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Performance Evaluation of Music and Minimum Norm Eigenvector Algorithms in Resolving Noisy Multiexponential Signals

Abdussamad U. Jibia, and Momoh-Jimoh E. Salami

Abstract—Eigenvector methods are gaining increasing acceptance in the area of spectrum estimation. This paper presents a successful attempt at testing and evaluating the performance of two of the most popular types of subspace techniques in determining the parameters of multiexponential signals with real decay constants buried in noise. In particular, MUSIC (Multiple Signal Classification) and minimum-norm techniques are examined. It is shown that these methods perform almost equally well on multiexponential signals with MUSIC displaying better defined peaks.

Keywords—Eigenvector, minimum norm, multiexponential, subspace.

I. INTRODUCTION

THE task of analyzing multiexponential signals with real decay constants has been the subject of many research efforts for more than five decades. This is because, straightforward as their analysis may appear, such signals do not form an orthogonal base. The signals are usually of the form:

$$S(\tau) = \sum_{k=1}^{M} A_k e^{-\lambda_k \tau} + n(\tau) \tag{1}$$

The unknown parameters are usually the number of components (M), the amplitudes (A_k) and the decay rates (λ_k). $n(\tau)$ represents the noise, considered to be white. The facts that these signals occur in many natural and artificial phenomena makes their analysis even important. Some of the areas they occur include: temperature modulation of Metal-oxide semiconductor (MOS) sensors [6], fluorescence decay analysis [8], Electromagnetic field analysis [7], nuclear magnetic resonance [3], transient spectroscopy [5], compartment analysis in physiology [4], etc.

Several methods have been documented for the analysis of

this class of signals. These methods range from theoretical exercises [9] to more practical efforts [1], [2], and [8]. In this paper, an attempt has been made to compare the performances of two eigenvector methods in the analysis of multiexponential signals buried in noise. Subspace methods were primarily developed for the analysis of complex exponentials buried in noise and do not therefore fit to our analysis in the first place. To make it suitable for analysis by eigenvector methods, multiexponential signal has been subjected to a preprocessing procedure that involves Gardeners' transformation followed by a deconvolution stage. The Gardeners' transformation transforms the signal into a convolution model which when deconvolved results in a sum of complex exponentials in noise assumed to be white. The subspace methods are then applied to the deconvolved data in turn.

The single most important property of eigenvector (also called subspace) methods is that they produce unbiased estimates with infinite resolution regardless of signal-to-noise ratios [14]. The methods considered in this paper are the Multiple Signal Classification (MUSIC) and the minimum-norm methods.

II. SIGNAL PREPROCESSING

The main objective of the signal preprocessing is to obtain an orthogonal representation of exponential data by applying Gardners' transformation to (1) followed by Fourier processing.

Initially, (1) is expressed as

$$S(\tau) = \sum_{k=1}^{M} A_k p(\lambda_k \tau) + n(\tau), \quad 0 < \tau < \infty \tag{2}$$

where the basis function $p(\tau) = \exp(-\tau)$. This equation can be rewritten as

$$S(\tau) = \int_{0}^{\infty} g(\lambda) p(\lambda \tau) d\lambda + n(\tau) \tag{3}$$

where

$$g(\lambda) = \sum_{k=1}^{M} A_k \delta(\lambda - \lambda_k) \tag{4}$$

and it contains all the parameters to be determined.

Multiplying both sides of (3) by τ^{α} and applying the

Gardner transformation, $\tau = e^t$ and $\lambda = e^{-r}$ results in a convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda + v(t)$$

(5)

where

$$y(t) = \exp(\alpha t)S\{\exp(t)\},$$

$$x(t) = \exp\{(\alpha-1)t\}g(e^{-t})\},$$

$$h(t) = \exp(\alpha t)p(e^t), and$$

$$v(t) = \exp(\alpha t)n(e^t)$$

Taking the Fourier transform of (5) and performing inverse filtering followed by inverse Fourier transformation yields

$$x(t) = \sum_{k=1}^{M} B_k \delta(t + \ln \lambda_k)$$

(6)

where $B_k = A_k(\lambda_k)^{-\alpha}$. Some of the drawbacks of this method have been highlighted in [18].

A discrete form of (5) is obtained by sampling $y(t)$ at a rate of $1/\Delta t$ Hz, yielding the discrete convolution

$$y[n] = \sum_{m=-n_{min}}^{n_{max}} x[m]h[n-m] + v[n]$$

(7)

where $N = n_{max}-n_{min}+1$, n_{max} and n_{min} represent respectively the upper and lower data cut-off points. The criteria for the selection of these sampling conditions have been thoroughly discussed in [2] and [4].

