Proof-Theoretical Observations of BI and BBI Base-Logic Interactions, and Development of Phased Sequent Calculus to Define Logical Combinations

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Dedicated to all those who intuit what separation logic has come to symbolise and all those who aim beyond.

Declaration

I declare that the substance of this thesis has not been already submitted for any degree and is not currently being submitted for any other degree or degrees. It is all my own work unless referenced to the contrary in the thesis.

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Abstract

I study sequent calculus of combined logics in this thesis. Two specific logics are looked at - Logic BI that combines intuitionistic logic and multiplicative intuitionistic linear logic and Logic BBI that combines classical logic and multiplicative linear logic. A proof-theoretical study into logical combinations themselves then follows.

To consolidate intuition about what this thesis is all about, let us suppose that we know about two different logics, Logic A developed for reasoning about Purpose A and Logic B developed for reasoning about Purpose B. Logic A serves Purpose A very well, but not Purpose B. Logic B serves Purpose B very well but not Purpose A. We wish to fulfill both Purpose A and Purpose B, but presently we can only afford to let one logic guide through our reasoning. What shall we do? One option is to be content with having Logic A with which we handle Purpose A efficiently and Purpose B rather inefficiently. Another option is to choose Logic B instead. But there is yet another option: we combine Logic A and Logic B to derive a new logic Logic C which is still one logic but which serves both Purpose A and Purpose B efficiently. The combined logic is synthetic of the strengths in more basic logics (Logic A and Logic B). As it nicely takes care of our requirements, it may be the best choice among all that have been so far considered. Yet this is not the end of the story. Depending on the manner Logic A and Logic B combine, Logic C may have extensions serving more purposes than just Purpose A and Purpose B. Ensuing is the following problem: we know about Logic A and Logic B, but we may not know about combined logics of the base logics. To understand the combined logics, we need to understand the extensions in which base logics interact each other. Analysis on the interesting parts tends to be non-trivial, however. The mentioned two specific combined logics `BI` and `BBI` do not make an exception, for which proof-theoretical development has been particularly slow. It has remained in obscurity how to properly handle base-logic interactions of the combined logics as appearing syntactically.

As one objective of this thesis, I provide analysis on the syntactic phenomena of the `BI` and `BBI` base-logic interactions within sequent calculus, to augment the knowledge. For `BI`, I deliver, through appropriate methodologies to reason about the syntactic phenomena of the base-logic interactions, the first `BI` sequent calculus free of any structural rules. Given its positive consequence to efficient proof searches, this is a significant step forward in further maturity of `BI` proof theory. Based on the calculus, I prove decidability of a fragment of `BI` purely syntactically. For `BBI` which is closely connected to application via separation logic, I develop adequate sequent calculus conventions and consider the implication of the underlying semantics onto syntax. Sound `BBI` sequent calculi result with a closer syntax-semantics correspondence than previously envisaged. From them, adaptation to separation logic is also considered.

To promote the knowledge of combined logics in general within computer science, it is also important that we be able to study logical combinations themselves. Towards this direction of generalisation, I present the concept of phased sequent calculus - sequent calculus which physically separates base logics, and in which a specific manner of logical combination to take place between them can be actually developed and analysed. For a demonstration, the said decidable `BI` fragment is formulated in phased sequent calculus, and the sense of logical combination in effect is analysed. A decision procedure is presented for the fragment.

Contents

2 Reviews of BI Proof Systems 46

3 Structural Interactions, Absorption of Structural Rules and Decidability in BI Sequent Calculus 65

List of Figures

Chapter 0

Prerequisites and Notations

Formal Reasoning

It can be error-prone to use an informal language for formal reasoning. Where there is a merit, I make use of meta-logical connectives as defined below for some object logic.

Definition 1 (Meta-logical connectives for formal reasoning)

1. $\wedge^\dagger$: *conjunction. Given two sentences* S_1 *and* S_2 *each of which is either a true or a false statement,* $S_1 \wedge^\dagger S_2$ *is a true statement if and only if both* S_1 *and* S_2 *are true statements.*
2. $\vee^\dagger$: *disjunction. Given two sentences* S_1 *and* S_2 *each of which is either a true or a false statement,* $S_1 \vee^\dagger S_2$ *is a true statement if and only if either* S_1 *or* S_2 *is a true statement.*
3. $\neg^\dagger$: *negation. Given a sentence* S_1*,* $\neg^\dagger S_1$ *is a true statement if and only if* S_1 *is a false statement.*
4. $\rightarrow^\dagger$: *material implication. Given two sentences* S_1 *and* S_2 *each of which is either a true or a false statement,* $S_1 \rightarrow^\dagger S_2$ *is a true statement if and only if either* S_1 *is a false statement or* S_2 *is a true statement.*

5. $\leftrightarrow^\dagger$*: equivalence. Given a sentence* S_1 *and* S_2 *each of which is either a true or a false statement,* $S_1 \leftrightarrow^\dagger S_2$ *is a true statement if and only if both* $S_1 \rightarrow^\dagger S_2$ *and* $S_2 \rightarrow^\dagger S_1$*.*

6. $\forall$*: universal quantification. Given a formula* $S(x)$ *with a free variable* x *(which may be occurring multiple times),* $\forall x.S(x)$ *is a true statement if and only if, for all the constants* c *that could replace* x*,* $S(c)$ *is a true statement.*

7. $\exists$*: existential quantification. Given a formula* $S(x)$ *with a free-variable* x *(which may be occurring multiple times),* $\exists x.S(x)$ *is a true statement if and only if there exists some constant* c *that could replace* x *such that* $S(c)$ *is a true statement.*

Binding Order

I adopt the following binding order in a decreasing precedence for all the - logical or meta-logical indiscriminate - connectives that appear in this thesis. Connectives in the same group have the same binding precedence.

1. $\neg$
2. $\wedge \quad \vee \quad *$
3. $\supset \quad -\!\!*$
4. $; \quad ,$
5. $\forall \quad \exists$
6. $\neg^\dagger$
7. $\wedge^\dagger \quad \vee^\dagger$
8. $\rightarrow^\dagger \quad \leftrightarrow^\dagger$

Example 1 $\neg^\dagger \forall a.\neg a \wedge b \supset c; d \rightarrow^\dagger f$ *is read as:* $(\neg^\dagger((\forall a).((((\neg a) \wedge b) \supset c); d))) \rightarrow^\dagger f$*.*

Notations Around Derivation Trees

A proof system defines a set of axioms and other inference rules. I define notations around derivation trees.

Definition 2 (Inference rules) *An inference rule **Inf** of some proof system PS is in one of the following forms:*

1. *One-premise inference rule:*

   $$\frac{\textit{Premise}}{\textit{Conclusion}}\ \textit{Label}$$

   in which are given one conclusion "Conclusion" and one premise "Premise". "Label" is the name given to the one-premise inference rule. Note that this and the rest in this definition are schemata. Each of "Premise", "Label" and "Conclusion" is instantiated appropriately in PS.

2. *Two-premise inference rule:*

   $$\frac{\textit{Premise1} \qquad \textit{Premise2}}{\textit{Conclusion}}\ \textit{Label}$$

   in which are given one conclusion "Conclusion" and two premises "Premise1" and "Premise2". "Label" is the name given to the two-premise inference rule.

3. *Axiom:*

   $$\frac{}{\textit{Conclusion}}\ \textit{Label}$$

   in which is given one conclusion "Conclusion" and no premise. "Label" is the name given to the axiom rule.

Definition 3 (Derivation trees) *Given some proof system PS with a set of inference rules **Infs**, a derivation tree with its conclusion Conclusion is defined by the following simultaneous induction. Hereafter, we denote by* $\Pi(\mathit{Conclusion})$ *a derivation tree whose root is* Conclusion*. By 'root' of a derivation tree, we mean that it is a conclusion which is not at the same time a premise in the derivation tree. Symmetrically, by 'leaf' of a derivation tree, we mean that it is a premise which is not at the same time a conclusion in the derivation tree.*

1. *If Conclusion is the conclusion of an axiom (with label Label), then*

   $$\frac{}{\mathit{Conclusion}}\ \mathit{Label}$$

   is a derivation tree. There are two nodes: an empty node in the premise of Label and Conclusion in the conclusion of Label, in the derivation tree.

2. *If Conclusion is the conclusion of some one-premise inference rule **Inf** with the premise Premise which is the root of* $\Pi(\mathit{Premise})$*, then;*

   $$\frac{\Pi(\mathit{Premise})}{\mathit{Conclusion}}\ \boldsymbol{\mathit{Inf}}$$

   is a derivation tree. This derivation tree comprises the root node Conclusion and all the nodes in $\Pi(\mathit{Premise})$*.*

3. *If Conclusion is the conclusion of some two-premise inference rule **Inf** with the left premise Premise1 and the right premise Premise2 each of which is the root node of a derivation tree, then;*

   $$\frac{\Pi(\mathit{Premise1}) \qquad \Pi(\mathit{Premise2})}{\mathit{Conclusion}}\ \boldsymbol{\mathit{Inf}}$$

   is a derivation tree. This derivation tree comprises the root node Conclusion and all the nodes in $\Pi(\mathit{Premise1})$ *and* $\Pi(\mathit{Premise2})$*.*

If all the leaf nodes in a derivation tree are empty nodes, we say that the derivation tree is closed.

Definition 4 (Derivation depth) *Derivation depth of a derivation tree* $\Pi(\mathit{Conclusion})$ *with the root Conclusion, denoted by* $\texttt{der\_depth}(\Pi(\mathit{Conclusion}))$*, is defined inductively:*

1. *If Conclusion is the conclusion of an axiom, it is 1.*

2. *If Conclusion is the conclusion of a one-premise inference rule with the premise Premise, then it is* $1 + \texttt{der\_depth}(\Pi(\mathit{Premise}))$*).*

3. *If Conclusion is the conclusion of a two-premise inference rule with the left premise Premise1 and the right premise Premise2, then it is*
   $1 + max(\texttt{der\_depth}(\Pi(\mathit{Premise1})), \texttt{der\_depth}(\Pi(\mathit{Premise2})))$.

Definition 5 (Transitions) *"$\leadsto$" is defined for two nodes D_1 and D_2 in a derivation tree such that $D_1 \leadsto D_2$ is a one-step transition via an inference rule **Inf**, satisfying (1) that D_2 is the premise (or one of the premises) of **Inf** and (2) that D_1 is the conclusion of **Inf**. The notation $D_1 \leadsto_{\textbf{\textit{Inf}}} D_2$ explicitly states which inference rule applies for the transition. A transition from D_1 to D_2 in zero (*i.e.* no transition) or more applications of inference rule(s) is denoted by $D_1 \leadsto^* D_2$. The notation $D_1 \leadsto^*_{\textbf{\textit{Infs}}} D_2$ explicitly states which inference rule(s) may apply for the transition. $D_1 \leadsto^+ D_2$ abbreviates $D_1 \leadsto D_3 \leadsto^* D_2$ for some D_3 in the derivation tree. $D_1 \leadsto^k D_2$ is a transition with exactly $k \geq 0$ steps.*

Definition 6 (Derivation length) *Given a derivation tree $\Pi(D)$ with the conclusion (root) D, derivation length of D_1 and D_2 denoted by $\texttt{der\_len}(D_1, D_2)$ is either undefined in case there exists no transition $D_1 \leadsto^* D_2$ or else defined inductively:*

1. *it is 0 if D_1 and D_2 refer to the same node in $\Pi(D)$.*
2. *it is $1 + \texttt{der\_len}(D_3, D_2)$ if there exists a node D_3 in $\Pi(D)$ such that $D_1 \leadsto D_3 \leadsto^* D_2$.*

The following three variations will be used frequently in this thesis. By a double line:

$$\frac{Premise}{\overline{Conclusion}}\ \textbf{Infs}$$

Premise upward is indicated to derive from Conclusion in zero or more steps making use of **Inf** $\in$ **Infs**. Similarly for when there are two premises. By a dotted line:

$$\genfrac{}{}{0pt}{}{Premise}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \atop Conclusion}\ \textbf{Inf}$$

Conclusion is indicated to be derivable from Premise without, in so doing, increasing the derivation depth. By a double-dotted line:

$$\genfrac{}{}{0pt}{}{Premise}{::::::::::::::: \atop Conclusion}\ \textbf{Inf}$$

Premise is indicated to be derivable from Conclusion just as Conclusion is from Premise. In another word, a double-dotted line is used to signify a bidirectionality of an inference rule.

Chapter 1

Introduction

Many problems we face are compositional. In fact, not many but save on rare occasions, it is harder to identify a problem that cannot be decomposed into smaller parts. If solutions to the sub-problems are known, the main problem can be answered in an incremental manner. But even in cases where solutions to some of them are presently unknown, they can be worked out and conjoined into the rest. It then appears that there is no reason why we should not focus on smaller, perceived-to-be easier problems - in order to maximise our productivity through the modular reasoning.

Separation logic (*Cf.* Ishtiaq and O'Hearn [2001]; Reynolds [2002]) may be a good example. Expressiveness power of classical logic and that of multiplicative (intuitionistic) linear logic combined, it can be used to efficiently reason about heap manipulating programs, allowing us to recognise portions of heap as disjoint resources. Full expressiveness power of classical logic is attainable on each separated resource. For instance, such an expression as "some fact p holds true on a part of heap and some fact $\neg q$ holds true on another part of heap such that the two heap portions do not overlap", can be stated in separation logic simply as "$p * \neg q$". Since the expression assumes that p and $\neg q$ hold true in disjoint parts of heap, a heap-manipulating program that alters information in either of them does not need have a side-effect on the information contained in the other: if p is updated to p' by some program command accessing only the portion of heap that contains the information, we have $p' * \neg q$ with no required modification on $\neg q$. This concept of local reasoning sparked inspiration and resulted in many applications (*Cf.* Bornat et al. [2005]; Calcagno et al. [2009]; Chin et al. [2012]; Distefano et al. [2006]; Parkinson and Bierman [2005]; Yang [2007] for example) subsequently.

What this line of research seems to suggest is that, with often synthesised problems around us to face, the vehicle for reasoning itself, *i.e.* logic, should be also moving towards accommodation of modularity, so that the manner by which we reason about a given problem can find a closer map to its underlying structure than to a view of it that a detour through many morphisms may provide. The idea to put together multiple logics itself has been around for quite a while, enquired for instance within the field of philosophy, as Caleiro et al. [2005]; Stanford Encyclopedia of Philosophy [2011] note. With the active evidence of separation logic we were a witness to in the last decade, it is amply suggested that pragmatic values lie in, and extend from, studies of combined logics.

Inseparable, however, are issues around the mechanism of interactions between the base logics to be so combined. Given that a combined logic with no base-logic overlaps is readily decomposable, it (the mechanism of interactions) is reasonably speaking the only part in a combined logic which is interesting and which hence merits a thorough investigation. Nevertheless, analysis on the *only part* tends to be non-trivial. Further, if the accumulated knowledge of combined logics is to be incorporated in practice, *e.g.* into theorem proving, there also arises a constructive (or computational) concern of how to formalise the knowledge in a way that is suitable for automation. Hence, with all the positive expectation notwithstanding, there are also problems that ought to be addressed before we may be able to see a fuller extent of their possibilities in application.

To promote the program, this thesis takes a reasonable approach of studying base-logic interactions within sequent calculus which, among many types of proof systems (formalisms as are often called) available, is particularly well-suited for an efficient automated theorem proving because potential curtailment of search-space explosion - hindrance to an efficient theorem proving - can be more easily and efficiently attempted. Two broad perspectives will be heeded: one that concerns specific combined logics, and one that focuses on generalisation, *i.e.* abstraction, of their logical characteristics in order to attain a higher standpoint. Both are complementary to the other and help mutually forge ahead the overall program of deepening our understanding about the nature of logical combinations and of logics so combined.

Into the first direction of specialisation, this thesis augments in constructive steps the knowledge of the syntactically observed base logic interactions for `BI` (a combined logic of intuitionistic logic and multiplicative intuitionistic linear logic; *Cf.* O'Hearn and Pym [1999]) and `BBI` (a combined logic of classical logic and multiplicative (intuitionistic) linear logic just as separation logic is, but more expressive). The reason for the choice of the specific logics is rather natural, as I itemise below:

1. In view of the practical implication, it is of a great interest that base-logic interactions within separation logic be better understood. However, there exists in literature no known adequate sequent calculus for separation logic which, had it been otherwise, could have offered a possibility of studying the syntactically occurring base-logic interactions *a posteriori*. Its sequent calculus needs to be developed first. Since separation logic is a specialised `BBI` (*Cf.* Larchey-Wendling and Galmiche [2012]), theoretical investigation into `BBI` is strongly relevant.

2. Meanwhile, as Galmiche and Larchey-Wendling [2006] indicate, `BBI` is strictly more expressive than `BI`. However, even though considered to be (conversely) strictly less expressive, `BI` still poses difficulty in analysis of the syntactically occurring base-logic interactions as inferrable from earlier work; so much so that there in fact exists no sensible analysis regarding the matter. It then seems natural that we first see to ourselves if we can at least analyse the easier problem with a success, that is, reasonably speaking, if we cannot analyse an easier problem, then hardly will there be any hope left for more difficult ones.

