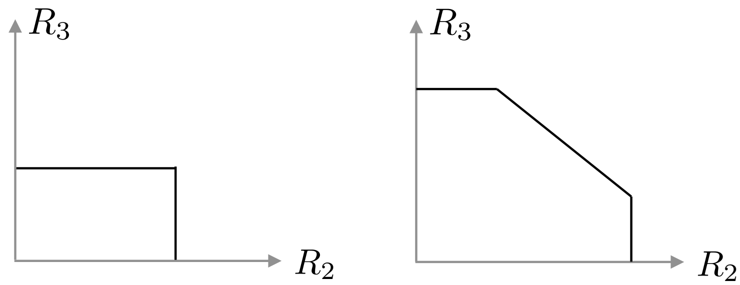
Starting from $\mathcal{R}_o$ from Table 3.2, we can express the two-dimensional outer bound region at R_3 as

$$\begin{aligned}
R_1 &\leq \min\left\{\frac{1}{2}\log\left(1+\frac{2P}{N_1}\right)-R_3, C_1\right\} \\
&\leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right)-R_3, \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{4}{3}\right)\right\} \\
R_2 &\leq \min\left\{\frac{1}{2}\log\left(1+\frac{2P}{N_2}\right)-R_3, C_2\right\} \\
&\leq \min\left\{\frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right)-R_3, \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{4}{3}\right)\right\}.
\end{aligned}$$

Depending on the bottleneck of $\min\{\cdot,\cdot\}$ expressions, there are two cases: $R_3 \leq \frac{1}{2}\log\left(\frac{7}{4}\right)$ and $R_3 \geq \frac{1}{2}\log\left(\frac{7}{4}\right)$. We assume that $R_3 \geq \frac{1}{2}\log\left(\frac{7}{4}\right)$, equivalently $\alpha \leq \frac{4}{7}$. We also assume that $R_3 \leq \frac{1}{2}\log\left(\frac{P}{N_3}\right)$, equivalently $\alpha P \geq N_3$. The other cases are trivial.

The two-dimensional outer bound region at $R_3 = \frac{1}{2}\log\left(\frac{P}{\alpha P}\right)$ is

$$\begin{aligned}
R_1 &\leq \min\left\{\frac{1}{2}\log\left(\frac{\alpha P}{N_1}\cdot\frac{7}{3}\right), \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{4}{3}\right)\right\} \\
R_2 &\leq \min\left\{\frac{1}{2}\log\left(\frac{\alpha P}{N_2}\cdot\frac{7}{3}\right), \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{4}{3}\right)\right\}.
\end{aligned}$$



(a) Large R_1 (b) Small R_1

Figure 3.4: The cross-section of the type 4 outer bound region.

For $\alpha \leq \frac{4}{7}$, the two-dimensional outer bound region is

$$R_1 \leq \frac{1}{2} \log \left(\frac{\alpha P}{N_1} \cdot \frac{7}{3} \right), \ R_2 \leq \frac{1}{2} \log \left(\frac{\alpha P}{N_2} \cdot \frac{7}{3} \right). \tag{3.81}$$

For $\alpha P \geq N_3$, the two-dimensional achievable rate region at $R_3 = \frac{1}{2} \log \left(\frac{P}{3\alpha P} \right)$ is

$$R_1 \leq \frac{1}{2} \log \left(\frac{\alpha P}{N_1} \right), \ R_2 \leq \frac{1}{2} \log \left(\frac{\alpha P}{N_2} \right). \tag{3.82}$$

By comparing the inner and outer bounds, we can see that $\delta_1 \leq \frac{1}{2} \log \left(\frac{7}{3} \right) < 0.62$, $\delta_2 \leq \frac{1}{2} \log \left(\frac{7}{3} \right) < 0.62$, and $\delta_3 \leq \frac{1}{2} \log (3) < 0.8$. We can conclude that the inner and outer bounds are to within one bit.

3.6 Inner Bound: Channel Type 4

The relaxed outer bound region $\mathcal{R}'_o$ given by

$$\begin{aligned}
R_k &\le \frac{1}{2}\log\left(\frac{P}{N_k}\right) + \frac{1}{2}\log\left(\frac{4}{3}\right), \; k = 1, 2, 3 \\
R_1 + R_2 &\le \frac{1}{2}\log\left(\frac{P}{N_1}\right) + \frac{1}{2}\log\left(\frac{7}{3}\right) \\
R_1 + R_3 &\le \frac{1}{2}\log\left(\frac{P}{N_1}\right) + \frac{1}{2}\log\left(\frac{7}{3}\right) \\
R_2 + R_3 &\le \frac{1}{2}\log\left(\frac{P}{N_2}\right) + \frac{1}{2}\log\left(\frac{7}{3}\right).
\end{aligned}$$

The cross-sectional region at a given R_1 is described by

$$\begin{aligned}
R_2 &\le \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - R_1, \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{4}{3}\right)\right\} \\
R_3 &\le \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - R_1, \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right)\right\} \\
R_2 + R_3 &\le \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right).
\end{aligned}$$

Depending on the bottleneck of $\min\{\cdot,\cdot\}$ expressions, there are three cases:

- $R_1 \le \frac{1}{2}\log\left(\frac{N_2}{N_1}\cdot\frac{7}{4}\right)$
- $\frac{1}{2}\log\left(\frac{N_2}{N_1}\cdot\frac{7}{4}\right) \le R_1 \le \frac{1}{2}\log\left(\frac{N_3}{N_1}\cdot\frac{7}{4}\right)$
- $R_1 \ge \frac{1}{2}\log\left(\frac{N_3}{N_1}\cdot\frac{7}{4}\right)$.

In this section, we focus on the third case. The other cases can be proved similarly. If the sum of the righthand sides of R_2 and R_3 bounds is smaller than the righthand side of $R_2 + R_3$ bound, i.e.,

$$\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - 2R_1 \le \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right), \tag{3.83}$$

then the R_2+R_3 bound is not active at the R_1. This condition can be expressed as a threshold on R_1 given by

$$\begin{aligned} R_1 > R_{1,th} &= \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - \frac{1}{4}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right) \\ &= \frac{1}{4}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) + \frac{1}{4}\log\left(\frac{N_2}{N_1}\right). \end{aligned} \tag{3.84}$$

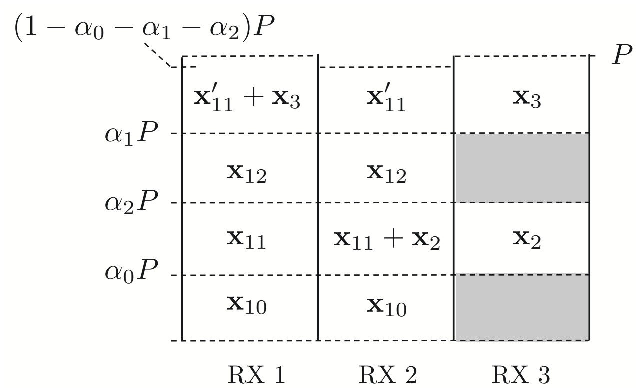
For this relatively large R_1, the cross-sectional region is a rectangle as described in Fig. 3.4 (a). In contrast, for a relatively small R_1, when the threshold condition does not hold, the cross-sectional region is a MAC-like region as described in Fig. 3.4 (b). In the rest of the section, we present achievable schemes for each case.

3.6.1 Achievable Scheme for Relatively Large R_1

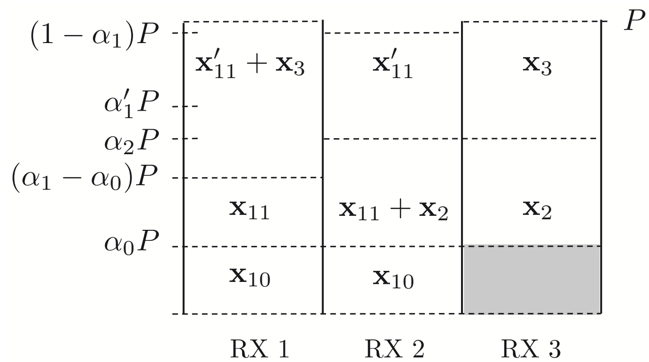
Theorem 3.15. *Given $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in [0,1]^3$, the region $\mathcal{R}_\alpha$ is defined by*

$$\begin{aligned} R_1 \le \min\Bigg\{&\frac{1}{2}\log^+\left(c_{11} + \frac{(1-\alpha_0-\alpha_1-\alpha_2)P}{(\alpha_0+\alpha_1+2\alpha_2)P+N_2}\right), \\ &\frac{1}{2}\log\left(1+\frac{\alpha_2 P}{\alpha_0 P + N_1}\right)\Bigg\} \\ &+\frac{1}{2}\log\left(1+\frac{\alpha_1 P}{(\alpha_0+\alpha_2)P+N_2}\right) \\ &+\frac{1}{2}\log\left(1+\frac{\alpha_0 P}{N_1}\right) \\ R_2 \le &\ \frac{1}{2}\log\left(1+\frac{\alpha_2 P}{\alpha_0 P + N_2}\right) \\ R_3 \le &\ \frac{1}{2}\log^+\left(c_3 + \frac{P}{(\alpha_0+\alpha_1+\alpha_2)P+N_3}\right) \end{aligned}$$

where $c_{11} = \frac{1-\alpha_0-\alpha_1-\alpha_2}{2-\alpha_0-\alpha_1-\alpha_2}$ and $c_3 = \frac{1}{2-\alpha_0-\alpha_1-\alpha_2}$, and $\mathcal{R} = \mathrm{CONV}\left(\bigcup_\alpha \mathcal{R}_\alpha\right)$ is achievable.



(a) Channel type 4: relatively large R_1



(b) Channel type 4: relatively small R_1

Figure 3.5: Signal scale diagram.

We present an achievable scheme for the case of $R_1 > R_{1,th}$. Message $M_1 \in \{1, 2, \ldots, 2^{nR_1}\}$ is split into three parts: $M_{10} \in \{1, 2, \ldots, 2^{nR_{10}}\}$, $M_{11} \in \{1, 2, \ldots, 2^{nR_{11}}\}$ and $M_{12} \in \{1, 2, \ldots, 2^{nR_{12}}\}$, so $R_1 = R_{10} + R_{11} + R_{12}$. We generate the signals in the following way: $\mathbf{x}_{11}$ and $\mathbf{x}'_{11}$ are differently coded signals of M_{11}, and $\mathbf{x}_{10}$ and $\mathbf{x}_{12}$ are coded signal of M_{10} and M_{12}, respectively. The transmit signal is the sum

$$\mathbf{x}_1 = \mathbf{x}_{10} + \mathbf{x}_{11} + \mathbf{x}_{12} + \mathbf{x}'_{11}.$$

The power allocation satisfies $\mathbb{E}[\|\mathbf{x}_{10}\|^2] = \alpha_0 nP$, $\mathbb{E}[\|\mathbf{x}_{11}\|^2] = \alpha_2 nP$, $\mathbb{E}[\|\mathbf{x}_{12}\|^2] = \alpha_1 nP$, and $\mathbb{E}[\|\mathbf{x}'_{11}\|^2] = (1 - \alpha_0 - \alpha_1 - \alpha_2)nP$.

The transmit signals $\mathbf{x}_2$ and $\mathbf{x}_3$ are coded signals of the messages $M_2 \in \{1, 2, \ldots, 2^{nR_2}\}$ and $M_3 \in \{1, 2, \ldots, 2^{nR_3}\}$, satisfying $\mathbb{E}[\|\mathbf{x}_2\|^2] = \alpha_2 nP$ and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$.

The signals $\mathbf{x}'_{11}$ and $\mathbf{x}_3$ are lattice-coded signals using the same coding lattice but different shaping lattices. As a result, the sum $\mathbf{x}'_{11} + \mathbf{x}_3$ is a dithered lattice codeword.

The received signals are

$$\begin{aligned}
\mathbf{y}_1 &= [\mathbf{x}'_{11} + \mathbf{x}_3] + \mathbf{x}_{12} + \mathbf{x}_{11} + \mathbf{x}_{10} + \mathbf{z}_1 \\
\mathbf{y}_2 &= \mathbf{x}'_{11} + \mathbf{x}_{12} + \mathbf{x}_{11} + \mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_2 \\
\mathbf{y}_3 &= \mathbf{x}_3 + \mathbf{x}_2 + \mathbf{z}_3.
\end{aligned}$$

The signal scale diagram at each receiver is shown in Fig. 3.5 (a). Decoding is performed in the following way.

- At receiver 1, $[\mathbf{x}'_{11} + \mathbf{x}_3]$ is first decoded while treating other signals as noise and removed from $\mathbf{y}_1$. Next, $\mathbf{x}_{12}$, $\mathbf{x}_{11}$, and $\mathbf{x}_{10}$ are decoded successively. For reliable decoding, the code rates should satisfy

$$\begin{aligned}
R_{11} &\le T'_{11} = \frac{1}{2}\log\left(c_{11} + \frac{(1-\alpha_0-\alpha_1-\alpha_2)P}{(\alpha_0+\alpha_1+\alpha_2)P+N_1}\right)\\
R_3 &\le T'_3 = \frac{1}{2}\log\left(c_3 + \frac{P}{(\alpha_0+\alpha_1+\alpha_2)P+N_1}\right)\\
R_{12} &\le T'_{12} = \frac{1}{2}\log\left(1 + \frac{\alpha_1 P}{(\alpha_0+\alpha_2)P+N_1}\right)\\
R_{11} &\le T''_{11} = \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{\alpha_0 P+N_1}\right)\\
R_{10} &\le T_{10} = \frac{1}{2}\log\left(1 + \frac{\alpha_0 P}{N_1}\right)
\end{aligned}$$

where $c_{11} = \frac{(1-\alpha_0-\alpha_1-\alpha_2)P}{(1-\alpha_0-\alpha_1-\alpha_2)P+P} = \frac{1-\alpha_0-\alpha_1-\alpha_2}{2-\alpha_0-\alpha_1-\alpha_2}$ and $c_3 = \frac{P}{(1-\alpha_0-\alpha_1-\alpha_2)P+P} = \frac{1}{2-\alpha_0-\alpha_1-\alpha_2}$. Note that $0 \le c_{11} \le \frac{1}{2}$, $c_{11}+c_3 = 1$, and $\frac{1}{2} \le c_3 \le 1$.

- At receiver 2, $\mathbf{x}'_{11}$ is first decoded while treating other signals as noise. Having successfully recovered M_{11}, receiver 2 can generate $\mathbf{x}_{11}$ and $\mathbf{x}'_{11}$, and cancel them from $\mathbf{y}_2$. Next, $\mathbf{x}_{12}$ is decoded from $\mathbf{x}_{12}+\mathbf{x}_2+\mathbf{x}_{10}+\mathbf{z}_2$. Finally, $\mathbf{x}_2$ is decoded from $\mathbf{x}_2+\mathbf{x}_{10}+\mathbf{z}_2$. For reliable decoding, the code rates should satisfy

$$\begin{aligned}
R_{11} &\le T'''_{11} = \frac{1}{2}\log\left(1 + \frac{(1-\alpha_0-\alpha_1-\alpha_2)P}{(\alpha_0+\alpha_1+2\alpha_2)P+N_2}\right)\\
R_{12} &\le T''_{12} = \frac{1}{2}\log\left(1 + \frac{\alpha_1 P}{(\alpha_0+\alpha_2)P+N_2}\right)\\
R_2 &\le T_2 = \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{\alpha_0 P+N_2}\right).
\end{aligned}$$

- At receiver 3, $\mathbf{x}_3$ is decoded while treating $\mathbf{x}_2+\mathbf{z}_3$ as noise. Reliable decoding is possible if

$$R_3 \le T''_3 = \frac{1}{2}\log\left(1 + \frac{P}{\alpha_2 P+N_3}\right). \tag{3.85}$$

Putting together, we can see that given $\alpha_0, \alpha_1, \alpha_2 \in [0, 1]$, the following rate region is achievable.

$$
\begin{aligned}
R_1 &\leq T_1 = \min\{T'_{11}, T''_{11}, T'''_{11}\} + \min\{T'_{12}, T''_{12}\} + T_{10} \\
R_2 &\leq T_2 \\
R_3 &\leq T_3 = \min\{T'_3, T''_3\}
\end{aligned}
$$

where

$$
\begin{aligned}
T_1 &= \min\{T'_{11}, T''_{11}, T'''_{11}\} + \min\{T'_{12}, T''_{12}\} + T_{10} \\
&= \min\{\min\{T'_{11}, T'''_{11}\}, T''_{11}\} + T''_{12} + T_{10} \\
&\geq \min\left\{\frac{1}{2}\log\left(c_{11} + \frac{(1-\alpha_0-\alpha_1-\alpha_2)P}{(\alpha_0+\alpha_1+2\alpha_2)P+N_2}\right),\right. \\
&\qquad\qquad\qquad \left.\frac{1}{2}\log\left(1+\frac{\alpha_2 P}{\alpha_0 P + N_1}\right)\right\} \\
&\quad + \frac{1}{2}\log\left(1 + \frac{\alpha_1 P}{(\alpha_0+\alpha_2)P+N_2}\right) \\
&\quad + \frac{1}{2}\log\left(1 + \frac{\alpha_0 P}{N_1}\right) \\
T_2 &= \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2}\right) \\
T_3 &\geq \frac{1}{2}\log\left(c_3 + \frac{P}{(\alpha_0+\alpha_1+\alpha_2)P+N_3}\right).
\end{aligned}
$$

3.6.2 The Gap for Relatively Large R_1

We choose α_0, α_1 and α_2 such that $\alpha_1 \leq \frac{3}{8}$, that $\alpha_1 \geq 3(\alpha_0 + \alpha_2)$, that $\alpha_2 P \geq 3N_3$, and that $\alpha_0 P = N_2$. It follows that $\alpha_0 + \alpha_1 + \alpha_2 \leq \frac{4}{3}\alpha_1 \leq \frac{1}{2}$, that $c_{11} \geq \frac{1}{3}$, and that $(\alpha_0 + \alpha_1 + 2\alpha_2)P + N_2 = 2(\alpha_0 + \alpha_2)P + \alpha_1 P \leq \frac{5}{3}\alpha_1 P$. We get the lower bounds for each term of T_1 expression above.

$$\begin{aligned}
&\min\{T'_{11}, T'''_{11}\} \\
&\geq \frac{1}{2}\log\left(c_{11} + \frac{(1-\alpha_0-\alpha_1-\alpha_2)P}{(\alpha_0+\alpha_1+2\alpha_2)P+N_2}\right) \\
&\geq \frac{1}{2}\log\left(\frac{1}{3} + \frac{(1-(4/3)\alpha_1)P}{(5/3)\alpha_1 P}\right) \\
&= \frac{1}{2}\log\left(\frac{P}{(5/3)\alpha_1 P} - \frac{7}{15}\right) \\
&= \frac{1}{2}\log\left(\frac{P}{(5/3)\alpha_1 P}\right) + \frac{1}{2}\log\left(1 - \frac{7}{15}\cdot\frac{5}{3}\alpha_1\right) \\
&\geq \frac{1}{2}\log\left(\frac{P}{(5/3)\alpha_1 P}\right) + \frac{1}{2}\log\left(\frac{17}{24}\right) \\
&\geq \frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\cdot\frac{17}{40}\right)
\end{aligned}$$

and

$$T''_{11} = \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_1}\right) \tag{3.86}$$

$$= \frac{1}{2}\log\left(\frac{(\alpha_0+\alpha_2)P+N_1}{\alpha_0 P + N_1}\right) \tag{3.87}$$

$$\geq \frac{1}{2}\log\left(\frac{(\alpha_0+\alpha_2)P}{\alpha_0 P + N_2}\right) \tag{3.88}$$

$$= \frac{1}{2}\log\left(\frac{(\alpha_0+\alpha_2)P}{2N_2}\right). \tag{3.89}$$

Since $(\alpha_0+\alpha_2)P \geq N_2 + 3N_3 \geq 4N_2$,

$$T''_{12} = \frac{1}{2}\log\left(1 + \frac{\alpha_1 P}{(\alpha_0+\alpha_2)P+N_2}\right) \tag{3.90}$$

$$\geq \frac{1}{2}\log\left(\frac{\alpha_1 P}{(5/4)(\alpha_0+\alpha_2)P}\right). \tag{3.91}$$

Putting together,

$$\begin{aligned}
T_1 &\geq \min\left\{\frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\cdot\frac{17}{40}\right), \frac{1}{2}\log\left(\frac{(\alpha_0+\alpha_2)P}{2N_2}\right)\right\} \\
&\quad +\frac{1}{2}\log\left(\frac{\alpha_1 P}{(5/4)(\alpha_0+\alpha_2)P}\right)+\frac{1}{2}\log\left(\frac{N_2}{N_1}\right) \\
&= \min\left\{\frac{1}{2}\log\left(\frac{P}{(\alpha_0+\alpha_2)P}\cdot\frac{N_2}{N_1}\cdot\frac{17}{40}\cdot\frac{4}{5}\right), \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{1}{2}\cdot\frac{4}{5}\right)\right\} \\
&= \min\left\{\frac{1}{2}\log\left(\frac{P}{(\alpha_0+\alpha_2)P}\cdot\frac{N_2}{N_1}\cdot\frac{17}{50}\right), \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{2}{5}\right)\right\}.
\end{aligned}$$

Given α_1, we choose α_2 that satisfies $\frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\cdot\frac{17}{40}\right)=\frac{1}{2}\log\left(\frac{(\alpha_0+\alpha_2)P}{2N_2}\right)$. As a result, we can write $T_1 \geq \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{2}{5}\right)$, and also

$$T_2 = \frac{1}{2}\log\left(1+\frac{\alpha_2 P}{\alpha_0 P+N_2}\right) \tag{3.92}$$

$$\geq \frac{1}{2}\log\left(\frac{(\alpha_0+\alpha_2)P}{2N_2}\right) \tag{3.93}$$

$$= \frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\cdot\frac{17}{40}\right). \tag{3.94}$$

Since $N_3 \leq \frac{1}{3}\alpha_2 P \leq \frac{1}{3}(\alpha_0+\alpha_2)P \leq \frac{1}{9}\alpha_1 P$,

$$\begin{aligned}
T_3 &\geq \frac{1}{2}\log\left(c_3+\frac{P}{(\alpha_0+\alpha_1+\alpha_2)P+N_3}\right) \\
&\geq \frac{1}{2}\log\left(\frac{1}{2}+\frac{P}{(4/3)\alpha_1 P+(1/9)\alpha_1 P}\right) \\
&\geq \frac{1}{2}\log\left(\frac{P}{(13/9)\alpha_1 P}\right).
\end{aligned}$$

The following rate region is achievable.

$$R_1 \leq \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{2}{5}\right) \tag{3.95}$$

$$R_2 \leq \frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\cdot\frac{17}{40}\right) \tag{3.96}$$

$$R_3 \leq \frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\cdot\frac{9}{13}\right). \tag{3.97}$$

For fixed α_1 and $R_1 = \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1} \cdot \frac{2}{5}\right)$, the two-dimensional rate region, given by

$$R_2 \leq \frac{1}{2}\log\left(\frac{P}{\alpha_1 P} \cdot \frac{17}{40}\right), \ R_3 \leq \frac{1}{2}\log\left(\frac{P}{\alpha_1 P} \cdot \frac{9}{13}\right)$$

is achievable.

In comparison, the two-dimensional outer bound region at $R_1 = \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1} \cdot \frac{2}{5}\right) + 1$, given by

$$\begin{aligned}
R_2 &\leq \frac{1}{2}\log\left(\frac{P}{N_1} \cdot \frac{7}{3}\right) - \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1} \cdot \frac{2}{5}\right) - 1 \\
&= \frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\right) + \frac{1}{2}\log\left(\frac{7}{3} \cdot \frac{5}{2} \cdot \frac{1}{4}\right) \\
R_3 &\leq \frac{1}{2}\log\left(\frac{P}{N_1} \cdot \frac{7}{3}\right) - \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1} \cdot \frac{2}{5}\right) - 1 \\
&= \frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\right) + \frac{1}{2}\log\left(\frac{7}{3} \cdot \frac{5}{2} \cdot \frac{1}{4}\right).
\end{aligned}$$

As discussed above, the sum-rate bound on $R_2 + R_3$ is loose for R_1 larger than the threshold, so the rate region is a rectangle. By comparing the inner and outer bound rate regions, we can see that $\delta_2 < \frac{1}{2}\log\left(\frac{40}{17} \cdot \frac{7}{3} \cdot \frac{5}{2} \cdot \frac{1}{4}\right) < 0.89$ and $\delta_3 < \frac{1}{2}\log\left(\frac{13}{9} \cdot \frac{7}{3} \cdot \frac{5}{2} \cdot \frac{1}{4}\right) < 0.54$. Therefore, we can conclude that the gap is to within one bit per message.