III. GENERATION OF THE DECONVOLVED DATA

Taking the DFT of (7) yields

$$Y(k) = X(k)H(k) + V(k)$$

from which the deconvolved data can be generated according to

$$\hat{X}(k) = \frac{Y(k)}{H(k)} = X(k) + \frac{V(k)}{H(k)},$$

(8)

for $0 \le k \le N$-1, where $Y(k)$, $X(k)$, $H(k)$, and $V(k)$ represent respectively the DFT of $y(n)$, $x(n)$, $h(n)$, and $v(n)$. This inverse filtering operation yields deconvolved data with decreasing SNR for increasing values of k. To alleviate this problem an optimal inverse filtering procedure is used for generating $X(k)$. In this approach $H(k)$ is modified so that the deconvolved data is generated according to [15] as

$$\hat{X}(k) = \frac{Y(k)H^*(k)}{[|H(k)|^2 + \mu]},$$

(9)

where the symbol * denotes complex conjugate. For high SNR, μ should be small and of the same order of magnitude as the attenuation of $H(k)$ at the cut-off frequency. However, as the SNR of the data decreases the choice of the optimum value of μ in (10) is best determined by experimental testing [3].

Denoting the truncated data as $f(k)$ and based on (6), (8) and (9), then we have

$$f(k) = \sum_{i=1}^{M} A_i \exp\{jk\Delta\omega\ln\lambda_i\} + \varepsilon(k),$$

(10)

For $k = 1, 2, \ldots, 2N_0 +1$; $N_0 \le (N/2)-1$ is the truncation point and $\varepsilon(k)$ is the deconvolved noise.

IV. SUBSPACE SIGNAL PROCESSING

Eigenvector (also called subspace) methods are based on a certain time-window of length P over which the signal model is characterized in form of a vector.

Consider the signal $f(k)$ from equation (10) at its current and future P- 1 values. The time-window can be written as

$$f(k) = [f(k)\ f(k+1)\ldots\ldots f(k+P-1)]$$

(11)

We can then write $f(k)$ in terms of length-P time-window as

$$f(k) = \sum_{i=1}^{M} A_i r(\ln\lambda_i)e^{jk\Delta\omega\ln\lambda_i} + w(k) = s(k) + w(k)$$

(12)

where $w(k) = [w(k)\ w(k+1)\ \ldots\ w(k+P-1)]^T$ is the time-window vector of white noise and $r(\ln\lambda) = [1\ e^{j\Delta\omega\ln\lambda}\ \ldots\ldots\ e^{j\Delta\omega\ln\lambda(P-1)}]^T$ is the time-window signal vector at each particular value of $\ln\lambda$.

The autocorrelation matrix of $f(k)$ may be expressed in terms of its eigendecomposition as

$$R_{ff} = \sum_{i=1}^{M} \lambda_i v_i v_i^H = V\Lambda V^H$$

(13)

where Λ is a diagonal matrix of the eigenvalues in descending order on the diagonal ($\lambda_1 \ge \lambda_2 \ge \ldots\ldots\ge \lambda_M$) while the columns of V are the corresponding eigenvectors. The M largest eigenvalues correspond to the signal and the remaining eigenvalues have equal values and correspond to the noise. R_{ff} can thus be partitioned into two portions, one due to the signal and the other due to the noise eigenvectors

$$R_{ff} = V_s\Lambda_s V_s + \sigma_w^2 V_w V_w^H$$

(14)

where

$$Vs = [v_1\ v_2\ \ldots\ldots v_M]\ and\ V_w = [v_{M+1}\ \ldots\ldots\ v_P]$$

(15)

are matrices whose columns consist of signal and noise eigenvectors respectively. Λ is an MxM diagonal matrix containing the signal eigenvalues. Thus the P-dimensional

subspace has been broken into two subspaces, i.e. the so-called signal and noise subspaces.

The matrices that project an arbitrary vector on the signal and noise subspaces are [12]:

$$P_s = V_s V_s^H \quad \text{and} \quad P_w = V_w V_w^H \tag{16}$$

Since the signal and noise subspaces are orthogonal,

$$P_w V_s = 0 \quad \text{and} \quad P_s V_w = 0 \tag{17}$$

all the $\ln\lambda$ vectors from equation (12) must lie completely in the signal subspace. This means

$$P_s r(\ln \lambda_i) = r(\ln \lambda_i) \; ; \; P_w r(\ln \lambda_i) = 0 \tag{18}$$

The above concepts are central to the two subspace methods presented below.