Thus elucidating the coverage of the specific combined logics to be studied, we may now proceed to see the main line of objectives into the direction of specialisation. Along with the other objectives, this thesis first solves a long-standing open problem in `BI` proof theory of analysis on the syntactic phenomena of base-logic interactions as occurring within `BI` sequent calculi; interactions between structural inference rules and logical inference rules (that is, *structural interactions*), specifically. Delivery of a practically significant contraction-free `BI` calculus, a hitherto encumbered attempt due to the lack of the knowledge, is for the first time made successfully through an adequate methodology that recognises the boundaries between one `BI` base logic and the other. Moving on to `BBI` proof theory, it presents a `BBI` sequent calculus. It is developed through contemplation over the syntactic implication of the `BBI` base-logic interactions. The knowledge of the structural interactions in `BI` from the first step then applies to the `BBI` calculus, resulting in a less non-deterministic sequent calculus. Both are sound with respect to the underlying `BBI` semantics. By taking into account a particularity of the heap semantics (semantics for separation logic), derivation of sound separation logic sequent calculi is immediate.

Into the direction of generalisation, this thesis develops an idea of sequent calculus in which a specific manner of base-logic interactions to take place within a combined logic can be actually developed and analysed. This idea I call phased sequent calculus in which a physical separation of base logics is expressible and in which a logical combination itself becomes as important a component to consider as the base logics.

The rest of this chapter is dedicated to introduction of semantics and proof systems of related logics as technical preliminaries, followed by descriptions of research problems and all the contributions: ones just mentioned and also the rest, in sufficiently technical terms.

1.1 Technical Preliminaries

We go through semantics and proof systems of related logics. Since no quantified logics find their way into main chapters of this thesis, it is (even if unstated) tacit that I look at propositional logics only. A set of propositional variables is denoted by $\mathcal{P}$. "if and only if" is abbreviated by "iff". For standard terminologies and philosophical aspects of those logics, readers are referred to introductory texts on mathematical logic such as Girard [1987]; Kleene [1952].

1.1.1 Classical logic

Every statement (sentence) is *considered already known* to be either true or false in classical logic, which is called the law of excluded middle, and it is by our attempts that the truth/falsity be found out. To prove some statement true, one may prove the fact directly by showing that it is true. Alternatively, one may prove that the negation of the same statement is false, thereupon follows the desired result by the law of the excluded middle.

Definition 7 (Formulas) *A formula* $A\ (, B, C)$ *in propositional classical logic is defined by:*

$A := p \mid \top \mid \mathbb{1} \mid A \wedge A \mid A \vee A \mid A \supset A$
where p *denotes a propositional variable (*$p \in \mathcal{P}$*),* $\top$ *a zero-place logical operator (or synonymously logical connective) signifying the truth, and* $\mathbb{1}$ *a zero-place logical operator signifying the falsity. In this thesis both* $\top$ *and* $\mathbb{1}$ *are primitive. The set of formulas in propositional classical logic (those that this grammar generates) is denoted by* $\mathfrak{F}_{\texttt{CL}}$*.* $\neg A$ *abbreviates* $A \supset \mathbb{1}$*.*

The following associativity and commutativity hold within $\mathfrak{F}_{\texttt{CL}}$.

Property 1 (Associativity and commutativity)

1. $(A_1 \wedge A_2) \wedge A_3 = A_1 \wedge (A_2 \wedge A_3)$.
2. $(A_1 \vee A_2) \vee A_3 = A_1 \vee (A_2 \vee A_3)$.
3. $A_1 \wedge A_2 = A_2 \wedge A_1$.
4. $A_1 \vee A_2 = A_2 \vee A_1$.

Semantics for (propositional[1]) classical logic is given in the following manner.

Definition 8 (Interpretation) *An interpretation* $\mathsf{I}_{\texttt{CL}}$ *is a function that maps propositional variables into either a logical truth or a logical falsity,* $\mathsf{I}_{\texttt{CL}} : \mathcal{P} \rightarrow \{\mathbf{T}, \mathbf{F}\}$*.*

Definition 9 (Semantics) *A model for classical logic is a tuple* $(\mathsf{I}_{\texttt{CL}}, \models)$ *for some* $\mathsf{I}_{\texttt{CL}}$*, satisfying the following forcing relations:*

- $\models p$ *iff* $\mathsf{I}_{\texttt{CL}}(p) = \mathbf{T}$.
- $\models \top$.
- $\neg^{\dagger}\, [\models \mathbb{1}]$.
- $\models A \wedge B$ *iff* $[\models A] \wedge^{\dagger} [\models B]$.
- $\models A \vee B$ *iff* $[\models A] \vee^{\dagger} [\models B]$.
- $\models A \supset B$ *iff* $\neg^{\dagger}[\models A] \vee^{\dagger} [\models B]$.

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[1] Assumed in the rest as such that I am speaking about propositional logics.

$$\frac{}{A\supset(B\supset A)}\text{Ax1}\qquad\frac{}{(A\supset(B\supset C))\supset((A\supset B)\supset(A\supset C))}\text{Ax2}\qquad\frac{}{A\supset A\vee B}\text{Ax3}$$

$$\frac{}{B\supset A\vee B}\text{Ax4}\qquad\frac{}{(A\supset C)\supset((B\supset C)\supset(A\vee B\supset C))}\text{Ax5}\qquad\frac{}{A\wedge B\supset A}\text{Ax6}$$

$$\frac{}{A\wedge B\supset B}\text{Ax7}\qquad\frac{}{A\supset(B\supset A\wedge B)}\text{Ax8}\qquad\frac{}{\mathbb{1}\supset A}\text{Ax9}$$

$$\frac{}{A\supset\top}\text{Ax10}\qquad\frac{}{((A\supset\mathbb{1})\supset\mathbb{1})\supset A}\text{Ax11}\qquad\frac{A\qquad A\supset B}{B}\text{MP}$$

Figure 1.1: `Hc`: a Hilbert system for propositional classical logic.

$$\frac{}{A\vdash A}\,id\qquad\frac{\Psi_1\vdash A;\Phi_1\qquad\Psi_2;A\vdash\Phi_2}{\Psi_1;\Psi_2\vdash\Phi_1;\Phi_2}\,\mathtt{Cut}\qquad\frac{}{\mathbb{1}\vdash\Phi}\,\mathbb{1}L$$

$$\frac{}{\Psi\vdash\top}\,\top R\qquad\frac{\Psi;A;B\vdash\Phi}{\Psi;A\wedge B\vdash\Phi}\,\wedge L\qquad\frac{\Psi;A\vdash\Phi\qquad\Psi;B\vdash\Phi}{\Psi;A\vee B\vdash\Phi}\,\vee L$$

$$\frac{\Psi\vdash A;\Phi\qquad\Psi;B\vdash\Phi}{\Psi;A\supset B\vdash\Phi}\,\supset L\qquad\frac{\Psi\vdash A;\Phi\qquad\Psi\vdash B;\Phi}{\Psi\vdash A\wedge B;\Phi}\,\wedge R\qquad\frac{\Psi\vdash A;B;\Phi}{\Psi\vdash A\vee B;\Phi}\,\vee R$$

$$\frac{\Psi;A\vdash B;\Phi}{\Psi\vdash A\supset B;\Phi}\,\supset R\qquad\frac{\Psi\vdash\Phi}{\Psi;A\vdash\Phi}\,WkL\qquad\frac{\Psi;A;A\vdash\Phi}{\Psi;A\vdash\Phi}\,CtrL$$

$$\frac{\Psi\vdash\Phi}{\Psi\vdash\Phi;A}\,WkR\qquad\frac{\Psi\vdash\Phi;A;A}{\Psi\vdash\Phi;A}\,CtrR$$

Figure 1.2: $\mathtt{G1c_p}$: a sequent calculus for propositional classical logic.

Of course, given $A\in\mathfrak{F}_{\mathtt{CL}}$ there are 2^n conceivable distinct interpretations for n occurrences of distinct propositional variables in A.

Definition 10 (Universal validity) *A formula $A\in\mathfrak{F}_{\mathtt{CL}}$ is said to be universally valid iff $\models A$ for all the conceivable distinct interpretations of propositional variables occurring in A.*

Classical logic can be formalised for example in Hilbert-style calculus (Figure 1.1)[1] with a finite number of axiom rules, and MP (modus ponens) that formalises the following: "if A is universally valid and if $A\supset B$ is universally valid, then B is universally valid".

Other formalisations are possible. In sequent calculus - the central interest of this

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[1] To align Hilbert-style systems with sequent-style systems, I explicitly consider Hilbert-style axioms as inference rules.

thesis - classical logic is formalised as in Figure 1.2 (an equivalent variant of the propositional part of LK that Gentzen [1934] originally presented). It postulates *structures* and *sequents*.

Definition 11 (Structures) *A structure* $\Psi(,\Phi)$ *of classical logic is defined by* $\Psi := A \mid \Psi;\Psi$. *The set of structures that this grammar generates is denoted by* $\mathfrak{S}_{\texttt{CL}}$.

Property 2 (Associativity and commutativity of structures)

1. $(\Psi_1;\Psi_2);\Psi_3 = \Psi_1;(\Psi_2;\Psi_3)$.
2. $(\Phi_1;\Phi_2);\Phi_3 = \Phi_1;(\Phi_2;\Phi_3)$.
3. $\Psi_1;\Psi_2 = \Psi_2;\Psi_1$.
4. $\Phi_1;\Phi_2 = \Phi_2;\Phi_1$.

Definition 12 (Sequents) *A sequent in classical logic is defined to be in the form:* $\Psi \vdash \Phi$ *for some* $\Psi \in \mathfrak{S}_{\texttt{CL}}$ *and some* $\Phi \in \mathfrak{S}_{\texttt{CL}}$. *The set of sequents in* $\texttt{G1c}_{\texttt{p}}$ *is denoted by* $\mathfrak{D}_{\texttt{CL}}$.

Definition 13 (Sequent calculus convention) *In any* $D \in \mathfrak{D}_{\texttt{CL}}$, *emptiness of an antecedent structure is identified with a* $\top$ *and that of a consequent structure with a* $\mathbb{1}$. *As is conventional, the left to the* $\vdash$ *is referred to as the antecedent, and the right to it as the consequent.*

Example 2 *Given* $\Psi_1;\Psi_2 \vdash \Phi_1;\Phi_2 \in \mathfrak{D}_{\texttt{CL}}$, *the antecedent part is identified with* Ψ_1 *(or* Ψ_2*) if* Ψ_2 *(or* Ψ_1*) is empty. Likewise the consequent part is identified with* Φ_1 *(or* Φ_2*) if* Φ_2 *(or* Φ_1*) is empty.*

In this thesis I classify every inference rule of a sequent calculus into one of three groups. One group comprises a single inference rule Cut which is a rule of transitivity. Another group comprises a set of rules which either are an axiom without a premise sequent or else act upon a logical connective (*e.g.* $id, \mathbb{1}L, \top R, \wedge L, \vee L, \supset L, \wedge R, \vee R, \supset R$ in Figure 1.2). They are termed logical (inference) rules. For each logical inference rule the formula - or formulas in case of id - in the conclusion sequent upon which the logical inference rule acts is termed the *principal formula(s)* or simply the *principal* of the logical inference rule. I also say that the principal of a logical inference rule is *active* in the rule. The last group consists of the remaining inference rules (*e.g.* $WkL, WkR, CtrL$ and $CtrR$ in Figure 1.2). Each inference rule in this group is called a structural (inference) rule acting on some structure in the conclusion sequent. I say that the structure which is acted upon is *active* in the rule.

But in any case why so many formalisations of the same logic, or any particular choice out of the many possibilities? There could be as many reasons as there are formalisms. A primary concern of this thesis is amenability of a proof system to automation. From that standpoint, the axiom-based Hilbert-style representation of a logic[1] is not the best due to the presence of MP which entails a non-trivial backward proof search, since what to appear in the premise(s) of MP is not necessarily inferrable from the conclusion. The use of sequent calculus by contrast has an advantage that, while the same problem can still arise through `Cut`, it is relatively simple to establish the equivalence in expressiveness power between a `Cut`-embedded sequent calculus SC_1 and its `Cut`-free version $[SC_1 - \texttt{Cut}]$. The equivalence usually follows from a cut elimination procedure, *i.e.* a procedure to eliminate `Cut` instances out of any given closed derivation tree through derivation tree permutations.

Theorem 1 (Equivalence of $\texttt{G1c}_\texttt{p}$ with $\texttt{G1c}_\texttt{p}$- `Cut` by Gentzen [1934]) *Any sequent in $\mathfrak{D}_{\texttt{CL}}$ derivable in $\texttt{G1c}_\texttt{p}$ is also derivable in $[\texttt{G1c}_\texttt{p}$ - `Cut`$]$ and vice versa.*

1.1.2 Intuitionistic logic

The law of the excluded middle that characterises classical logic is not universally accepted, disallowed for example by intuitionistic schools (*Cf.* Brouwer [1908]; Heyting [1930]; van Dalen [2002] but also Critique of Pure Reason by Kant for his earlier remarks). The division is in their viewpoint about infinity. In intuitionistic logic, what are true are only what have been or guaranteed to be verified true by some constructive means. The rest remain in the realm of becoming.

Definition 14 (Formulas) *A formula A of intuitionistic logic is a set element of $\mathfrak{F}_{\texttt{CL}}$.*

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[1] One particular Hilbert-style representation of classical logic was indicated in Figure 1.1; there are shorter possibilities.

We denote the set of intuitionistic logic formulas by $\mathfrak{F}_{\mathtt{IL}}$*. Clearly* $\mathfrak{F}_{\mathtt{CL}} = \mathfrak{F}_{\mathtt{IL}}$*. We assume that both associativity and commutativity within* $\mathfrak{F}_{\mathtt{IL}}$ *carry over from* $\mathfrak{F}_{\mathtt{CL}}$*.*

Kripke [1965] introduces the concept of possible worlds to define models for intuitionistic logic. Originally for a set of modal logics (*Cf.* Kripke [1959]), it captures the nature of becoming in an intuitive manner.

Definition 15 (Frame) *A frame for intuitionistic logic is a tuple* $(W, \leq)$ *with a non-empty set* W *of possible worlds partially ordered by* $\leq$*.*

Definition 16 (Interpretation) *An interpretation* $\mathsf{I}_{\mathtt{IL}}$ *is a function that maps propositional variables into a set of possible worlds,* $\mathsf{I}_{\mathtt{IL}} : \mathcal{P} \rightarrow \mathbb{P}(W)$*, satisfying the following monotonicity:* $\forall w_1, w_2 \in W. \forall p \in \mathcal{P}.[w_1 \leq w_2] \wedge^{\dagger} [w_1 \in \mathsf{I}_{\mathtt{IL}}(p)] \rightarrow^{\dagger} [w_2 \in \mathsf{I}_{\mathtt{IL}}(p)]$.

Definition 17 (Semantics) *A Kripke model for intuitionistic logic is a 4-tuple* $(W, \leq, \mathsf{I}_{\mathtt{IL}}, \models)$ *for some frame* $(W, \leq)$ *and some interpretation* $\mathsf{I}_{\mathtt{IL}}$*, satisfying, for all* $p \in \mathcal{P}$ *and for all* $w \in W$*:*

- $w \models p$ *iff* $w \in \mathsf{I}_{\mathtt{IL}}(p)$.
- $w \models \top$.
- $\neg^{\dagger}[w \models \mathbb{1}]$.
- $w \models A \wedge B$ *iff* $[w \models A] \wedge^{\dagger} [w \models B]$.
- $w \models A \vee B$ *iff* $[w \models A] \vee^{\dagger} [w \models B]$.
- $w \models A \supset B$ *iff* $\forall w' \in W.[w \leq w'] \wedge^{\dagger} [w' \models A] \rightarrow^{\dagger} [w' \models B]$.

Definition 18 (Universal validity) *Let* A *be an intuitionistic logic formula. Then it is said to be valid in some intuitionistic Kripke model* $(W, \leq, \mathsf{I}_{\mathtt{IL}}, \models)$ *iff* $\forall w \in W.[w \models A]$*. It is said to be universally valid iff it is valid in all the conceivable intuitionistic Kripke models.*

Gentzen [1934] first formulates intuitionistic logic `LJ` in sequent calculus. Figure 1.3 shows a propositional fragment of its equivalent variant `G1i`.

$$\frac{}{A \vdash A}\,id \qquad \frac{\Psi \vdash A \qquad \Phi; A \vdash B}{\Phi; \Psi \vdash B}\,\texttt{Cut} \qquad \frac{}{\mathbb{1} \vdash A}\,\mathbb{1}L$$

$$\frac{}{\Psi \vdash \top}\,\top R \qquad \frac{\Psi; A; B \vdash C}{\Psi; A \wedge B \vdash C}\,\wedge L \qquad \frac{\Psi; A \vdash C \qquad \Psi; B \vdash C}{\Psi; A \vee B \vdash C}\,\vee L$$

$$\frac{\Psi \vdash A \qquad \Psi; B \vdash C}{\Psi; A \supset B \vdash C}\,\supset L \qquad \frac{\Psi \vdash A \qquad \Psi \vdash B}{\Psi \vdash A \wedge B}\,\wedge R \qquad \frac{\Psi \vdash A_i \quad (i \in \{1, 2\})}{\Psi \vdash A_1 \vee A_2}\,\vee R$$

$$\frac{\Psi; A \vdash B}{\Psi \vdash A \supset B}\,\supset R \qquad \frac{\Psi \vdash B}{\Psi; A \vdash B}\,WkL \qquad \frac{\Psi; A; A \vdash B}{\Psi; A \vdash B}\,CtrL$$

Figure 1.3: $\texttt{G1i}_{\texttt{p}}$: a sequent calculus for propositional intuitionistic logic.