3.6.3 Achievable Scheme for Relatively Small R_1

Theorem 3.16. *Given $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in [0,1]^3$, the region $\mathcal{R}_\alpha$ is defined by*

$$\begin{aligned}
R_1 \leq \min &\left\{ \frac{1}{2}\log^+\left(c_{11} + \frac{(1-\alpha_1)P}{(\alpha_1+\alpha_2)P+N_2}\right),\right. \\
&\left.\frac{1}{2}\log\left(1+\frac{(\alpha_1-\alpha_0)P}{\alpha_0 P+N_1}\right)\right\} + \frac{1}{2}\log\left(1+\frac{\alpha_0 P}{N_1}\right) \\
R_2 \leq &\ \frac{1}{2}\log\left(1+\frac{\alpha_2 P}{\alpha_0 P+N_2}\right) \\
R_3 \leq &\ \frac{1}{2}\log^+\left(c_3 + \frac{P}{\max\{\alpha_1,\alpha_2\}P+N_3}\right)
\end{aligned}$$

where $c_{11} = \frac{1-\alpha_1}{2-\alpha_1}$ and $c_3 = \frac{1}{2-\alpha_1}$, and $\mathcal{R} = \text{CONV}\left(\bigcup_\alpha \mathcal{R}_\alpha\right)$ is achievable.

For the case of $R_1 < R_{1,th}$, we present the following achievable scheme. At transmitter 1, we split M_1 into M_{10} and M_{11}, so $R_1 = R_{10} + R_{11}$. The transmit signal is the sum

$$\mathbf{x}_1 = \mathbf{x}_{10} + \mathbf{x}_{11} + \mathbf{x}'_{11}.$$

The power allocation satisfies $\mathbb{E}[\|\mathbf{x}_{10}\|^2] = \alpha_0 nP$, $\mathbb{E}[\|\mathbf{x}_{11}\|^2] = (\alpha_1 - \alpha_0)nP$, and $\mathbb{E}[\|\mathbf{x}'_{11}\|^2] = (1-\alpha_1)nP$ at receiver 1, $\mathbb{E}[\|\mathbf{x}_2\|^2] = \alpha_2 nP$ at receiver 2, and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$ at receiver 3.

The signals $\mathbf{x}'_{11}$ and $\mathbf{x}_3$ are lattice codewords using the same coding lattice but different shaping lattices. As a result, the sum $\mathbf{x}'_{11} + \mathbf{x}_3$ is a lattice codeword.

The received signals are

$$\begin{aligned}
\mathbf{y}_1 &= [\mathbf{x}'_{11} + \mathbf{x}_3] + \mathbf{x}_{11} + \mathbf{x}_{10} + \mathbf{z}_1 \\
\mathbf{y}_2 &= \mathbf{x}'_{11} + \mathbf{x}_{11} + \mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_2 \\
\mathbf{y}_3 &= \mathbf{x}_3 + \mathbf{x}_2 + \mathbf{z}_3.
\end{aligned}$$

The signal scale diagram at each receiver is shown in Fig. 3.5 (b). Decoding is performed in the following way.

- At receiver 1, $[\mathbf{x}'_{11} + \mathbf{x}_3]$ is first decoded while treating other signals as noise and removed from $\mathbf{y}_1$. Next, $\mathbf{x}_{11}$ and then $\mathbf{x}_{10}$ is decoded successively. For reliable decoding, the code rates should satisfy

$$
\begin{aligned}
R_{11} &\le T'_{11} = \frac{1}{2}\log\left(c_{11} + \frac{(1-\alpha_1)P}{\alpha_1 P + N_1}\right) \\
R_3 \ &\le T'_3 \ = \frac{1}{2}\log\left(c_3 + \frac{P}{\alpha_1 P + N_1}\right) \\
R_{11} &\le T''_{11} = \frac{1}{2}\log\left(1 + \frac{(\alpha_1-\alpha_0)P}{\alpha_0 P + N_1}\right) \\
R_{10} &\le T_{10} = \frac{1}{2}\log\left(1 + \frac{\alpha_0 P}{N_1}\right)
\end{aligned}
$$

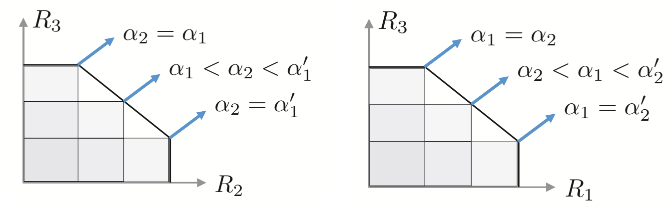
  where $c_{11} = \frac{(1-\alpha_1)P}{(1-\alpha_1)P+P} = \frac{1-\alpha_1}{2-\alpha_1}$ and $c_3 = \frac{P}{(1-\alpha_1)P+P} = \frac{1}{2-\alpha_1}$. Note that $0 \le c_{11} \le \frac{1}{2}$, $c_{11} + c_3 = 1$, and $\frac{1}{2} \le c_3 \le 1$.

- At receiver 2, $\mathbf{x}'_{11}$ is first decoded while treating other signals as noise. Having successfully recovered M_{11}, receiver 1 can generate $\mathbf{x}_{11}$ and $\mathbf{x}'_{11}$, and cancel them from $\mathbf{y}_2$. Next, $\mathbf{x}_2$ is decoded from $\mathbf{x}_2 + \mathbf{x}_{10} + \mathbf{z}_2$. At receiver 2, $\mathbf{x}_{10}$ is not decoded. For reliable decoding, the code rates should satisfy

$$
\begin{aligned}
R_{11} &\le T'''_{11} = \frac{1}{2}\log\left(1 + \frac{(1-\alpha_1)P}{(\alpha_1+\alpha_2)P + N_2}\right) \\
R_2 \ &\le T_2 \ = \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2}\right).
\end{aligned}
$$

- At receiver 3, $\mathbf{x}_3$ is decoded while treating $\mathbf{x}_2 + \mathbf{z}_3$ as noise. Reliable decoding is possible if

$$
R_3 \ \le T''_3 = \frac{1}{2}\log\left(1 + \frac{P}{\alpha_2 P + N_3}\right). \tag{3.98}
$$



(a) Channel type 4: small R_1

(b) Channel type 5: small R_2

Figure 3.6: MAC-like region.

Putting together, we can see that given $\alpha_0, \alpha_1\alpha_2 \in [0, 1]$, the following rate region is achievable.

$$R_1 \leq T_1 = \min\{T_{11}', T_{11}'', T_{11}'''\} + T_{10} \tag{3.99}$$

$$R_2 \leq T_2 \tag{3.100}$$

$$R_3 \leq T_3 = \min\{T_3', T_3''\} \tag{3.101}$$

where

$$\begin{aligned}
T_1 &= \min\{T_{11}', T_{11}'', T_{11}'''\} + T_{10} \\
&= \min\{\min\{T_{11}', T_{11}'''\}, T_{11}''\} + T_{10} \\
&\geq \min\left\{\frac{1}{2}\log\left(c_{11} + \frac{(1-\alpha_1)P}{(\alpha_1+\alpha_2)P + N_2}\right),\right. \\
&\quad \left.\frac{1}{2}\log\left(1 + \frac{(\alpha_1-\alpha_0)P}{\alpha_0 P + N_1}\right)\right\} + \frac{1}{2}\log\left(1 + \frac{\alpha_0 P}{N_1}\right) \\
T_2 &= \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{\alpha_0 P + N_2}\right) \\
T_3 &\geq \frac{1}{2}\log\left(c_3 + \frac{P}{\max\{\alpha_1, \alpha_2\}P + N_3}\right).
\end{aligned}$$

3.6.4 The Gap for Relatively Small R_1

We choose α_0, α_1, and α_2 such that $\alpha_1 \leq \alpha_2 \leq \frac{1}{2}$, that $\alpha_1 P \geq 3N_2$, that $\alpha_2 P \geq 3N_3$, and that $\alpha_0 P = \frac{4}{5}N_2$. It follows that $c_{11} \geq \frac{1}{3}$ and that $(\alpha_1 + \alpha_2)P + N_2 \leq \frac{4}{3}\alpha_1 P + \alpha_2 P \leq \frac{7}{3}\alpha_2 P$.

$$\begin{aligned}
&\min\{T'_{11}, T'''_{11}\} \\
&= \frac{1}{2}\log\left(c_{11} + \frac{(1-\alpha_1)P}{(\alpha_1+\alpha_2)P + N_2}\right) \\
&\geq \frac{1}{2}\log\left(\frac{1}{3} + \frac{(1-\alpha_2)P}{(7/3)\alpha_2 P}\right) \\
&= \frac{1}{2}\log\left(\frac{P}{(7/3)\alpha_2 P} - \frac{2}{21}\right) \\
&= \frac{1}{2}\log\left(\frac{P}{(7/3)\alpha_2 P}\right) + \frac{1}{2}\log\left(1 - \frac{2}{21}\cdot\frac{7}{3}\alpha_2\right) \\
&\geq \frac{1}{2}\log\left(\frac{P}{(7/3)\alpha_2 P}\right) + \frac{1}{2}\log\left(\frac{8}{9}\right) \\
&\geq \frac{1}{2}\log\left(\frac{P}{\alpha_2 P}\cdot\frac{8}{21}\right)
\end{aligned}$$

and

$$\begin{aligned}
T''_{11} &= \frac{1}{2}\log\left(1 + \frac{(\alpha_1 - \alpha_0)P}{\alpha_0 P + N_1}\right) && (3.102)\\
&= \frac{1}{2}\log\left(\frac{\alpha_1 P + N_1}{\alpha_0 P + N_1}\right) && (3.103)\\
&\geq \frac{1}{2}\log\left(\frac{\alpha_1 P}{\alpha_0 P + N_2}\right) && (3.104)\\
&= \frac{1}{2}\log\left(\frac{\alpha_1 P}{(9/5)N_2}\right). && (3.105)
\end{aligned}$$

Putting together,

$$\begin{aligned}
T_1 \geq \min&\left\{\frac{1}{2}\log\left(\frac{P}{\alpha_2 P}\cdot\frac{8}{21}\right), \frac{1}{2}\log\left(\frac{\alpha_1 P}{(9/5)N_2}\right)\right\} \\
&+\frac{1}{2}\log\left(\frac{N_2}{N_1}\cdot\frac{4}{5}\right).
\end{aligned}$$

Let us define α_1' by the equality $\frac{1}{2}\log\left(\frac{P}{\alpha_1' P}\cdot\frac{8}{21}\right) = \frac{1}{2}\log\left(\frac{\alpha_1 P}{(9/5)N_2}\right)$. If we choose $\alpha_2 \le \alpha_1'$, then $\frac{1}{2}\log\left(\frac{P}{\alpha_2 P}\cdot\frac{8}{21}\right) \ge \frac{1}{2}\log\left(\frac{\alpha_1 P}{(9/5)N_2}\right)$, and

$$T_1 \ge \frac{1}{2}\log\left(\frac{\alpha_1 P}{(9/5)N_2}\cdot\frac{N_2}{N_1}\cdot\frac{4}{5}\right) = \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{4}{9}\right).$$

We can see that the following rate region is achievable.

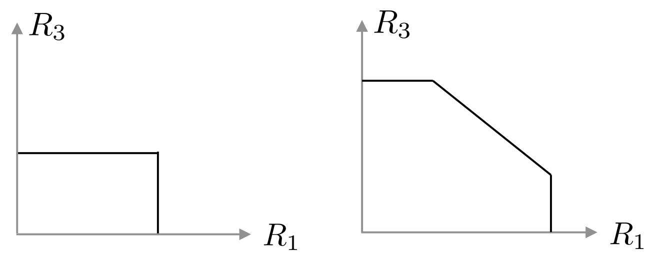
$$R_1 \le \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{4}{9}\right) \tag{3.106}$$
$$R_2 \le \frac{1}{2}\log\left(\frac{\alpha_2 P}{(9/5)N_2}\right) \tag{3.107}$$
$$R_3 \le \frac{1}{2}\log\left(\frac{P}{(4/3)\alpha_2 P}\right). \tag{3.108}$$

For fixed $\alpha_2 \in [\alpha_1, \alpha_1']$ and $R_1 = \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{4}{9}\right)$, the two-dimensional rate region $\mathcal{R}_\alpha$, given by

$$R_2 \le \frac{1}{2}\log\left(\frac{\alpha_2 P}{(9/5)N_2}\right) \tag{3.109}$$
$$R_3 \le \frac{1}{2}\log\left(\frac{P}{(4/3)\alpha_2 P}\right) \tag{3.110}$$

is achievable. The union $\bigcup_{\alpha_2\in[\alpha_1,\alpha_1']}\mathcal{R}_\alpha$ is a MAC-like region, given by

$$R_2 \le \frac{1}{2}\log\left(\frac{\alpha_1' P}{(9/5)N_2}\right) \tag{3.111}$$
$$\le \frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\cdot\frac{8}{21}\right) \tag{3.112}$$
$$R_3 \le \frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\cdot\frac{3}{4}\right) \tag{3.113}$$
$$R_2 + R_3 \le \frac{1}{2}\log\left(\frac{\alpha_2 P}{(9/5)N_2}\cdot\frac{P}{(4/3)\alpha_2 P}\right) \tag{3.114}$$
$$\le \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{15}{36}\right). \tag{3.115}$$



(a) Large R_2 (b) Small R_2

Figure 3.7: The cross-section of the type 5 outer bound region.

This region is described in Fig. 3.6 (a).

In comparison, the two-dimensional outer bound region at $R_1 = \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{4}{9}\right) + 1$, given by

$$\begin{aligned}
R_2 &\leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{4}{9}\right) - 1\\
&= \frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\right) + \frac{1}{2}\log\left(\frac{7}{3}\cdot\frac{9}{4}\cdot\frac{1}{4}\right)\\
R_3 &\leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\cdot\frac{4}{9}\right) - 1\\
&= \frac{1}{2}\log\left(\frac{P}{\alpha_1 P}\right) + \frac{1}{2}\log\left(\frac{7}{3}\cdot\frac{9}{4}\cdot\frac{1}{4}\right)\\
R_2 + R_3 &\leq \frac{1}{2}\log\left(\frac{P}{N_2}\right) + \frac{1}{2}\log\left(\frac{7}{3}\right).
\end{aligned}$$

Since $\delta_2 < \frac{1}{2}\log\left(\frac{21}{8}\cdot\frac{7}{3}\cdot\frac{9}{4}\cdot\frac{1}{4}\right) < 0.90$, $\delta_3 < \frac{1}{2}\log\left(\frac{4}{3}\cdot\frac{7}{3}\cdot\frac{9}{4}\cdot\frac{1}{4}\right) < 0.41$ and $\delta_{23} < \frac{1}{2}\log\left(\frac{36}{15}\cdot\frac{7}{3}\right) < 1.25 < \sqrt{2}$, we can conclude that the gap is to within one bit per message.

3.7 Inner Bound: Channel Type 5

Let us consider the relaxed outer bound region $\mathcal{R}'_o$ given by

$$\begin{aligned}
R_k &\le \frac{1}{2}\log\left(\frac{P}{N_k}\right) + \frac{1}{2}\log\left(\frac{4}{3}\right), \;\; k = 1,2,3 \\
R_1 + R_2 &\le \frac{1}{2}\log\left(\frac{P}{N_1}\right) + \frac{1}{2}\log\left(\frac{7}{3}\right) \\
R_2 + R_3 &\le \frac{1}{2}\log\left(\frac{P}{N_2}\right) + \frac{1}{2}\log\left(\frac{7}{3}\right) \\
R_1 + R_3 &\le \frac{1}{2}\log\left(\frac{P}{N_1}\right) + \frac{1}{2}\log\left(\frac{7}{3}\right).
\end{aligned}$$

The cross-sectional region at a given R_2 is described by

$$\begin{aligned}
R_1 &\le \min\left\{\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - R_2, \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{4}{3}\right)\right\} \\
R_3 &\le \min\left\{\frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right) - R_2, \frac{1}{2}\log\left(\frac{P}{N_3}\cdot\frac{4}{3}\right)\right\} \\
R_1 + R_3 &\le \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right).
\end{aligned}$$

Depending on the bottleneck of $\min\{\cdot,\cdot\}$ expressions, there are three cases:

- $R_2 \le \frac{1}{2}\log\left(\frac{7}{4}\right)$
- $\frac{1}{2}\log\left(\frac{7}{4}\right) \le R_2 \le \frac{1}{2}\log\left(\frac{N_3}{N_2}\cdot\frac{7}{4}\right)$
- $R_2 \ge \frac{1}{2}\log\left(\frac{N_3}{N_2}\cdot\frac{7}{4}\right)$.

In this section, we focus on the third case. The other cases can be proved similarly. If the sum of the righthand sides of R_1 and R_3 bounds is smaller than the righthand side of $R_1 + R_3$ bound, i.e.,

$$\frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) + \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right) - 2R_2 \le \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right),$$

then the $R_1 + R_3$ bound is not active at the R_2. By rearranging, the threshold condition is given by

$$R_2 > R_{2,th} = \frac{1}{4}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right). \tag{3.116}$$

Note that $R_{2,th}$ is roughly half of C_2. For this relatively large R_2, the cross-sectional region is a rectangle as described in Fig. 3.7 (a). In contrast, for a relatively small R_1, when the threshold condition does not hold, the cross-sectional region is a MAC-like region as described in Fig. 3.7 (b). In the following subsections, we present achievable schemes for each case.

3.7.1 Achievable Scheme for Relatively Large R_2

Theorem 3.17. *Given $\alpha = (\alpha_1, \alpha_2, \alpha_2') \in [0,1]^3$, the region $\mathcal{R}_\alpha$ is defined by*

$$\begin{aligned}
R_1 &\le \frac{1}{2}\log\left(1+\frac{\alpha_1 P}{N_1}\right)\\
R_2 &\le \min\left\{\frac{1}{2}\log^+\left(c_{21}+\frac{(1-\alpha_2-\alpha_2')P}{(\alpha_1+\alpha_2+\alpha_2')P+N_2}\right),\right.\\
&\qquad \left.\frac{1}{2}\log\left(1+\frac{\alpha_2' P}{N_2}\right)\right\}+\frac{1}{2}\log\left(1+\frac{\alpha_2 P}{\alpha_2' P+N_2}\right)\\
R_3 &\le \frac{1}{2}\log^+\left(c_3+\frac{P}{\max\{\alpha_1,\alpha_2+\alpha_2'\}P+N_3}\right)
\end{aligned}$$

where $c_{21} = \frac{1-\alpha_2-\alpha_2'}{2-\alpha_2-\alpha_2'}$ and $c_3 = \frac{1}{2-\alpha_2-\alpha_2'}$, and $\mathcal{R} = \mathrm{CONV}\left(\bigcup_\alpha \mathcal{R}_\alpha\right)$ is achievable.

We present an achievable scheme for the case of $R_2 > R_{2,th}$. Message $M_2 \in \{1,2,\ldots,2^{nR_2}\}$ for receiver 2 is split into two parts: $M_{21} \in \{1,2,\ldots,2^{nR_{21}}\}$ and $M_{22} \in \{1,2,\ldots,2^{nR_{22}}\}$, so $R_2 = R_{21} + R_{22}$. We generate the signals in the following way: $\mathbf{x}_{21}$ and $\mathbf{x}_{21}'$ are differently coded signals of M_{21}, and $\mathbf{x}_{22}$ is a coded signal of M_{22}. The transmit signal is the sum

$$\mathbf{x}_2 = \mathbf{x}_{21} + \mathbf{x}_{22} + \mathbf{x}_{21}'.$$

The power allocation satisfies $\mathbb{E}[\|\mathbf{x}_1\|^2] = \alpha_1 nP$, at receiver 1, $\mathbb{E}[\|\mathbf{x}_{21}\|^2] = \alpha_2' nP$, $\mathbb{E}[\|\mathbf{x}_{22}\|^2] = \alpha_2 nP$, and $\mathbb{E}[\|\mathbf{x}_{21}'\|^2] = (1-\alpha_2-\alpha_2')P$ at receiver 2, and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$ at receiver 3.

The signals $\mathbf{x}_{21}'$ and $\mathbf{x}_3$ are lattice codewords using the same coding lattice but different shaping lattices. As a result, the sum $\mathbf{x}_{21}' + \mathbf{x}_3$ is a lattice codeword.

The received signals are

$$\begin{aligned}
\mathbf{y}_1 &= \mathbf{x}_{21}' + \mathbf{x}_{22} + \mathbf{x}_{21} + \mathbf{x}_1 + \mathbf{z}_1\\
\mathbf{y}_2 &= [\mathbf{x}_{21}' + \mathbf{x}_3] + \mathbf{x}_{22} + \mathbf{x}_{21} + \mathbf{z}_2\\
\mathbf{y}_3 &= \mathbf{x}_3 + \mathbf{x}_1 + \mathbf{z}_3.
\end{aligned}$$

The signal scale diagram at each receiver is shown in Fig. 3.8 (a). Decoding is performed in the following way.

- At receiver 1, $\mathbf{x}_{21}'$ is first decoded while treating other signals as noise. Having successfully recovered M_{21}, receiver 1 can generate $\mathbf{x}_{21}$ and $\mathbf{x}_{21}'$, and cancel them from $\mathbf{y}_1$. Next, $\mathbf{x}_{22}$ is decoded from $\mathbf{x}_{22}+\mathbf{x}_1+\mathbf{z}_1$. Finally, $\mathbf{x}_1$ is decoded from $\mathbf{x}_1 + \mathbf{z}_1$. For reliable decoding, the code rates should satisfy

$$\begin{aligned}
R_{21} \le T_{21}' &= \frac{1}{2}\log\left(1+\frac{(1-\alpha_2-\alpha_2')P}{(\alpha_1+\alpha_2+\alpha_2')P+N_1}\right)\\
R_{22} \le T_{22}' &= \frac{1}{2}\log\left(1+\frac{\alpha_2 P}{\alpha_1 P+N_1}\right)\\
R_1 \le T_1 &= \frac{1}{2}\log\left(1+\frac{\alpha_1 P}{N_1}\right).
\end{aligned}$$

- At receiver 2, $[\mathbf{x}_{21}'+\mathbf{x}_3]$ first decoded while treating other signals as noise and removed from $\mathbf{y}_2$. Next, $\mathbf{x}_{22}$ and $\mathbf{x}_{21}$ are decoded successively. For reliable decoding, the code rates should satisfy

$$\begin{aligned}
R_{21} &\le T''_{21} = \frac{1}{2}\log\left(c_{21} + \frac{(1-\alpha_2-\alpha'_2)P}{(\alpha_2+\alpha'_2)P+N_2}\right)\\
R_3 &\le T'_3 \;= \frac{1}{2}\log\left(c_3 + \frac{P}{(\alpha_2+\alpha'_2)P+N_2}\right)\\
R_{22} &\le T''_{22} = \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{\alpha'_2 P+N_2}\right)\\
R_{21} &\le T'''_{21} = \frac{1}{2}\log\left(1 + \frac{\alpha'_2 P}{N_2}\right)
\end{aligned}$$

where $c_{21} = \frac{(1-\alpha_2-\alpha'_2)P}{(1-\alpha_2-\alpha'_2)P+P} = \frac{1-\alpha_2-\alpha'_2}{2-\alpha_2-\alpha'_2}$ and $c_3 = \frac{P}{(1-\alpha_2-\alpha'_2)P+P} = \frac{1}{2-\alpha_2-\alpha'_2}$. Note that $0 \le c_{21} \le \frac{1}{2}$, $c_{21}+c_3 = 1$, and $\frac{1}{2} \le c_3 \le 1$.

- At receiver 3, $\mathbf{x}_3$ is decoded while treating $\mathbf{x}_1+\mathbf{z}_3$ as noise. Reliable decoding is possible if

$$R_3 \le T''_3 = \frac{1}{2}\log\left(1+\frac{P}{\alpha_1 P+N_3}\right). \tag{3.117}$$

Putting together, we can see that given $\alpha_1, \alpha_2, \alpha'_2 \in [0,1]$, the following rate region is achievable.

$$\begin{aligned}
R_1 &\le T_1\\
R_2 &\le T_2 = \min\{T'_{21}, T''_{21}, T'''_{21}\} + \min\{T'_{22}, T''_{22}\}\\
R_3 &\le T_3 = \min\{T'_3, T''_3\}
\end{aligned}$$

where

$$
\begin{aligned}
T_1 &= \frac{1}{2}\log\left(1+\frac{\alpha_1 P}{N_1}\right)\\
T_2 &= \min\{T'_{21}, T''_{21}, T'''_{21}\} + T''_{22}\\
&= \min\{\min\{T'_{21}, T''_{21}\}, T'''_{21}\} + T''_{22}\\
&\geq \min\left\{\frac{1}{2}\log\left(c_{21}+\frac{(1-\alpha_2-\alpha'_2)P}{(\alpha_1+\alpha_2+\alpha'_2)P+N_2}\right),\right.\\
&\qquad \left.\frac{1}{2}\log\left(1+\frac{\alpha'_2 P}{N_2}\right)\right\} + \frac{1}{2}\log\left(1+\frac{\alpha_2 P}{\alpha'_2 P+N_2}\right)\\
T_3 &\geq \frac{1}{2}\log\left(c_3+\frac{P}{\max\{\alpha_1, \alpha_2+\alpha'_2\}P+N_3}\right).
\end{aligned}
$$

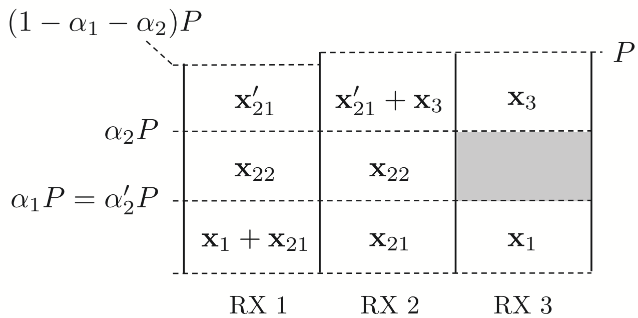
3.7.2 The Gap for Relatively Large R_2

We choose α_1 and α_2 such that $\alpha_1 P \geq N_2$, that $\alpha_2 P \geq N_3$, that $\alpha_1 = \alpha'_2 \leq \alpha_2$, and that $\alpha_1 + \alpha_2 \leq \frac{1}{2}$. It follows that $c_{21} \geq \frac{1}{3}$. We get the lower bounds for each term of T_2 expression above.