A. MUSIC Algorithm

The MUSIC algorithm was proposed [10] as an improvement on the Pisaranko harmonic decomposition (phd) which limited the length of P to P= M+1. In the MUSIC method, the time window is allowed to be P> M+1, thus giving a subspace of dimension greater than 1.

For each eigenvector v_p in the noise subspace ($M<p\leq P$),

$$r^H (\ln \lambda_m) v_p = \sum_{k=1}^{P} v_m(k) e^{-j\Delta\omega \ln(\lambda_m)(k-1)} = 0 \tag{19}$$

for all the M values of $\ln\lambda$.

Thus, if we compute the pseudospectrum of each noise eigenvector as

$$\overline{R}_P(e^{j\Delta\omega \ln \lambda}) = \frac{1}{\left| r^H(\ln \lambda) v_p \right|^2} = \frac{1}{\left| V_p(e^{j\Delta\omega \ln \lambda}) \right|^2} \tag{20}$$

the polynomial $V_p(e^{j\Delta\omega \ln \lambda})$ has P-1 roots, M of which corresponds to the $\ln\lambda$ of the complex exponentials. These roots produce M peaks in the pseudospectrum. The pseudospectra of all P-M noise eigenvectors share these roots that are due to the signal subspace. The remaining roots of the noise eigenvectors occur at different frequencies and may produce extra peaks in the pseudospectrum if close to the unit circle. The solution to this is to average out the P-M pseudospectra of the individual noise eigenvectors:

$$\overline{R}_{music}(e^{j\Delta\omega \ln \lambda}) = \frac{1}{\sum_{p=M+1}^{P} \left| r^H(\ln \lambda) v_p \right|^2} = \frac{1}{\sum_{p=M+1}^{P} \left| V_p(e^{j\Delta\omega \ln \lambda}) \right|^2} \tag{21}$$

This is known as the MUSIC pseudospectrum [12].

B. Minimum Norm Algorithm

Like the MUSIC method, the time-window P is allowed to be P> M+1.

The minimum-norm method uses an arbitrary vector

$$b = [b(1) \; b(2) \; \dots\dots\dots \; b(P)]^T \tag{22}$$

constrained to lie on the noise subspace.

For any arbitrary vector that lies in the noise subspace

$$P_w b = b \quad \text{and} \quad P_s b = \mathbf{0} \tag{23}$$

where $\mathbf{0}$ is the length-M zero vector.

The minimum norm seeks to minimize the norm of $\mathbf{b}$ in order to avoid spurious peaks in the pseudospectrum. From equation (16), the norm of a vector b contained in the noise subspace is

$$\|b\|^2 = b^H b = b^H P_w b \tag{24}$$

Since an unconstrained minimization of this norm will produce the zero vector, the first element of b is constrained to be unity, i.e.

$$\delta_1^H b = 1 \tag{25}$$

The solution to this can be found by using Lagrange multipliers [12] as

$$b_{mn} = \frac{P_w \delta_1}{\delta_1^H P_w \delta_1} \tag{26}$$

The $\ln\lambda$ estimates are then obtained from the peaks in the pseudospectrum of the minimum norm vector.

V. SIMULATION RESULTS

A MATLAB programme was written and run for the above algorithms and the performance of the above subspace techniques in analyzing multiexponential signals examined using the signals $S_1(\tau)$ and $S_2(\tau)$, where

$$S_1(\tau) = 0.5e^{-0.5\tau} + e^{-\tau} + 2e^{-2\tau} + 5e^{-5\tau} + 10e^{-10\tau} + n(\tau) \tag{27}$$