Definition 19 (Structures and sequents) *A structure Ψ for intuitionistic logic is a set element of $\mathfrak{S}_{\texttt{CL}}$. The set of the structures is denoted by $\mathfrak{S}_{\texttt{IL}}$. Clearly $\mathfrak{S}_{\texttt{IL}} = \mathfrak{S}_{\texttt{CL}}$. We assume that both associativity and commutativity within $\mathfrak{S}_{\texttt{IL}}$ carry over from $\mathfrak{S}_{\texttt{CL}}$. A sequent in intuitionistic logic is defined to be in the form: $\Psi \vdash A$ for some $\Psi \in \mathfrak{S}_{\texttt{IL}}$ and some $A \in \mathfrak{F}_{\texttt{IL}}$. The set of the sequents is denoted by $\mathfrak{D}_{\texttt{IL}}$. We assume that the sequent calculus convention within $\mathfrak{D}_{\texttt{IL}}$ carries over from $\mathfrak{D}_{\texttt{CL}}$.*

It is given in his work that `LJ` and [`LJ` - `Cut`] are equivalent in expressiveness. Decidability, however, is not immediate in the set of [$\texttt{G1i}_{\texttt{p}}$- `Cut`] inference rules due to the presence of $CtrL$ which may still stretch a [$\texttt{G1i}_{\texttt{p}}$- `Cut`]-derivation branch to infinity. The question as to whether $CtrL$ can be shown admissible in [$\texttt{G1i}_{\texttt{p}}$- `Cut`] becomes relevant.

As the first step of $CtrL$ elimination, it is a commonplace to attempt absorption of its effect into logical inference rules. Reasoning towards the objective is usually simpler in weakening-free [$\texttt{G1i}_{\texttt{p}}$- `Cut`] (*i.e.* some equivalent proof system to [$\texttt{G1i}_{\texttt{p}}$- `Cut`] in which the effect of WkL is absorbed within available logical inference rules). Figure 1.4 shows the resultant calculus, $\texttt{G3i}_{\texttt{p}}$ (*Cf.* Beth [1955]; Kleene [1952]; Troelstra and Schwichtenberg [2000]). A so-called inversion lemma holds in $\texttt{G3i}_{\texttt{p}}$.

Lemma 1 (Inversion lemma) *For the following sequent pairs, if the sequent on the left is $\texttt{G3i}_{\texttt{p}}$-derivable with the derivation depth of k or less, then so is (are) the se-*

$$\frac{}{\Psi; p \vdash p}\,id \qquad \frac{}{\Psi; \mathbb{1} \vdash A}\,\mathbb{1}L \qquad \frac{}{\Psi \vdash \top}\,\top R$$

$$\frac{\Psi; A; B \vdash C}{\Psi; A \wedge B \vdash C}\,\wedge L \qquad \frac{\Psi; A \vdash C \qquad \Psi; B \vdash C}{\Psi; A \vee B \vdash C}\,\vee L \qquad \frac{\Psi; A \supset B \vdash A \qquad \Psi; B \vdash C}{\Psi; A \supset B \vdash C}\supset L$$

$$\frac{\Psi \vdash A \qquad \Psi \vdash B}{\Psi \vdash A \wedge B}\,\wedge R \qquad \frac{\Psi \vdash A_i \quad (i \in \{1,2\})}{\Psi \vdash A_1 \vee A_2}\,\vee R \qquad \frac{\Psi; A \vdash B}{\Psi \vdash A \supset B}\supset R$$

Figure 1.4: $\texttt{G3i}_\texttt{p}$: a contraction-free sequent calculus for propositional intuitionistic logic.

quent(s) on the right. That is, the result is depth-preserving.

$$\begin{aligned}
\Psi; A \wedge B \vdash C &\quad , \quad \Psi; A; B \vdash C \\
\Psi; A \vee B \vdash C &\quad , \quad \textit{both } \Psi; A \vdash C \textit{ and } \Psi; B \vdash C \\
\Psi; A \supset B \vdash C &\quad , \quad \Psi; B \vdash C \\
\Psi \vdash A \wedge B &\quad , \quad \textit{both } \Psi \vdash A \textit{ and } \Psi \vdash B \\
\Psi \vdash A \supset B &\quad , \quad \Psi; A \vdash B
\end{aligned}$$

Proof. Details are found in Curry [1963]; Shütte [1950]; Troelstra and Schwichtenberg [2000]. ☐

Lemma 2 (Admissibility of WkL and $CtrL$) *$WkL_{\texttt{G1i}_\texttt{p}}$ and $CtrL_{\texttt{G1i}_\texttt{p}}$ are both depth-preserving admissible in [$\texttt{G3i}_\texttt{p}$+ $WkL_{\texttt{G1i}_\texttt{p}}$ + $CtrL_{\texttt{G1i}_\texttt{p}}$].*

Proof. Lemma 1 simplifies the proof of $CtrL_{\texttt{G1i}_\texttt{p}}$ admissibility. Proof approaches are found in Troelstra and Schwichtenberg [2000]. ☐

Proposition 1 (Equivalence of $\texttt{G1i}_\texttt{p}$ with $\texttt{G3i}_\texttt{p}$) *Any $\Psi \vdash A \in \mathfrak{D}_{\texttt{IL}}$ which is derivable in $\texttt{G3i}_\texttt{p}$ is also derivable in [$\texttt{G1i}_\texttt{p}$- $\texttt{Cut}$] and vice versa.*

$$\frac{}{\Psi; p \vdash p}\ id \qquad \frac{}{\Psi; \mathbb{1} \vdash A}\ \mathbb{1}L \qquad \frac{}{\Psi \vdash \top}\ \top R$$

$$\frac{\Psi; A; B \vdash C}{\Psi; A \wedge B \vdash C}\ \wedge L \qquad \frac{\Psi; A \vdash C \qquad \Psi; B \vdash C}{\Psi; A \vee B \vdash C}\ \vee L \qquad \frac{\Psi; p; A \vdash B}{\Psi; p; p{\supset}A \vdash B}\ \supset L_p$$

$$\frac{\Psi; A \vdash B}{\Psi; \top{\supset}A \vdash B}\ \supset L_\top \qquad \frac{\Psi; A_1{\supset}(A_2{\supset}A_3) \vdash C}{\Psi; (A_1 \wedge A_2){\supset}A_3 \vdash C}\ \supset L_\wedge \qquad \frac{\Psi; A_1{\supset}A_3; A_2{\supset}A_3 \vdash C}{\Psi; (A_1 \vee A_2){\supset}A_3 \vdash C}\ \supset L_\vee$$

$$\frac{\Psi; A_2{\supset}A_3 \vdash A_1{\supset}A_2 \qquad \Psi; A_3 \vdash C}{\Psi; (A_1{\supset}A_2){\supset}A_3 \vdash C}\ \supset L_\supset \qquad \frac{\Psi \vdash A_i \quad (i \in \{1, 2\})}{\Psi \vdash A_1 \vee A_2}\ \vee R$$

$$\frac{\Psi \vdash A \qquad \Psi \vdash B}{\Psi \vdash A \wedge B}\ \wedge R \qquad \frac{\Psi; A \vdash B}{\Psi \vdash A \supset B}\ \supset R$$

Figure 1.5: $\texttt{G4i}_\texttt{p}$: a contraction-free sequent calculus for propositional intuitionistic logic. Implicit contraction does not occur in any inference rule.

Proof. A detailed proof methodology is found in Chapter 3 and Chapter 4 of this thesis; it is, however, recommended to interested readers who are not yet familiar with sequent calculi that the proof be attempted with a reference of only Troelstra and Schwichtenberg [2000]. Lemma 2 for where contraction and weakening are required. □

In $\texttt{G3i}_\texttt{p}$, any formula duplicate upwards may only occur within $\supset L$, which is a visible improvement over $\texttt{G1i}_\texttt{p}$ towards a more efficient backward theorem proving. As far as the propositional fragment is concerned, it is actually possible to eliminate the lingering implicit contraction altogether out of the inference rule, as shown in Figure 1.5. $\texttt{G4i}_\texttt{p}$ is an equivalent variant of `LJT` (by Dyckhoff [1992]) in which decidability of propositional intuitionistic logic is readily established. Here I reformulate the results and the proofs in a more detailed and clearer - so do they appear to myself - manner than are found in Dyckhoff [1992]. Lemma 6 below about the behaviour of intuitionistic implication should be adequately understood before readers move onto Chapter 3.

Definition 20 (Irreducible sequents) *A structure $\Psi \in \mathfrak{S}_{\texttt{IL}}$ is said to be irreducible if it contains as its sub-structure*[1] *none of the following:*

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[1] In an ordinary sense. Ψ itself is also a sub-structure of Ψ.

1. $p; p \supset A$

2. $\top \supset A$

3. $\mathbb{1}$

4. $A_1 \wedge A_2$

5. $A_1 \vee A_2$

$\Psi \vdash A \in \mathfrak{D}_{\mathtt{IL}}$ is said to be irreducible iff Ψ is.

Lemma 3 (Normalisation) *Any $\Psi \vdash A \in \mathfrak{D}_{\mathtt{IL}}$ which is not irreducible can be normalised into a set of irreducible sequents such that it be $\mathtt{G3i_p}$-derivable iff they are.*

Proof. Follows from inversion lemma (Lemma 1) and a good observation of the semantics of $\mathbb{1}$. □

We now establish (not necessarily depth-preserving) equivalences that hold in $\mathtt{G3i_p}$-space, utilising the following depth-preserving results.

Lemma 4 (Preparatory observations)

1. *If $D : \Psi; A_2 \supset A_3 \vdash A_2$ is $\mathtt{G3i_p}$-derivable, then so is $D' : \Psi; (A_1 \wedge A_2) \supset A_3 \vdash A_2$, preserving the derivation depth.*

2. *If $D : \Psi; A_1 \supset A_3; A_2 \supset A_3 \vdash A_1 \vee A_2$ is $\mathtt{G3i_p}$-derivable, then so is at least either $D_1 : \Psi; A_1 \supset A_3; A_2 \supset A_3 \vdash A_1$ or $D_2 : \Psi; A_1 \supset A_3; A_2 \supset A_3 \vdash A_2$, preserving the derivation depth.*

Proof. By induction on derivation depth.

1. For the first case, the base case is trivial. For the inductive cases, assume that it holds true for all the derivation depths of up to k and show that it still holds true for the derivation depth of $k + 1$. Consider what the principal is for the last inference rule to (forwardly) derive D. If it is some formula in Ψ or the consequent formula, then induction hypothesis and the same inference rule conclude. Otherwise, if it is the antecedent formula $A_2 \supset A_3$, then $\Pi(D)$ looks like:

$$\frac{D_1 : \Psi; A_2 \supset A_3 \vdash A_2 \qquad D_2 : \Psi; A_3 \vdash A_2}{D : \Psi; A_2 \supset A_3 \vdash A_2} \supset L$$

Induction hypothesis on D_1 concludes.

2. For the second case, the base case is trivial. For the inductive cases, assume that it holds true for all the derivation depths of up to k and show that it still holds true for the derivation depth of $k + 1$. Consider what the principal is for the last inference rule to (forwardly) derive D. If it is some formula in Ψ, then induction hypothesis and the same inference rule conclude. If it is the antecedent formula of either $A_1 \supset A_3$ or $A_2 \supset A_3$, or the consequent formula $A_1 \vee A_2$, then induction hypothesis concludes.

□

Lemma 5 (Equivalences in `G3ip`)

1. *$D : \Psi; A \vdash B$ is `G3ip`-derivable iff $D' : \Psi; \top \supset A \vdash B$ is.*

2. *$D : \Psi; A_1 \supset (A_2 \supset A_3) \vdash C$ is `G3ip`-derivable iff $D' : \Psi; (A_1 \wedge A_2) \supset A_3 \vdash C$ is.*

3. *$D : \Psi; A_1 \supset A_3; A_2 \supset A_3 \vdash C$ is `G3ip`-derivable iff $D' : \Psi; (A_1 \vee A_2) \supset A_3 \vdash C$ is.*

4. *$D : \Psi; A_2 \supset A_3 \vdash A_1 \supset A_2$ is `G3ip`-derivable iff $D' : \Psi; (A_1 \supset A_2) \supset A_3 \vdash A_1 \supset A_2$ is.*

Proof. By induction on derivation depth into both directions. By Lemma 3 we only consider irreducible sequents.

1. First case is trivial.

2. For the second case, base cases are trivial into both directions. Consider the inductive cases now. Into the *if* direction, consider what the principal is for the last inference rule applied to derive D'. If it is some formula in Ψ or C, then induction hypothesis and the same inference rule conclude. On the other hand, if it is the antecedent formula $(A_1 \wedge A_2) \supset A_3$, then $\Pi(D')$ looks like:

$$\dfrac{D_1' : \Psi; (A_1 \wedge A_2) \supset A_3 \vdash A_1 \wedge A_2 \qquad D_2' : \Psi; A_3 \vdash C}{D' : \Psi; (A_1 \wedge A_2) \supset A_3 \vdash C} \supset L$$

By induction hypothesis on D_1', it holds that $D_x : \Psi; A_1 \supset (A_2 \supset A_3) \vdash A_1 \wedge A_2$ is $\mathtt{G3i_p}$-derivable. By inversion lemma on D_x, $D_y : \Psi; A_2 \supset A_3 \vdash A_2$ and $D_z : \Psi; A_1 \supset (A_2 \supset A_3) \vdash A_1$ are both $\mathtt{G3i_p}$-derivable. Then by $\supset L$ on D_y (as the left premise sequent) and on D_2' (as the right premise sequent), $D_a : \Psi; A_2 \supset A_3 \vdash C$ is $\mathtt{G3i_p}$-derivable. Then $\supset L$ on D_z (as the left premise sequent) and on D_a (as the right premise sequent) concludes. Into the *only if* direction, consider what the principal is for the last inference rule applied to derive D. If it is some formula in Ψ or C, then induction hypothesis and the same inference rule conclude. On the other hand, if it is the antecedent formula $A_1 \supset (A_2 \supset A_3)$, then we have the following partial derivation for D:

$$\dfrac{D_1 : \Psi; A_1 \supset (A_2 \supset A_3) \vdash A_1 \qquad \dfrac{D_3 : \Psi; A_2 \supset A_3 \vdash A_2 \qquad D_4 : \Psi; A_3 \vdash C}{D_2 : \Psi; A_2 \supset A_3 \vdash C} \supset L}{D : \Psi; A_1 \supset (A_2 \supset A_3) \vdash C} \supset L$$

By induction hypothesis on D_1, $D_x^* : \Psi; (A_1 \wedge A_2) \supset A_3 \vdash A_1$ is $\mathtt{G3i_p}$-derivable. Meanwhile, by Lemma 4 on D_3, $D_y^* : \Psi; (A_1 \wedge A_2) \supset A_3 \vdash A_2$ is $\mathtt{G3i_p}$-derivable. By $\wedge R$ on D_x^* and on D_y^*, $D_z^* : \Psi; (A_1 \wedge A_2) \supset A_3 \vdash A_1 \wedge A_2$ is $\mathtt{G3i_p}$-derivable. Then $\supset L$ on D_z^* (as the left premise sequent) and on D_4 (as the right premise sequent) concludes. To be exhaustive, however, we must have of couse considered the possibility where the $A_2 \supset A_3$ is not the principal on D_2. Suppose we actually had the following partial derivation for D:

$$\dfrac{D_1 : \Psi; A_1 \supset (A_2 \supset A_3) \vdash A_1 \qquad \dfrac{}{D_2}\,\textbf{Inf}}{D : \Psi; A_1 \supset (A_2 \supset A_3) \vdash C} \supset L$$

where **Inf** $\in \{id, \mathbb{1}L, \top R\}$. Then $\Psi; (A_1 \wedge A_2) \supset A_3 \vdash C$ would be clearly derivable with **Inf**. Suppose, instead, that we actually had the following partial derivation for D:

$$\dfrac{D_1 : \Psi; A_1 \supset (A_2 \supset A_3) \vdash A_1 \qquad \dfrac{D_3 : C_1; \Psi; A_2 \supset A_3 \vdash C_2}{D_2 : \Psi; A_2 \supset A_3 \vdash C_1 \supset C_2} \supset R}{D : \Psi; A_1 \supset (A_2 \supset A_3) \vdash C_1 \supset C_2} \supset L$$

But since we have inversion lemma, that D is derivable implies that $C_1; \Psi; A_1 \supset (A_2 \supset A_3) \vdash C_2$ is derivable, which would eliminate this particular $\supset R$ application on the right premise sequent of the $\supset L$. Similarly for when it is $\wedge R$ that applies instead of $\supset R$. Finally, suppose we had the following partial derivation for D:

$$\dfrac{D_1 : \Psi; A_1 \supset (A_2 \supset A_3) \vdash A_1 \qquad \dfrac{D_3 : \Psi; A_2 \supset A_3 \vdash C_i}{D_2 : \Psi; A_2 \supset A_3 \vdash C_1 \vee C_2}\vee R}{D : \Psi; A_1 \supset (A_2 \supset A_3) \vdash C_1 \vee C_2}\supset L$$

where $i \in \{1, 2\}$. Then because D_1 and D_3 are assumed derivable, so is $D' : \Psi; A_1 \supset (A_2 \supset A_3) \vdash C_i$, which would eliminate (that is, push down) this particular $\vee R$ application on the right premise sequent of the $\supset L$.