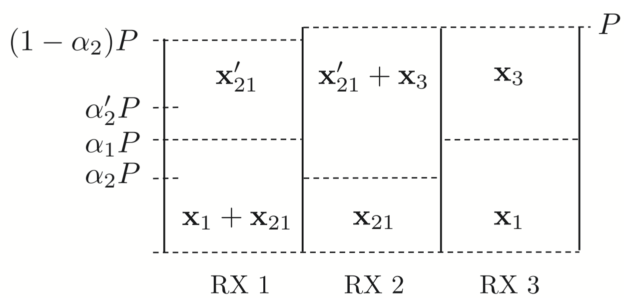
$$
\begin{aligned}
&\min\{T'_{21}, T''_{21}\} && (3.118)\\
&\geq \frac{1}{2}\log\left(c_{21}+\frac{(1-\alpha_1-\alpha_2)P}{(2\alpha_1+\alpha_2)P+N_2}\right) && (3.119)\\
&\geq \frac{1}{2}\log\left(\frac{1}{3}+\frac{(1-\alpha_1-\alpha_2)P}{(3\alpha_1+\alpha_2)P}\right) && (3.120)\\
&\geq \frac{1}{2}\log\left(\frac{P}{(3\alpha_1+\alpha_2)P}\right). && (3.121)
\end{aligned}
$$

The first entry of $\min\{\cdot,\cdot\}$ in

$$
T_2 = \min\{\min\{T'_{21}, T''_{21}\} + T''_{22}, T'''_{21} + T''_{22}\}
$$



(a) Channel type 5: relatively large R_2



(b) Channel type 5: relatively small R_2

Figure 3.8: Signal scale diagram.

is lower bounded as follows.

$$
\begin{aligned}
&\min\{T'_{21}, T''_{21}\} + T''_{22} \\
&\geq \frac{1}{2}\log\left(\frac{P}{(3\alpha_1+\alpha_2)P}\right) + \frac{1}{2}\log\left(\frac{(\alpha_1+\alpha_2)P+N_2}{\alpha_1 P+N_2}\right) \\
&= \frac{1}{2}\log\left(\frac{P}{\alpha_1 P+N_2}\cdot\frac{(\alpha_1+\alpha_2)P+N_2}{(3\alpha_1+\alpha_2)P}\right) \\
&\geq \frac{1}{2}\log\left(\frac{P}{3(\alpha_1 P+N_2)}\right) \\
&\geq \frac{1}{2}\log\left(\frac{P}{6\alpha_1 P}\right).
\end{aligned}
$$

The second entry of $T_2 = \min\{\cdot,\cdot\}$ is lower bounded as follows.

$$
\begin{aligned}
&T'''_{21} + T''_{22} \\
&= \frac{1}{2}\log\left(1+\frac{\alpha_1 P}{N_2}\right) + \frac{1}{2}\log\left(1+\frac{\alpha_2 P}{\alpha_1 P+N_2}\right) \\
&= \frac{1}{2}\log\left(1+\frac{(\alpha_1+\alpha_2)P}{N_2}\right) \\
&\geq \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right).
\end{aligned}
$$

Putting together, we get the lower bound

$$
T_2 \geq \min\left\{\frac{1}{2}\log\left(\frac{P}{6\alpha_1 P}\right), \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right)\right\}.
$$

Given α_2, we choose α_1 that satisfies $\frac{1}{2}\log\left(\frac{P}{6\alpha_1 P}\right) = \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right)$. As a result, we can write $T_2 \geq \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right)$. We also have

$$
T_3 \geq \frac{1}{2}\log\left(\frac{P}{(\alpha_1+\alpha_2)P+N_3}\right) \geq \frac{1}{2}\log\left(\frac{P}{3\alpha_2 P}\right).
$$

Putting together, we can see that the following rate region is achievable.

$$R_1 \leq \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\right) \tag{3.122}$$

$$R_2 \leq \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right) \tag{3.123}$$

$$R_3 \leq \frac{1}{2}\log\left(\frac{P}{3\alpha_2 P}\right). \tag{3.124}$$

For fixed α_2 and $R_2 = \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right)$, the two-dimensional rate region, given by

$$R_1 \leq \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\right) \tag{3.125}$$

$$= \frac{1}{2}\log\left(\frac{P}{6\alpha_2 P}\cdot\frac{N_2}{N_1}\right) \tag{3.126}$$

$$R_3 \leq \frac{1}{2}\log\left(\frac{P}{3\alpha_2 P}\right) \tag{3.127}$$

is achievable.

In comparison, the two-dimensional outer bound region at $R_2 = \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right) + 1$ is given by

$$\begin{aligned}
R_1 &\leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right) - 1 \\
&= \frac{1}{2}\log\left(\frac{P}{\alpha_2 P}\cdot\frac{N_2}{N_1}\right) + \frac{1}{2}\log\left(\frac{7}{3}\cdot\frac{1}{4}\right) \\
R_3 &\leq \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right) - \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right) - 1 \\
&= \frac{1}{2}\log\left(\frac{P}{\alpha_2 P}\right) + \frac{1}{2}\log\left(\frac{7}{3}\cdot\frac{1}{4}\right).
\end{aligned}$$

As discussed above, the sum-rate bound on $R_1 + R_3$ is loose for R_2 larger than the threshold, so the rate region is a rectangle.

By comparing the inner and outer bound rate regions, we can see that $\delta_1 < \frac{1}{2}\log\left(6\cdot\frac{7}{3}\cdot\frac{1}{4}\right) < 0.91$ and $\delta_3 < \frac{1}{2}\log\left(3\cdot\frac{7}{3}\cdot\frac{1}{4}\right) < 0.41$. Therefore, we can conclude that the gap is to within one bit per message.

3.7.3 Achievable Scheme for Relatively Small R_2

Theorem 3.18. *Given $\alpha = (\alpha_1, \alpha_2) \in [0,1]^2$, the region $\mathcal{R}_\alpha$ is defined by*

$$
\begin{aligned}
R_1 &\le \frac{1}{2}\log\left(1+\frac{\alpha_1 P}{N_1}\right)\\
R_2 &\le \min\left\{\frac{1}{2}\log^+\left(c_{21}+\frac{(1-\alpha_2)P}{(\alpha_1+\alpha_2)P+N_2}\right), \frac{1}{2}\log\left(1+\frac{\alpha_2 P}{N_2}\right)\right\}\\
R_3 &\le \frac{1}{2}\log^+\left(c_3+\frac{P}{\max\{\alpha_1,\alpha_2\}P+N_3}\right)
\end{aligned}
$$

where $c_{21} = \frac{1-\alpha_2}{2-\alpha_2}$ and $c_3 = \frac{1}{2-\alpha_2}$, and $\mathcal{R} = \textsc{conv}\left(\bigcup_\alpha \mathcal{R}_\alpha\right)$ is achievable.

For the case of $R_2 < R_{2,th}$, we present the following scheme. At transmitter 2, rate splitting is not necessary. The transmit signal is the sum

$$\mathbf{x}_2 = \mathbf{x}_{21} + \mathbf{x}'_{21}$$

where $\mathbf{x}_{21}$ and $\mathbf{x}'_{21}$ are differently coded versions of the same message $M_2 \in \{1, 2, \ldots, 2^{nR_2}\}$.

The power allocation: $\mathbb{E}[\|\mathbf{x}_1\|^2] = \alpha_1 nP$ at receiver 1, $\mathbb{E}[\|\mathbf{x}_{21}\|^2] = \alpha_2 nP$, and $\mathbb{E}[\|\mathbf{x}'_{21}\|^2] = (1-\alpha_2)nP$ at receiver 2, and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$ at receiver 3.

The signals $\mathbf{x}'_{21}$ and $\mathbf{x}_3$ are lattice codewords using the same coding lattice but different shaping lattices. As a result, the sum $\mathbf{x}'_{21} + \mathbf{x}_3$ is a lattice codeword.

The received signals are

$$
\begin{aligned}
\mathbf{y}_1 &= \mathbf{x}'_{21} + \mathbf{x}_{21} + \mathbf{x}_1 + \mathbf{z}_1\\
\mathbf{y}_2 &= [\mathbf{x}'_{21} + \mathbf{x}_3] + \mathbf{x}_{21} + \mathbf{z}_2\\
\mathbf{y}_3 &= \mathbf{x}_3 + \mathbf{x}_1 + \mathbf{z}_3.
\end{aligned}
$$

The signal scale diagram at each receiver is shown in Fig. 3.8 (b). Decoding is performed in the following way.

- At receiver 1, $\mathbf{x}'_{21}$ is first decoded while treating other signals as noise. Having successfully recovered M_{21}, receiver 1 can generate $\mathbf{x}_{21}$ and $\mathbf{x}'_{21}$, and cancel them from $\mathbf{y}_1$. Next, $\mathbf{x}_1$ is decoded from $\mathbf{x}_1 + \mathbf{z}_1$. For reliable decoding, the code rates should satisfy

$$\begin{aligned} R_{21} &\leq T'_{21} = \frac{1}{2}\log\left(1 + \frac{(1-\alpha_2)P}{(\alpha_1+\alpha_2)P + N_1}\right) \\ R_1 &\leq T_1 \;\; = \frac{1}{2}\log\left(1 + \frac{\alpha_1 P}{N_1}\right). \end{aligned}$$

- At receiver 2, $[\mathbf{x}'_{21} + \mathbf{x}_3]$ first decoded while treating other signals as noise and removed from $\mathbf{y}_2$. Next, $\mathbf{x}_{21}$ is decoded from $\mathbf{x}_{21} + \mathbf{z}_2$. For reliable decoding, the code rates should satisfy

$$\begin{aligned} R_{21} &\leq T''_{21} = \frac{1}{2}\log\left(c_{21} + \frac{(1-\alpha_2)P}{\alpha_2 P + N_2}\right) \\ R_3 &\leq T'_3 \;\; = \frac{1}{2}\log\left(c_3 + \frac{P}{\alpha_2 P + N_2}\right) \\ R_{21} &\leq T'''_{21} = \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{N_2}\right) \end{aligned}$$

  where $c_{21} = \frac{(1-\alpha_2)P}{(1-\alpha_2)P+P} = \frac{1-\alpha_2}{2-\alpha_2}$ and $c_3 = \frac{P}{(1-\alpha_2)P+P} = \frac{1}{2-\alpha_2}$. Note that $0 \leq c_{21} \leq \frac{1}{2}$, $c_{21} + c_3 = 1$, and $\frac{1}{2} \leq c_3 \leq 1$.

- At receiver 3, $\mathbf{x}_3$ is decoded while treating $\mathbf{x}_1 + \mathbf{z}_3$ as noise. Reliable decoding is possible if

$$R_3 \;\leq T''_3 = \frac{1}{2}\log\left(1 + \frac{P}{\alpha_1 P + N_3}\right). \tag{3.128}$$

Putting together, we get

$$R_1 \leq T_1 \tag{3.129}$$

$$R_2 \leq T_2 = \min\{T'_{21}, T''_{21}, T'''_{21}\} \tag{3.130}$$

$$R_3 \leq T_3 = \min\{T'_3, T''_3\} \tag{3.131}$$

where

$$\begin{aligned}
T_1 &= \frac{1}{2}\log\left(1 + \frac{\alpha_1 P}{N_1}\right) \\
T_2 &= \min\{T'_{21}, T''_{21}, T'''_{21}\} \\
&= \min\{\min\{T'_{21}, T''_{21}\}, T'''_{21}\} \\
&\geq \min\left\{\frac{1}{2}\log\left(c_{21} + \frac{(1-\alpha_2)P}{(\alpha_1+\alpha_2)P + N_2}\right), \frac{1}{2}\log\left(1 + \frac{\alpha_2 P}{N_2}\right)\right\} \\
T_3 &\geq \frac{1}{2}\log\left(c_3 + \frac{P}{\max\{\alpha_1, \alpha_2\}P + N_3}\right).
\end{aligned}$$

3.7.4 The Gap for Relatively Small R_2

We choose α_1 and α_2 such that $\alpha_1 P \geq N_2$, that $\alpha_2 P \geq N_3$, that $\alpha_1 + \alpha_2 \leq \frac{1}{2}$, and that $\alpha_1 \geq \alpha_2$. It follows that $c_{21} \geq \frac{1}{3}$. We get the lower bound

$$\min\{T'_{21}, T''_{21}\} \tag{3.132}$$

$$= \frac{1}{2}\log\left(c_{21} + \frac{(1-\alpha_2)P}{(\alpha_1+\alpha_2)P + N_2}\right) \tag{3.133}$$

$$\geq \frac{1}{2}\log\left(\frac{1}{3} + \frac{(1-\alpha_1)P}{3\alpha_1 P}\right) \tag{3.134}$$

$$= \frac{1}{2}\log\left(\frac{P}{3\alpha_1 P}\right) \tag{3.135}$$

and

$$T_2 \geq \min\left\{\frac{1}{2}\log\left(\frac{P}{3\alpha_1 P}\right), \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right)\right\}.$$

Let us define α_2' by the equality $\frac{1}{2}\log\left(\frac{P}{3\alpha_2' P}\right) = \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right)$. If we choose $\alpha_1 \le \alpha_2'$, then $T_2 \ge \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right)$. We can see that the following rate region is achievable.

$$R_1 \le \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\right) \tag{3.136}$$

$$R_2 \le \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right) \tag{3.137}$$

$$R_3 \le \frac{1}{2}\log\left(\frac{P}{2\alpha_1 P}\right). \tag{3.138}$$

For fixed $\alpha_1 \in [\alpha_2, \alpha_2']$ and $R_2 = \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right)$, the two-dimensional rate region $\mathcal{R}_\alpha$, given by

$$R_1 \le \frac{1}{2}\log\left(\frac{\alpha_1 P}{N_1}\right) \tag{3.139}$$

$$R_3 \le \frac{1}{2}\log\left(\frac{P}{2\alpha_1 P}\right) \tag{3.140}$$

is achievable. The union $\bigcup_{\alpha_1 \in [\alpha_2, \alpha_2']} \mathcal{R}_\alpha$ is a MAC-like region, given by

$$R_1 \le \frac{1}{2}\log\left(\frac{\alpha_2' P}{N_1}\right) \tag{3.141}$$

$$= \frac{1}{2}\log\left(\frac{P}{3\alpha_2 P}\cdot\frac{N_2}{N_1}\right) \tag{3.142}$$

$$R_3 \le \frac{1}{2}\log\left(\frac{P}{2\alpha_2 P}\right) \tag{3.143}$$

$$R_1 + R_3 = \frac{1}{2}\log\left(\frac{P}{2N_1}\right). \tag{3.144}$$

In comparison, the two-dimensional outer bound region at $R_2 = \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right) +$

1 is given by

$$\begin{aligned}
R_1 &\leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{7}{3}\right) - \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right) - 1 \\
&= \frac{1}{2}\log\left(\frac{P}{\alpha_2 P}\cdot\frac{N_2}{N_1}\right) + \frac{1}{2}\log\left(\frac{7}{3}\cdot\frac{1}{4}\right) \\
R_3 &\leq \frac{1}{2}\log\left(\frac{P}{N_2}\cdot\frac{7}{3}\right) - \frac{1}{2}\log\left(\frac{\alpha_2 P}{N_2}\right) - 1 \\
&= \frac{1}{2}\log\left(\frac{P}{\alpha_2 P}\right) + \frac{1}{2}\log\left(\frac{7}{3}\cdot\frac{1}{4}\right) \\
R_1 + R_3 &\leq \frac{1}{2}\log\left(\frac{P}{N_1}\cdot\frac{8}{3}\right).
\end{aligned}$$

Since $\delta_1 < \frac{1}{2}\log\left(3\cdot\frac{7}{3}\cdot\frac{1}{4}\right) < 0.41$, $\delta_3 < \frac{1}{2}\log\left(2\cdot\frac{7}{3}\cdot\frac{1}{4}\right) < 0.12$ and $\delta_{13} < \frac{1}{2}\log\left(2\cdot\frac{7}{3}\right) < 1.12 < \sqrt{2}$, we can conclude that the gap is to within one bit per message.

3.8 Random Coding Achievability: Channel Type 4

At transmitter 1, message M_1 is split into three parts (M_{12}, M_{11}, M_{10}), and the transmit signal is $\mathbf{x}_1 = \mathbf{x}_{12} + \mathbf{x}_{11} + \mathbf{x}_{10}$. The signals satisfy $\mathbb{E}[\|\mathbf{x}_{12}\|^2] = n(P - N_2 - N_3)$, $\mathbb{E}[\|\mathbf{x}_{11}\|^2] = nN_3$, and $\mathbb{E}[\|\mathbf{x}_{10}\|^2] = nN_2$.

At transmitter 2, message M_2 is split into three parts (M_{21}, M_{20}), and the transmit signal is $\mathbf{x}_2 = \mathbf{x}_{21} + \mathbf{x}_{20}$. The signals satisfy $\mathbb{E}[\|\mathbf{x}_{21}\|^2] = n(P - N_3)$ and $\mathbb{E}[\|\mathbf{x}_{20}\|^2] = nN_3$. Rate-splitting is not performed at transmitter 3, and $\mathbb{E}[\|\mathbf{x}_3\|^2] = nP$.

The top layer codewords $(\mathbf{x}_{12}, \mathbf{x}_{21}, \mathbf{x}_3)$ are from a joint random codebook for (M_{12}, M_{21}, M_3). The mid-layer codewords $(\mathbf{x}_{11}, \mathbf{x}_{20})$ are from a joint random codebook for (M_{11}, M_{20}). The bottom layer codeword $\mathbf{x}_{10}$ is from a single-user random codebook for M_{10}.

The received signals are

$$
\begin{aligned}
\mathbf{y}_1 &= (\mathbf{x}_{12} + \mathbf{x}_3) + \mathbf{x}_{11} + \mathbf{x}_{10} + \mathbf{z}_1 \\
\mathbf{y}_2 &= (\mathbf{x}_{12} + \mathbf{x}_{21}) + (\mathbf{x}_{11} + \mathbf{x}_{20}) + \mathbf{x}_{10} + \mathbf{z}_2 \\
\mathbf{y}_3 &= (\mathbf{x}_{21} + \mathbf{x}_3) + \mathbf{x}_{20} + \mathbf{z}_3.
\end{aligned}
$$

Decoding is performed from the top layer to the bottom layer. At receiver 1, simultaneous decoding of $(\mathbf{x}_{12}, \mathbf{x}_3)$ is performed while treating other signals as noise. And then, $\mathbf{x}_{11}$ and $\mathbf{x}_{10}$ are decoded successively. At receiver 2, simultaneous decoding of $(\mathbf{x}_{12}, \mathbf{x}_{21})$ is performed while treating other signals as noise. And then, simultaneous decoding of $(\mathbf{x}_{11}, \mathbf{x}_{20})$ is performed. At receiver 3, simultaneous decoding of $(\mathbf{x}_{21}, \mathbf{x}_3)$ is performed while treating other signals as noise. For reliable decoding, code rates should satisfy

$$
\begin{aligned}
R_{12} &\le I_1 = \frac{1}{2}\log\left(1 + \frac{P - N_2 - N_3}{N_1 + N_2 + N_3}\right) \\
R_3 &\le I_2 = \frac{1}{2}\log\left(1 + \frac{P}{N_1 + N_2 + N_3}\right) \\
R_{12} + R_3 &\le I_3 = \frac{1}{2}\log\left(1 + \frac{2P - N_2 - N_3}{N_1 + N_2 + N_3}\right) \\
R_{11} &\le I_4 = \frac{1}{2}\log\left(1 + \frac{N_3}{N_1 + N_2}\right) \\
R_{10} &\le I_5 = \frac{1}{2}\log\left(1 + \frac{N_2}{N_1}\right)
\end{aligned}
$$

at receiver 1,

$$
\begin{aligned}
R_{12} &\leq I_6 = \frac{1}{2}\log\left(1+\frac{P-N_2-N_3}{2N_2+2N_3}\right)\\
R_{21} &\leq I_7 = \frac{1}{2}\log\left(1+\frac{P-N_3}{2N_2+2N_3}\right)\\
R_{12}+R_{21} &\leq I_8 = \frac{1}{2}\log\left(1+\frac{2P-N_2-2N_3}{2N_2+2N_3}\right)\\
R_{11} &\leq I_9 = \frac{1}{2}\log\left(1+\frac{N_3}{2N_2}\right)\\
R_{20} &\leq I_{10} = \frac{1}{2}\log\left(1+\frac{N_3}{2N_2}\right)\\
R_{11}+R_{20} &\leq I_{11} = \frac{1}{2}\log\left(1+\frac{2N_3}{2N_2}\right)
\end{aligned}
$$

at receiver 2,

$$
\begin{aligned}
R_{21} &\leq I_{12} = \frac{1}{2}\log\left(1+\frac{P-N_3}{2N_3}\right)\\
R_3 \ &\leq I_{13} = \frac{1}{2}\log\left(1+\frac{P}{2N_3}\right)\\
R_{21}+R_3 \ &\leq I_{14} = \frac{1}{2}\log\left(1+\frac{2P-N_3}{2N_3}\right)
\end{aligned}
$$

at receiver 3. Putting together,

$$
\begin{aligned}
R_{12} &\leq T_1 = \min\{I_1, I_6\} = I_6\\
R_{21} &\leq T_2 = \min\{I_7, I_{12}\} = I_7\\
R_3 \ &\leq T_3 = \min\{I_2, I_{13}\}\\
R_{12}+R_{21} &\leq T_4 = I_8\\
R_{12}+R_3 \ &\leq T_5 = I_3\\
R_{21}+R_3 \ &\leq T_6 = I_{14}
\end{aligned}
$$

at the top layer,

$$R_{11} \leq T_7 = \min\{I_4, I_9\} = I_9$$
$$R_{20} \leq T_8 = I_{10}$$
$$R_{11} + R_{20} \leq T_9 = I_{11}$$

at the mid-layer,

$$R_{10} \leq T_{10} = I_5$$

at the bottom layer. Note that the rate variables are not coupled between layers. We get the achievable rate region

$$\begin{aligned} R_1 &= R_{12} + R_{11} + R_{10} \leq T_1 + T_7 + T_{10} \\ R_2 &= R_{21} + R_{20} \leq T_2 + T_8 \\ R_3 &\leq T_3 \\ R_1 + R_2 &\leq T_4 + T_9 + T_{10} \\ R_1 + R_3 &\leq T_5 + T_7 + T_{10} \\ R_2 + R_3 &\leq T_6 + T_8. \end{aligned}$$

This region includes the following region.

$$\begin{aligned} R_1 &\leq \frac{1}{2}\log\left(2 + \frac{P}{N_1}\right) - 1 \\ R_2 &\leq \frac{1}{2}\log\left(3 + \frac{P}{N_2}\right) - 1 \\ R_3 &\leq \frac{1}{2}\log\left(3 + \frac{P}{N_3}\right) - \frac{1}{2}\log(3) \\ R_1 + R_2 &\leq \frac{1}{2}\log\left(1 + \frac{2P}{N_1}\right) - \frac{1}{2} \\ R_1 + R_3 &\leq \frac{1}{2}\log\left(1 + \frac{2P}{N_1}\right) - 1 \\ R_2 + R_3 &\leq \frac{1}{2}\log\left(1 + \frac{2P}{N_2}\right) - 1. \end{aligned}$$

Therefore, we can conclude the capacity region to within one bit.