$$S_2(\tau) = 0.5e^{-0.5\tau} + e^{-\tau} + n(\tau) \tag{28}$$

The pseudospectrum of $S_1(\tau)$ and $S_2(\tau)$ are shown in Fig. 1 through Fig. 4 for the three techniques and Tables I and II give the estimate for $\ln\lambda_i$ for $S_1(\tau)$ and $S_2(\tau)$ respectively. The results show that the MUSIC algorithm produces more defined peaks. For the two algorithms, it is evident that the estimates do not vary significantly with SNR and they are accurate within the tolerance of about ±0.02. We may therefore conclude that for multiexponential signals in noise, MUSIC is the better algorithm even though the minimum norm performs almost equally well.
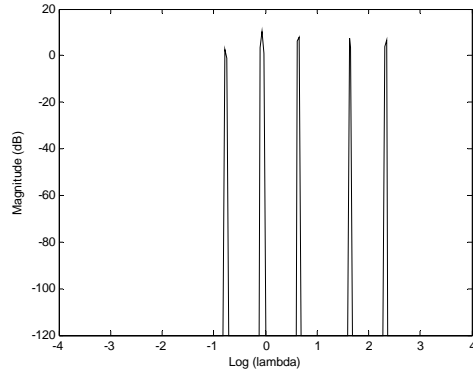
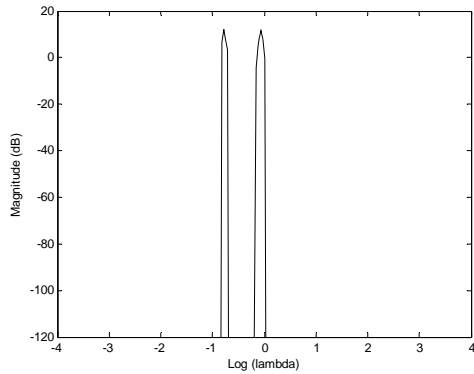Fig. 1 MUSIC pseudospectrum of $S_1(\tau)$ with SNR of 40dB

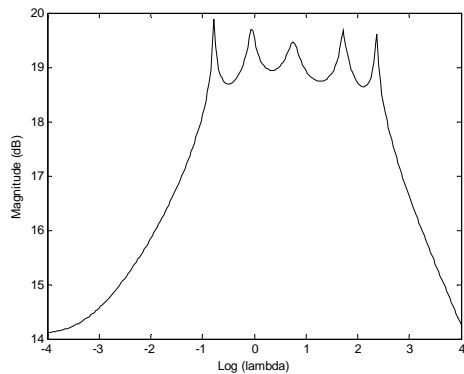Fig. 2 MUSIC pseudospectrum for $S_2(\tau)$ at SNR of 40dB

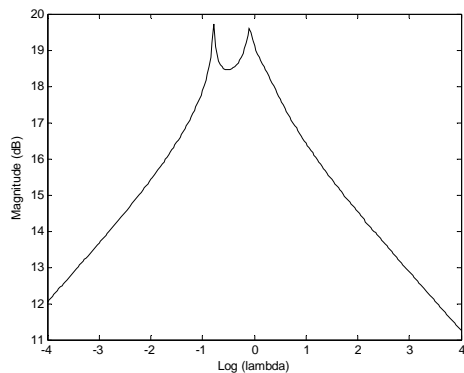Fig. 3 Min-Norm pseudospectrum of $S_1(\tau)$ with SNR of 40dB

Fig. 4 Min-norm pseudospectrum for $S_2(\tau)$ at SNR of 40dB

TABLE I
VALUES OF $\ln\lambda_i$ FOR DIFFERENT SNR FOR $S_1(\tau)$

method	SNR (dB)	Lnλ_1	Lnλ_2	Lnλ_3	Lnλ_4	Lnλ_5
MUSIC	40	-0.75	-0.0625	0.6563	1.625	2.344
	60	-0.75	-0.0625	0.6563	1.625	2.344
	100	-0.75	-0.0625	0.6563	1.625	2.344
Minimum Norm	40	-0.75	-0.0625	0.6563	1.625	2.313
	60	-0.75	-0.0625	0.6563	1.625	2.313
	100	-0.75	-0.0625	0.6563	1.625	2.313
Expected value		-0.6931	0	0.6931	1.609	2.302

TABLE II
VALUES OF $\ln\lambda_i$ FOR DIFFERENT SNR FOR $S_2(\tau)$

method	SNR (dB)	Lnλ_1	Lnλ_2
MUSIC	40	-0.7813	-0.0625
	60	-0.7813	-0.0625
	100	-0.7813	-0.0625
Minimum Norm	40	-0.7813	-0.0625
	60	-0.7813	-0.0625
	100	-0.7813	-0.0625
Expected value		-0.69315	0

VI. CONCLUSION

The performance of two eigenvector methods has been examined in the analysis of multiexponential signals in noise. It has been shown that both MUSIC and Minimum norm subspace techniques perform equally well with MUSIC resulting in better defined peaks at the $\ln\lambda$ values. It has also been shown that the performance of the two methods does not deteriorate with decreasing SNR.

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