3. For the third case, suppose that $(A_1 \vee A_2) \supset A_3$ becomes the principal in D':

$$\dfrac{D_1' : \Psi; (A_1 \vee A_2) \supset A_3 \vdash A_1 \vee A_2 \qquad D_2' : \Psi; A_3 \vdash C}{D' : \Psi; (A_1 \vee A_2) \supset A_3 \vdash C}\supset L$$

In the meantime, we have the following partial derivation for D with the principal of either $A_1 \supset A_3$ or $A_2 \supset A_3$:

$$\dfrac{D_1 : \Psi; A_1 \supset A_3; A_2 \supset A_3 \vdash A_i \quad (i \in \{1, 2\}) \qquad D_2 : \Psi; A_j \supset A_3; A_3 \vdash C \quad (j = mod_2(i) + 1)}{D : \Psi; A_1 \supset A_3; A_2 \supset A_3 \vdash C}\supset L$$

where $\mathrm{mod}\;_2(x) \equiv x\ (mod\, 2)$ and $\mathrm{mod}\;_2(x) \in \{0, 1\}$. Base cases are trivial into both directions. We now consider the inductive cases. Into one direction, it holds, by induction hypothesis on D_1', that $D_x : \Psi; A_1 \supset A_3; A_2 \supset A_3 \vdash A_1 \vee A_2$ is $\mathtt{G3i_p}$-derivable. By Lemma 4 on D_x, $D_y : \Psi; A_1 \supset A_3; A_2 \supset A_3 \vdash A_i$ is also $\mathtt{G3i_p}$-derivable for at least either $i = 1$ or $i = 2$. Let us assume with no loss of generality that $i = 1$ here. Meanwhile, it holds, by induction hypothesis on D_2' and Lemma 2, that $D_z : \Psi; A_2 \supset A_3; A_3 \vdash C$ is $\mathtt{G3i_p}$-derivable. Then $\supset L$ on D_y (as the left premise sequent) and D_z (as the right premise sequent) conclude. Into the other direction, by induction hypothesis on D_1, we have $D_a : \Psi; (A_1 \vee A_2) \supset A_3 \vdash A_1$ as $\mathtt{G3i_p}$-derivable.[1] Then $D_b : \Psi; (A_1 \vee A_2) \supset A_3 \vdash A_1 \vee A_2$ is also $\mathtt{G3i_p}$-derivable via $\vee R$. Meanwhile, it holds, by induction on D_2, that $D_c : \Psi; A_2 \supset A_3; A_3 \vdash C$ is $\mathtt{G3i_p}$-derivable. By inversion lemma on D_c and

---

[1] I assume that $i = 1$ without a loss of generality.

Lemma 2, $D_d : \Psi; A_3 \vdash C$ is also $\texttt{G3i}_\texttt{p}$-derivable. $\supset L$ on D_b (as the left premise sequent) and D_d (as the right premise sequent) then concludes.

4. For the fourth case, by inversion lemma on both D and D', we need only show the following: $D'' : \Psi; A_2 \supset A_3; A_1 \vdash A_2$ is $\texttt{G3i}_\texttt{p}$-derivable iff $D''' : \Psi; (A_1 \supset A_2) \supset A_3; A_1 \vdash A_2$ is. Therefore it suffices to prove (more generally) that $D_a : \Psi; A_2 \supset A_3; A_1 \vdash C$ is $\texttt{G3i}_\texttt{p}$-derivable iff $D_b : \Psi; (A_1 \supset A_2) \supset A_3; A_1 \vdash C$ is. Suppose that $(A_1 \supset A_2) \supset A_3$ becomes the principal in D_b:

$$\frac{D_1 : \Psi; (A_1 \supset A_2) \supset A_3; A_1 \vdash A_1 \supset A_2 \qquad D_2 : \Psi; A_3; A_1 \vdash C}{D_b : \Psi; (A_1 \supset A_2) \supset A_3; A_1 \vdash C} \supset L$$

By inversion lemma, D_1 is $\texttt{G3i}_\texttt{p}$-derivable iff $D_1^* : \Psi; (A_1 \supset A_2) \supset A_3; A_1; A_1 \vdash A_2$ is iff $D_1^{**} : \Psi; (A_1 \supset A_2) \supset A_3; A_1 \vdash A_2$ is (contraction admissibility due to Lemma 2). Meanwhile, we have the following for D_a with the principal $A_2 \supset A_3$:

$$\frac{D_1' : \Psi; A_2 \supset A_3; A_1 \vdash A_2 \qquad D_2' : \Psi; A_3; A_1 \vdash C}{D_a : \Psi; A_2 \supset A_3; A_1 \vdash C} \supset L$$

D_2' is identical to D_2. By induction hypothesis (into both directions), D_1' is $\texttt{G3i}_\texttt{p}$-derivable iff D_1^{**} is. $\square$

Now follows the main contribution of Dyckhoff [1992]. Its purpose is to establish that, if $p \supset A$ for some $p \in \mathcal{P}$ and some $A \in \mathfrak{F}_{\text{IL}}$ is in a sequent, then $\supset L$ does not need apply unless the sequent has $p; p \supset A$ on the antecedent.

Lemma 6 (Behaviour of intuitionistic implication)

Let D denote a $\texttt{G3i}_\texttt{p}$-derivable irreducible sequent $\Psi \vdash C \in \mathfrak{D}_{\text{IL}}$, then D has a closed derivation in which the principal of the last inference rule to derive D is not in the form: $p \supset A$ for $p \in \mathcal{P}$ and $A \in \mathfrak{F}_{\text{IL}}$.

Proof. Proof is by contradiction, that is, by classical reasoning that a counter-example to the current lemma cannot exist.[1] Suppose, by way of showing contradiction, that

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[1] We do not need to know in advance whether propositional intuitionistic logic is decidable for this proof to (classically) go through since the derivation tree is assumed closed. *Cf.* Davey and Priestley [2002] for example.

there cannot exist any other derivation trees for D with a shorter (in derivation length) leftmost derivation branch than ones ending in $\supset L$ with a formula in the form: $p \supset A$ as its principal, then one such derivation tree $\Pi(D)$ would look like:

$$\dfrac{\dfrac{\begin{array}{cc}\overline{D_L} & \\ \vdots & \vdots \\ D_1 & D_2\end{array}}{D_3 : \Psi';p \supset A \vdash p}\,\textbf{Inf} \qquad \begin{array}{c}\vdots \\ D_4 : \Psi';A \vdash C\end{array}}{D : \Psi';p \supset A \vdash C}\supset L$$

where $\Psi = \Psi';p \supset A$ or alternatively $\Psi \equiv \Psi';p \supset A$ (up to associativity and commutativity of ";"), and D_L is the conclusion sequent of an axiom rule in the leftmost derivation branch. As D is irreducible, so is D_3 which, therefore, cannot be the conclusion sequent of an axiom. Then, since a propositional variable can be active only for id, the consequent part of D_3 cannot be active for some $\mathtt{G3i_p}$ inference rule **Inf**. **Inf** is hence known to be $\supset L$. Furthermore, that the leftmost derivation branch is shortest has to dictate that the principal for **Inf** is not in the form: "$p' \supset A'$" for some $p' \in \mathcal{P}$ and some $A' \in \mathfrak{F}_{\mathtt{IL}}$.

These points taken into account, D, D_1, D_2, D_3 and D_4 are actually seen taking the following forms for some other $B, B' \in \mathfrak{F}_{\mathtt{IL}}$ for $B \notin \mathcal{P}$:

- $D : \Psi'';B \supset B';p \supset A \vdash C$
- $D_1 : \Psi'';B \supset B';p \supset A \vdash B$
- $D_2 : \Psi'';B';p \supset A \vdash p$
- $D_3 : \Psi'';B \supset B';p \supset A \vdash p$
- $D_4 : \Psi'';B \supset B';A \vdash C$

But, then, this perforce implies the existence of an alternative derivation tree $\Pi'(D)$ which results by permuting $\Pi(D)$:

$$\dfrac{\begin{array}{c}\overline{D_L} \\ \vdots \\ D_1\end{array} \qquad \dfrac{\begin{array}{c}\vdots \\ D_2\end{array} \qquad \begin{array}{c}\vdots \\ D_4' : \Psi'';B';A \vdash C\end{array}}{\Psi'';B';p \supset A \vdash C}\supset L}{D}\supset L$$

D_4' derives from applying inversion lemma (Lemma 1) on D_4. A direct contradiction to the supposition has been drawn, for the leftmost branch in $\Pi'(D)$ is shorter, *i.e.* $\texttt{der\_len}_{\Pi'(D)}(D, D_L) < \texttt{der\_len}_{\Pi(D)}(D, D_L)$.[1] □

The equivalence of $\texttt{G3i}_\texttt{p}$ to $\texttt{G4i}_\texttt{p}$ is proved with the induction measure of sequent weight.

Definition 21 (Sequent weight) *Given a sequent* $\Psi \vdash A \in \mathfrak{D}_{\texttt{IL}}$*, its sequent weight is defined to be the sum of the formula weights of all the formulas occurring within. The formula weight of* $A \in \mathfrak{F}_{\texttt{IL}}$*, which we here denote by* $\texttt{f\_weight}(A)$*, is defined as follows:*

- $\texttt{f\_weight}(A) = 2$ *if* $A \in \{\top, \mathbb{1}, p\}$.
- $\texttt{f\_weight}(A) = \texttt{f\_weight}(A_1)(1 + \texttt{f\_weight}(A_2))$ *if* $A = A_1 \wedge A_2$.
- $\texttt{f\_weight}(A) = 1 + \texttt{f\_weight}(A_1) + \texttt{f\_weight}(A_2)$ *if* $A = A_1 \vee A_2$.
- $\texttt{f\_weight}(A) = 1 + \texttt{f\_weight}(A_1)\texttt{f\_weight}(A_2)$ *if* $A = A_1 \supset A_2$.

Proposition 2 (Equivalence of $\texttt{G4i}_\texttt{p}$ with $\texttt{G3i}_\texttt{p}$ Dyckhoff [1992]) *Any* $\Psi \vdash A \in \mathfrak{D}_{\texttt{IL}}$ *which are derivable in* $\texttt{G4i}_\texttt{p}$ *are also derivable in* $\texttt{G3i}_\texttt{p}$ *and vice versa.*

Proof. One direction, to show that what $\texttt{G4i}_\texttt{p}$ derive are derivable also in $\texttt{G3i}_\texttt{p}$, is straightforward since all the inference rules in $\texttt{G4i}_\texttt{p}$ are derivable in $\texttt{G3i}_\texttt{p}$ (*Cf.* Lemma 5 for $\texttt{G4i}_\texttt{p}$ implication rules). Proof into the other direction of showing what $\texttt{G3i}_\texttt{p}$ derive are also derivable in $\texttt{G4i}_\texttt{p}$ is by induction on sequent weight. First of all note that all the $\texttt{G3i}_\texttt{p}$ inference rules are identical to a corresponding $\texttt{G4i}_\texttt{p}$ inference rule except for $\supset L$, and that the sequent weight of premise sequent(s) is lower than that of the consequent sequent. For $\supset L$, consider what the actual instance for A is in $A \supset B$.

1. $A = \top$: trivial.
2. $A = \mathbb{1}$: Straightforward:

$$\frac{D_1 : \Psi; \mathbb{1} \supset B \vdash \mathbb{1} \qquad \Psi; B \vdash C}{D : \Psi; \mathbb{1} \supset B \vdash C} \supset L_{\texttt{G3i}_\texttt{p}}$$

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[1] *Cf.* Definition 6 for $\texttt{der\_len}$, that is, derivation length. The sub-script is used to specify the particular derivation tree and nodes in the tree.

The consequent part of D_1 is $\mathbb{1}$, and so for D to be $\texttt{G3i}_\texttt{p}$-derivable, it must be the case that, for all $A \in \mathfrak{F}_{\texttt{IL}}$, $\Psi \vdash A$ is $\texttt{G3i}_\texttt{p}$-derivable. In particular $A = C$ (or alternatively $A \equiv C$ up to assoc. and commut. as in Property 1).

3. $A = A_1 \wedge A_2$: Lemma 5.
4. $A = A_1 \vee A_2$: Lemma 5.
5. $A = A_1 \supset A_2$: Lemma 5.
6. $A = p$: Lemma 6. □

$\texttt{G4i}_\texttt{p}$ suggests an efficient decision procedure for propositional intuitionistic logic since it exhibits a property that the weight of any premise sequent in each $\texttt{G4i}_\texttt{p}$ inference rule is strictly smaller than that of the conclusion sequent. The lingering implicit contaction on the left premise sequent of $\supset L_{\texttt{G3i}_\texttt{p}}$ is no longer visible.

Along with $\texttt{G3i}_\texttt{p}$ inversions (*Cf.* Lemma 1), the following inversion results also hold in $\texttt{G4i}_\texttt{p}$.

Lemma 7 (Additional inversions for $\texttt{G4i}_\texttt{p}$) *For the following sequent pairs, if the sequent on the left is $\texttt{G4i}_\texttt{p}$-derivable with the derivation depth of k or less, then so is the sequent shown on the right.*

$$\begin{array}{lcl} \Psi; (A_1 \wedge A_2) \supset A_3 \vdash C & , & \Psi; A_1 \supset (A_2 \supset A_3) \vdash C \\ \Psi; (A_1 \vee A_2) \supset A_3 \vdash C & , & \Psi; A_1 \supset A_3; A_2 \supset A_3 \vdash C \end{array}$$

Proof. Straightforward. (Note, for the first, that the antecedent formula $(A_1 \wedge A_2) \supset A_3$ can become the principal only for $\supset L_\wedge$, and, for the second, that the antecedent formula $(A_1 \vee A_2) \supset A_3$ can become the principal only for $\supset L_\vee$.) □

1.1.3 Multiplicative intuitionistic linear logic without exponentials

According to Girard [1987], neither classical logic nor intuitionistic logic captures real-life causation adequately. He discountenanced the following example which is enforced in classical/intuitionistic logic: if a proposition p holds true and a proposition $p \supset q$ holds true, then q holds true, but p still holds true. What Girard [1987] finds problematic is the monotonic presence of propositions ("stable truths" in Girard [1987]). But the permanency of something that currently exists is capitalistically denied for instance in the principle of exchange: if we have a sufficient amount of money that buys a car, then we can exchange our money for the car, but the money so spent no longer remains with us. This concept of resource exchange is described nicely in linear logic (*Cf.* Girard [1987]). In addition to the material implication in classical/intuitionistic logic, *linear implication*[1] $\mathrel{-\!\!*}$ is used in order to describe; if a proposition p holds true and another proposition $p \mathrel{-\!\!*} q$ holds true, then if the true proposition p is consumed, q becomes a truth. To say that there are such resources $p, q, r \cdots$, a logical connective 'times' $*$[2] is used: $p * q * r * \cdots$. To say that there is a 'zero' resource, a nullary logical connective ${}^{*}\!\top$[3] is used. Although this thesis requires only these three (which form multiplicative intuitionistic lienar logic without exponentials), other features of linear logic can be learned from Girard [1987].

Definition 22 (Formulas/Structures) *A formula in multiplicative intuitionistic linear logic without exponentials* $J(, K, L)$ *is defined by* $J := p \mid {}^{*}\!\top \mid J * J \mid J \mathrel{-\!\!*} J$. *By* $\mathfrak{F}_{\mathtt{MILL}}$ *the set of formulas that this grammar generates is denoted. A structure in multiplicative intuitionistic linear logic without exponentials* $\Upsilon(, \Lambda)$ *is defined by* $\Upsilon := J \mid J, J$. *By* $\mathfrak{S}_{\mathtt{MILL}}$ *the set of structures that this grammar generates is denoted.*

The following associativity and commutativity hold.

Property 3 (Associativity and commutativity)

1. $(J_1 * J_2) * J_3 = J_1 * (J_2 * J_3)$.

2. $J_1 * J_2 = J_2 * J_1$.

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[1] As Girard [1987] calls. In his article, the symbol $\multimap$ is used for linear implication.

[2] In Girard [1987], the symbol $\otimes$ is instead used.