3.9 Random Coding Achievability: Channel Type 5

Transmit signal construction is the same as the one for channel type 4. The received signals are

$$\mathbf{y}_1 = (\mathbf{x}_{12} + \mathbf{x}_{21}) + (\mathbf{x}_{11} + \mathbf{x}_{20}) + \mathbf{x}_{10} + \mathbf{z}_1$$
$$\mathbf{y}_2 = (\mathbf{x}_{21} + \mathbf{x}_3) + \mathbf{x}_{20} + \mathbf{z}_2$$
$$\mathbf{y}_3 = (\mathbf{x}_{12} + \mathbf{x}_3) + \mathbf{x}_{11} + \mathbf{x}_{10} + \mathbf{z}_3.$$

Decoding is performed from the top layer to the bottom layer. At receiver 1, simultaneous decoding of $(\mathbf{x}_{12}, \mathbf{x}_{21})$ is performed while treating other signals as noise. And then, simultaneous decoding of $\mathbf{x}_{11}$ and $\mathbf{x}_{20}$ is performed. Lastly, $\mathbf{x}_{10}$ is decoded. At receiver 2, simultaneous decoding of $(\mathbf{x}_{21}, \mathbf{x}_3)$ is performed while treating other signals as noise. And then, $\mathbf{x}_{20}$ is decoded. At receiver 3, simultaneous decoding of $(\mathbf{x}_{12}, \mathbf{x}_3)$ is performed while treating other signals as noise. And then, $\mathbf{x}_{11}$ and $\mathbf{x}_{10}$ are decoded successively. For reliable decoding, code rates should satisfy

$$\begin{aligned}
R_{12} &\le I_1 = \frac{1}{2}\log\left(1+\frac{P-N_2-N_3}{N_1+N_2+2N_3}\right)\\
R_{21} &\le I_2 = \frac{1}{2}\log\left(1+\frac{P-N_3}{N_1+N_2+2N_3}\right)\\
R_{12}+R_{21} &\le I_3 = \frac{1}{2}\log\left(1+\frac{2P-N_2-2N_3}{N_1+N_2+2N_3}\right)\\
R_{11} &\le I_4 = \frac{1}{2}\log\left(1+\frac{N_3}{N_1+N_2}\right)\\
R_{20} &\le I_5 = \frac{1}{2}\log\left(1+\frac{N_3}{N_1+N_2}\right)\\
R_{11}+R_{20} &\le I_6 = \frac{1}{2}\log\left(1+\frac{2N_3}{N_1+N_2}\right)\\
R_{10} &\le I_7 = \frac{1}{2}\log\left(1+\frac{N_2}{N_1}\right)
\end{aligned}$$

at receiver 1,

$$\begin{aligned}
R_{21} &\le I_8 = \frac{1}{2}\log\left(1+\frac{P-N_3}{N_2+N_3}\right)\\
R_3 \ &\le I_9 = \frac{1}{2}\log\left(1+\frac{P}{N_2+N_3}\right)\\
R_{21}+R_3 \ &\le I_{10} = \frac{1}{2}\log\left(1+\frac{2P-N_3}{N_2+N_3}\right)\\
R_{20} &\le I_{11} = \frac{1}{2}\log\left(1+\frac{N_3}{N_2}\right)
\end{aligned}$$

at receiver 2,

$$\begin{aligned}
R_{12} &\le I_{12} = \frac{1}{2}\log\left(1+\frac{P-N_2-N_3}{N_2+2N_3}\right)\\
R_3 \ &\le I_{13} = \frac{1}{2}\log\left(1+\frac{P}{N_2+2N_3}\right)\\
R_{12}+R_3 \ &\le I_{14} = \frac{1}{2}\log\left(1+\frac{2P-N_2-N_3}{N_2+2N_3}\right)
\end{aligned}$$

at receiver 3. Putting together,

$$
\begin{aligned}
R_{12} &\le T_1 = \min\{I_1, I_{12}\} = I_1 \\
R_{21} &\le T_2 = \min\{I_2, I_8\} = I_2 \\
R_3 \ &\le T_3 = \min\{I_9, I_{13}\} = I_{13} \\
R_{12} + R_{21} &\le T_4 = I_3 \\
R_{12} + R_3 \ &\le T_5 = I_{14} \\
R_{21} + R_3 \ &\le T_6 = I_{10}
\end{aligned}
$$

at the top layer,

$$
\begin{aligned}
R_{11} &\le T_7 = I_4 \\
R_{20} &\le T_8 = \min\{I_5, I_{11}\} = I_5 \\
R_{11} + R_{20} &\le T_9 = I_6
\end{aligned}
$$

at the mid-layer,

$$
R_{10} \le T_{10} = I_7
$$

at the bottom layer. Note that the rate variables are not coupled between layers. We get the achievable rate region

$$
\begin{aligned}
R_1 = R_{12} + R_{11} + R_{10} &\le T_1 + T_7 + T_{10} \\
R_2 = R_{21} + R_{20} &\le T_2 + T_8 \\
R_3 &\le T_3 \\
R_1 + R_2 &\le T_4 + T_9 + T_{10} \\
R_1 + R_3 &\le T_5 + T_7 + T_{10} \\
R_2 + R_3 &\le T_6 + T_8.
\end{aligned}
$$

This region includes the following region.

$$\begin{aligned}
R_1 &\leq \frac{1}{2}\log\left(2+\frac{P}{N_1}\right)-\frac{1}{2} \\
R_2 &\leq \frac{1}{2}\log\left(2+\frac{P}{N_2}\right)-1 \\
R_3 &\leq \frac{1}{2}\log\left(3+\frac{P}{N_3}\right)-\frac{1}{2}\log(3) \\
R_1+R_2 &\leq \frac{1}{2}\log\left(1+\frac{2P}{N_1}\right) \\
R_1+R_3 &\leq \frac{1}{2}\log\left(1+\frac{2P}{N_1}\right)-\frac{1}{2} \\
R_2+R_3 &\leq \frac{1}{2}\log\left(1+\frac{2P}{N_2}\right)-\frac{1}{2}.
\end{aligned}$$

Therefore, we can conclude the capacity region to within one bit.

Chapter 4

The Symmetric Gaussian X-Channel[1]

4.1 Channel Model

The symmetric Gaussian X channel, denoted by (h, SNR), is defined by

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}_1 + h\mathbf{x}_2 + \mathbf{z}_1, \\ \mathbf{y}_2 &= h\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{z}_2, \end{aligned}$$

where $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$, the power constraint is $\|\mathbf{x}_k\|^2 \leq n\mathrm{SNR}$ for $k = 1, 2$, and the noise $\mathbf{z}_j \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ for $j = 1, 2$. There are four independent messages for each source-destination pairs: $V_1 \in \{1, 2, \ldots, 2^{nR_{v1}}\}$ from transmitter 1 to receiver 1, $V_2 \in \{1, 2, \ldots, 2^{nR_{v2}}\}$ from transmitter 2 to receiver 1, $W_1 \in \{1, 2, \ldots, 2^{nR_{w1}}\}$ from transmitter 1 to receiver 2, $W_2 \in \{1, 2, \ldots, 2^{nR_{w2}}\}$ from transmitter 2 to receiver 2. We assume that $h \in \mathbb{R}$ is not varying over time or frequency and is perfectly known at transmitters and receivers. Without loss of generality, we assume that h is positive.

The interference level parameter $\alpha \geq 0$ is a function of SNR and h, defined by

$$\alpha = \frac{\log(h^2\mathrm{SNR})}{\log(\mathrm{SNR})}.$$

[1] The result in this chapter was presented in part at the IEEE ISIT 2015 [30]. Muryong Kim as the first author performed the research and generated the main results in theorems.

	The ranges of α	The ranges of h
$\mathcal{B}_1$	$0 \leq \alpha < \frac{1}{2}$	$\mathrm{SNR}^{-\frac{1}{2}} \leq h < \mathrm{SNR}^{-\frac{1}{4}}$
$\mathcal{B}_2'$	$\frac{1}{2} \leq \alpha < \frac{2}{3}$	$\mathrm{SNR}^{-\frac{1}{4}} \leq h < \mathrm{SNR}^{-\frac{1}{6}}$
$\mathcal{B}_2''$	$\frac{2}{3} \leq \alpha < \frac{3}{4}$	$\mathrm{SNR}^{-\frac{1}{6}} \leq h < \mathrm{SNR}^{-\frac{1}{8}}$
$\mathcal{B}_3$	$\frac{3}{4} \leq \alpha < 1$	$\mathrm{SNR}^{-\frac{1}{8}} \leq h < 1$
$\mathcal{B}_4$	$1 < \alpha \leq \frac{4}{3}$	$1 < h \leq \mathrm{SNR}^{\frac{1}{6}}$
$\mathcal{B}_5''$	$\frac{4}{3} < \alpha \leq \frac{3}{2}$	$\mathrm{SNR}^{\frac{1}{6}} < h \leq \mathrm{SNR}^{\frac{1}{4}}$
$\mathcal{B}_5'$	$\frac{3}{2} < \alpha \leq 2$	$\mathrm{SNR}^{\frac{1}{4}} < h \leq \mathrm{SNR}^{\frac{1}{2}}$
$\mathcal{B}_6$	$\alpha > 2$	$h > \mathrm{SNR}^{\frac{1}{2}}$

Table 4.1: Different regimes of h.

Throughout the paper, we assume SNR > 1 and $h^2\mathrm{SNR} \geq 1$ unless stated otherwise. We can express h in terms of SNR and α, i.e., $h = \mathrm{SNR}^{\frac{\alpha-1}{2}}$. The GDOF of the symmetric Gaussian X channel is defined by

$$d(\alpha) = \lim_{\mathrm{SNR}\to\infty} \frac{C_{sum}}{\frac{1}{2}\log(\mathrm{SNR})}$$

where

$$C_{sum} = \sup\{R_{v1} + R_{v2} + R_{w1} + R_{w2} : (R_{v1}, R_{v2}, R_{w1}, R_{w2}) \in \mathcal{C}\}$$

is the sum-rate capacity, and the capacity region $\mathcal{C}$ is the closure of the set of achievable rate tuples. Given SNR, we divide the range of h into eight regimes $\mathcal{B}_i$ as described in Table 4.1.

In [9], the GDOF of the symmetric Gaussian X channel is characterized

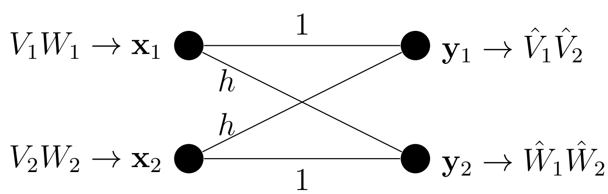


Figure 4.1: The X channel.

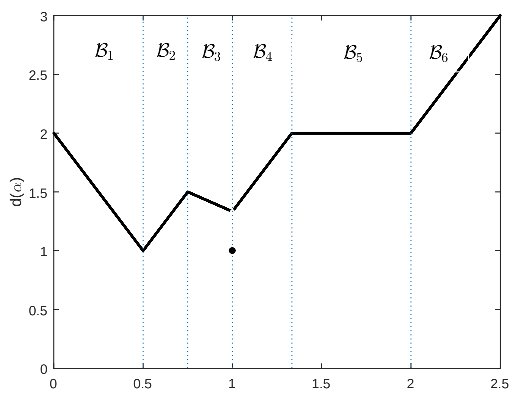


Figure 4.2: GDOF of the symmetric Gaussian X channel.

as

$$d(\alpha) = \begin{cases} 2 - 2\alpha, & 0 \le \alpha < \frac{1}{2}, & (\mathcal{B}_1) \\ 2\alpha, & \frac{1}{2} \le \alpha < \frac{3}{4}, & (\mathcal{B}_2) \\ 2 - \frac{2}{3}\alpha, & \frac{3}{4} \le \alpha < 1, & (\mathcal{B}_3) \\ 1, & \alpha = 1, & \\ 2\alpha - \frac{2}{3}, & 1 < \alpha \le \frac{4}{3}, & (\mathcal{B}_4) \\ 2, & \frac{4}{3} < \alpha \le 2, & (\mathcal{B}_5) \\ 2\alpha - 2, & \alpha > 2, & (\mathcal{B}_6) \end{cases}$$

where $\mathcal{B}_2 = \mathcal{B}_2' \cup \mathcal{B}_2''$ and $\mathcal{B}_5 = \mathcal{B}_5' \cup \mathcal{B}_5''$.

4.2 Naïve Schemes

If both transmitters send to receiver 1 for a fraction of time and to receiver 2 for the rest of time, i.e., timesharing multiple access channel (MAC), we can achieve the sum-rate

$$R_{MAC} = \frac{1}{2}\log(1 + (1 + h^2)\mathrm{SNR}).$$

As the same expression appears in upper bounds, we use R_{MAC} as a shorthand notation in this paper.

If transmitter 1 sends to one of the receivers with higher channel gain (greater of 1 and h) for half the time, and transmitter 2 sends to one of the receivers with higher channel gain for the rest of the time, i.e, time-division multiplexing (TDM), we can achieve the sum-rate

$$R_{TDM} = \frac{1}{2}\log(1 + \max\{1, h^2\}(2\mathrm{SNR}))$$

where 2SNR appears in the expression since each transmitter sends for half the time. Note that $R_{MAC} \le R_{TDM}$ for any h and SNR.

If transmitter 1 sends a single message to receiver 1, transmitter 2 sends a single message to receiver 2, and each receiver decodes its desired signal while treating interference as noise (IAN), we can achieve

$$R_{IAN} = \log\left(1 + \frac{\mathrm{SNR}}{1 + h^2\mathrm{SNR}}\right)$$

for $h \leq 1$. For $h > 1$, transmitters can swap their destination receivers and achieve

$$R_{IAN} = \log\left(1 + \frac{h^2\mathrm{SNR}}{1 + \mathrm{SNR}}\right).$$

4.3 Main Results

For the regimes $\mathcal{B}_1$,$\mathcal{B}_2$,$\mathcal{B}_5$,$\mathcal{B}_6$, we characterize the sum-rate capacity C_{sum} of the symmetric Gaussian X channel to within two bits.

Theorem 4.1 (Constant gap for $\mathcal{B}_1$,$\mathcal{B}_2$,$\mathcal{B}_5$,$\mathcal{B}_6$)**.**
For $h \leq SNR^{-\frac{1}{8}}$ and $h \geq SNR^{\frac{1}{6}}$,

$$R_{ETW} - 2 \leq C_{sum} \leq R_{ETW}$$

where

$$R_{ETW} = \begin{cases} \log\left(1 + h^2\mathrm{SNR} + \frac{\mathrm{SNR}}{1+h^2\mathrm{SNR}}\right), & h \leq 1, \\ \log\left(1 + \mathrm{SNR} + \frac{h^2\mathrm{SNR}}{1+\mathrm{SNR}}\right), & h > 1. \end{cases} \tag{4.1}$$

Proof. The proof is given in a later section. □

The upper bound R_{ETW} was originally derived for the two-user Gaussian interference channel in [1]. In [9], it was shown that this bound can be used as an upper bound for the symmetric Gaussian X channel. The achievability part of the theorem is based on layered lattice coding, interference alignment, and layer-by-layer successive decoding.

For the regimes $\mathcal{B}_3$ and $\mathcal{B}_4$ where $\text{SNR}^{-\frac{1}{8}} \leq h \leq \text{SNR}^{\frac{1}{6}}$, we develop achievable schemes based on compute-and-forward framework [16, 7]. We also derive a new upper bound that is useful for $\mathcal{B}_3$ and $\mathcal{B}_4$.

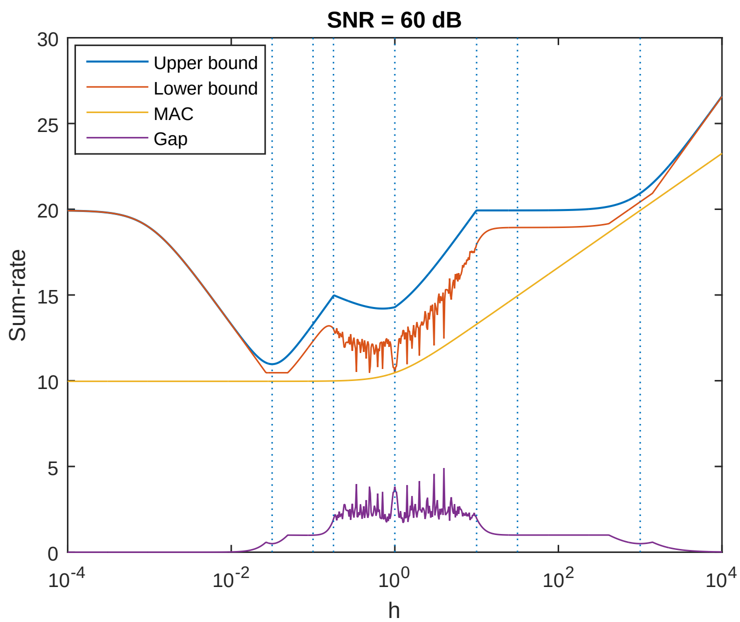
Theorem 4.2 (Upper bound)**.** *The sum-rate capacity C_{sum} is upper bounded by*

$$C_{sum} \leq \begin{cases} \frac{4}{3}R_{MAC} + \frac{1}{3}\log\left(1+\frac{SNR}{1+h^2SNR}\right), & h \leq 1, \\ \frac{4}{3}R_{MAC} + \frac{1}{3}\log\left(1+\frac{h^2SNR}{1+SNR}\right), & h > 1, \end{cases} \tag{4.2}$$

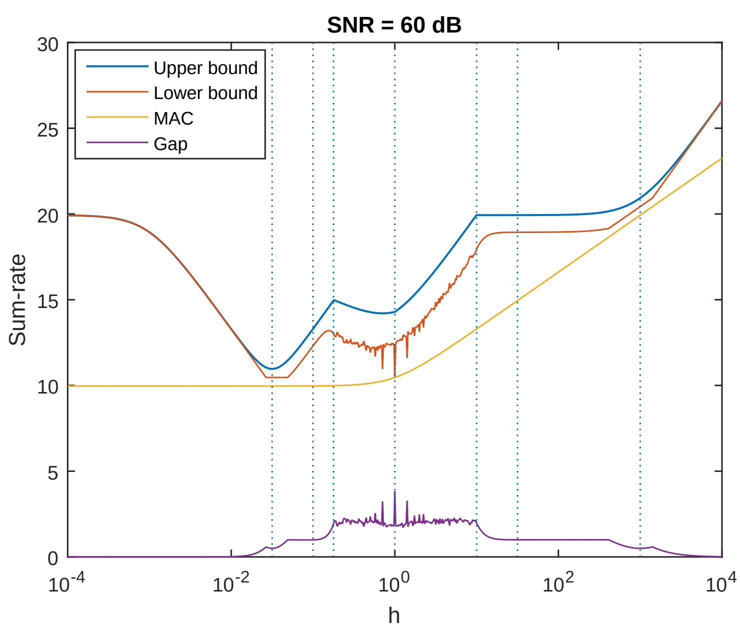
for any h and SNR.

Proof. The proof is given in a later section. □

Fig. 4.3 shows the sum-rate capacity upper and lower bounds at SNR = 60 dB. At this SNR, the boundary $h = \text{SNR}^{-\frac{1}{8}}$ in Theorem 1 corresponds to $h^2\text{SNR} = 45$ dB. Thus, the result in Theorem 1 can be interpreted as the approximate sum-rate capacity for the case where the direct-link and cross-link have at least 15 dB gap in received SNR. At SNR = 30 dB, it corresponds to the case with at least 7.5 dB gap in received SNR.



(a) No channel steering ($g = 1$)



(b) Channel steering (optimized over g)

Figure 4.3: Sum-rate capacity lower and upper bounds and the gap.

4.4 Sum-rate Capacity Upper Bound

In this section, we prove the upper bound in Theorem 2. In [9], the following inequalities were derived.

$$R_{v1} + R_{v2} + R_{w2} \leq R_{MAC} + \frac{1}{2}\log\left(1 + \frac{\mathrm{SNR}}{1 + h^2\mathrm{SNR}}\right), \tag{4.3}$$

$$R_{w1} + R_{w2} + R_{v1} \leq R_{MAC} + \frac{1}{2}\log\left(1 + \frac{\mathrm{SNR}}{1 + h^2\mathrm{SNR}}\right), \tag{4.4}$$

$$R_{v1} + R_{v2} + R_{w1} \leq R_{MAC} + \frac{1}{2}\log\left(1 + \frac{h^2\mathrm{SNR}}{1 + \mathrm{SNR}}\right), \tag{4.5}$$

$$R_{w1} + R_{w2} + R_{v2} \leq R_{MAC} + \frac{1}{2}\log\left(1 + \frac{h^2\mathrm{SNR}}{1 + \mathrm{SNR}}\right), \tag{4.6}$$

where $R_{v1}, R_{v2}, R_{w1}, R_{w2}$ are the code rates for the messages V_1, V_2, W_1, W_2, respectively, and

$$R_{MAC} = \frac{1}{2}\log(1 + (1 + h^2)\mathrm{SNR}).$$

By adding the four inequalities, we get the upper bound

$$R_{sum} \leq \frac{4}{3}R_{MAC} + \frac{1}{3}\log\left(1 + \frac{h^2\mathrm{SNR}}{1 + \mathrm{SNR}}\right) + \frac{1}{3}\log\left(1 + \frac{\mathrm{SNR}}{1 + h^2\mathrm{SNR}}\right)$$

for any h and SNR. Here, $R_{sum} = R_{v1} + R_{v2} + R_{w1} + R_{w2}$. We improve the bound by tightening (4.3) and (4.4) for $h > 1$ and by tightening (4.5) and (4.6) for $h \leq 1$.

4.4.1 The case of $h > 1$

We tighten (4.3) and (4.4) to

$$R_{v1} + R_{v2} + R_{w2} \leq R_{MAC}, \tag{4.7}$$

$$R_{w1} + R_{w2} + R_{v1} \leq R_{MAC}. \tag{4.8}$$

By combining these two inequalities with (4.5) and (4.6), we get

$$R_{sum} \leq \frac{4}{3}R_{MAC} + \frac{1}{3}\log\left(1 + \frac{h^2\mathrm{SNR}}{1+\mathrm{SNR}}\right). \tag{4.9}$$

In the following, we show the derivation of (4.7), and the derivation of (4.8) is similar due to symmetry of the channel. Let X^n denote a length-n sequence of random variables $(X_1, X_2, \cdots, X_n)$. The following inequalities hold.

$$\begin{aligned}
& n(R_{v1} + R_{v2} + R_{w2} - \epsilon_n) \\
& \leq I(V_1, V_2; Y_1^n) + I(W_2; Y_2^n) && (4.10) \\
& \leq I(V_1, V_2; Y_1^n, W_1) + I(W_2; Y_2^n, V_1, V_2, W_1) && (4.11) \\
& = I(V_1, V_2; Y_1^n|W_1) + I(W_2; Y_2^n|V_1, V_2, W_1) && (4.12) \\
& \leq I(V_1, V_2; Y_1^n|W_1) + I(W_2; Y_1^n|V_1, V_2, W_1) && (4.13) \\
& = I(V_1, V_2, W_2; Y_1^n|W_1) && (4.14) \\
& = H(Y_1^n|W_1) - H(Y_1^n|V_1, W_1, V_2, W_2) && (4.15) \\
& = H(Y_1^n|W_1) - H(Y_1^n|X_1, X_2, V_1, W_1, V_2, W_2) && (4.16) \\
& = H(Y_1^n|W_1) - H(Z_1^n) && (4.17) \\
& \leq H(Y_1^n) - H(Z_1^n) && (4.18) \\
& \leq nH(Y_1) - nH(Z_1) && (4.19) \\
& \leq nH(Y_{1G}) - nH(Z_1) && (4.20) \\
& = \frac{n}{2}\log(1 + (1+h^2)\mathrm{SNR}) && (4.21)
\end{aligned}$$

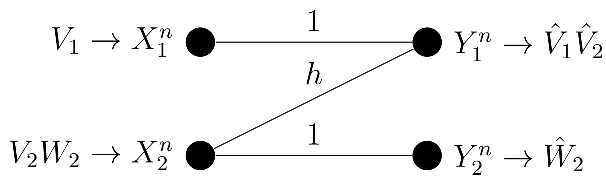


Figure 4.4: The upper-bound Z channel for Eq. (4.7).

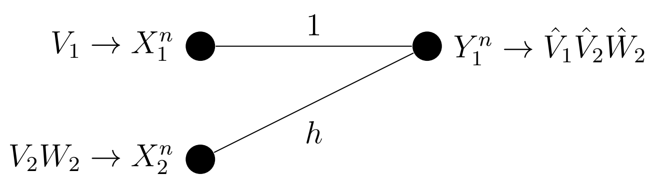


Figure 4.5: The upper-bound MAC for Eq. (4.7).