[3] In Girard [1987], it is I.

$$\frac{}{J \vdash J}\,id \qquad \frac{\Upsilon \vdash J \qquad \Lambda, J \vdash K}{\Upsilon, \Lambda \vdash K}\,\mathtt{Cut} \qquad \frac{}{\vdash {}^{*}\top}\,{}^{*}\top R$$

$$\frac{\Upsilon, J, K \vdash L}{\Upsilon, J * K \vdash L}\,*L \qquad \frac{\Upsilon \vdash J \qquad \Lambda \vdash K}{\Upsilon, \Lambda \vdash J * K}\,*R \qquad \frac{\Upsilon \vdash J \qquad \Lambda, K \vdash L}{\Upsilon, \Lambda, J \mathrel{-\!\!*} K \vdash L}\,\mathrel{-\!\!*}L$$

$$\frac{\Upsilon, J \vdash K}{\Upsilon \vdash J \mathrel{-\!\!*} K}\,\mathrel{-\!\!*}R$$

Figure 1.6: $\mathtt{MILL_p}$: A sequent calculus for propositional multiplicative intuitionistic linear logic without exponentials.

3. $(\Upsilon_1, \Upsilon_2), \Upsilon_3 = \Upsilon_1, (\Upsilon_2, \Upsilon_3)$.
4. $\Upsilon_1, \Upsilon_2 = \Upsilon_2, \Upsilon_1$.

A sequent calculus formulation of multiplicative intuitionistic linear logic without exponentials, $\mathtt{MILL_p}$, is found in Figure 1.6.

Lemma 8 (Sequents) *A sequent in multiplicative intuitionistic linear logic without exponentials is defined to be in the form:* $\Upsilon \vdash J$ *for some* $\Upsilon \in \mathfrak{S}_{\mathtt{MILL}}$ *and some* $J \in \mathfrak{F}_{\mathtt{MILL}}$. *The set of the sequents in multiplicative intuitionistic linear logic is denoted by* $\mathfrak{D}_{\mathtt{MILL}}$.

Definition 23 (Sequent calculus convention) *For any sequent in* $\mathfrak{D}_{\mathtt{MILL}}$ *in the form:* $\Upsilon_1, \Upsilon_2 \vdash J$, *the antecedent structure "*Υ_1, Υ_2*" is identified with* Υ_1 *(or* Υ_2*) if* Υ_2 *(or* Υ_1*) is empty.*

Example 3 $p \vdash p * {}^{*}\top \in \mathfrak{D}_{\mathtt{MILL}}$ *is derivable in* $\mathtt{MILL}_p$ *as follows.*

$$\frac{\dfrac{}{\vdash {}^{*}\top}\,{}^{*}\top R \qquad \dfrac{}{p \vdash p}\,id}{p \vdash p * {}^{*}\top}\,*R$$

1.1.4 `BI`

`BI` is a combined logic of intuitionistic logic and multiplicative intuitionistic linear logic without exponentials. A fragment of `BI` is simply intuitionistic logic and another fragment simply multiplicative intuitionistic linear logic without exponentials. As a whole, however, `BI` exhibits distinct logical characteristics, allowing any logical connectives available to either of the base logics to appear at any part of a `BI` formula.

Definition 24 (Formulas) *A* `BI` *formula* $F, (G, H)$ *is defined by:*

$$F := p \mid \top \mid \mathbb{1} \mid {}^{*}\top \mid F \wedge F \mid F \vee F \mid F \supset F \mid F * F \mid F \mathrel{-\!\!*} F.$$

The set of formulas that this grammar generates is denoted by $\mathfrak{F}_{\texttt{BI}}$.

The following associativity and commutativity hold within $\mathfrak{F}_{\texttt{BI}}$.

Property 4 (Associativity and commutativity)

1. $(F_1 * F_2) * F_3 = F_1 * (F_2 * F_3)$.
2. $(F_1 \wedge F_2) \wedge F_3 = F_1 \wedge (F_2 \wedge F_3)$.
3. $(F_1 \vee F_2) \vee F_3 = F_1 \vee (F_2 \vee F_3)$.
4. $F_1 * F_2 = F_2 * F_1$.
5. $F_1 \wedge F_2 = F_2 \wedge F_1$.
6. $F_1 \vee F_2 = F_2 \vee F_1$.

Semantics of `BI` based on relational models is developed in Galmiche et al. [2005].

Definition 25 (BI frame Galmiche et al. [2005]) *A* `BI` *frame is a 4-tuple* $(W, \epsilon, \pi, \mathcal{R})$ *with a set* W *of possible worlds, a neutral element* $\epsilon \in W$*, a greatest element* $\pi \in W$*, and a ternary relation* $\mathcal{R}$*, satisfying:*

1. $\forall x \in W.\mathcal{R}\epsilon x x$.
2. $\forall x, y, z \in W.\mathcal{R}xyz \leftrightarrow^{\dagger} \mathcal{R}yxz$.
3. $\forall x, y, z, t \in W.(\exists u \in W.\mathcal{R}xyu \wedge^{\dagger} \mathcal{R}uzt) \leftrightarrow^{\dagger} (\exists v \in W.\mathcal{R}yzv \wedge^{\dagger} \mathcal{R}xvt)$.
4. $\forall x, x', y, z \in W.\mathcal{R}xyz \wedge^{\dagger} \mathcal{R}\epsilon x' x \rightarrow^{\dagger} \mathcal{R}x' yz$.
5. $\forall x, y, z, z' \in W.\mathcal{R}xyz \wedge^{\dagger} \mathcal{R}\epsilon z z' \rightarrow^{\dagger} \mathcal{R}xyz'$.
6. $\forall x, y \in W.\mathcal{R}xy\pi$.

$$\frac{}{F \vdash F}\ \text{id} \qquad \frac{\Gamma_1 \vdash G \qquad \Gamma(G) \vdash H}{\Gamma(\Gamma_1) \vdash H}\ \text{Cut} \qquad \frac{}{\Gamma(\mathbb{1}) \vdash H}\ \mathbb{1}L \qquad \frac{}{\Gamma \vdash \top}\ \top R$$

$$\frac{\Gamma(\emptyset_a) \vdash H}{\Gamma(\top) \vdash H}\ \top L \qquad \frac{\Gamma(\emptyset_m) \vdash H}{\Gamma({}^{*}\top) \vdash H}\ {}^{*}\top L \qquad \frac{}{\emptyset_m \vdash {}^{*}\top}\ {}^{*}\top R \qquad \frac{\Gamma(F;G) \vdash H}{\Gamma(F \wedge G) \vdash H}\ \wedge L$$

$$\frac{\Gamma(F) \vdash H \qquad \Gamma(G) \vdash H}{\Gamma(F \vee G) \vdash H}\ \vee L \qquad \frac{\Gamma_1 \vdash F \qquad \Gamma(\Gamma_1; G) \vdash H}{\Gamma(\Gamma_1; F \supset G) \vdash H}\ \supset L \qquad \frac{\Gamma(F,G) \vdash H}{\Gamma(F * G) \vdash H}\ *L$$

$$\frac{\Gamma_1 \vdash F \qquad \Gamma(G) \vdash H}{\Gamma(\Gamma_1, F \mathrel{-\!\!*} G) \vdash H}\ \mathrel{-\!\!*}L \qquad \frac{\Gamma \vdash F \qquad \Gamma \vdash G}{\Gamma \vdash F \wedge G}\ \wedge R \qquad \frac{\Gamma \vdash F_i}{\Gamma \vdash F_1 \vee F_2}\ \vee R \qquad \frac{\Gamma; F \vdash G}{\Gamma \vdash F \supset G}\ \supset R$$

$$\frac{\Gamma_1 \vdash F \qquad \Gamma_2 \vdash G}{\Gamma_1, \Gamma_2 \vdash F * G}\ *R \qquad \frac{\Gamma, F \vdash G}{\Gamma \vdash F \mathrel{-\!\!*} G}\ \mathrel{-\!\!*}R \qquad \frac{\Gamma(\Gamma_1) \vdash H}{\Gamma(\Gamma_1; \Gamma_2) \vdash H}\ \text{Wk L} \qquad \frac{\Gamma(\Gamma_1; \Gamma_1) \vdash H}{\Gamma(\Gamma_1) \vdash H}\ \text{Ctr L}$$

$$\frac{\Gamma(\Gamma_1; \emptyset_a) \vdash H}{\Gamma(\Gamma_1) \vdash H}\ EqAnt_1 \qquad \frac{\Gamma(\Gamma_1, \emptyset_m) \vdash H}{\Gamma(\Gamma_1) \vdash H}\ EqAnt_2$$

Figure 1.7: `LBI`: a `BI` sequent calculus. $i \in \{1, 2\}$.

7. $\forall x, y \in W.\mathcal{R}\pi xy \rightarrow^{\dagger} [\pi = y]$.

Definition 26 (Interpretation) *An interpretation* $\mathsf{I}_{\texttt{BI}}$ *is a function that maps propositional variables to the power-set of* W*,* i.e. $\mathsf{I}_{\texttt{BI}} : \mathcal{P} \rightarrow \mathbb{P}(W)$*. The following monotonicity holds:* $\forall m, n \in W\ \forall p \in \mathcal{P}.\mathcal{R}\epsilon mn \wedge^{\dagger} [m \in \mathsf{I}_{\texttt{BI}}(p)] \rightarrow^{\dagger} [n \in \mathsf{I}_{\texttt{BI}}(p)]$.

Definition 27 (`BI` Kripke relational model) *A* `BI` *Kripke relational model is a 6-tuple* $(W, \epsilon, \pi, \mathcal{R}, \mathsf{I}_{\texttt{BI}}, \models)$ *for some* `BI` *frame* $(W, \epsilon, \pi, \mathcal{R})$ *and some interpretation* $\mathsf{I}_{\texttt{BI}}$*, satisfying, for all* $p \in \mathcal{P}$ *and for all* $m \in W$*:*

$m \models p$ *iff* $m \in \mathsf{I}_{\texttt{BI}}(p)$.
$m \models \top$.
$m \models \mathbb{1}$ *iff* $m = \pi$.
$m \models {}^{*}\top$ *iff* $\mathcal{R}\epsilon\epsilon m$.
$m \models F_1 \wedge F_2$ *iff* $[m \models F_1] \wedge^{\dagger} [m \models F_2]$.
$m \models F_1 \vee F_2$ *iff* $[m \models F_1] \vee^{\dagger} [m \models F_2]$.
$m \models F_1 \supset F_2$ *iff* $\forall m' \in W.\mathcal{R}\epsilon mm' \rightarrow^{\dagger} ([m' \models F_1] \rightarrow^{\dagger} [m' \models F_2])$.
$m \models F_1 * F_2$ *iff* $\exists m_1, m_2 \in W.\mathcal{R}m_1 m_2 m \wedge^{\dagger} [m_1 \models F_1] \wedge^{\dagger} [m_2 \models F_2]$.
$m \models F_1 \mathrel{-\!\!*} F_2$ *iff* $\forall m_1, m_2 \in W.[m_1 \models F_1] \rightarrow^{\dagger} (\mathcal{R}mm_1m_2 \rightarrow^{\dagger} [m_2 \models F_2])$.

Definition 28 (Universal validity) *A formula* $F \in \mathfrak{F}_{\texttt{BI}}$ *is said to be valid in some* BI *Kripke relational model* $(W, \epsilon, \pi, \mathfrak{R}, \mathsf{I}_{\texttt{BI}}, \models)$ *iff* $\forall m \in W.[m \models F]$*. It is said to be universally valid iff it is valid in all the conceivable* BI *Kripke relational models.*

Pym [2002] presents the first sequent calculus for BI, as shown in Figure 1.7.

Definition 29 (Structures) *A structure* Γ *is defined by:*

$\Gamma := F \mid \emptyset_a \mid \emptyset_m \mid \Gamma;\Gamma \mid \Gamma,\Gamma$

where $F \in \mathfrak{F}_{\texttt{BI}}$. $\emptyset_a$ *is an additive structural unit and* $\emptyset_m$ *a multiplicative structural unit. The set of structures that this grammar generates is denoted by* $\mathfrak{S}_{\texttt{BI}}$.

Property 5 (Associativity and commutativity of structures)
The following hold within $\mathfrak{S}_{\texttt{BI}}$*:*

1. $\Gamma_1;(\Gamma_2;\Gamma_3) = (\Gamma_1;\Gamma_2);\Gamma_3$.
2. $\Gamma_1,(\Gamma_2,\Gamma_3) = (\Gamma_1,\Gamma_2),\Gamma_3$.
3. $\Gamma_1;\Gamma_2 = \Gamma_2;\Gamma_1$.
4. $\Gamma_1,\Gamma_2 = \Gamma_2,\Gamma_1$.

Distributivity over the two structural connectives, however, is limited, which prompts us into specifically defining contexts.

Definition 30 (Contexts) *A* BI *context* $\Gamma(-)$ *is defined by:*

$\Gamma(-) := - \mid -;\Gamma \mid \Gamma;- \mid \Gamma(-);\Gamma \mid \Gamma;\Gamma(-) \mid \Gamma(-),\Gamma \mid \Gamma,\Gamma(-)$.

Given any BI *context* $\Gamma_1(-)$ *and any* $\Gamma_2 \in \mathfrak{S}_{\texttt{BI}}$*,* $\Gamma_1(\Gamma_2)$ *is some* BI *structure* $\Gamma_3 \in \mathfrak{S}_{\texttt{BI}}$ *that results from replacing the occurrence of "-" in* $\Gamma_1(-)$ *with* Γ_2. $\Gamma(\Gamma_1)(\Gamma_2)$ *abbreviates* $(\Gamma(\Gamma_1))(\Gamma_2)$.

Definition 31 (Sequents) *All the sequents appearing in* LBI *derivations are a set element of* $\mathfrak{D}_{\texttt{BI}} := \{\Gamma \vdash F | [\Gamma \in \mathfrak{S}_{\texttt{BI}}] \wedge^{\dagger} [F \in \mathfrak{F}_{\texttt{BI}}]\}$.

Theorem 2 (Soundness and completeness Galmiche et al. [2005])
Any $F \in \mathfrak{F}_{\texttt{BI}}$ *derivable in* LBI *is universally valid (soundness). Any universally valid* $F \in \mathfrak{F}_{\texttt{BI}}$ *is derivable in* LBI *(completeness).*

1.1.5 `BBI` and Separation Logic

1.1.5.1 `BBI`

`BBI` is a combined logic of classical logic and multiplicative (intuitionistic) linear logic without exponentials. Galmiche and Larchey-Wendling [2006] present semantics of `BBI` based on non-deterministic monoids.

Definition 32 (Formulas) *A* `BBI` *formula $F(, G, H)$ is a set element of $\mathfrak{F}_{\mathtt{BI}}$. The set of* `BBI` *formulas is denoted by $\mathfrak{F}_{\mathtt{BBI}}$. Clearly $\mathfrak{F}_{\mathtt{BBI}} = \mathfrak{F}_{\mathtt{BI}}$. Both associativity and commutativity within $\mathfrak{F}_{\mathtt{BI}}$ carry over to $\mathfrak{F}_{\mathtt{BBI}}$.*

Definition 33 (Non-deterministic monoid) *A non-deterministic monoid is defined by a 3-tuple $(W, \circ, \epsilon)$ with a set W of possible worlds, a binary function $\circ : \mathbb{P}(W) \times \mathbb{P}(W) \to \mathbb{P}(W)$ and a neutral element $\epsilon \in W$, satisfying the following:*

1. $\forall w \in W.\{\epsilon\} \circ \{w\} = \{w\}$ *(neutrality).*
2. $\forall w_1, w_2 \in W.\{w_1\} \circ \{w_2\} = \{w_2\} \circ \{w_1\}$ *(commutativity).*
3. $\forall w_1 \in \mathbb{P}(W)\, \forall w_2 \in W.w_1 \circ \{w_2\} = \{\{w_3\} \circ \{w_2\} \mid w_3 \in w_1\}$.
4. $\forall w_2 \in \mathbb{P}(W)\, \forall w_1 \in W.\{w_1\} \circ w_2 = \{\{w_1\} \circ \{w_3\} \mid w_3 \in w_2\}$.
5. $\forall w_1, w_2, w_3 \in W.\{w_1\} \circ (\{w_2\} \circ \{w_3\}) = (\{w_1\} \circ \{w_2\}) \circ \{w_3\}$ *(associativity).*
6. $\forall w \in \mathbb{P}(W).\emptyset \circ w = \emptyset$ *(composition of undefinedness).*[1]

Definition 34 (Interpretation) *An interpretation $\mathsf{I}_{\mathtt{BBI}}$ is a function that maps propositional variables into the power-set of W,* i.e. $\mathsf{I}_{\mathtt{BBI}} : \mathcal{P} \to \mathbb{P}(W)$.

Definition 35 (`BBI` Kripke semantics) *A* `BBI` *non-deterministic Kripke model is defined to be a 5-tuple $(W, \circ, \epsilon, \mathsf{I}_{\mathtt{BBI}}, \models)$ for some non-deterministic monoid $(W, \circ, \epsilon)$, some interpretation $\mathsf{I}_{\mathtt{BBI}}$ and a forcing relation $\models$, satisfying, for all $m \in W$:*

- $m \models p$ *iff* $m \in \mathsf{I}_{\mathtt{BBI}}(p)$.
- $m \models \top$.

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[1] $\emptyset$ denotes an empty set elsewhere in this thesis.

$$\dfrac{}{F \supset (G \supset F)}\,\text{Ax1} \qquad \dfrac{}{(F \supset (G \supset H)) \supset ((F \supset G) \supset (F \supset H))}\,\text{Ax2} \qquad \dfrac{}{F{\supset}F \vee G}\,\text{Ax3}$$

$$\dfrac{}{G{\supset}F \vee G}\,\text{Ax4} \qquad \dfrac{}{(F{\supset}H){\supset}((G{\supset}H){\supset}(F \vee G{\supset}H))}\,\text{Ax5} \qquad \dfrac{}{F \wedge G{\supset}F}\,\text{Ax6}$$

$$\dfrac{}{F \wedge G{\supset}G}\,\text{Ax7} \qquad \dfrac{}{F{\supset}(G{\supset}(F \wedge G))}\,\text{Ax8} \qquad \dfrac{}{\mathbb{1} \supset F}\,\text{Ax9}$$

$$\dfrac{}{F \supset \top}\,\text{Ax10} \qquad \dfrac{}{((F \supset \mathbb{1}) \supset \mathbb{1}) \supset F}\,\text{Ax11} \qquad \dfrac{}{F \supset ({}^{*}\top * F)}\,\text{Ax12}$$

$$\dfrac{}{({}^{*}\top * F) \supset F}\,\text{Ax13}$$

$$\dfrac{F_1 \supset G_1 \qquad F_2 \supset G_2}{(F_1 * F_2) \supset (G_1 * G_2)}\,* \qquad \dfrac{F \supset (G \mathrel{-\!\!*} H)}{(F * G) \supset H}\,{-\!\!*}_1 \qquad \dfrac{(F * G) \supset H}{F \supset (G \mathrel{-\!\!*} H)}\,{-\!\!*}_2$$

$$\dfrac{F \qquad F \supset G}{G}\,\text{MP}$$

Figure 1.8: `HBBI`: a `BBI` Hilbert-style system.

- $\neg^{\dagger}[m \models \mathbb{1}]$.
- $m \models {}^{*}\top$ *iff* $m = \epsilon$.
- $m \models F_1 \wedge F_2$ *iff* $[m \models F_1] \wedge^{\dagger} [m \models F_2]$.
- $m \models F_1 \vee F_2$ *iff* $[m \models F_1] \vee^{\dagger} [m \models F_2]$.
- $m \models F_1 \supset F_2$ *iff* $\neg^{\dagger}[m \models F_1] \vee^{\dagger} [m \models F_2]$.
- $m \models F_1 * F_2$ *iff* $\exists m_1, m_2 \in W.[m \in m_1 \circ m_2] \wedge^{\dagger} [m_1 \models F_1] \wedge^{\dagger} [m_2 \models F_2]$.
- $m \models F_1 \mathrel{-\!\!*} F_2$ *iff* $\forall m_1, m_2 \in W.[m_2 \in m \circ m_1] \rightarrow^{\dagger} ([m_1 \models F_1] \rightarrow^{\dagger} [m_2 \models F_2])$.

Definition 36 (Universal validity) *A formula* $F \in \mathfrak{F}_{\texttt{BBI}}$ *is said to be valid in some* `BBI` *non-deterministic Kripke model* $(W, \circ, \epsilon, \mathsf{I}_{\texttt{BBI}}, \models)$ *iff* $\forall m \in W.[m \models F]$*. It is said to be universally valid iff it is valid in all the conceivable* `BBI` *non-deterministic Kripke models.*

They present a `BBI` Hilbert-style system as shown in Figure 1.8. I call the proof system `HBBI` in this thesis.

Theorem 3 (Soundness and completeness) *Any formula* $F \in \mathfrak{F}_{\texttt{BBI}}$ *which is derivable in* `HBBI` *is universally valid (soundness). Any formula* $F \in \mathfrak{F}_{\texttt{BBI}}$ *which is universally valid is derivable in* `HBBI` *(completeness).*

Proof. Proofs are found in Galmiche and Larchey-Wendling [2006]. □

Theorem 4 (Undecidability of `BBI`) `BBI` *is undecidable.*

Proof. Proofs are found in Brotherston and Kanovich [2010]; Larchey-Wendling and Galmiche [2012]. □

1.1.5.2 Separation logic

Separation logic, a prominent logic in program analysis, is closely related to `BBI`. To define the heap model of separation logic, first assume the following:

- a countable set of variables Var $(= \{x_1, x_2, \cdots\})$.
- a countable set of locations L $(= \{l_1, l_2, \cdots\})$.
- a countable set of constants $Const$ $(= \{c_1, c_2, \cdots\})$.
- a countable set of values $V = L \bigcup Const$ $(= \{v_1, v_2, \cdots\})$.
- an expression Ex which can be an element either of Var or of V.
- a set of finite partial functions mapping a subset of locations into values `Heap` $= \bigcup_{L' \subseteq^{fin} L}(L' \rightarrow V)$.

Definition 37 (Formulas) *A separation logic formula* FF $(, GG, HH)$ *is defined by:* $FF \ := \ Ex_1 {\mapsto} Ex_2 \mid \top \mid \mathbb{1} \mid {}^{*}\!\top \mid FF_1 \wedge FF_2 \mid FF_1 \vee FF_2 \mid FF_1 {\supset} FF_2 \mid FF_1 * FF_2 \mid FF_1 {-\!\!*} FF_2$. *The set of separation logic formulas is denoted by* $\mathfrak{F}_{sep}$.

Definition 38 (Heap monoid) *A heap monoid is a 3-tuple* $(\mathtt{Heap}, \circ, \{\mathtt{emp}\})$ *with a neutral element* $\mathtt{emp} = \bullet \rightarrow v$ *with an empty domain as an element of* $\mathtt{Heap}$*, and a binary function* $\circ : \mathtt{Heap} \times \mathtt{Heap} \rightarrow \mathtt{Heap}$*, satisfying:*

1. $\forall h_1, h_2 \in \mathtt{Heap}.[dom(h_1) \cap dom(h_2) \neq \emptyset] \rightarrow^{\dagger} [h_1 \circ h_2 = \emptyset]$ *(disjointness).*
2. $\forall h_1, h_2 \in \mathtt{Heap}.[dom(h_1) \cap dom(h_2) = \emptyset] \rightarrow^{\dagger} [h_1 \circ h_2 = h_1 \cup h_2]$ *(disjoint union).*
3. $\forall h \in \mathtt{Heap}.\{\mathtt{emp}\} \circ h = h$ *(neutrality).*
4. $\forall h_1, h_2 \in \mathtt{Heap}.h_1 \circ h_2 = h_2 \circ h_1$ *(commutativity).*
5. $\forall h_1, h_2, h_3 \in \mathtt{Heap}.h_1 \circ (h_2 \circ h_3) = (h_1 \circ h_2) \circ h_3$ *(associativity).*

In the condition of disjoint union, the set union $h_1 \cup h_2 \in \mathtt{Heap}$ *of the two functions* $h_1 \in \mathtt{Heap}$ *and* $h_2 \in \mathtt{Heap}$ *is in the following sense:*

1. $dom(h_1 \cup h_2) = dom(h_1) \cup dom(h_2)$.
2. $\forall l \in dom(h_1 \cup h_2).[l \in dom(h_1)] \rightarrow^{\dagger} [(h_1 \cup h_2)(l) = h_1(l)]$.
3. $\forall l \in dom(h_1 \cup h_2).[l \in dom(h_2)] \rightarrow^{\dagger} [(h_1 \cup h_2)(l) = h_2(l)]$.

Definition 39 (Interpretation) *An interpretation* I_{sep} *is a function that maps elements of* V *into themselves (identity map) and elements of* Var *into* V*. That is, with a function* $\mathtt{Stack} : Var \rightarrow V$*, it satisfies:*

1. $\forall v \in V.\mathsf{I}_{sep}(v) = v$.
2. $\forall x \in Var.\mathsf{I}_{sep}(x) = \mathtt{Stack}(x)$.

Definition 40 (Semantics)
A heap model is defined to be a 5-tuple $(\mathtt{Heap}, \circ, \{\mathtt{emp}\}, \mathsf{I}_{sep}, \models)$ *for some heap monoid* $(\mathtt{Heap}, \circ, \{\mathtt{emp}\})$*, some interpretation* I_{sep} *and a forcing relation* $\models$*, satisfying, for all* $h \in \mathtt{Heap}$*:*

- $h \models Ex_1 \mapsto Ex_2$ *iff*
  $[|dom(h)| = 1] \wedge^{\dagger} (\forall l \in dom(h).[\mathsf{I}_{sep}(Ex_1) = l] \wedge^{\dagger} [h(l) = \mathsf{I}_{sep}(Ex_2)])$.

- $h \models \top$.
- $\neg^\dagger[h \models \mathbb{1}]$.
- $h \models {}^{*}\top$ *iff* $h = \{\texttt{emp}\}$.
- $h \models FF_1 \wedge FF_2$ *iff* $[h \models FF_1] \wedge^\dagger [h \models FF_2]$.
- $h \models FF_1 \vee FF_2$ *iff* $[h \models FF_1] \vee^\dagger [h \models FF_2]$.
- $h \models FF_1 \supset FF_2$ *iff* $\neg^\dagger[h \models FF_1] \vee^\dagger [h \models FF_2]$.
- $h \models FF_1 * FF_2$ *iff* $\exists h_1, h_2 \in \texttt{Heap}.[\emptyset \neq h_1 \circ h_2] \wedge^\dagger [h = h_1 \circ h_2] \wedge^\dagger [h_1 \models FF_1] \wedge^\dagger [h_2 \models FF_2]$.
- $h \models FF_1 \mathrel{-\!\!*} FF_2$ *iff* $\forall h_1 \in \texttt{Heap}.[\emptyset \neq h \circ h_1] \wedge^\dagger [h_1 \models FF_1] \rightarrow^\dagger [h \circ h_1 \models FF_2]$.

Definition 41 (Unversal validity) *A formula* $FF \in \mathfrak{F}_{sep}$ *is said to be valid in some heap model* $(\texttt{Heap}, \circ, \{\texttt{emp}\}, \mathsf{I}_{sep}, \models)$ *iff* $\forall h \in \texttt{Heap}.[h \models FF]$. *It is said to be universally valid iff it is valid in all the conceivable heap models.*

As indicated for example in Larchey-Wendling and Galmiche [2012], with a suitable function $\texttt{Translate} : \mathfrak{F}_{\texttt{BBI}} \rightarrow \mathfrak{F}_{sep}$, it holds that if $F \in \mathfrak{F}_{\texttt{BBI}}$ is universally valid, then $\texttt{Translate}(F) \in \mathfrak{F}_{sep}$ is universally valid. This fact makes the study of logical properties of `BBI` a worthwhile for theoretical investigation into the nature of separation logic.

1.2 Technical Descriptions of Research Problems and Contributions

Having covered background materials in sufficient details, we shall now go through an overview of research problems and a list of contributions in technical terms.

1.2.1 `BI` proof theory

One of the distinct logical characteristics of `BI` is shaped by the two adjoint relations that co-exist in the logic: $[F \wedge G \vdash H] \simeq [F \vdash G \supset H]$ and $[F * G \vdash H] \simeq [F \vdash G \mathrel{-\!\!*} H]$, the former taken from intuitionistic logic, and the latter from multiplicative intuitionistic linear logic without exponentials (hereafter simply multiplicative intuitionistic linear logic as no confusion is likely to arise). Insofar as `BI` comes with additive components (which derive from intuitionistic logic) and multiplicative components (which derive from multiplicative intuitionistic linear logic), it is not so remote from linear logic in the underlying idea of enriching expressiveness of a logic by composition. However, the nature as a combined logic is more salient in `BI` having intuitionistic logic and multiplicative intuitionistic linear logic as its base logics.

A `BI` proof system usually distinguishes additive contexts from multiplicative ones by defining two structural connectives ";" and ",": the semi-colon reserved for additive structure formation and the comma for multiplicative structure formation. This differentiation helps insulate additive structures (those connected with ";", *e.g.* $\Gamma_1; \Gamma_2$) from multiplicative ones (Γ_1, Γ_2) and vice versa in a syntactically unambiguous way, *e.g.*

$$\frac{\Gamma; F \vdash G}{\Gamma \vdash F \supset G} \supset R \quad \frac{\Gamma, F \vdash G}{\Gamma \vdash F \mathrel{-\!\!*} G} \mathrel{-\!\!*} R$$

The same differentiation is convenient also for formulation of structural rules in `BI` sequent calculi, weakening and contraction in particular, which are available in the context of `BI` additive structures only:

$$\frac{\Gamma(\Gamma_1; \Gamma_1) \vdash F}{\Gamma(\Gamma_1) \vdash F} \text{Contraction} \quad \frac{\Gamma(\Gamma_1) \vdash F}{\Gamma(\Gamma_1; \Gamma_1) \vdash F} \text{Weakening}$$

1.2.1.1 Research problems

There, however, emerges somewhat a curious phenomenon around the base-logic interactions as observed syntactically. For example in contraction, not simply an additive structure: "$\Gamma_1; \Gamma_2$" but also a multiplicative one: "Γ_1, Γ_2" may duplicate. Given a `LBI` sequent $\Gamma(F, G) \vdash H$, the following three are all derivable.

$$\frac{\Gamma((F; F), G) \vdash H}{\Gamma(F, G) \vdash H} Ctr_1 \quad \frac{\Gamma(F, (G; G)) \vdash H}{\Gamma(F, G) \vdash H} Ctr_2 \quad \frac{\Gamma((F, G); (F, G)) \vdash H}{\Gamma(F, G) \vdash H} Ctr_3$$

Whereas, if it can be proved that only formula contractions like the Ctr_1 or the Ctr_2 above are required in `LBI` derivations, it will suffice to have;

$$\frac{\Gamma(G;G) \vdash H}{\Gamma(G) \vdash H}$$

in place of the general contraction rule Contraction, if not, there cannot be any restriction that can be imposed on the size of what to duplicate, and consequently contraction analysis will be non-trivial. Is it possible to ascertain that the structural contraction of the following sort:

$$\frac{\Gamma((\Gamma_1,\Gamma_2);(\Gamma_1,\Gamma_2)) \vdash H}{\Gamma(\Gamma_1,\Gamma_2) \vdash H}\text{MContraction}$$

is admissible,[1] or should there be any situations where it must take place? And if it is not admissible, what exactly is demanding the presence of MContraction? To achieve contraction restriction, one must first answer these questions by studying the way it behaves within `LBI`, paying a particular attention to syntactically occurring base-logic interactions (simply structural interactions hereafter).

Two issues stand in the way of a successful `LBI` contraction analysis, however. The first issue is the structural equivalence $\Gamma, \emptyset_m = \Gamma = \Gamma; \emptyset_a$ (where $\emptyset_a$ denotes the additive structural unit and $\emptyset_m$ the multiplicative structural unit) which is by nature bidirectional:

$$\frac{\Gamma(\Gamma_1) \vdash F}{\Gamma(\Gamma_1;\emptyset_a) \vdash F} \quad \frac{\Gamma(\Gamma_1;\emptyset_a) \vdash F}{\Gamma(\Gamma_1) \vdash F} \quad \frac{\Gamma(\Gamma_1) \vdash F}{\Gamma(\Gamma_1,\emptyset_m) \vdash F} \quad \frac{\Gamma(\Gamma_1,\emptyset_m) \vdash F}{\Gamma(\Gamma_1) \vdash F}$$

Apart from being an obvious source of non-termination, it obscures the core mechanism of structural interactions by a free-transformation of an additive structure into a multiplicative one and vice versa: a structure "Γ" can be additive because it is equivalent to "$\Gamma; \emptyset_a$" but it can be also multiplicative because it is equivalent to "$\Gamma, \emptyset_m$", which implies that any structure may be both additive and multiplicative. The second issue is the difficulty of isolating the effect of contraction from that of weakening. An earlier work Donnelly et al. [2004] for example attempted absorption of the effect of structural weakening into the other inference rules for a subset of `BI`. But their approach does not fully eliminate the effect of structural weakening because they absorb it also into structural contraction. Structural weakening still occurs through the modified structural contraction in their system. But isolation of weakening and contraction is not the only one difficulty. In addition, it is not so straightforward to know whether, first of all, either weakening or contraction is immune to the effect of the structural

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[1] An inference rule in a sequent calculus is admissible iff any sequent which is derivable in the calculus is derivable without the particular rule.

equivalence. As the result, contraction-free `BI` sequent calculi, be the contraction-freeness in the sense of `G3i` or of `G4i` (*Cf.* 1.1.2), have remained in obscurity, the multitude of technical complications around interactions among `LBI` structural rules hindering the emergence.

The current lack of knowledge about structural interactions within `BI` proof systems is not so desirable, theoretically but also from a practical viewpoint. Into theorem proving for example, the presence of bidirectional structural rules and contraction as explicit structural rules in `LBI` means that it is difficult to actually prove that an invalid[1] `BI` formula is underivable within the calculus. This is because `LBI` by itself does not provide termination conditions for a derivation of a given `BI` formula unless the derivation tree actually closes.[2] That is, the only case in which no more backward derivation on a `LBI` sequent is possible is when it is empty; the only case in which it can be empty is when it is the premise of an axiom.