4.4.2 The case of $h \leq 1$

We tighten (4.5) and (4.6) to

$$R_{v1} + R_{v2} + R_{w1} \leq R_{MAC}, \tag{4.22}$$

$$R_{w1} + R_{w2} + R_{v2} \leq R_{MAC}. \tag{4.23}$$

By combining these two inequalities with (4.3) and (4.4), we get

$$R_{sum} \leq \frac{4}{3}R_{MAC} + \frac{1}{3}\log\left(1 + \frac{\text{SNR}}{1 + h^2\text{SNR}}\right). \tag{4.24}$$

In the following, we show the derivation of (4.22), and the derivation of (4.23) is similar due to symmetry of the channel. By the Fano's inequality, we get

$$n(R_{v1} + R_{v2} + R_{w1} - \epsilon_n) \leq I(V_1, V_2; Y_1^n|W_2) + I(W_1; Y_2^n|W_2) \tag{4.25}$$

Note that (4.25) is an upper bound on the sum-rate capacity of the Z channel where the communication link between transmitter 2 and receiver 2 is removed as shown in Fig. 4.6. By the chain rule of mutual information,

$$I(V_1, V_2; Y_1^n|W_2) = I(V_1, V_2, W_1; Y_1^n|W_2) - I(W_1; Y_1^n|V_1, V_2, W_2).$$

In what follows, we show that

$$n(R_{v1} + R_{v2} + R_{w1} - \epsilon_n) \leq I(V_1, V_2, W_1; Y_1^n|W_2) \tag{4.26}$$

by showing that

$$I(W_1; Y_2^n|W_2) \leq I(W_1; Y_1^n|V_1, V_2, W_2). \tag{4.27}$$

We start by upper bounding the left-hand side,

$$I(W_1; Y_2^n|W_2) \leq I(W_1; Y_2^n|V_1, V_2, W_2) \tag{4.28}$$

where the inequality is due to the fact that conditioning reduces entropy. By using stochastic degradedness argument similar to the one used for $h > 1$ case, we further upper bound (4.28) by

$$I(W_1; Y_2^n|V_1, V_2, W_2) \leq I(W_1; Y_1^n|V_1, V_2, W_2).$$

Here, Y_2^n is a degraded version of Y_1^n since $h \leq 1$. From this inequality and (4.28), we conclude that the inequality (4.27) holds. We proceed from (4.26) to

$$\begin{aligned} n(R_{v1} + R_{v2} + R_{w1} - \epsilon_n) &\leq I(V_1, V_2, W_1; Y_1^n|W_2) \\ &\leq \frac{n}{2}\log(1 + (1 + h^2)\mathrm{SNR}) \end{aligned}$$

Thus, we conclude that the inequality in (4.22). The derivation of (4.23) is almost identical, thus omitted. This completes the proof of the theorem.

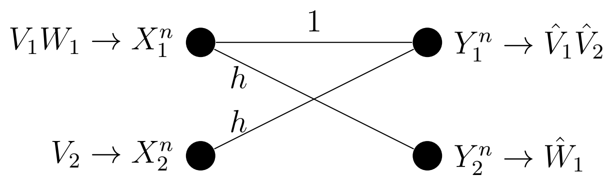


Figure 4.6: The upper-bound Z channel for Eq. (4.22).

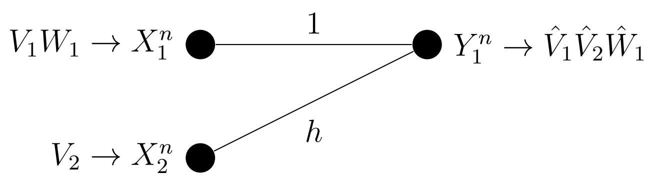


Figure 4.7: The upper-bound MAC for Eq. (4.22).

4.5 Layered Lattice Coding

Encoding and decoding strategies vary for different regimes of h. The choices of different transmit signals for different regimes and the resulting received signals are given in Table 4.2 where **v**-signals carry desired messages for receiver 1 and **w**-signals for receiver 2. The parameter g is the channel steering parameter, we refer to the $g = 1$ case as no channel steering and to the $g \neq 1$ cases as channel steering. In this section, we focus on the no channel steering case. We give high-level description of encoding and decoding strategies for different regimes as follows.

- $\mathcal{B}_1$ (single layer transmission with single-user decoding): Each transmitter sends a single message. Receiver 1 decodes $\mathbf{v}_p$ while treating $\mathbf{w}_p$ as noise. Receiver 2 decodes $\mathbf{w}_p$ while treating $\mathbf{v}_p$ as noise. This scheme is often called *treating interference as noise* (IAN) in the literature.

- $\mathcal{B}_2$ (multi-layer transmission with successive decoding): Each transmitter sends three independent messages. Transmitter 1 splits $\mathbf{v}$-signal into two parts: $\mathbf{v}_d$ to be decoded at both receivers and $\mathbf{v}_p$ to be decoded only at receiver 1. Transmitter 2 splits $\mathbf{w}$-signal into two parts in the same way. Receiver 1 decodes successively in the order $\mathbf{v}_d \to \mathbf{w}_f \to \mathbf{v}_c \to \mathbf{v}_p$ while treating remaining signals as noise in each step where $\mathbf{w}_f = \mathbf{w}_d + \mathbf{w}_c$ is aligned interference. Receiver 2 decodes in the order $\mathbf{w}_d \to \mathbf{v}_f \to \mathbf{w}_c \to \mathbf{w}_p$ where $\mathbf{v}_f = \mathbf{v}_d + \mathbf{v}_c$.

- $\mathcal{B}_3$ (multi-layer transmission with compute-and-forward decoding): Transmit signals are similar to those for $\mathcal{B}_2$ with slight change in the coefficients for $\mathbf{v}_p$ and $\mathbf{w}_p$. Receiver 1 decodes in the order $(\mathbf{v}_d, \mathbf{w}_f, \mathbf{v}_c) \to \mathbf{v}_p$, i.e., first decode three integer linear combinations of $\mathbf{v}_d$, $\mathbf{v}_c$, $\mathbf{w}_f$ by compute-and-forward while treating $\mathbf{v}_p$ and $\mathbf{w}_p$ signals as noise. After removing $\mathbf{v}_d$, $\mathbf{v}_c$, $\mathbf{w}_f$, receiver decodes $\mathbf{v}_p$ while treating $\mathbf{w}_p$ as noise. Receiver 2 performs decoding in the order $(\mathbf{w}_d, \mathbf{v}_f, \mathbf{w}_c) \to \mathbf{w}_p$ in the same way.

- $\mathcal{B}_4$ (multi-layer transmission with compute-and-forward decoding): Each transmitter sends three independent messages. Transmitter 1 splits $\mathbf{w}$-signal into two parts: $\mathbf{w}_c$ to be decoded at both receivers and $\mathbf{w}_p$ to be decoded only at receiver 2. Transmitter 2 splits $\mathbf{v}$-signal into two parts in the same way. Decoding procedure is the same as the one for $\mathcal{B}_3$.

- $\mathcal{B}_5$ (multi-layer transmission with successive decoding): Transmit signals are similar to those for $\mathcal{B}_4$ with slight change in the coefficients for $\mathbf{v}_p$ and $\mathbf{w}_p$. Receiver 1 decodes successively in the order $\mathbf{v}_c \to \mathbf{w}_f \to \mathbf{v}_d \to \mathbf{v}_p$ while treating remaining signals as noise in each step. Receiver 2 decodes in the order $\mathbf{w}_c \to \mathbf{v}_f \to \mathbf{w}_d \to \mathbf{w}_p$ in the same way.

	Transmit signals $\mathbf{x}_j$	Received signals $\bar{\mathbf{y}}_j = \mathbf{y}_j - \mathbf{z}_j$	$\sigma^2(\Lambda) = P$
$\mathcal{B}_1$	$\mathbf{x}_1 = \mathbf{v}_p$ $\mathbf{x}_2 = \mathbf{w}_p$	$\bar{\mathbf{y}}_1 = \mathbf{v}_p + h\mathbf{w}_p$ $\bar{\mathbf{y}}_2 = \mathbf{w}_p + h\mathbf{v}_p$	SNR
$\mathcal{B}_2$	$\mathbf{x}_1 = \mathbf{v}_d + h\mathbf{w}_c + h^3\mathbf{v}_p$ $\mathbf{x}_2 = \mathbf{w}_d + h\mathbf{v}_c + h^3\mathbf{w}_p$	$\bar{\mathbf{y}}_1 = \mathbf{v}_d + h\mathbf{w}_f + h^2\mathbf{v}_c + h^3\mathbf{v}_p + h^4\mathbf{w}_p$ $\bar{\mathbf{y}}_2 = \mathbf{w}_d + h\mathbf{v}_f + h^2\mathbf{w}_c + h^3\mathbf{w}_p + h^4\mathbf{v}_p$	$\frac{\text{SNR}}{1+h^2+h^6}$
$\mathcal{B}_3$	$\mathbf{x}_1 = \mathbf{v}_d + h\mathbf{w}_c + \frac{1}{\sqrt{h^2P}}\mathbf{v}_p$ $\mathbf{x}_2 = \mathbf{w}_d + h\mathbf{v}_c + \frac{1}{\sqrt{h^2P}}\mathbf{w}_p$	$\bar{\mathbf{y}}_1 = \mathbf{v}_d + h\mathbf{w}_f + h^2\mathbf{v}_c + \frac{1}{\sqrt{h^2P}}\mathbf{v}_p + \frac{1}{\sqrt{P}}\mathbf{w}_p$ $\bar{\mathbf{y}}_2 = \mathbf{w}_d + h\mathbf{v}_f + h^2\mathbf{w}_c + \frac{1}{\sqrt{h^2P}}\mathbf{w}_p + \frac{1}{\sqrt{P}}\mathbf{v}_p$	$\frac{\text{SNR}-h^{-2}}{1+h^2}$
$\mathcal{B}_4$	$\mathbf{x}_1 = \mathbf{w}_c + h^{-1}\mathbf{v}_d + \frac{1}{\sqrt{P}}\mathbf{w}_p$ $\mathbf{x}_2 = \mathbf{v}_c + h^{-1}\mathbf{w}_d + \frac{1}{\sqrt{P}}\mathbf{v}_p$	$\bar{\mathbf{y}}_1 = h\mathbf{v}_c + \mathbf{w}_f + h^{-1}\mathbf{v}_d + \frac{h}{\sqrt{P}}\mathbf{v}_p + \frac{1}{\sqrt{P}}\mathbf{w}_p$ $\bar{\mathbf{y}}_2 = h\mathbf{w}_c + \mathbf{v}_f + h^{-1}\mathbf{w}_d + \frac{h}{\sqrt{P}}\mathbf{w}_p + \frac{1}{\sqrt{P}}\mathbf{v}_p$	$\frac{\text{SNR}-1}{1+h^{-2}}$
$\mathcal{B}_5$	$\mathbf{x}_1 = \mathbf{w}_c + h^{-1}\mathbf{v}_d + h^{-3}\mathbf{w}_p$ $\mathbf{x}_2 = \mathbf{v}_c + h^{-1}\mathbf{w}_d + h^{-3}\mathbf{v}_p$	$\bar{\mathbf{y}}_1 = h\mathbf{v}_c + \mathbf{w}_f + h^{-1}\mathbf{v}_d + h^{-2}\mathbf{v}_p + h^{-3}\mathbf{w}_p$ $\bar{\mathbf{y}}_2 = h\mathbf{w}_c + \mathbf{v}_f + h^{-1}\mathbf{w}_d + h^{-2}\mathbf{w}_p + h^{-3}\mathbf{v}_p$	$\frac{\text{SNR}}{1+h^{-2}+h^{-6}}$
$\mathcal{B}_6$	$\mathbf{x}_1 = \mathbf{w}_p$ $\mathbf{x}_2 = \mathbf{v}_p$	$\bar{\mathbf{y}}_1 = h\mathbf{v}_p + \mathbf{w}_p$ $\bar{\mathbf{y}}_2 = h\mathbf{w}_p + \mathbf{v}_p$	SNR

Table 4.2: Transmit and received signals for each regime.

- $\mathcal{B}_6$ (single layer transmission with single-user decoding): Each transmitter sends a single message. The achievable scheme is similar to the one for $\mathcal{B}_1$, but the roles of direct link and cross link are reversed.

We explain the structure of lattice signals that we use for lattice interference alignment. The following standard definitions [12] are used.

4.5.1 Lattice Signaling for Interference Alignment

Lattice signals are defined as follows.

- Shaping lattice Λ with $\sigma^2(\Lambda) = P$ and $G(\Lambda) = \frac{1}{2\pi e}$, thus $V(\Lambda) = (2\pi e P)^{\frac{n}{2}}$,

- Lattice codewords

$$\bar{\mathbf{v}}_d \in \Lambda_{vd} \cap \mathcal{V}(\Lambda), \qquad \bar{\mathbf{w}}_d \in \Lambda_{wd} \cap \mathcal{V}(\Lambda),$$
$$\bar{\mathbf{v}}_c \in \Lambda_{vc} \cap \mathcal{V}(\Lambda), \qquad \bar{\mathbf{w}}_c \in \Lambda_{wc} \cap \mathcal{V}(\Lambda),$$
$$\bar{\mathbf{v}}_p \in \Lambda_{vp} \cap \mathcal{V}(\Lambda), \qquad \bar{\mathbf{w}}_p \in \Lambda_{wp} \cap \mathcal{V}(\Lambda),$$

- Dither signals

$$\mathbf{d}_{vd}, \mathbf{d}_{vc}, \mathbf{d}_{vp}, \mathbf{d}_{wd}, \mathbf{d}_{wc}, \mathbf{d}_{wp} \sim \mathrm{Unif}(\mathcal{V}(\Lambda))$$

  and dithered codewords

$$\mathbf{v}_d = [\bar{\mathbf{v}}_d + \mathbf{d}_{vd}] \bmod \Lambda, \qquad \mathbf{w}_d = [\bar{\mathbf{w}}_d + \mathbf{d}_{wd}] \bmod \Lambda,$$
$$\mathbf{v}_c = [\bar{\mathbf{v}}_c + \mathbf{d}_{vc}] \bmod \Lambda, \qquad \mathbf{w}_c = [\bar{\mathbf{w}}_c + \mathbf{d}_{wc}] \bmod \Lambda,$$
$$\mathbf{v}_p = [\bar{\mathbf{v}}_p + \mathbf{d}_{vp}] \bmod \Lambda, \qquad \mathbf{w}_p = [\bar{\mathbf{w}}_p + \mathbf{d}_{wp}] \bmod \Lambda,$$

  where $\frac{1}{n}\|\mathbf{v}_d\|^2 = \frac{1}{n}\|\mathbf{v}_c\|^2 = \frac{1}{n}\|\mathbf{v}_p\|^2 = \frac{1}{n}\|\mathbf{w}_d\|^2 = \frac{1}{n}\|\mathbf{w}_c\|^2 = \frac{1}{n}\|\mathbf{w}_p\|^2 = P$.

- Code rates

$$R_j = \frac{1}{n}\log\left(\frac{V(\Lambda)}{V(\Lambda_j)}\right)$$

  for $j \in \{vd, vc, vp, wd, wc, wp\}$.

If the transmitters send

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{v}_d + h\mathbf{w}_c, \\ \mathbf{x}_2 &= \mathbf{w}_d + h\mathbf{v}_c, \end{aligned} \tag{4.29}$$

with transmit power $\frac{1}{n}\|\mathbf{x}_j\|^2 = (1+h^2)P = \mathrm{SNR}$, each receiver observes an equivalent three-user MAC,

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{v}_d + h\mathbf{w}_f + h^2\mathbf{v}_c + \mathbf{z}_1, \\ \mathbf{y}_2 &= \mathbf{w}_d + h\mathbf{v}_f + h^2\mathbf{w}_c + \mathbf{z}_2, \end{aligned}$$

where $\mathbf{v}_f = \mathbf{v}_c + \mathbf{v}_d$ and $\mathbf{w}_f = \mathbf{w}_c + \mathbf{w}_d$ are the aligned interference signals. Note that $\frac{1}{n}\|\mathbf{w}_f\|^2 = \frac{1}{n}\|\mathbf{w}_c\|^2 + \frac{1}{n}\|\mathbf{w}_d\|^2 = 2P$ and also $\frac{1}{n}\|\mathbf{v}_f\|^2 = 2P$. The dithered and undithered signals are related by

$$[\mathbf{v}_f - \mathbf{d}_{vd} - \mathbf{d}_{vc}] \bmod \Lambda \quad = \quad \bar{\mathbf{v}}_f \bmod \Lambda. \tag{4.30}$$

where $\bar{\mathbf{v}}_f = \bar{\mathbf{v}}_c + \bar{\mathbf{v}}_d$. If we choose nested lattices $\Lambda \subseteq \Lambda_{vc} \subseteq \Lambda_{vd}$, then $\bar{\mathbf{v}}_f \in \Lambda_{vd}$. In contrast, if $\Lambda \subseteq \Lambda_{vd} \subseteq \Lambda_{vc}$, then $\bar{\mathbf{v}}_f \in \Lambda_{vc}$.

4.5.2 Successive Decoding for $\mathcal{B}_2$

For $\mathcal{B}_2$, the transmit signals are formed by LLC

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{v}_d + h\mathbf{w}_c + h^3\mathbf{v}_p, \\
\mathbf{x}_2 &= \mathbf{w}_d + h\mathbf{v}_c + h^3\mathbf{w}_p,
\end{aligned} \tag{4.31}$$

with transmit power $\frac{1}{n}\|\mathbf{x}_j\|^2 = (1 + h^2 + h^6)P = \mathrm{SNR}$.

The received signal at receiver 1 is

$$\mathbf{y}_1 = \mathbf{v}_d + h\mathbf{w}_f + h^2\mathbf{v}_c + h^3\mathbf{v}_p + h^4\mathbf{w}_p + \mathbf{z}_1.$$

Successive decoding is performed in four steps with the decoding order $\mathbf{v}_d \to \mathbf{w}_f \to \mathbf{v}_c \to \mathbf{v}_p$. Decoding of desired signals $\mathbf{v}_d$, $\mathbf{v}_c$, $\mathbf{v}_p$ are similar to the lattice decoding in [12] while decoding of aligned interference signals $\mathbf{w}_f = \mathbf{w}_c + \mathbf{w}_d$ is similar to the decoding at the relay in [13, 14].

In the first step of successive decoding, we decode $\bar{\mathbf{v}}_d$ from the mod-Λ channel $\mathbf{y}_1^{(1)}$. After linear scaling, dither removal, and mod-Λ operation, we get

$$\mathbf{y}_1^{(1)} = [\beta\mathbf{y}_1 - \mathbf{d}_{vd}] \bmod \Lambda = \left[\bar{\mathbf{v}}_d + \mathbf{z}_1^{(1)}\right] \bmod \Lambda \tag{4.32}$$

where the effective noise is

$$\mathbf{z}_1^{(1)} = (\beta - 1)\mathbf{v}_d + \beta(h\mathbf{w}_f + h^2\mathbf{v}_c + h^3\mathbf{v}_p + h^4\mathbf{w}_p + \mathbf{z}_1)$$

and its variance

$$\sigma_e^2 = \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{z}_1^{(1)}\right\|^2\right] = (\beta-1)^2P + \beta^2 N_e$$

where $N_e = (2h^2+h^4+h^6+h^8)P+1$. With the MMSE scaling factor $\beta = \frac{P}{P+N_e}$ plugged in, we get $\sigma_e^2 = \beta N_e = \frac{PN_e}{P+N_e}$. The capacity of the mod-Λ channel [12] between $\bar{\mathbf{v}}_d$ and $\mathbf{y}_1^{(1)}$ is

$$\begin{aligned}
&\frac{1}{n}I\left(\bar{\mathbf{v}}_d;\mathbf{y}_1^{(1)}\right)\\
&\geq \frac{1}{2}\log\left(\frac{P}{\beta N_e}\right)\\
&= \frac{1}{2}\log\left(1+\frac{P}{N_e}\right)\\
&= \frac{1}{2}\log\left(1+\frac{P}{(2h^2+h^4+h^6+h^8)P+1}\right)\\
&= \frac{1}{2}\log\left(1+\frac{\mathrm{SNR}}{\sigma_{vd}^2}\right)\\
&= C_{vd}
\end{aligned}$$

where $\sigma_{vd}^2 = (2h^2+h^4+h^6+h^8)\mathrm{SNR}+1+h^2+h^6$. For reliable decoding of $\bar{\mathbf{v}}_d$ at receiver 1, we have the code rate constraint $R_{vd} \leq C_{vd}$.

In the second step of successive decoding, we decode the aligned interference $\bar{\mathbf{w}}_f$ in the same way as we did in the first step. After canceling $\mathbf{v}_d$, the mod-Λ channel is given by

$$\begin{aligned}
&\mathbf{y}_1^{(2)} = [\beta h^{-1}(\mathbf{y}_1 - \mathbf{v}_d) - \mathbf{d}_{wf}] \bmod \Lambda\\
&= [\beta(\mathbf{w}_f + h\mathbf{v}_c + h^2\mathbf{v}_p + h^3\mathbf{w}_p + h^{-1}\mathbf{z}_1) - \mathbf{d}_{wf}] \bmod \Lambda\\
&= \left[\bar{\mathbf{w}}_f + \mathbf{z}_1^{(2)}\right] \bmod \Lambda
\end{aligned}$$

where

$$\mathbf{z}_1^{(2)} = (\beta - 1)\mathbf{w}_f + \beta(h\mathbf{v}_c + h^2\mathbf{v}_p + h^3\mathbf{w}_p + h^{-1}\mathbf{z}_1)$$

and the effective noise variance

$$\sigma_e^2 = \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{z}_1^{(2)}\right\|^2\right] = (\beta - 1)^2(2P) + \beta^2 N_e$$

where $N_e = (h^2 + h^4 + h^6)P + h^{-2}$. Note that for simplicity, we use the same notations β, σ_e^2 and N_e in every decoding step although their values are different in different steps. With the MMSE scaling factor $\beta = \frac{2P}{2P+N_e}$ plugged in, we get $\sigma_e^2 = \beta N_e = \frac{2PN_e}{2P+N_e}$. The capacity of the mod-Λ channel between $\bar{\mathbf{w}}_f$ and $\mathbf{y}_1^{(2)}$ is

$$\begin{aligned}
&\frac{1}{n}I\left(\bar{\mathbf{w}}_f; \mathbf{y}_1^{(2)}\right) \\
&\geq \frac{1}{2}\log\left(\frac{P}{\beta N_e}\right) \\
&= \frac{1}{2}\log\left(\frac{2P + N_e}{2N_e}\right) \\
&= \frac{1}{2}\log\left(\frac{1}{2} + \frac{P}{N_e}\right) \\
&= \frac{1}{2}\log\left(\frac{1}{2} + \frac{P}{(h^2 + h^4 + h^6)P + h^{-2}}\right) \\
&= \frac{1}{2}\log\left(\frac{1}{2} + \frac{\mathrm{SNR}}{\sigma_{wf}^2}\right) \\
&= C_{wf}
\end{aligned}$$

where $\sigma_{wf}^2 = (h^2 + h^4 + h^6)\mathrm{SNR} + h^{-2}(1 + h^2 + h^6)$. For reliable decoding of $\bar{\mathbf{w}}_f$ at receiver 1, we have the code rate constraint $R_{wf} = \max\{R_{wc}, R_{wd}\} \leq C_{wf}$. By lattice decoding, we can recover the *modulo* sum of interference codewords $[\bar{\mathbf{w}}_f] \bmod \Lambda = [\bar{\mathbf{w}}_c + \bar{\mathbf{w}}_d] \bmod \Lambda$ from $\mathbf{y}_1^{(2)}$. Then, we can recover the *real* sum $\mathbf{w}_f = \mathbf{w}_c + \mathbf{w}_d$ in the following way [17].

- Recover $[\mathbf{w}_f] \bmod \Lambda$ by adding back dither signals,

$$
\begin{aligned}
&[[\bar{\mathbf{w}}_f] \bmod \Lambda + \mathbf{d}_{wc} + \mathbf{d}_{wd}] \bmod \Lambda \\
&= [\bar{\mathbf{w}}_c + \bar{\mathbf{w}}_d + \mathbf{d}_{wc} + \mathbf{d}_{wd}] \bmod \Lambda \\
&= [\mathbf{w}_c + \mathbf{w}_d] \bmod \Lambda \\
&= [\mathbf{w}_f] \bmod \Lambda
\end{aligned}
$$

- Subtract it from the received signal,

$$
\begin{aligned}
&h^{-1}(\mathbf{y}_1 - \mathbf{v}_d) - [\mathbf{w}_f] \bmod \Lambda \\
&= \mathbf{w}_f - [\mathbf{w}_f] \bmod \Lambda + h\mathbf{v}_c + h^2\mathbf{v}_p + h^3\mathbf{w}_p + h^{-1}\mathbf{z}_1 \\
&= Q_\Lambda(\mathbf{w}_f) + \mathbf{z}_1'
\end{aligned}
$$

  where $\mathbf{z}_1' = h\mathbf{v}_c + h^2\mathbf{v}_p + h^3\mathbf{w}_p + h^{-1}\mathbf{z}_1$.