For the antithesis of the intricacy of analysis on the structural interactions and the need for more scalable a calculus, the currently established practice is in fact not to face the difficulty (*Cf.* Galmiche et al. [2005]) but instead to turn to semantics. While it is largely thanks to this judicious decision that we are aware of the indication of `BI` decidability (*Cf.* Galmiche et al. [2005]) which assures that a decision procedure be extractable, its practical significance has been less significant, as attested by the long absence of the actual procedure. The given proof for `BI` decidability in Galmiche et al. [2005] is in fact paradoxical, as to be stated in Chapter 2.

1.2.1.2 Contribution: development of α`LBI` and then `LBIZ` through study of structural interactions

I present a rigorous study of structural interactions within `LBI`, which is intended to be a pathway for resolution of the dilemma that hindered earlier work. I first of all deliver a new `BI` sequent calculus α`LBI`, proving all the following:

- Admissibility of the structural equivalence.
- Admissibility of weakening.

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[1] In a given semantics by a class of models, a formula that is not universally valid is invalid.

[2] *Cf.* Definition 3.

- Admissibility of contraction.

Since those three are all that appear as structural rules in `LBI`, it is a new `BI` sequent calculus comprising logical inference rules only. These admissibility results are depth-preserving, as to be subsequently shown. Also, answering to the earlier questions as to whether structural contractions of the sort of MContraction are admissible, they must be absorbed into the left rule of the two `BI` implications, but not needed in the rest. In fact, it follows immediately by eye inspection on the available α`LBI` inference rules that they are the only α`LBI` inference rules in which any kind of contraction needs absorbed.

Two concepts hold a key to the solutions. One is what I term the essence of structures which recognises, in a sequent, a set of structures that are intrinsically connected but which may appear dispersed within the sequent by the presence of redundant decorative artifacts that the multiplicative unit and also weakening collectively create. It is a notational invention that takes care of interactions between `LBI` logical inference rules, weakening and the structural equivalence around the multiplicative unit. Another is *deep absorption* of `LBI` weakening into `LBI` logical inference rules. It gives rise to a critical observation of incremental weakening. With it, the effect of contraction in `BI` sequent calculi is for the first time demonstrated separable from that of weakening. I also prove admissibility of `Cut` directly within [α`LBI`+ `Cut`].

The rigorous analysis within the development of α`LBI` about the structural interactions prompts a tidying-up of the foundation of `BI` proof theory. I read out a message inscribed in the set of α`LBI` inference rules - the positing of the structural units $\emptyset_a$ and $\emptyset_m$, hitherto sources of complexity, has no substance in `BI` proof systems. Coherent equivalence as a set of structural rules (*Cf.* Pym [2002]), one of the long favoured ideas adopted in earlier `BI` proof systems (*Cf.* Brotherston [2012]; Donnelly et al. [2004]; O'Hearn and Pym [1999]; Pym [2002]), is finally placed under examination of its adequacy, and, as far as the structural equivalence - one of the conditions in the coherent equivalence - is concerned, removed. The result is a new `BI` sequent calculus comprising logical inference rules only.

1.2.1.3 Contribution: decidability of a `BI` fragment - a purely syntactical demonstration

Upon derivation of `LBIZ` begins a search for a syntactically demonstrable decidable fragment of `BI`. I will show that [`BI` - the multiplicative implication - the multiplicative unit] is decidable. This fragment is termed $\mathtt{BI}_{\mathtt{base}}$.

Concerning the how, my approach is to utilise an implicit contraction elimination method widely known in intuitionistic logic (*Cf.* 1.1.2). $\mathtt{BI}_{\mathtt{base}}$ is for now the largest `BI` fragment which can be purely syntactically proved decidable, and for which there actually is a proof.

1.2.1.4 Contribution: reasoning BI as BI with structural layers

The present thesis is hoped to draw attention the following: wherever reasoning about `BI` proof theory (and also `BBI` proof theory) is concerned, a particular choice of the representation of a `BI` structure (also `BBI` structure) is not indifferent to the other candidates, and there is every reason to be meticulous about it if an accurate reasoning is to be sought after. Earlier syntactic studies on `BI` proof theory saw the two constituents: intuitionistic logic and multiplicative intuitionistic linear logic, but hardly any of them the boundary between the two (in which the distinct logical characteristics of `BI` lie). I reason about `BI` by considering a structure as a nesting of structural layers.

1.2.2 `BBI` proof theory

Boolean `BI` (`BBI`) is a combined logic of classical logic and multiplicative intuitionistic linear logic as its base logics. Classical logic forms the additive sub-logic of `BBI` and multiplicative intuitionistic linear logic the multiplicative sub-logic. One immediate difference from `BI` is in the availability of classical negation. Pym [2002] for instance considers a prototypical logic `BBI` as an extension of logic `BI` with the law of the excluded middle. The more recent `BBI` semantics as developed by Galmiche and Larchey-Wendling [2006] makes use of non-deterministic monoids, which is strictly more general, as Larchey-Wendling and Galmiche [2009] note, than the class of heap models (*Cf.* Reynolds [2002] for the initial heap semantics). It is known (noted for instance by Larchey-Wendling and Galmiche [2012]) that the set of `BBI` formulas universally valid in the class of non-deterministic models are in turn universally valid in the class of heap models, whose fact makes it worthwhile to study `BBI`. With a firm theoretical foundation of Logic `BBI`, a promising research direction of adapting the knowledge of `BBI` proof theory into separation logic theory in an incremental manner comes in scope. It is reasonable to suppose that maturity in `BBI` proof theory would aid farther progresses into decision problems for separation logic (*Cf.* Berdine et al. [2004]; Brochenin et al. [2012]; Calcagno et al. [2001]; Iosif et al. [2013] for the current status) and/or development of efficient separation logic theorem proving techniques by consolidating currently available tools (*Cf.* Berdine et al. [2005, 2011]; Chang and Rival [2008]; Chin et al. [2012]; Distefano and Parkinson [2008]; Distefano et al. [2006]; Jacobs et al. [2011]; Magill et al. [2008]; Villard et al. [2010]).

The first tool towards this goal was shown by Park et al. [2013] based on an earlier `BBI` display calculus (*Cf.* Brotherston [2012]). A formal system in semantic tableaux is also known (*Cf.* Larchey-Wendling and Galmiche [2009]).

1.2.2.1 Research problems

The large enthusiasm around application (via separation logic) notwithstanding, the core mechanism of base-logic interactions in `BBI` is still not thoroughly understood, which is in fact even harder to analyse than the `BI` base-logic interactions. `BBI` proof theory, in which semantic characteristics need finitely formalised, is particularly hard-hit by the lack of the comprehension. Even for propositional `BBI`, no proof systems suitable for an efficient proof search are so far available.

A few `BBI` proof systems are nonetheless known such as a Hilbert-style system (by Galmiche and Larchey-Wendling [2006]), a display calculus (by Brotherston [2012]) and its envisaged optimisation (by Park et al. [2013]). These are, however, not very suitable for a scalable proof search: the axiomatic Hilbert-style system for the obvious reason of the presence of modus ponens, and the display(-like) `BBI` proof systems for extra structural rules postulated (display postulates Belnap [1982]) some of which are `Cut` in sequent calculus sense. Both modus ponens and display postulates, allowing an infinite introduction of new constructs, break down the property that a `Cut`-free sequent calculus usually (though probably not always) promises: analyticity of a proof system which guarantees the need of at most a finite number of distinct constructs required in the course of a proof search - no matter how long derivations are to be.

Definition 42 (Distinct new constructs and Analyticity) *Given a proof system, an inference rule available in the proof system is said to be introducing a distinct new construct (resp. distinct new constructs) if and only if it introduces in premise(s) some structure (resp. some structures) which is (resp. are) not equivalent to any structure(s) in the conclusion up to relations that are defined to hold among structures in the proof system (such as associativity and commutativity). A given proof system is said to be analytic if and only if (A) there are only finitely many inference rules in the proof system and (B) it holds, for any derivation constructable (finitely or infinitely) with the set of inference rules made available within the system, that the number of distinct new constructs to be introduced is finitely bounded.*

Of the two measures towards a demonstration of decidability of a (propositional) logic within a proof system: analyticity (which an infinite introduction of distinct constructs breaks) and duplication-freeness (which the presence of contraction breaks), an analytic proof system ensures the first, thereby restricting the cause of decidability/undecidability to only one measure than two. In the context of `BBI`, it is impossible to also eliminate the need for duplications since such would prove the decidability of `BBI`, contradicting the earlier undecidability result (*Cf.* Brotherston and Kanovich [2010]; Larchey-Wendling and Galmiche [2012]); however, permitting both infinite duplication and infinite production of new constructs, overhead to proof searches is immense. Proof-theoretical investigation into decidable fragments of the logic is also tricky with a non-analytic proof system such as can be figured from an earlier attempt (by Kracht [1996]) towards the goal. Therefore a finding of an analytic `BBI` proof system would be a significant progress forward in the emerging research of `BBI` theorem-proving and its adaptation to separation logic, as Park et al. [2013] also note.

There are two major technical difficulties that stand in the way of `BBI` sequent calculus inception. The first - just as in `BI` - is the limited distributivity between the base logics. The partial distributivity makes it very hard to see the condition under which the base-logic interactions occur. The second is a semantic peculiarity of the `BBI` multiplicative unit in that it is judged point-wise. For example, the `BBI` non-deterministic semantics gives rise to the following result.

Lemma 9 (Brotherston and Kanovich [2010])
$({}^{*}\top \wedge F) \supset (F * F) \in \mathfrak{F}_{\texttt{BBI}}$ *is universally valid.*

The same formula which is also in $\mathfrak{F}_{\texttt{BI}}$, however, is not generally universally valid in `BI` Kripke relational semantics, *i.e.* there exists a formula for which it is not universally valid, precisely because that some possible world m in the `BI` Kripke relational semantics forces ${}^{*}\top$ does not perforce dictate that it be ϵ. The implication that the semantic difference has on syntax must be closely studied.

1.2.2.2 Contribution: development of `BBI` sequent calculi

I first present a `BBI` sequent calculus $\texttt{LBBI}_p$ by heeding the underlying semantics. The following are examined specifically: (1) behaviour of classical implication within `BBI` proof systems, (2) collapsing of multiplicative conjuncts, and (3) the non-intuitionistic multiplicative unit in `BBI`. For (1), I develop adequate sequent calculus conventions to take into account the way classical logic is captured within the other `BBI` sub-logic. This consideration is similar to the one taken in Park et al. [2013] except that I explicitly formulate a one-sided calculus. For (2), it is to be noted that a multiplicative conjunct may exhibit certain coupling with other multiplicative conjuncts. This phenomenon is handled in $\texttt{LBBI}_p$ with a special distribution rule. It is also to be identified that a naive use of the distribution rule could result in an infinite introduction of new constructs. This problem is also addressed. Its consequence to `Cut` is then studied. For (3) which is a lesser-heeded point in `BBI` proof theory, it is to be observed that a `BBI` proof system can have a close semantic-syntax correspondence only if the behaviour of the multiplicative unit is properly captured within the system. The way it interacts with multiplicative components must be adequately reflected onto syntax. Dedicated inference rules around the multiplicative unit are defined, to initiate further investigation of its peculiarity and impact on `BBI` sequent calculi.

From $\texttt{LBBI}_p$, I derive a variant $\alpha\texttt{LBBI}_p$ by adapting the knowledge of `BI` structural interactions (which is to be covered in Chapter 3). Despite uncertainty to still remain over admissibility of `Cut` in [$\alpha\texttt{LBBI}_p$+ `Cut`], it is expected to mark the beginning of study into `BBI` sequent calculi from the direction opposite the earlier work: instead of the top-down methodology starting from a highly expressive display calculus, these sequent calculi derive from a bottom-up approach. Sound separation logic sequent calculi follow immediately from `LBBI`$_p$ and α`LBBI`$_p$. Admissibility of `Cut` in a conservative fragment of [α`LBBI`$_p$+ `Cut`] is also shown.

1.2.3 Studies into Logical Combinations and Combined Logics

Just as much as it is important to understand the mechanism of base-logic interactions in some specific combined logic, it is also important that we look at the general problem of logical combinations themselves. Into this direction, I initiate highly constructive proof-theoretical studies into logical combinations, delivering the concept of phased sequent calculus in which interactions between a given set of base logics can be actually developed and analysed. Being a sequent calculus, analysis of amenability to automation also comes in scope.

When we reflect upon logical combinations and consequently combined logics themselves, we are first faced with the following question; “What does it mean by combining logics?” The posed question is properly answerable only if the definition of a logical combination is known. Another question soon follows; “Who is giving the definition?”

It is none other than ourselves who are trying to combine logics, and who, by the intention, becomes a mediator on behalf of the base logics. In phased sequent calculus, the mediator is formalised as a set of interaction inference rules, the mediation strength of which determines how base logics are allowed to interact. For a demonstration of the phased sequent calculus, I formulate `BI`$_{\texttt{base}}$, and propose the use of state diagrams to develop and analyse base-logic interactions. As an exhibition of a basic proof search in phased sequent calculus, I also present a decision procedure for the `BI` fragment. The locality embedded within the phased sequent calculus simplifies the termination argument. It is anticipated that phased sequent calculus is to encompass theory and application of combined logics.

1.3 Synopsis of the Remaining Chapters

- In Chapter 2, related `BI` proof systems will be reviewed, some briefly, some more critically. A cut admissibility proof in `LBI` by means of `BI-MultiCut` that appears in this chapter is part based on Arisaka and Qin [2012].

- In Chapter 3, reasoning techniques about `BI` base logic interactions will be introduced and new `BI` sequent calculi free of any structural rules will be presented. Decidability of a `BI` fragment will be proved purely syntactically. No earlier syntactic proofs for a sizable `BI` fragment beyond intuitionistic logic or multiplicative intuitionistic linear logic without exponentials are for now known.

- In Chapter 4, `BBI` sequent calculi will be presented through adequate sequent calculus conventions and semantic observations. The knowledge of structural interactions in `BI` sequent calculus is adapted. In the same chapter will be also found a direction into separation logic sequent calculi, though completeness will be left open. A cut elimination procedure for a conservative fragment will be shown. Related work will then be compared. This chapter will conclude studies into the direction of specialisation.

- In Chapter 5, phased sequent calculus, an idea to promote farther studies into base-logic interactions and consequently combined logics themselves, will be introduced. The decidable fragment of `BI` will be used for an illustration of the idea. A methodology to engineer a particular sense of logical combination with abstract state diagrams will be proposed. A decision procedure for the fragment will be presented. Related work will be then compared.

- In Chapter 6, some future work will be suggested, following a summary of contributions in earlier chapters.

Chapters 2 and 3 form the technical foundations for Chapter 4 and Chapter 5. They should be therefore read in the written order and should be sufficiently understood before readers move on farther. Chapter 5 has no technical dependencies on Chapter 4, and may be read just after Chapter 3.

Chapter 2

Reviews of BI Proof Systems

In this chapter I go through earlier `BI` proof systems that have a close connection to my own and, where relevant, provide a closer review for some of them.

2.1 BI Proof Systems

Several `BI` proof systems are found in literature: a natural deduction system (*Cf.* O'Hearn and Pym [1999]), sequent calculi (*Cf.* Donnelly et al. [2004]; Harland and Pym [2003]; Pym [2002]), a Hilbert-type system (*Cf.* Pym et al. [2004]), semantic tableaux (*Cf.* Galmiche and Méry [2003]; Galmiche et al. [2005]), a display calculus (*Cf.* Brotherston [2012]), a deep inference (*Cf.* Horsfall [2006]). The history of `BI` began in O'Hearn and Pym [1999] as a logic represented by a proof system. Corresponding semantics were developed subsequently in Galmiche and Méry [2003]; Galmiche et al. [2005]; Harland and Pym [2003]; Pym [2002]; Pym et al. [2004]. The rest of this section illustrates those that are relevant to this thesis with appropriate comparisons.

2.1.1 LBI

The first `BI` sequent calculus by Pym [2002] was introduced in the previous chapter. Along with soundness and completeness, admissibility of `Cut` also holds in `LBI`. The fact, however, does not follow from the suggested proof approach in Pym [2002]. I briefly illustrate an unaddressed issue in the suggested cut elimination procedure. The standard notations of the cut rank and the cut level are given first.

Definition 43 (Formula size) *The size of a formula* $F \in \mathfrak{F}_{\mathtt{BI}}$, $\mathtt{f\_size}(F)$, *is defined as follows: it is 1 if no binary logical operators occur in* F*; is* $\mathtt{f\_size}(F_1) + \mathtt{f\_size}(F_2) + 1$ *if* $F = F_1 \star F_2$ *for* $\star \in \{\wedge, \vee, \supset, *, -\!\!*\}$.

Definition 44 (Cut rank and cut level) *The level of a* `Cut` *instance is the sum of the derivation depths of both of its premise sequents. The rank of a* `Cut` *instance is the size of the cut formula* F.