- Quantize it on the shaping lattice Λ to recover $Q_\Lambda(\mathbf{w}_f)$,

$$
Q_\Lambda\left(Q_\Lambda(\mathbf{w}_f) + \mathbf{z}_1'\right) = Q_\Lambda(\mathbf{w}_f)
$$

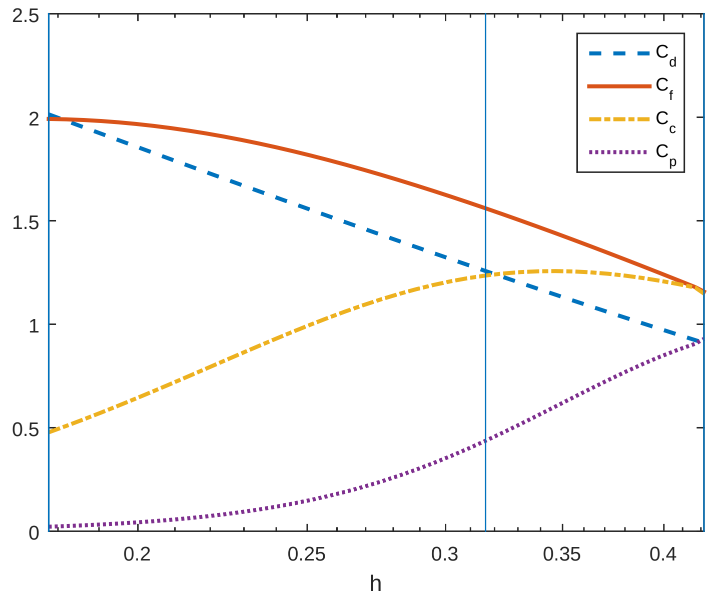
  with probability $1 - P_e$ where

$$
P_e = \Pr[Q_\Lambda\left(Q_\Lambda(\mathbf{w}_f) + \mathbf{z}_1'\right) \neq Q_\Lambda(\mathbf{w}_f)]
$$

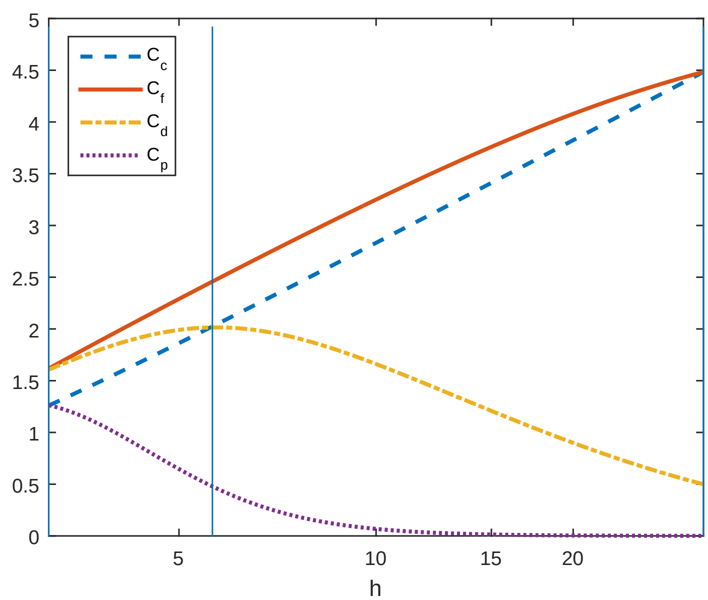
  is the probability of decoding error. Since we chose Λ to be simultaneously Rogers-good and Poltyrev-good, and $V(\Lambda) \geq V(\Lambda_f)$, $P_e \to 0$ as $n \to \infty$.

- Recover $\mathbf{w}_f$ by adding two vectors,

$$
[\mathbf{w}_f] \bmod \Lambda + Q_\Lambda(\mathbf{w}_f) = \mathbf{w}_f.
$$



(a) $\mathcal{B}_2$



(b) $\mathcal{B}_5$

Figure 4.8: The LLC code rate constraints C_d, C_f, C_c, C_p for $\mathcal{B}_2$ and $\mathcal{B}_5$.

Now, we proceed to the third step where we decode $\mathbf{v}_c$ from

$$\begin{aligned}
&\mathbf{y}_1^{(3)} = [\beta h^{-2}(\mathbf{y}_1 - \mathbf{v}_d - h\mathbf{w}_f) - \mathbf{d}_{vc}] \bmod \Lambda \\
&= [\beta(\mathbf{v}_c + h\mathbf{v}_p + h^2\mathbf{w}_p + h^{-2}\mathbf{z}_1) - \mathbf{d}_{vc}] \bmod \Lambda \\
&= \left[\bar{\mathbf{v}}_c + \mathbf{z}_1^{(3)}\right] \bmod \Lambda
\end{aligned}$$

where

$$\mathbf{z}_1^{(3)} = (\beta - 1)\mathbf{v}_c + \beta(h\mathbf{v}_p + h^2\mathbf{w}_p + h^{-2}\mathbf{z}_1)$$

and the effective noise variance

$$\sigma_e^2 = \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{z}_1^{(3)}\right\|^2\right] = (\beta - 1)^2 P + \beta^2 N_e$$

where $N_e = (h^2 + h^4)P + h^{-4}$. With the MMSE scaling factor $\beta = \frac{P}{P+N_e}$ plugged in, we get $\sigma_e^2 = \beta N_e = \frac{PN_e}{P+N_e}$. The capacity of the mod-Λ channel between $\bar{\mathbf{v}}_c$ and $\mathbf{y}_1^{(3)}$ is

$$\begin{aligned}
&\frac{1}{n} I\left(\bar{\mathbf{v}}_c; \mathbf{y}_1^{(3)}\right) \\
&\geq \frac{1}{2}\log\left(\frac{P}{\beta N_e}\right) \\
&= \frac{1}{2}\log\left(\frac{P + N_e}{N_e}\right) \\
&= \frac{1}{2}\log\left(1 + \frac{P}{N_e}\right) \\
&= \frac{1}{2}\log\left(1 + \frac{P}{(h^2 + h^4)P + h^{-4}}\right) \\
&= \frac{1}{2}\log\left(1 + \frac{\mathrm{SNR}}{\sigma_{vc}^2}\right) \\
&= C_{vc}
\end{aligned}$$

where $\sigma_{vc}^2 = (h^2 + h^4)\mathrm{SNR} + h^{-4}(1 + h^2 + h^6)$. For reliable decoding of $\bar{\mathbf{v}}_c$ at receiver 1, we have the code rate constraint $R_{vc} \leq C_{vc}$.

In the fourth step, we decode $\mathbf{v}_p$ from

$$\mathbf{y}_1^{(4)} = [\beta h^{-3}(\mathbf{y}_1 - \mathbf{v}_d - h\mathbf{w}_f - h^2\mathbf{v}_c) - \mathbf{d}_{vp}] \bmod \Lambda$$
$$= [\beta(\mathbf{v}_p + h\mathbf{w}_p + h^{-3}\mathbf{z}_1) - \mathbf{d}_{vp}] \bmod \Lambda$$
$$= \left[\bar{\mathbf{v}}_p + \mathbf{z}_1^{(4)}\right] \bmod \Lambda$$

where

$$\mathbf{z}_1^{(4)} = (\beta - 1)\mathbf{v}_p + \beta(h\mathbf{w}_p + h^{-3}\mathbf{z}_1)$$

and the effective noise variance

$$\sigma_e^2 = \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{z}_1^{(4)}\right\|^2\right] = (\beta - 1)^2 P + \beta^2 N_e$$

where $N_e = h^2 P + h^{-6}$. With the MMSE scaling factor $\beta = \frac{P}{P+N_e}$ plugged in, we get $\sigma_e^2 = \beta N_e = \frac{PN_e}{P+N_e}$. The capacity of the mod-Λ channel between $\bar{\mathbf{v}}_p$ and $\mathbf{y}_1^{(4)}$ is

$$\frac{1}{n}I\left(\bar{\mathbf{v}}_p; \mathbf{y}_1^{(4)}\right)$$
$$\geq \frac{1}{2}\log\left(\frac{P}{\beta N_e}\right)$$
$$= \frac{1}{2}\log\left(\frac{P + N_e}{N_e}\right)$$
$$= \frac{1}{2}\log\left(1 + \frac{P}{N_e}\right)$$
$$= \frac{1}{2}\log\left(1 + \frac{P}{h^2 P + h^{-6}}\right)$$
$$= \frac{1}{2}\log\left(1 + \frac{\mathrm{SNR}}{\sigma_{vp}^2}\right)$$
$$= C_{vp}$$

where $\sigma_{vp}^2 = h^2\mathrm{SNR} + h^{-6}(1 + h^2 + h^6)$. For reliable decoding of $\bar{\mathbf{v}}_p$ at receiver 1, we have the code rate constraint $R_{vp} \leq C_{vp}$.

Since received signal $\mathbf{y}_2$ has the equivalent signal structure,

$$\mathbf{y}_2 = \mathbf{w}_d + h\mathbf{v}_f + h^2\mathbf{w}_c + h^3\mathbf{w}_p + h^4\mathbf{v}_p + \mathbf{z}_2,$$

we can see that $C_{wd} = C_{vd}$, $C_{vf} = C_{wf}$, $C_{wc} = C_{vc}$, $C_{wp} = C_{vp}$. In summary, we have the following set of code rate constraints for reliable decoding at the receivers:

$$\begin{aligned}
R_{vd} &\leq C_{vd} \quad \text{at receiver 1,} \\
R_{vc} &\leq C_{vc} \quad \text{at receiver 1,} \\
\max\{R_{vd}, R_{vc}\} &\leq C_{vf}, \quad \text{at receiver 2,} \\
R_{wd} &\leq C_{wd} \quad \text{at receiver 2,} \\
R_{wc} &\leq C_{wc} \quad \text{at receiver 2,} \\
\max\{R_{wd}, R_{wc}\} &\leq C_{wf} \quad \text{at receiver 1,} \\
R_{vp} &\leq C_{vp} \quad \text{at receiver 1,} \\
R_{wp} &\leq C_{wp} \quad \text{at receiver 2.}
\end{aligned}$$

After rearranging, we get

$$\begin{aligned}
R_{vd} &\leq \min\{C_d, C_f\}, \\
R_{vc} &\leq \min\{C_c, C_f\}, \\
R_{wc} &\leq \min\{C_c, C_f\}, \\
R_{wd} &\leq \min\{C_d, C_f\}, \\
R_{vp} &\leq C_p, \\
R_{wp} &\leq C_p,
\end{aligned}$$

where

$$C_d = C_{vd} = C_{wd} = \frac{1}{2}\log\left(1+\frac{\mathrm{SNR}}{\sigma_d^2}\right),$$
$$C_f = C_{vf} = C_{wf} = \frac{1}{2}\log\left(\frac{1}{2}+\frac{\mathrm{SNR}}{\sigma_f^2}\right),$$
$$C_c = C_{vc} = C_{wc} = \frac{1}{2}\log\left(1+\frac{\mathrm{SNR}}{\sigma_c^2}\right),$$
$$C_p = C_{vp} = C_{wp} = \frac{1}{2}\log\left(1+\frac{\mathrm{SNR}}{\sigma_p^2}\right),$$

and

$$\sigma_d^2 = (2h^2+h^4+h^6+h^8)\mathrm{SNR}+1+h^2+h^6,$$
$$\sigma_f^2 = (h^2+h^4+h^6)\mathrm{SNR}+h^{-2}+1+h^4,$$
$$\sigma_c^2 = (h^2+h^4)\mathrm{SNR}+h^{-4}+h^{-2}+h^2,$$
$$\sigma_p^2 = h^2\mathrm{SNR}+h^{-6}+h^{-4}+1.$$

Fig. 4.8 shows the curves of C_d, C_f, C_c, C_p for $\mathcal{B}_2$ at SNR $= 156$ and at SNR $= 10^6$.

The sum-rate achievable by layered lattice coding (LLC) is given by

$$\begin{aligned} R_{LLC} &= R_{vd}+R_{vc}+R_{vp}+R_{wd}+R_{wc}+R_{wp} \\ &= 2\cdot\left(\min\{C_d,C_f\}+\min\{C_c,C_f\}+C_p\right). \end{aligned}$$

It can be check that the case $C_f < C_d$ happens when h is close to the left-boundry of $\mathcal{B}_2$ where TDM outperforms LLC (see Fig. 4.8 for example). Thus, we only consider the case $C_d \le C_f$. The bottleneck of $\min\{C_c,C_f\}$ depends on SNR and h. The LLC sum-rate can be expressed as

$$R_{LLC} = \min\{R_{dcp}, R_{dfp}\}$$

where

$$R_{dcp} = 2\left(C_d + C_c + C_p\right),$$
$$R_{dfp} = 2\left(C_d + C_f + C_p\right).$$

To get some intuition about R_{LLC}, let us first calculate an approximation of R_{LLC} up to a constant, but not with a specified constant. In $\mathcal{B}_2$, $h^2\text{SNR}$ is the largest term in σ_d^2, σ_f^2, σ_c^2, and h^{-6} is the largest term in σ_p^2. We can see that $\sigma_d^2 \simeq h^2\text{SNR}$, $\sigma_f^2 \simeq h^2\text{SNR}$, $\sigma_c^2 \simeq h^2\text{SNR}$, and $\sigma_p^2 \simeq h^{-6}$. Therefore,

$$C_d \simeq \frac{1}{2}\log\left(h^{-2}\right) = \frac{1}{2}\log\left(\text{SNR}^{1-\alpha}\right),$$
$$C_f \simeq \frac{1}{2}\log\left(h^{-2}\right) = \frac{1}{2}\log\left(\text{SNR}^{1-\alpha}\right),$$
$$C_c \simeq \frac{1}{2}\log\left(h^{-2}\right) = \frac{1}{2}\log\left(\text{SNR}^{1-\alpha}\right),$$
$$C_p \simeq \frac{1}{2}\log\left(h^{6}\text{SNR}\right) = \frac{1}{2}\log\left(\text{SNR}^{3\alpha-2}\right),$$

and

$$\begin{aligned} R_{LLC} &= 2\cdot\left(\min\{C_d, C_f\} + \min\{C_c, C_f\} + C_p\right) \\ &\simeq 2(1-\alpha)\log\left(\text{SNR}\right) + (3\alpha-2)\log\left(\text{SNR}\right) \\ &= \alpha\log\left(\text{SNR}\right) \end{aligned}$$

for any $\text{SNR} > 1$ as long as $\frac{1}{2} \leq \alpha \leq \frac{3}{4}$. In $\mathcal{B}_2$,

$$\begin{aligned} R_{ETW} &= \log\left(1 + \text{SNR}^{\alpha} + \frac{\text{SNR}}{1+\text{SNR}^{\alpha}}\right) \\ &\leq \log\left(1 + \text{SNR}^{\alpha} + \text{SNR}^{1-\alpha}\right) \\ &\leq \log\left(3\text{SNR}^{\alpha}\right) \end{aligned} \tag{4.33}$$

where the last inequality follows since $\text{SNR}^{\alpha} \geq \text{SNR}^{1-\alpha} > 1$. Thus, we can see that $R_{LLC} \simeq R_{ETW}$ in $\mathcal{B}_2$ for any SNR where the approximation is up to a constant. Now, let us show tight constant-gap characterization for $\mathcal{B}_2$.

4.5.3 Constant Gap for $\mathcal{B}_2'$

We characterize the gap

$$\Delta = R_{ETW} - R_{LLC} = R_{ETW} - \min\{R_{dcp}, R_{dfp}\}.$$

In sufficiently low SNR, TDM achieves the sum-rate capacity to within constant bits. Excluding such low SNR cases can simplify the LLC constant-gap achievability proof. We use the following lemmas.

Lemma 4.3. *For $\mathcal{B}_2'$ and SNR ≤ 156, TDM achieves the sum-rate capacity to within one bit, i.e., $R_{ETW} - R_{TDM} < 1$. For $\mathcal{B}_2''$ and SNR ≤ 875, TDM achieves the sum-rate capacity to within two bits, i.e., $R_{ETW} - R_{TDM} < 2$.*

Proof. The proof is straightforward, thus omitted. □

Thus, we need to consider LLC only for SNR > 156. We show that for $\mathcal{B}_2'$,

$$\Delta = \max\{\Delta_{dcp}, \Delta_{dfp}\} \leq 1$$

where $\Delta_{dcp} = R_{ETW} - R_{dcp}$ and $\Delta_{dfp} = R_{ETW} - R_{dfp}$. We first arrange Δ_{dcp} in the form

$$\begin{aligned}
\Delta_{dcp} &= R_{ETW} - R_{dcp} \\
&= \log\left(\frac{N(h, \mathrm{SNR})}{D(h, \mathrm{SNR})}\right) \\
&= \log\left(2 + \frac{N(h, \mathrm{SNR}) - 2D(h, \mathrm{SNR})}{D(h, \mathrm{SNR})}\right)
\end{aligned}$$

where $N(h, \mathrm{SNR})$ and $D(h, \mathrm{SNR})$ are some positive polynomials. And, we show that $\Delta_{dcp} \leq 1$ by showing that $N(h, \mathrm{SNR}) - 2D(h, \mathrm{SNR}) \leq 0$. Here, we repeatedly use the fact that SNR $< h^{-6}$ in $\mathcal{B}_2'$ to upper bound positive terms in $N(h, \text{SNR}) - 2D(h, \text{SNR})$. We then cancel out positive terms with negative terms. As a result, only negative terms remain, thus $N(h, \text{SNR}) - 2D(h, \text{SNR}) \leq 0$ and $\Delta_{dcp} \leq 1$.

We repeat similar steps for Δ_{dfp}. In the last step, we use the fact that $h < \text{SNR}^{-\frac{1}{6}} < 156^{-\frac{1}{6}} < \frac{1}{2}$ to upper bound the remaining positive terms to a constant, 5.11. Thus $N(h, \text{SNR}) - 2D(h, \text{SNR}) \leq 0$ and $\Delta_{dfp} \leq 1$, and the constant gap for $\mathcal{B}_2'$ is proved.

4.5.4 Constant Gap for $\mathcal{B}_2''$

Due to Lemma 1, we need to consider LLC only for SNR > 875. We show that for $\mathcal{B}_2''$,

$$\Delta = \max\{\Delta_{dcp}, \Delta_{dfp}\} \leq 2.$$

We first arrange Δ_{dcp} in the form

$$\begin{aligned}
\Delta_{dcp} &= R_{ETW} - R_{dcp} \\
&= \log\left(\frac{N(h, \text{SNR})}{D(h, \text{SNR})}\right) \\
&= \log\left(4 + \frac{N(h, \text{SNR}) - 4D(h, \text{SNR})}{D(h, \text{SNR})}\right)
\end{aligned}$$

And, we show that $\Delta_{dcp} \leq 2$ by showing that $N(h, \text{SNR}) - 4D(h, \text{SNR}) \leq 0$. Here, we use the fact that SNR $< h^{-8}$ in $\mathcal{B}_2''$ to upper bound positive terms in $N(h, \text{SNR}) - 4D(h, \text{SNR})$. We then cancel out positive terms with negative terms. As a result, only negative terms remain, thus $N(h, \text{SNR}) - 4D(h, \text{SNR}) \leq 0$, and $\Delta_{dcp} \leq 2$.

We repeat similar steps for Δ_{dfp}. In the last step, we use the fact that $h < 1$ to upper bound the remaining positive terms and cancel them out with negative terms. Thus, $N(h, \mathrm{SNR}) - 4D(h, \mathrm{SNR}) \leq 0$ and $\Delta_{dcp} \leq 2$, and the constant gap for $\mathcal{B}_2''$ is proved.

4.5.5 Successive Decoding for $\mathcal{B}_5$

The achievable sum-rate derivations and constant-gap proof for $\mathcal{B}_5$ are almost identical to those for $\mathcal{B}_2$. As pointed out in [9], any achievable sum-rate for channel (h, SNR) is also achievable for channel $(h', \mathrm{SNR}') = (h^{-1}, h^2\mathrm{SNR})$ by simply switching the roles of receivers. Thus, sum-rate expressions derived for $h < 1$ can be translated to sum-rate expressions for $h > 1$ by replacing h with h^{-1} and then replacing SNR with h^2SNR, i.e.,

$$R_{sum}(h, \mathrm{SNR}) = R_{sum}(h^{-1}, h^2\mathrm{SNR}).$$

This is also true for upper bound expressions. Any upper bound for $h < 1$ can be translated to a valid upper bound for $h > 1$ by replacing h with h^{-1} and then replacing SNR with h^2SNR.

The sum-rate achievable by layered lattice coding is given by

$$R_{LLC} = 2 \cdot (\min\{C_d, C_f\} + \min\{C_c, C_f\} + C_p)$$

where

$$C_c = \frac{1}{2}\log\left(1 + \frac{\mathrm{SNR}}{\sigma_c^2}\right), \quad C_f = \frac{1}{2}\log\left(\frac{1}{2} + \frac{\mathrm{SNR}}{\sigma_f^2}\right),$$
$$C_d = \frac{1}{2}\log\left(1 + \frac{\mathrm{SNR}}{\sigma_d^2}\right), \quad C_p = \frac{1}{2}\log\left(1 + \frac{\mathrm{SNR}}{\sigma_p^2}\right),$$

and

$$\sigma_c^2 = (2h^{-2} + h^{-4} + h^{-6} + h^{-8})\mathrm{SNR} + h^{-2} + h^{-4} + h^{-8},$$
$$\sigma_f^2 = (h^{-2} + h^{-4} + h^{-6})\mathrm{SNR} + 1 + h^{-2} + h^{-6},$$
$$\sigma_d^2 = (h^{-2} + h^{-4})\mathrm{SNR} + h^2 + 1 + h^{-4},$$
$$\sigma_p^2 = h^{-2}\mathrm{SNR} + h^4 + h^2 + h^{-2}.$$

The expressions for C_c, C_f, C_d, C_p and R_{LLC} are identical to those for $\mathcal{B}_2$, but the expressions for σ_c^2, σ_f^2, σ_d^2, σ_p^2 are changed. For completeness, the full derivations are given below.

For $\mathcal{B}_5$, the transmit signals are formed by layered lattice coding

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{w}_c + h^{-1}\mathbf{v}_d + h^{-3}\mathbf{w}_p, \\ \mathbf{x}_2 &= \mathbf{v}_c + h^{-1}\mathbf{w}_d + h^{-3}\mathbf{v}_p, \end{aligned} \tag{4.34}$$

with transmit power $\frac{1}{n}\|\mathbf{x}_j\|^2 = (1 + h^{-2} + h^{-6})P = \mathrm{SNR}$.

The received signal at receiver 1 is

$$\mathbf{y}_1 = h\mathbf{v}_c + \mathbf{w}_f + h^{-1}\mathbf{v}_d + h^{-2}\mathbf{v}_p + h^{-3}\mathbf{w}_p + \mathbf{z}_1.$$

Successive decoding is performed in four steps with the decoding order $\mathbf{v}_c \to \mathbf{w}_f \to \mathbf{v}_d \to \mathbf{v}_p$.

In the first step of successive decoding, we decode $\bar{\mathbf{v}}_d$ from the mod-Λ channel $\mathbf{y}_1^{(1)}$. After linear scaling, dither removal, and mod-Λ operation, we get

$$\mathbf{y}_1^{(1)} = [\beta h^{-1}\mathbf{y}_1 - \mathbf{d}_{vc}] \bmod \Lambda = \left[\bar{\mathbf{v}}_c + \mathbf{z}_1^{(1)}\right] \bmod \Lambda \tag{4.35}$$

where the effective noise is

$$\mathbf{z}_1^{(1)} = (\beta - 1)\mathbf{v}_c + \beta(h^{-1}\mathbf{w}_f + h^{-2}\mathbf{v}_c + h^{-3}\mathbf{v}_p + h^{-4}\mathbf{w}_p + h^{-1}\mathbf{z}_1)$$

and its variance

$$\sigma_e^2 = \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{z}_1^{(1)}\right\|^2\right] = (\beta-1)^2 P + \beta^2 N_e$$

where $N_e = (2h^{-2}+h^{-4}+h^{-6}+h^{-8})P + h^{-2}$. With the MMSE scaling factor $\beta = \frac{P}{P+N_e}$ plugged in, we get $\sigma_e^2 = \beta N_e = \frac{PN_e}{P+N_e}$. The capacity of the mod-Λ channel between $\bar{\mathbf{v}}_c$ and $\mathbf{y}_1^{(1)}$ is

$$\begin{aligned}
&\frac{1}{n} I\left(\bar{\mathbf{v}}_c; \mathbf{y}_1^{(1)}\right) \\
&\geq \frac{1}{2}\log\left(\frac{P}{\beta N_e}\right) \\
&= \frac{1}{2}\log\left(1+\frac{P}{N_e}\right) \\
&= \frac{1}{2}\log\left(1+\frac{P}{(2h^{-2}+h^{-4}+h^{-6}+h^{-8})P+h^{-2}}\right) \\
&= \frac{1}{2}\log\left(1+\frac{\mathrm{SNR}}{\sigma_{vc}^2}\right) \\
&= C_{vc}
\end{aligned}$$

where $\sigma_{vc}^2 = (2h^{-2}+h^{-4}+h^{-6}+h^{-8})\mathrm{SNR} + h^{-2}(1+h^{-2}+h^{-6})$. For reliable decoding of $\bar{\mathbf{v}}_c$ at receiver 1, we have a code rate constraint $R_{vc} \leq C_{vc}$.

In the second step of successive decoding, we decode the aligned interference $\bar{\mathbf{w}}_f$. After canceling $\mathbf{v}_c$, the mod-Λ channel is given by

$$\begin{aligned}
&\mathbf{y}_1^{(2)} = [\beta(\mathbf{y}_1 - h\mathbf{v}_c) - \mathbf{d}_{wf}] \bmod \Lambda \\
&= [\beta(\mathbf{w}_f + h^{-1}\mathbf{v}_d + h^{-2}\mathbf{v}_p + h^{-3}\mathbf{w}_p + \mathbf{z}_1) - \mathbf{d}_{wf}] \bmod \Lambda \\
&= \left[\bar{\mathbf{w}}_f + \mathbf{z}_1^{(2)}\right] \bmod \Lambda
\end{aligned}$$

where

$$\mathbf{z}_1^{(2)} = (\beta-1)\mathbf{w}_f + \beta(h^{-1}\mathbf{v}_d + h^{-2}\mathbf{v}_p + h^{-3}\mathbf{w}_p + \mathbf{z}_1)$$

and the effective noise variance

$$\sigma_e^2 = \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{z}_1^{(2)}\right\|^2\right] = (\beta-1)^2(2P) + \beta^2 N_e$$

where $N_e = (h^{-2}+h^{-4}+h^{-6})P+1$. With the MMSE scaling factor $\beta = \frac{2P}{2P+N_e}$ plugged in, we get $\sigma_e^2 = \beta N_e = \frac{2PN_e}{2P+N_e}$. The capacity of the mod-Λ channel between $\bar{\mathbf{w}}_f$ and $\mathbf{y}_1^{(2)}$ is

$$\begin{aligned}
&\frac{1}{n}I\left(\bar{\mathbf{w}}_f;\mathbf{y}_1^{(2)}\right)\\
&\geq \frac{1}{2}\log\left(\frac{P}{\beta N_e}\right)\\
&= \frac{1}{2}\log\left(\frac{2P+N_e}{2N_e}\right)\\
&= \frac{1}{2}\log\left(\frac{1}{2}+\frac{P}{N_e}\right)\\
&= \frac{1}{2}\log\left(\frac{1}{2}+\frac{P}{(h^{-2}+h^{-4}+h^{-6})P+1}\right)\\
&= \frac{1}{2}\log\left(\frac{1}{2}+\frac{\mathrm{SNR}}{\sigma_{wf}^2}\right)\\
&= C_{wf}
\end{aligned}$$

where $\sigma_{wf}^2 = (h^{-2}+h^{-4}+h^{-6})\mathrm{SNR}+1+h^{-2}+h^{-6}$. For reliable decoding of $\bar{\mathbf{w}}_f$ at receiver 1, we have the code rate constraint $R_{wf} = \max\{R_{wc}, R_{wd}\} \leq C_{wf}$.

In the third step, we decode $\mathbf{v}_d$ from

$$\begin{aligned}
\mathbf{y}_1^{(3)} &= [\beta h(\mathbf{y}_1 - h\mathbf{v}_c - \mathbf{w}_f) - \mathbf{d}_{vd}] \bmod \Lambda\\
&= [\beta(\mathbf{v}_d + h^{-1}\mathbf{v}_p + h^{-2}\mathbf{w}_p + h\mathbf{z}_1) - \mathbf{d}_{vd}] \bmod \Lambda\\
&= \left[\bar{\mathbf{v}}_d + \mathbf{z}_1^{(3)}\right] \bmod \Lambda
\end{aligned}$$

where

$$\mathbf{z}_1^{(3)} = (\beta-1)\mathbf{v}_d + \beta(h^{-1}\mathbf{v}_p + h^{-2}\mathbf{w}_p + h\mathbf{z}_1)$$

and the effective noise variance

$$\sigma_e^2 = \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{z}_1^{(3)}\right\|^2\right] = (\beta-1)^2 P + \beta^2 N_e$$

where $N_e = (h^{-2}+h^{-4})P + h^2$. With the MMSE scaling factor $\beta = \frac{P}{P+N_e}$ plugged in, we get $\sigma_e^2 = \beta N_e = \frac{PN_e}{P+N_e}$. The capacity of the mod-Λ channel between $\bar{\mathbf{v}}_d$ and $\mathbf{y}_1^{(3)}$ is

$$\begin{aligned}
&\frac{1}{n} I\left(\bar{\mathbf{v}}_d; \mathbf{y}_1^{(3)}\right) \\
&\geq \frac{1}{2}\log\left(\frac{P}{\beta N_e}\right) \\
&= \frac{1}{2}\log\left(\frac{P+N_e}{N_e}\right) \\
&= \frac{1}{2}\log\left(1+\frac{P}{N_e}\right) \\
&= \frac{1}{2}\log\left(1+\frac{P}{(h^{-2}+h^{-4})P+h^2}\right) \\
&= \frac{1}{2}\log\left(1+\frac{\mathrm{SNR}}{\sigma_{vd}^2}\right) \\
&= C_{vd}
\end{aligned}$$

where $\sigma_{vd}^2 = (h^{-2}+h^{-4})\mathrm{SNR} + h^2(1+h^{-2}+h^{-6})$. For reliable decoding of $\bar{\mathbf{v}}_d$ at receiver 1, we have the code rate constraint $R_{vd} \leq C_{vd}$.

In the fourth step, we decode $\mathbf{v}_p$ from

$$\begin{aligned}
&\mathbf{y}_1^{(4)} = [\beta h^2(\mathbf{y}_1 - h\mathbf{v}_c - \mathbf{w}_f - h^{-1}\mathbf{v}_d) - \mathbf{d}_{vp}] \bmod \Lambda \\
&= [\beta(\mathbf{v}_p + h^{-1}\mathbf{w}_p + h^2\mathbf{z}_1) - \mathbf{d}_{vp}] \bmod \Lambda \\
&= \left[\bar{\mathbf{v}}_p + \mathbf{z}_1^{(4)}\right] \bmod \Lambda
\end{aligned}$$

where

$$\mathbf{z}_1^{(4)} = (\beta-1)\mathbf{v}_p + \beta(h^{-1}\mathbf{w}_p + h^2\mathbf{z}_1)$$

and the effective noise variance

$$\sigma_e^2 = \frac{1}{n}\mathbb{E}\left[\left\|\mathbf{z}_1^{(4)}\right\|^2\right] = (\beta-1)^2 P + \beta^2 N_e$$

where $N_e = h^{-2}P + h^4$. With the MMSE scaling factor $\beta = \frac{P}{P+N_e}$ plugged in, we get $\sigma_e^2 = \beta N_e = \frac{PN_e}{P+N_e}$. The capacity of the mod-Λ channel between $\bar{\mathbf{v}}_p$ and $\mathbf{y}_1^{(4)}$ is

$$\begin{aligned}
&\frac{1}{n} I\left(\bar{\mathbf{v}}_p; \mathbf{y}_1^{(4)}\right)\\
&\geq \frac{1}{2}\log\left(\frac{P}{\beta N_e}\right)\\
&= \frac{1}{2}\log\left(\frac{P+N_e}{N_e}\right)\\
&= \frac{1}{2}\log\left(1+\frac{P}{N_e}\right)\\
&= \frac{1}{2}\log\left(1+\frac{P}{h^{-2}P+h^4}\right)\\
&= \frac{1}{2}\log\left(1+\frac{\mathrm{SNR}}{\sigma_{vp}^2}\right)\\
&= C_{vp}
\end{aligned}$$

where $\sigma_{vp}^2 = h^{-2}\mathrm{SNR} + h^4(1+h^{-2}+h^{-6})$. For reliable decoding of $\bar{\mathbf{v}}_p$ at receiver 1, we have the code rate constraint $R_{vp} \leq C_{vp}$.

Since received signal $\mathbf{y}_2$ has an equivalent signal structure,

$$\mathbf{y}_2 = h\mathbf{w}_c + \mathbf{v}_f + h^{-1}\mathbf{w}_d + h^{-2}\mathbf{w}_p + h^{-3}\mathbf{v}_p + \mathbf{z}_2,$$

we can see that $C_{wd} = C_{vd}$, $C_{vf} = C_{wf}$, $C_{wc} = C_{vc}$, $C_{wp} = C_{vp}$. In summary, we have the following set of code rate constraints for reliable decoding at the receivers:

$$\begin{aligned}
R_{vd} &\leq \min\{C_d, C_f\}, \\
R_{vc} &\leq \min\{C_c, C_f\}, \\
R_{wc} &\leq \min\{C_c, C_f\}, \\
R_{wd} &\leq \min\{C_d, C_f\}, \\
R_{vp} &\leq C_p, \\
R_{wp} &\leq C_p,
\end{aligned}$$

where

$$\begin{aligned}
C_c = C_{vc} = C_{wc} = \frac{1}{2}\log\left(1 + \frac{\mathrm{SNR}}{\sigma_c^2}\right), \\
C_f = C_{vf} = C_{wf} = \frac{1}{2}\log\left(\frac{1}{2} + \frac{\mathrm{SNR}}{\sigma_f^2}\right), \\
C_d = C_{vd} = C_{wd} = \frac{1}{2}\log\left(1 + \frac{\mathrm{SNR}}{\sigma_d^2}\right), \\
C_p = C_{vp} = C_{wp} = \frac{1}{2}\log\left(1 + \frac{\mathrm{SNR}}{\sigma_p^2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\sigma_c^2 = (2h^{-2} + h^{-4} + h^{-6} + h^{-8})\mathrm{SNR} + h^{-2} + h^{-4} + h^{-8}, \\
&\sigma_f^2 = (h^{-2} + h^{-4} + h^{-6})\mathrm{SNR} + 1 + h^{-2} + h^{-6}, \\
&\sigma_d^2 = (h^{-2} + h^{-4})\mathrm{SNR} + h^2 + 1 + h^{-4}, \\
&\sigma_p^2 = h^{-2}\mathrm{SNR} + h^4 + h^2 + h^{-2}.
\end{aligned}$$

4.5.6 Constant Gap for $\mathcal{B}_1$ and $\mathcal{B}_6$

As pointed out in [9], in these regimes, the capacity of the symmetric Gaussian X channel is not significantly different from the capacity of the symmetric two-user Gaussian interference channel, and the GDOF is identical for the two channels. In these regimes, ETW upper bound is tight, and IAN achieves the sum-rate capacity to within one bit. Note that in near the boundary between $\mathcal{B}_1$ and $\mathcal{B}_2$ and the boundary between $\mathcal{B}_5$ and $\mathcal{B}_6$, either TDM or timesharing between IAN and TDM slightly outperforms IAN, especially for lower SNR.

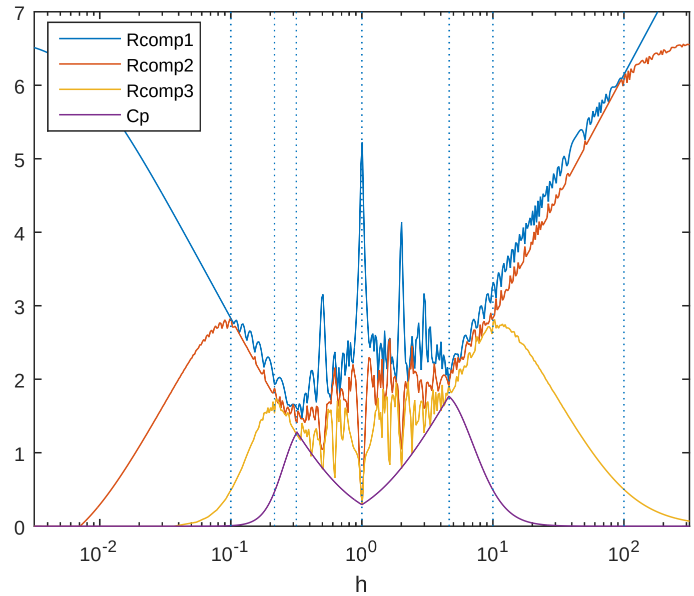
4.5.7 Limitation of Successive Decoding in $\mathcal{B}_3$ and $\mathcal{B}_4$

For $h \leq 1$, the number of layers above noise level is $L = \lceil \frac{1}{1-\alpha} \rceil$. Thus, given SNR, L grows unbounded as h approaches 1. This motivates us to apply compute-and-forward decoding in these regimes while keeping the number of layers to be small.

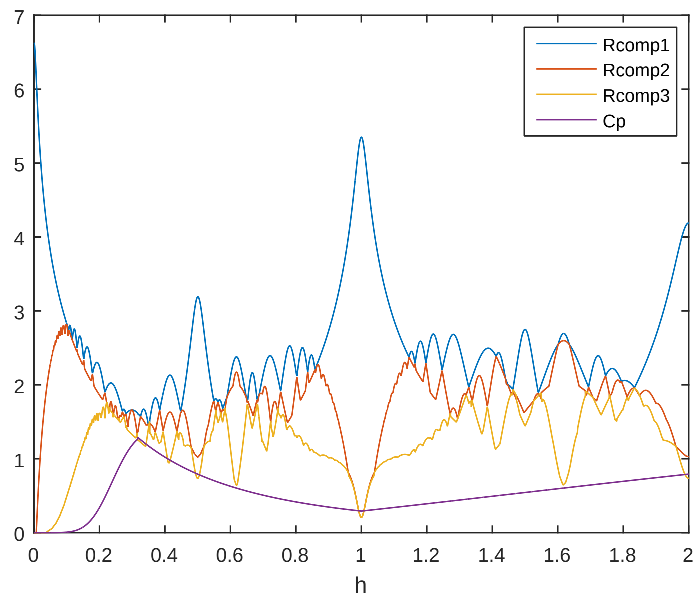
4.6 Compute-and-forward decoding

Although applicable to any regime, compute-and-forward decoding is useful for regimes $\mathcal{B}_3$ and $\mathcal{B}_4$ where h is relatively close to 1. At each receiver, we first decode three integer linear combinations of lattice codewords with linearly independent coefficient vectors. We refer to lattice equations as a set of integer linear combinations of lattice codewords. Upon successful decoding, we can solve the lattice equations for individual codewords: two desired codewords and one aligned interference codeword.

We start by explaining compute-and-forward decoding for $\mathcal{B}_3$. Decod-



(a) Log scale



(b) Linear scale

Figure 4.9: Computaion rates at SNR = 40 dB.

ing procedures for other regimes are similar. The transmit signals are

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{v}_d + h\mathbf{w}_c + \frac{1}{\sqrt{h^2P}}\mathbf{v}_p, \\
\mathbf{x}_2 &= \mathbf{w}_d + h\mathbf{v}_c + \frac{1}{\sqrt{h^2P}}\mathbf{w}_p,
\end{aligned} \tag{4.36}$$

with transmit power $\frac{1}{n}\|\mathbf{x}_j\|^2 = (1+h^2)P + h^{-2} = \text{SNR}$. The received signals are

$$\begin{aligned}
\mathbf{y}_1 &= \mathbf{v}_d + h\mathbf{w}_f + h^2\mathbf{v}_c + \frac{1}{\sqrt{h^2P}}\mathbf{v}_p + \frac{1}{\sqrt{P}}\mathbf{w}_p + \mathbf{z}_1, \\
\mathbf{y}_2 &= \mathbf{w}_d + h\mathbf{v}_f + h^2\mathbf{w}_c + \frac{1}{\sqrt{h^2P}}\mathbf{w}_p + \frac{1}{\sqrt{P}}\mathbf{v}_p + \mathbf{z}_2.
\end{aligned}$$

At receiver 1, decoding is performed in the following two steps:

- Decode $p_d\bar{\mathbf{v}}_d + p_f\bar{\mathbf{w}}_f + p_c\bar{\mathbf{v}}_c$ three times with linearly independent coefficient vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ while treating

  $$\mathbf{z}_1' = \frac{1}{\sqrt{h^2P}}\mathbf{v}_p + \frac{1}{\sqrt{P}}\mathbf{w}_p + \mathbf{z}_1$$

  as noise. Solve lattice equations to recover individual codewords: $\bar{\mathbf{v}}_d$, $\bar{\mathbf{w}}_f$, $\bar{\mathbf{v}}_c$ and remove the effect of $\mathbf{v}_d + h\mathbf{w}_f + h^2\mathbf{v}_c$ from $\mathbf{y}_1$.

- Decode $\mathbf{v}_p$ from $\sqrt{h^2P}\mathbf{z}_1' = \mathbf{v}_p + h\mathbf{w}_p + \sqrt{h^2P}\mathbf{z}_1$ while treating the other signal as noise. It is straightforward to show that

  $$R_p = \frac{1}{2}\log\left(1 + \frac{P}{2h^2P}\right) = \frac{1}{2}\log\left(1 + \frac{1}{2h^2}\right)$$

  is achievable.

In the first step, after normalized by the noise variance $\frac{1}{n}\mathbb{E}[\|\mathbf{z}_1'\|^2] = h^{-2} + 2$, the equivalent channel can be expressed as

$$\mathbf{y}_1^{(1)} = \frac{\mathbf{y}_1}{\sqrt{h^{-2}+2}} = \mathbf{h}^T \begin{bmatrix} \mathbf{v}_d \\ \mathbf{w}_f \\ \mathbf{v}_c \end{bmatrix} + \mathbf{z}_1^{(1)}$$

where the effective channel vector

$$\mathbf{h} = \frac{1}{\sqrt{h^{-2}+2}}[1 \ h \ h^2]^T$$

and $\frac{1}{n}\mathbb{E}[\|\mathbf{z}_1^{(1)}\|^2] = 1$. After linear scaling, dither removal, and mod-Λ operation, we get

$$\begin{aligned} & [\beta \mathbf{y}_1' - p_d \mathbf{d}_{vd} - p_f(\mathbf{d}_{wc} + \mathbf{d}_{wd}) - p_c \mathbf{d}_{vc}] \bmod \Lambda \\ & = [p_d \bar{\mathbf{v}}_d + p_f \bar{\mathbf{w}}_f + p_c \bar{\mathbf{v}}_c + \mathbf{z}_{1e}] \bmod \Lambda \end{aligned} \tag{4.37}$$

where the effective noise is

$$\begin{aligned} \mathbf{z}_{1e} &= \left(\frac{\beta}{\sqrt{h^{-2}+2}} - p_d \right) \mathbf{v}_d + \left(\frac{\beta h}{\sqrt{h^{-2}+2}} - p_f \right) \mathbf{w}_f \\ & \quad + \left(\frac{\beta h^2}{\sqrt{h^{-2}+2}} - p_c \right) \mathbf{v}_c + \beta \mathbf{z}_1^{(1)}. \end{aligned}$$

The effective noise variance is given by

$$\sigma_e^2(\mathbf{h}, \mathbf{M}, \mathbf{p}, \beta) = \frac{1}{n}\mathbb{E}[\|\mathbf{z}_{1e}\|^2] = \left\| \mathbf{M}\left(\beta \mathbf{h} - \mathbf{p}\right) \right\|^2 P + \beta^2$$

where $\mathbf{p} = [p_d \ p_f \ p_c]^T$, $\mathbf{M} = \mathrm{diag}(1, \sqrt{2}, 1)$, and the MMSE scaling parameter $\beta = \frac{P\mathbf{h}^T\mathbf{M}^2\mathbf{p}}{1+P\mathbf{h}^T\mathbf{M}^2\mathbf{h}}$. With the optimal β plugged in, we get

$$\begin{aligned} \sigma_e^2(\mathbf{h}, \mathbf{M}, \mathbf{p}) &= P \left(\mathbf{p}^T \mathbf{M}^2 \mathbf{p} - \frac{P(\mathbf{h}^T\mathbf{M}^2\mathbf{p})^2}{1 + P\mathbf{h}^T\mathbf{M}^2\mathbf{h}} \right) \\ &= \mathbf{p}^T \left(P^{-1}\mathbf{M}^{-2} + \mathbf{h}\mathbf{h}^T \right)^{-1} \mathbf{p} \end{aligned}$$

and the computation rate [7, 16] is defined by

$$R_{comp}(\mathbf{h}, \mathbf{M}, \mathbf{p}) = \frac{1}{2} \log^+ \left(\frac{P}{\sigma_e^2(\mathbf{h}, \mathbf{M}, \mathbf{p})} \right) \tag{4.38}$$

where $\log^+(x) = \max\{x, 0\}$. The optimization problem

$$\min_{\mathbf{p} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}} \sigma_e^2(\mathbf{h}, \mathbf{M}, \mathbf{p})$$

	P	$\mathbf{h}$	R_p
$\mathcal{B}_1 \cup \mathcal{B}_2$	$\frac{\text{SNR}}{1+h^2+h^6}$	$\frac{1}{\sqrt{h^6P+h^8P+1}}[1\ h\ h^2]^T$	$\frac{1}{2}\log\left(1+\frac{h^6P}{h^8P+1}\right)$
$\mathcal{B}_3$	$\frac{\text{SNR}-h^{-2}}{1+h^2}$	$\frac{1}{\sqrt{h^{-2}+2}}[1\ h\ h^2]^T$	$\frac{1}{2}\log\left(1+\frac{1}{2h^2}\right)$
$\mathcal{B}_4$	$\frac{\text{SNR}-1}{1+h^{-2}}$	$\frac{1}{\sqrt{h^2+2}}[h\ 1\ h^{-1}]^T$	$\frac{1}{2}\log\left(1+\frac{h^2}{2}\right)$
$\mathcal{B}_5 \cup \mathcal{B}_6$	$\frac{\text{SNR}}{1+h^{-2}+h^{-2}}$	$\frac{1}{\sqrt{h^{-4}P+h^{-6}P+1}}[h\ 1\ h^{-1}]^T$	$\frac{1}{2}\log\left(1+\frac{h^{-4}P}{h^{-6}P+1}\right)$

Table 4.3: The effective channel vectors for compute-and-forward

is equivalent to a shortest lattice vector (SLV) problem and can be solved by using well-known algorithms such as the LLL algorithm. See also [18] for recent results.

The *optimal* integer vectors **a**, **b**, **c** are such that they are non-zero, linearly independent, and

$$R_{comp,1} \geq R_{comp,2} \geq R_{comp,3}$$

where

$$R_{comp,1} = R_{comp}(\mathbf{h}, \mathbf{M}, \mathbf{a}),$$

$$R_{comp,2} = R_{comp}(\mathbf{h}, \mathbf{M}, \mathbf{b}),$$

$$R_{comp,3} = R_{comp}(\mathbf{h}, \mathbf{M}, \mathbf{c}),$$

are the highest computation rates. We can always find such vectors, and they are not unique.

The expressions of R_p and the parameters P and $\mathbf{h}$ to calculate computation rates $R_{comp,j}$ vary in different regimes and are given in Table 4.3. $\mathbf{M}$ is the same for every regime. Fig. 4.9 shows the computation rates as well as R_p at SNR $= 10^4$.

In [7], based on Minkowski's successive minima theorem, it was shown that the sum of K highest computation rates for the *effective* K-user MAC is within constant gap from the sum-rate capacity, i.e.,

$$\sum_{k=1}^{K} R_{comp,k} \;\; \geq \;\; \frac{1}{2}\log(1+\|\mathbf{h}\|^2 P) - \frac{1}{2}\log(\det \mathbf{M}^2) - \frac{K}{2}\log(K) \quad (4.39)$$

where $\mathbf{h}$, $\mathbf{M}$, P, K depend on the lattice alignment scenario as well as the underlying physical channel.

4.6.1 Compute-and-forward achievable sum-rate

Let us denote the integer matrix by $\mathbf{A} = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^T$ or

$$\mathbf{A} = \begin{bmatrix} a_d & a_f & a_c \\ b_d & b_f & b_c \\ c_d & c_f & c_c \end{bmatrix}.$$

Since integer vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are linearly independent, $\mathbf{A}$ is full rank. Although we can always find a full rank $\mathbf{A}$, it is not guaranteed that all three computation rates are strictly positive. Due to symmetry of the channel, two receivers observe the same channel vector:

$$\mathbf{y}_1^{(1)} = \mathbf{h}^T \begin{bmatrix} \mathbf{v}_d \\ \mathbf{w}_f \\ \mathbf{v}_c \end{bmatrix} + \mathbf{z}_1^{(1)}, \quad \mathbf{y}_2^{(1)} = \mathbf{h}^T \begin{bmatrix} \mathbf{w}_d \\ \mathbf{v}_f \\ \mathbf{w}_c \end{bmatrix} + \mathbf{z}_2^{(1)}. \qquad (4.40)$$

Therefore, the optimal integer vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ at receiver 1 are the same as those at receiver 2. We use *algebraic successive cancellation* (ASC) [7] with the following two different cancellation orders:

- ASC order I: $\bar{\mathbf{v}}_d \to \bar{\mathbf{w}}_f$ at receiver 1 and $\bar{\mathbf{w}}_d \to \bar{\mathbf{v}}_f$ at receiver 2. We get the effective coefficient matrices

$$\mathbf{A}_1 = \begin{bmatrix} a_d & a_f & a_c \\ 0 & b'_f & b'_c \\ 0 & 0 & c''_c \end{bmatrix}, \; \mathbf{A}_2 = \begin{bmatrix} a_d & a_f & a_c \\ 0 & b'_f & b'_c \\ 0 & 0 & c''_c \end{bmatrix} \qquad (4.41)$$

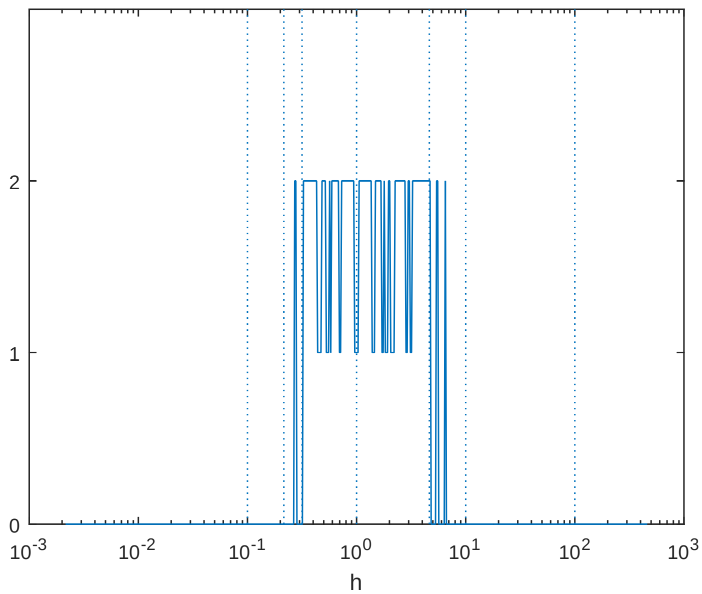


Figure 4.10: ASC feasibility pattern.

at receiver 1 and 2, respectively. The matrices have zeros for the canceled variables.

- ASC order II: $\bar{\mathbf{v}}_d \to \bar{\mathbf{v}}_c$ at receiver 1 and $\bar{\mathbf{v}}_f \to \bar{\mathbf{w}}_c$ at receiver 2. The resulting coefficient matrices are

$$\mathbf{A}_1 = \begin{bmatrix} a_d & a_f & a_c \\ 0 & b'_f & b'_c \\ 0 & c''_f & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} a_d & a_f & a_c \\ b'_d & 0 & b'_c \\ c''_d & 0 & 0 \end{bmatrix}. \tag{4.42}$$

Depending on h and SNR, the ASC orders can be feasible or infeasible. For each ASC order to be feasible, the matrix $\mathbf{A}$ must satisfy a set of conditions. We state the feasibility conditions in the following lemma.

Lemma 4.4. *(ASC feasibility conditions)*

- *ASC order I is feasible if* $a_d, a_f, a_c \neq 0$ *and* $a_d b_f \neq a_f b_d$.

- *ASC order II is feasible if* $a_d, a_f, a_c \neq 0$, $a_d b_c \neq a_c b_d$, *and* $a_f b_c \neq a_c b_f$.

Proof. ASC order I: If the conditions are satisfied, $\mathbf{A}$ can be pseudo-triangularized (up to column permutation) in the following way.

$$
\begin{aligned}
\mathbf{A} \quad \rightarrow \quad & \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}'^T \\ \mathbf{c}'^T \end{bmatrix} = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T - \frac{b_d}{a_d}\mathbf{a}^T \\ \mathbf{c}^T - \frac{c_d}{a_d}\mathbf{a}^T \end{bmatrix} = \begin{bmatrix} a_d & a_f & a_c \\ 0 & b_f - \frac{b_d}{a_d}a_f & b_c - \frac{b_d}{a_d}a_c \\ 0 & c_f - \frac{c_d}{a_d}a_f & c_c - \frac{c_d}{a_d}a_c \end{bmatrix} \\
\rightarrow \quad & \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}'^T \\ \mathbf{c}''^T \end{bmatrix} = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T - \frac{b_d}{a_d}\mathbf{a}^T \\ \mathbf{c}'^T - \frac{c_f-(c_d/a_d)a_f}{b_f-(b_d/a_d)a_f}\mathbf{b}'^T \end{bmatrix} \\
= \; & \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T - \frac{b_d}{a_d}\mathbf{a}^T \\ \mathbf{c}^T - \frac{c_d}{a_d}\mathbf{a}^T - \frac{c_f-(c_d/a_d)a_f}{b_f-(b_d/a_d)a_f}(\mathbf{b}^T - \frac{b_d}{a_d}\mathbf{a}^T) \end{bmatrix} \\
= \; & \begin{bmatrix} a_d & a_f & a_c \\ 0 & b'_f & b'_c \\ 0 & 0 & c''_c \end{bmatrix} = \mathbf{A}_1 = \mathbf{A}_2.
\end{aligned}
$$

ASC order II: If the conditions are satisfied, $\mathbf{A}$ can be pseudo-triangularized in the following two distinct forms:

$$
\begin{aligned}
\mathbf{A} \;\rightarrow\; & \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}'^T \\ \mathbf{c}'^T \end{bmatrix} = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T - \frac{b_d}{a_d}\mathbf{a}^T \\ \mathbf{c}^T - \frac{c_d}{a_d}\mathbf{a}^T \end{bmatrix} = \begin{bmatrix} a_d & a_f & a_c \\ 0 & b_f - \frac{b_d}{a_d}a_f & b_c - \frac{b_d}{a_d}a_c \\ 0 & c_f - \frac{c_d}{a_d}a_f & c_c - \frac{c_d}{a_d}a_c \end{bmatrix} \\
\rightarrow\; & \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}'^T \\ \mathbf{c}''^T \end{bmatrix} = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T - \frac{b_d}{a_d}\mathbf{a}^T \\ \mathbf{c}'^T - \frac{c_c-(c_d/a_d)a_c}{b_c-(b_d/a_d)a_c}\mathbf{b}'^T \end{bmatrix} \\
= & \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T - \frac{b_d}{a_d}\mathbf{a}^T \\ \mathbf{c}^T - \frac{c_d}{a_d}\mathbf{a}^T - \frac{c_c-(c_d/a_d)a_c}{b_c-(b_d/a_d)a_c}(\mathbf{b}^T - \frac{b_d}{a_d}\mathbf{a}^T) \end{bmatrix} \\
= & \begin{bmatrix} a_d & a_f & a_c \\ 0 & b'_f & b'_c \\ 0 & c''_f & 0 \end{bmatrix} = \mathbf{A}_1,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{A} \;\rightarrow\; & \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}'^T \\ \mathbf{c}'^T \end{bmatrix} = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T - \frac{b_f}{a_f}\mathbf{a}^T \\ \mathbf{c}^T - \frac{c_f}{a_f}\mathbf{a}^T \end{bmatrix} = \begin{bmatrix} a_d & a_f & a_c \\ b_d - \frac{b_f}{a_f}a_d & 0 & b_c - \frac{b_f}{a_f}a_c \\ c_d - \frac{c_f}{a_f}a_d & 0 & c_c - \frac{c_f}{a_f}a_c \end{bmatrix} \\
\rightarrow\; & \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}'^T \\ \mathbf{c}''^T \end{bmatrix} = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T - \frac{b_f}{a_f}\mathbf{a}^T \\ \mathbf{c}'^T - \frac{c_c-(c_f/a_f)a_c}{b_c-(b_f/a_f)a_c}\mathbf{b}'^T \end{bmatrix} \\
= & \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T - \frac{b_f}{a_f}\mathbf{a}^T \\ \mathbf{c}^T - \frac{c_f}{a_f}\mathbf{a}^T - \frac{c_c-(c_f/a_f)a_c}{b_c-(b_f/a_f)a_c}(\mathbf{b}^T - \frac{b_f}{a_f}\mathbf{a}^T) \end{bmatrix} \\
= & \begin{bmatrix} a_d & a_f & a_c \\ b'_d & 0 & b'_c \\ c''_d & 0 & 0 \end{bmatrix} = \mathbf{A}_2.
\end{aligned}
\tag{4.43}
$$

□

Depending on h and the resulting matrix A, we can achieve different combinations of computation rates. Fig. 4.10 shows ASC feasibility pattern over h. Based on compute-and-forward decoding and ASC feasibility, we state the following achievability.

Theorem 4.5 (Compute-and-forward achievable sum-rate)**.** *The theorem is stated in three parts:*

- *If* $\mathbf{A}$ *satisfy ASC order I feasibility condition,*

$$R_{comp,2323} = 2R_{comp,2} + 2R_{comp,3} + 2R_p$$

  is achievable.

- *If* $\mathbf{A}$ *satisfy ASC order II feasibility condition,*

$$R_{comp,1233} = R_{comp,1} + R_{comp,2} + 2R_{comp,3} + 2R_p$$

  is achievable.

- *For the other cases,*

$$R_{comp,3333} = 4R_{comp,3} + 2R_p$$

  is achievable.

The expressions of R_p *and the parameters* P *and* $\mathbf{h}$ *for computation rates are given in Table 4.3.*

Proof. If ASC order I is feasible, the code rate constraints are

$$\begin{aligned}
R_{vd} &\leq \min\{R_{comp,1}, R_{comp,2}\} = R_{comp,2}, \\
R_{vc} &\leq \min\{R_{comp,1}, R_{comp,2}, R_{comp,3}\} = R_{comp,3} \\
R_{wc} &\leq \min\{R_{comp,1}, R_{comp,2}, R_{comp,3}\} = R_{comp,3} \\
R_{wd} &\leq \min\{R_{comp,1}, R_{comp,2}\} = R_{comp,2}
\end{aligned}$$

since

- $\bar{\mathbf{v}}_d$ is involved in equation **a** at receiver 1 and in equations **a**, **b** at receiver 2 via $\bar{\mathbf{v}}_f$,
- $\bar{\mathbf{v}}_c$ is involved in equations **a**, **b**, **c** at receiver 1 and in equations **a**, **b** at receiver 2 via $\bar{\mathbf{v}}_f$,
- $\bar{\mathbf{w}}_c$ is involved in equations **a**, **b** at receiver 1 via $\bar{\mathbf{w}}_f$ and in equations **a**, **b** at receiver 2,
- $\bar{\mathbf{w}}_d$ is involved in equations **a**, **b** at receiver 1 via $\bar{\mathbf{w}}_f$ and in equations **a** at receiver 2.

Thus, the sum-rate

$$\begin{aligned} R_{LLC} &= R_{vd} + R_{vc} + R_{vp} + R_{wd} + R_{wc} + R_{wp} \\ &\leq 2R_{comp,2} + 2R_{comp,3} + 2R_p \end{aligned}$$

is achievable.

If ASC order II is feasible, the code rate constraints are

$$\begin{aligned} R_{vd} &\leq R_{comp,1}, \\ R_{vc} &\leq \min\{R_{comp,1}, R_{comp,2}\} = R_{comp,2} \\ R_{wc} &\leq \min\{R_{comp,1}, R_{comp,2}, R_{comp,3}\} = R_{comp,3} \\ R_{wd} &\leq \min\{R_{comp,1}, R_{comp,2}, R_{comp,3}\} = R_{comp,3} \end{aligned}$$

since

- $\bar{\mathbf{v}}_d$ is involved in equation **a** at receiver 1 and in equation **a** at receiver 2 via $\bar{\mathbf{v}}_f$,

- $\bar{\mathbf{v}}_c$ is involved in equations **a**, **b** at receiver 1 and in equation **a** at receiver 2 via $\bar{\mathbf{v}}_f$,
- $\bar{\mathbf{w}}_c$ is involved in equations **a**, **b**, **c** at receiver 1 via $\bar{\mathbf{w}}_f$ and in equations **a** and **b** at receiver 2,
- $\bar{\mathbf{w}}_d$ is involved in equations **a**, **b**, **c** at receiver 1 via $\bar{\mathbf{w}}_f$ and in equations **a**, **b**, **c** at receiver 2.

Thus, the sum-rate

$$\begin{aligned} R_{LLC} &= R_{vd} + R_{vc} + R_{vp} + R_{wd} + R_{wc} + R_{wp} \\ &\leq R_{comp,1} + R_{comp,2} + 2R_{comp,3} + 2R_p \end{aligned}$$

is achievable.

For the other cases, the achievability of

$$\begin{aligned} R_{LLC} &= R_{vd} + R_{vc} + R_{vp} + R_{wd} + R_{wc} + R_{wp} \\ &\leq 4R_{comp,3} + 2R_p \end{aligned}$$

is straightforward. $\square$

If we apply (4.39) to our case, we get

$$R_{comp,1} + R_{comp,2} + R_{comp,3} \geq R_{MAC,e} - \frac{c}{2} \tag{4.44}$$

where $R_{MAC,e} = \frac{1}{2}\log\left(1 + \|\mathbf{h}\|^2 P\right)$ and $c = 1 + 3\log 3$. $R_{comp,j}$ and $R_{MAC,e}$ can be calculated with the parameters P and $\mathbf{h}$ in Table 4.3.

If ASC order II is feasible, we derive the following lower bound on the sum-rate.

Lemma 4.6 (Sum of computation rates for ASC order II)**.** *The sum of computation rates $R_{comp,1233}$ is lower bounded by*

$$R_{sum,1233} \geq 2(R_{MAC,e} - R_{comp,1}) + 2R_p - c$$

where $R_{MAC,e} = \frac{1}{2}\log\left(1 + \|\mathbf{h}\|^2 P\right)$ and $c = 1 + 3\log 3$. The expressions of R_p are given in Table 4.3, and $R_{comp,1}$ and $R_{MAC,e}$ can be calculated with the parameters P and $\mathbf{h}$ in the table.

Proof. Due to (4.44),

$$R_{sum,1233} \geq R_{MAC,e} + R_{comp,3} + 2R_p - \frac{c}{2}. \tag{4.45}$$

By rearranging (4.44), we can lower bound $R_{comp,3}$ in terms of $R_{comp,1}$,

$$\begin{aligned} R_{comp,3} &\geq R_{MAC,e} - R_{comp,1} - R_{comp,2} - \frac{c}{2} \\ &\geq R_{MAC,e} - 2R_{comp,1} - \frac{c}{2}. \end{aligned} \tag{4.46}$$

By combining (4.45) and (4.46), we get the lower bound in the lemma statement. $\square$

This lemma result can be useful since it depends on $R_{comp,1}$ but not on $R_{comp,2}$ and $R_{comp,3}$.

It is obvious that $R_{comp,1}$ is in the range,

$$\frac{1}{3}\left(R_{MAC,e} - \frac{c}{2}\right) \leq R_{comp,1} \leq R_{MAC,e}$$

since $R_{comp,1} \geq R_{comp,2} \geq R_{comp,3}$, and any code sent over a MAC cannot be reliably decoded if its code rate is greater than the sum-rate capacity of the MAC. For ease of discussion, let us use the following definition of an outage event.

Definition 4.7 (Outage event)**.** *Given P and $\mathbf{M}$, an effective MAC with $\mathbf{h}$ is said to be in k-outage if*

$$R_{comp,1} = R_{comp}(\mathbf{h}, \mathbf{M}, \mathbf{a}) > \frac{1}{3} R_{MAC,e} + k,$$

for some constant k, and let $\mathcal{S}_k$ denote the k-outage set, the set of such $\mathbf{h}$ in k-outage.

If $\mathbf{h} \notin \mathcal{S}_k$, it follows that

$$\begin{aligned} R_{comp,2} + R_{comp,3} &\geq R_{MAC,e} - R_{comp,1} - \frac{c}{2} \\ &\geq \frac{2}{3} R_{MAC,e} - k - \frac{c}{2}, \end{aligned} \tag{4.47}$$

and

$$\begin{aligned} R_{sum,1233} &\geq 2(R_{MAC,e} - R_{comp,1}) + 2R_p - c \\ &\geq \frac{4}{3} R_{MAC,e} + 2R_p - 2k - c, \end{aligned} \tag{4.48}$$

and

$$R_{sum,2323} \geq \frac{4}{3} R_{MAC,e} + 2R_p - 2k - c. \tag{4.49}$$

4.6.2 Channel steering

Channel steering is a method to reduce the sensitivity of computation rates to the variation of h. In the following explanation, for simplicity, we do not consider the signals $\mathbf{v}_p$ and $\mathbf{w}_p$ but only focus on the messages $\mathbf{v}_d, \mathbf{v}_c, \mathbf{w}_c, \mathbf{w}_d$ that are involved in compute-and-forward decoding. For $\mathcal{B}_3$, the transmitters send the signals,

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{v}_d + gh\mathbf{w}_c, \\ \mathbf{x}_2 &= g\mathbf{w}_d + h\mathbf{v}_c, \end{aligned} \tag{4.50}$$

	Transmit signals $\mathbf{x}_j$
$\mathcal{B}_1 \cup \mathcal{B}_2$	$\mathbf{x}_1 = \mathbf{v}_d + gh\mathbf{w}_c + h^3\mathbf{v}_p$ $\mathbf{x}_2 = g\mathbf{w}_d + h\mathbf{v}_c + h^3\mathbf{w}_p$
$\mathcal{B}_3$	$\mathbf{x}_1 = \mathbf{v}_d + gh\mathbf{w}_c + (h^2P)^{-\frac{1}{2}}\mathbf{v}_p$ $\mathbf{x}_2 = g\mathbf{w}_d + h\mathbf{v}_c + (h^2P)^{-\frac{1}{2}}\mathbf{w}_p$
$\mathcal{B}_4$	$\mathbf{x}_1 = \mathbf{w}_c + g^{-1}h^{-1}\mathbf{v}_d + P^{-\frac{1}{2}}\mathbf{w}_p$ $\mathbf{x}_2 = g^{-1}\mathbf{v}_c + h^{-1}\mathbf{w}_d + P^{-\frac{1}{2}}\mathbf{v}_p$
$\mathcal{B}_5 \cup \mathcal{B}_6$	$\mathbf{x}_1 = \mathbf{w}_c + g^{-1}h^{-1}\mathbf{v}_d + h^{-3}\mathbf{w}_p$ $\mathbf{x}_2 = g^{-1}\mathbf{v}_c + h^{-1}\mathbf{w}_d + h^{-3}\mathbf{v}_p$

Table 4.4: Transmit signals.

	Received signals $\bar{\mathbf{y}}_j = \mathbf{y}_j - \mathbf{z}_j$
$\mathcal{B}_1 \cup \mathcal{B}_2$	$\bar{\mathbf{y}}_1 = \mathbf{v}_d + gh\mathbf{w}_f + h^2\mathbf{v}_c + h^3\mathbf{v}_p + h^4\mathbf{w}_p$ $\bar{\mathbf{y}}_2 = g\mathbf{w}_d + h\mathbf{v}_f + gh^2\mathbf{w}_c + h^3\mathbf{w}_p + h^4\mathbf{v}_p$
$\mathcal{B}_3$	$\bar{\mathbf{y}}_1 = \mathbf{v}_d + gh\mathbf{w}_f + h^2\mathbf{v}_c + (h^2P)^{-\frac{1}{2}}\mathbf{v}_p + P^{-\frac{1}{2}}\mathbf{w}_p$ $\bar{\mathbf{y}}_2 = g\mathbf{w}_d + h\mathbf{v}_f + gh^2\mathbf{w}_c + (h^2P)^{-\frac{1}{2}}\mathbf{w}_p + P^{-\frac{1}{2}}\mathbf{v}_p$
$\mathcal{B}_4$	$\bar{\mathbf{y}}_1 = g^{-1}h\mathbf{v}_c + \mathbf{w}_f + g^{-1}h^{-1}\mathbf{v}_d + hP^{-\frac{1}{2}}\mathbf{v}_p + P^{-\frac{1}{2}}\mathbf{w}_p$ $\bar{\mathbf{y}}_2 = h\mathbf{w}_c + g^{-1}\mathbf{v}_f + h^{-1}\mathbf{w}_d + hP^{-\frac{1}{2}}\mathbf{w}_p + P^{-\frac{1}{2}}\mathbf{v}_p$
$\mathcal{B}_5 \cup \mathcal{B}_6$	$\bar{\mathbf{y}}_1 = g^{-1}h\mathbf{v}_c + \mathbf{w}_f + g^{-1}h^{-1}\mathbf{v}_d + h^{-2}\mathbf{v}_p + h^{-3}\mathbf{w}_p$ $\bar{\mathbf{y}}_2 = h\mathbf{w}_c + g^{-1}\mathbf{v}_f + h^{-1}\mathbf{w}_d + h^{-2}\mathbf{w}_p + h^{-3}\mathbf{v}_p$

Table 4.5: Received signals.

	P_1	P_2
$\mathcal{B}_1 \cup \mathcal{B}_2$	$\frac{\mathrm{SNR}}{1+g^2h^2+h^6}$	$\frac{\mathrm{SNR}}{g^2+h^2+h^6}$
$\mathcal{B}_3$	$\frac{\mathrm{SNR}-h^{-2}}{1+g^2h^2}$	$\frac{\mathrm{SNR}-h^{-2}}{g^2+h^2}$
$\mathcal{B}_4$	$\frac{\mathrm{SNR}-1}{1+g^{-2}h^{-2}}$	$\frac{\mathrm{SNR}-1}{g^{-2}+h^{-2}}$
$\mathcal{B}_5 \cup \mathcal{B}_6$	$\frac{\mathrm{SNR}}{1+g^{-2}h^{-2}+h^{-6}}$	$\frac{\mathrm{SNR}}{g^{-2}+h^{-2}+h^{-6}}$

Table 4.6: Signal power and the effective channel vectors

	$\mathbf{h}_1$	$\mathbf{h}_2$
$\mathcal{B}_1 \cup \mathcal{B}_2$	$\frac{1}{\sqrt{h^6P+h^8P+1}}[1\ gh\ h^2]^T$	$\frac{1}{\sqrt{h^6P+h^8P+1}}[g\ h\ gh^2]^T$
$\mathcal{B}_3$	$\frac{1}{\sqrt{h^{-2}+2}}[1\ gh\ h^2]^T$	$\frac{1}{\sqrt{h^{-2}+2}}[g\ h\ gh^2]^T$
$\mathcal{B}_4$	$\frac{1}{\sqrt{h^2+2}}[g^{-1}h\ 1\ g^{-1}h^{-1}]^T$	$\frac{1}{\sqrt{h^2+2}}[h\ g^{-1}\ h^{-1}]^T$
$\mathcal{B}_5 \cup \mathcal{B}_6$	$\frac{1}{\sqrt{h^{-4}P+h^{-6}P+1}}[g^{-1}h\ 1\ g^{-1}h^{-1}]^T$	$\frac{1}{\sqrt{h^{-4}P+h^{-6}P+1}}[h\ g^{-1}\ h^{-1}]^T$

Table 4.7: Signal power and the effective channel vectors

with the shaping lattice Λ with $\sigma^2(\Lambda) = \min\{P_1, P_2\}$, and the transmit power $(1+g^2h^2)P_1 = \text{SNR}$ and $(g^2+h^2)P_2 = \text{SNR}$, respectively. The received signals are

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{v}_d + gh\mathbf{w}_f + h^2\mathbf{v}_c + \mathbf{z}_1, \\ \mathbf{y}_2 &= g\mathbf{w}_d + h\mathbf{v}_f + gh^2\mathbf{w}_c + \mathbf{z}_2. \end{aligned} \tag{4.51}$$

Roughly speaking, the sum-rate of the four messages, $R_{sum} = R_{vd} + R_{vc} + R_{wc} + R_{wd}$ becomes close to $\frac{4}{3}R_{MAC,e}$ when

$$R_{comp,1} \approx R_{comp,2} \approx R_{comp,3} \approx \frac{1}{3}R_{MAC,e}.$$

Channel steering helps achieve this as close as possible by introducing asymmetry between effective channel vectors that each receiver observes. Since the receivers observe slightly different channel vectors $\mathbf{h}_1 = [1\ gh\ h^2]^T$ and $\mathbf{h}_2 = [g\ h\ gh^2]^T$, their best three integer coefficient vectors may become different: $\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1$ for receiver 1, and $\mathbf{a}_2, \mathbf{b}_2, \mathbf{c}_2$ for receiver 2. If ASC order II is feasible, the code rates have to satisfy

$$\begin{aligned} R_{vd} &\le \min\{R_{comp}(\mathbf{h}_1, \mathbf{M}, \mathbf{a}_1), R_{comp}(\mathbf{h}_2, \mathbf{M}, \mathbf{a}_2)\} \\ R_{vc} &\le \min\{R_{comp}(\mathbf{h}_1, \mathbf{M}, \mathbf{b}_1), R_{comp}(\mathbf{h}_2, \mathbf{M}, \mathbf{a}_2)\} \\ R_{wc} &\le \min\{R_{comp}(\mathbf{h}_1, \mathbf{M}, \mathbf{c}_1), R_{comp}(\mathbf{h}_2, \mathbf{M}, \mathbf{b}_2))\} \\ R_{wd} &\le \min\{R_{comp}(\mathbf{h}_1, \mathbf{M}, \mathbf{c}_1), R_{comp}(\mathbf{h}_2, \mathbf{M}, \mathbf{c}_2))\} \end{aligned}$$

The achievable sum-rate can be optimized over $g > 0$, i.e.,

$$R_{sum} = \max_{g}(R_{vd} + R_{vc} + R_{wc} + R_{wd}). \tag{4.52}$$

Note that we can optimize over the set of g that results in $\mathbf{A}$ satisfying the ASC order feasibility condition. Fig. 4.3 shows the numerical evaluations of lower and upper bounds with and without channel steering.

Chapter 5

Conclusions

We presented approximate capacity region of some important special cases of partially connected interference channels. The outer bounds based on Z-channel type argument are derived. Achievable schemes are developed and shown to approximately achieve the capacity to within a constant bit. For future work, the channels with fully general coefficients may be considered. In this dissertation, we presented different schemes for each channel type although they share some principle. A universal scheme is to be developed for unified capacity characterization of all possible topologies. The connection between interference channel and index coding problems is much to explore.

We also developed achievable sum rate expressions for the Gaussian X-channel at finite SNR using layered lattice coding with interference alignment. For different regimes of channel parameter h, different decoding strategies including successive decoding and compute-and-forward decoding were used. For some regimes of h, we characterized the sum-rate capacity to within constant bits by using successive decoding. For a set of h that satisfy certain feasibility conditions, we showed that compute-and-forward decoding outperforms a timesharing MAC-based lower bound. The systematic methods for channel steering to reduce the sensitivity of achievable rates to channel gains are to be studied in the future.

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Vita

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