2.1.1.1 Issue

Following Pym [2002], the result is supposed to hold and be provable by making use of only `MultiCut` (*Cf.* some standard text such as Troelstra and Schwichtenberg [2000]):

$$\dfrac{\Gamma_1 \vdash F \qquad \Gamma_2(F; \cdots ; F) \vdash G}{\Gamma_2(\Gamma_1) \vdash G}\,\mathtt{MultiCut}$$

However, there is certain issue in the approach: `MultiCut` does not take care of the effect of structural contraction that `LBI` permits. As it turns out, this issue is just as much unresolvable as `Cut` in `G1i` derivations without `MultiCut` if only a means of local permutation (*Cf.* von Plato [2001] for a global permutation) is adopted.[1] Pym [2002] indicates that the use of local permutation suffices for the proof of admissibility of `Cut` in `LBI`. But it fails to take into account the following case:

$$\dfrac{\Gamma_1 \vdash F \qquad \dfrac{\Gamma_3((\Gamma_2, F); (\Gamma_2, F)) \vdash H}{\Gamma_3(\Gamma_2, F) \vdash H}\,CtrL}{\Gamma_3(\Gamma_2, \Gamma_1) \vdash H}\,\mathtt{Cut}$$

Call the partial derivation Π_1. Then a permutation attempt:

$$\dfrac{\dfrac{\Gamma_1 \vdash F \qquad \dfrac{\Gamma_1 \vdash F \qquad \Gamma_3((\Gamma_2, F); (\Gamma_2, F)) \vdash H}{\Gamma_3((\Gamma_2, \Gamma_1); (\Gamma_2, F)) \vdash H}\,\mathtt{Cut}}{\Gamma_3((\Gamma_2, \Gamma_1); (\Gamma_2, \Gamma_1)) \vdash H}\,\mathtt{Cut}}{\Gamma_3(\Gamma_2, \Gamma_1) \vdash H}\,CtrL$$

entails the irreducibility of the cut level of the lower `Cut` instance (that which is just above $CtrL$).

---

[1] *Cf.* Troelstra and Schwichtenberg [2000] for the details.

2.1.1.2 LBI Cut Elimination Proof With BI-MultiCut

To actually prove the `Cut` admissibility by means of local permutation, it hence is necessary that the effect of structural contraction be encoded into `Cut`:

$$\frac{\Gamma_1 \vdash F \qquad \Gamma_3((\Gamma_2(F))^{\times n}) \vdash H}{\Gamma_3(\Gamma_2(\Gamma_1)) \vdash H}\,\texttt{BI-MultiCut}$$

where $(\Gamma)^{\times n}$ abbreviates $\underbrace{\Gamma; \Gamma; \cdots; \Gamma}_{n}$. With `BI-MultiCut`, Π_1 is transformed cleanly:

$$\frac{\Gamma_1 \vdash F \qquad \Gamma_3((\Gamma_2, F)^{\times 2}) \vdash H}{\Gamma_3(\Gamma_2, \Gamma_1) \vdash H}\,\texttt{BI-MultiCut}$$

An admissibility proof of `Cut` in `LBI` is then achieved.

Theorem 5 (Admissibility of `Cut` in `LBI`) `Cut` *is admissible in* `LBI`. *There exists a cut elimination procedure that transforms any closed* `LBI`*-derivation into a closed [*`LBI-Cut`*]-derivation.*

Proof. By induction on the cut rank and a sub-induction on the cut level. In the below, (U, V) denotes, for `LBI` inference rules U and V, that one of the premises has been just derived with U and the other with V. "$\cdots \Rightarrow \cdots$" denotes a derivation permutation strategy where the left hand side of the "$\Rightarrow$" is the given (partial) derivation whereas the right hand side is the permuted (partial) derivation. I abbreviate "$(\Gamma(\Gamma_1))(\Gamma_2)$" by "$\Gamma(\Gamma_1)(\Gamma_2)$", as noted in Chapter 1.

$(id,\ id)$**:** $$\frac{\dfrac{}{F \vdash F}\,id \qquad \dfrac{}{F \vdash F}\,id}{F \vdash F}\,\texttt{Cut} \quad \Rightarrow \quad \frac{}{F \vdash F}\,id$$

$(id, \top R)$**:** $$\frac{\dfrac{}{F \vdash F}\,id \qquad \dfrac{}{\Gamma_2(F) \vdash \top}\,\top R}{\Gamma_2(F) \vdash \top}\,\texttt{Cut} \quad \Rightarrow \quad \frac{}{\Gamma_2(F) \vdash \top}\,\top R$$

$(id, \mathbb{1}L)$**:** $$\frac{\dfrac{}{F \vdash F}\,id \qquad \dfrac{}{\Gamma_2(F)(\mathbb{1}) \vdash H}\,\mathbb{1}L}{\Gamma_2(F)(\mathbb{1}) \vdash H}\,\texttt{Cut} \quad \Rightarrow \quad \frac{}{\Gamma_2(F)(\mathbb{1}) \vdash H}\,\mathbb{1}L$$

$(id, \wedge L)$**:** $$\frac{\dfrac{}{F_1 \wedge F_2 \vdash F_1 \wedge F_2}\,id \qquad \dfrac{\Gamma_1(F_1; F_2) \vdash H}{\Gamma_1(F_1 \wedge F_2) \vdash H}\,\wedge L}{\Gamma_1(F_1 \wedge F_2) \vdash H}\,\texttt{Cut} \quad \Rightarrow \quad \frac{\Gamma_1(F_1; F_2) \vdash H}{\Gamma_1(F_1 \wedge F_2) \vdash H}\,\wedge L$$

(id, U)**:** Similar. Straightforward.

$(\top R, \top R)$:

$$\dfrac{\dfrac{}{\Gamma_1 \vdash \top}\,\top R \qquad \dfrac{}{\Gamma_2(\top) \vdash \top}\,\top R}{\Gamma_2(\Gamma_1) \vdash \top}\,\texttt{Cut} \quad \Rightarrow \quad \dfrac{}{\Gamma_2(\Gamma_1) \vdash \top}\,\top R$$

$(\top R, \mathbb{1}L)$:

1. $$\dfrac{\dfrac{}{\Gamma_1 \vdash \top}\,\top R \qquad \dfrac{}{\Gamma_2(\top)(\mathbb{1}) \vdash H}\,\mathbb{1}L}{\Gamma_2(\Gamma_1)(\mathbb{1}) \vdash H}\,\texttt{Cut} \quad \Rightarrow \quad \dfrac{}{\Gamma_2(\Gamma_1)(\mathbb{1}) \vdash H}\,\mathbb{1}L$$

2. $$\dfrac{\dfrac{}{\Gamma_1(\mathbb{1}) \vdash H}\,\mathbb{1}L \qquad \dfrac{}{\Gamma_2(H) \vdash \top}\,\top R}{\Gamma_2(\Gamma_1(\mathbb{1})) \vdash \top}\,\texttt{Cut} \quad \Rightarrow \quad \dfrac{}{\Gamma_2(\Gamma_1(\mathbb{1})) \vdash \top}\,\top R \text{ or } \mathbb{1}L$$

$(\top R, U)$: Straightforward.

$(\mathbb{1}L, \mathbb{1}L)$:

$$\dfrac{\dfrac{}{\Gamma_1(\mathbb{1}) \vdash H_1}\,\mathbb{1}L \qquad \dfrac{}{\Gamma_2(H_1)(\mathbb{1}) \vdash H_2}\,\mathbb{1}L}{\Gamma_2(\Gamma_1(\mathbb{1}))(\mathbb{1}) \vdash H_2}\,\texttt{Cut} \quad \Rightarrow \quad \dfrac{}{\Gamma_2(\Gamma_1(\mathbb{1}))(\mathbb{1}) \vdash H_2}\,\mathbb{1}L$$

$(\bot L, U)$: Straightforward.

$(\wedge R, U)$: excluding the Us already examined:

1. $$\dfrac{\dfrac{D_1 : \Gamma_1 \vdash F_1 \qquad D_2 : \Gamma_1 \vdash F_2}{D_4 : \Gamma_1 \vdash F_1 \wedge F_2}\,\wedge R \qquad \dfrac{\dfrac{D_3 : \Gamma_2((\Gamma_3(F_1 \wedge F_2))^{\times n-1}; \Gamma_3(F_1; F_2)) \vdash H}{\Gamma_2((\Gamma_3(F_1 \wedge F_2))^{\times n}) \vdash H}\,\wedge L}{\Gamma_2(\Gamma_3(F_1 \wedge F_2)) \vdash H}\,CtrL}{\Gamma_2(\Gamma_1) \vdash H}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{D_1 \qquad \dfrac{D_2 \qquad \dfrac{D_4 \qquad D_3}{\Gamma_2(\Gamma_3(\Gamma_1); \Gamma_3(F_1; F_2)) \vdash H}\,\texttt{BI-MultiCut}}{\Gamma_2(\Gamma_3(\Gamma_1); \Gamma_3(F_1; \Gamma_1)) \vdash H}\,\texttt{Cut}}{\Gamma_2(\Gamma_3(\Gamma_1); \Gamma_3(\Gamma_1; \Gamma_1)) \vdash H}\,\texttt{Cut}}{\Gamma_2(\Gamma_3(\Gamma_1)) \vdash H}\,CtrL$$

2. $$\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash F_1 \wedge F_2}\,\wedge R \qquad \dfrac{\dfrac{\Gamma_2'((\Gamma_3'(F_1 \wedge F_2))^{\times n}) \vdash H'}{\Gamma_2((\Gamma_3(F_1 \wedge F_2))^{\times n}) \vdash H}\,U}{\Gamma_2(\Gamma_3(F_1 \wedge F_2)) \vdash H}\,CtrL}{\Gamma_2(\Gamma_3(\Gamma_1)) \vdash H}\,\texttt{Cut} \quad \Rightarrow$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash F_1 \wedge F_2}\,\wedge R \qquad \Gamma_2'((\Gamma_3'(F_1 \wedge F_2))^{\times n}) \vdash H'}{\Gamma_2'(\Gamma_3'(\Gamma_1)) \vdash H'}\,\texttt{BI-MultiCut}}{\Gamma_2(\Gamma_3(\Gamma_1)) \vdash H}\,U$$

3. Also straightforward when U is instead a two-premise inference rule.

$(\lor R, U)$: excluding the Us already examined:

1.
$$\dfrac{\dfrac{D_1 : \Gamma_1 \vdash F_i}{D_4 : \Gamma_1 \vdash F_1 \lor F_2}\lor R \qquad \dfrac{\dfrac{D_2 \qquad D_3}{D_5 : \Gamma_2((\Gamma_3(F_1 \lor F_2))^{\times n}) \vdash H}\lor L}{\Gamma_2(\Gamma_3(F_1 \lor F_2)) \vdash H}\mathit{CtrL}}{\Gamma_2(\Gamma_3(\Gamma_1)) \vdash H}\texttt{Cut}$$

where $[D_2 : \Gamma_2((\Gamma_3(F_1 \lor F_2))^{\times n-1}; \Gamma_3(F_1)) \vdash H]$ and $[D_3 : \Gamma_2((\Gamma_3(F_1 \lor F_2))^{\times n-1}; \Gamma_3(F_2)) \vdash H]$.

$\Rightarrow$
$$\dfrac{\dfrac{D_1(i=1) \qquad \dfrac{D_4 \qquad D_2}{\Gamma_2((\Gamma_3(\Gamma_1)); \Gamma_3(F_1)) \vdash H}\texttt{BI-MultiCut}}{\Gamma_2((\Gamma_3(\Gamma_1)); \Gamma_3(\Gamma_1)) \vdash H}\texttt{Cut}}{\Gamma_2(\Gamma_3(\Gamma_1)) \vdash H}\mathit{CtrL}$$

2. Straightforward, otherwise.

$(\supset R, U)$: excluding the Us already examined:

1.
$$\dfrac{\dfrac{D_1 : \Gamma_1; F_1 \vdash F_2}{D_4 : \Gamma_1 \vdash F_1 \supset F_2}\supset R \qquad \dfrac{\dfrac{D_2^i \qquad D_3}{\Gamma_2((\Gamma_4(\Gamma_3; F_1 \supset F_2))^{\times n}) \vdash H}\supset L}{\Gamma_2(\Gamma_4(\Gamma_3; F_1 \supset F_2)) \vdash H}\mathit{Ctr\,L}}{\Gamma_2(\Gamma_4(\Gamma_3; \Gamma_1)) \vdash H}\texttt{Cut}$$

where $[D_2^1 : \Gamma_3 \vdash F_1]$ (or $[D_2^2 : (\Gamma_3; F_1 \supset F_2)^{\times n-1}; \Gamma_3 \vdash F_1]$ in case $[\Gamma_4(\Gamma_3; F_1 \supset F_2) \equiv \Gamma_3; F_1 \supset F_2]$ up to assoc. and commut.) and $[D_3 : \Gamma_2((\Gamma_4(\Gamma_3; F_1 \supset F_2))^{\times n-1}; \Gamma_4(\Gamma_3; F_2)) \vdash H]$.

$$\Rightarrow \dfrac{\dfrac{D_2^* \qquad \dfrac{D_1 \qquad \dfrac{D_4 \qquad D_3}{\Gamma_2(\Gamma_4(\Gamma_3; \Gamma_1); \Gamma_4(\Gamma_3; F_2)) \vdash H}\texttt{BI-MultiCut}}{\Gamma_2(\Gamma_4(\Gamma_3; \Gamma_1); \Gamma_4(\Gamma_3; \Gamma_1; F_1)) \vdash H}\texttt{Cut}}{D'}\texttt{Cut}}{\Gamma_2(\Gamma_4(\Gamma_3; \Gamma_1)) \vdash H}\mathit{Ctr\,L}$$

where D' is $[\Gamma_2(\Gamma_4(\Gamma_3; \Gamma_1); \Gamma_4(\Gamma_3; \Gamma_1; \Gamma_3)) \vdash H]$ for $i = 1$ or $[\Gamma_2(\Gamma_4(\Gamma_3; \Gamma_1); \Gamma_4(\Gamma_3; \Gamma_1; \Gamma_3; \Gamma_1; \Gamma_3)) \vdash H]$ for $i = 2$, and D_2^* is D_2^1 for $i = 1$, or the following for $i = 2$.

$$\dfrac{D_4 \qquad D_2^2}{\Gamma_3;\Gamma_1;\Gamma_3 \vdash F_1}\ \texttt{BI-MultiCut}$$

$(*R, U)$: excluding the Us already examined:

1. $$\dfrac{\dfrac{D_1 \qquad D_2}{D_4 : \Gamma_2,\Gamma_3 \vdash F_1 * F_2}\ {*R} \qquad \dfrac{\dfrac{D_3 : \Gamma((\Gamma_1(F_1 * F_2))^{\times n-1};\Gamma_1(F_1,F_2)) \vdash H}{\Gamma((\Gamma_1(F_1 * F_2))^{\times n}) \vdash H}\ {*L}}{\Gamma(\Gamma_1(F_1 * F_2)) \vdash H}\ CtrL}{\Gamma(\Gamma_1(\Gamma_2,\Gamma_3)) \vdash H}\ \texttt{Cut}$$

   where $[D_1 : \Gamma_2 \vdash F_1]$ and $[D_2 : \Gamma_3 \vdash F_2]$.

   $\Rightarrow$
   $$\dfrac{\dfrac{D_1 \qquad \dfrac{D_2 \qquad \dfrac{D_4 \qquad D_3}{\Gamma(\Gamma_1(\Gamma_2,\Gamma_3);\Gamma_1(F_1,F_2)) \vdash H}\ \texttt{BI-MultiCut}}{\Gamma(\Gamma_1(\Gamma_2,\Gamma_3);\Gamma_1(F_1,\Gamma_3)) \vdash H}\ \texttt{Cut}}{\Gamma(\Gamma_1(\Gamma_2,\Gamma_3);\Gamma_1(\Gamma_2,\Gamma_3)) \vdash H}\ \texttt{Cut}}{\Gamma(\Gamma_1(\Gamma_2,\Gamma_3)) \vdash H}\ CtrL$$

2. Straightforward, otherwise.

$(-\!\!*R, U)$: excluding the Us already examined:

1. $$\dfrac{\dfrac{D_1 : \Gamma_1, F_1 \vdash F_2}{D_4 : \Gamma_1 \vdash F_1 -\!\!* F_2}\ {-\!\!*R} \qquad \dfrac{\dfrac{D_2 \qquad D_3}{\Gamma_2((\Gamma_4(\Gamma_3, F_1 -\!\!* F_2))^{\times n}) \vdash H}\ {-\!\!*L}}{\Gamma_2(\Gamma_4(\Gamma_3, F_1 -\!\!* F_2)) \vdash H}\ CtrL}{\Gamma_2(\Gamma_4(\Gamma_1,\Gamma_3)) \vdash H}\ \texttt{Cut}$$

   where $[D_2 : \Gamma_3 \vdash F_1]$ and $[D_3 : \Gamma_2(\Gamma_4(F_2);(\Gamma_4(\Gamma_3, F_1 -\!\!* F_2))^{\times n-1}) \vdash H]$.

   $\Rightarrow$
   $$\dfrac{\dfrac{D_2 \qquad \dfrac{D_1 \qquad \dfrac{D_4 \qquad D_3}{\Gamma_2(\Gamma_4(F_2);\Gamma_4(\Gamma_3,\Gamma_1)) \vdash H}\ \texttt{BI-MultiCut}}{\Gamma_2(\Gamma_4(\Gamma_1,F_1);\Gamma_4(\Gamma_3,\Gamma_1)) \vdash H}\ \texttt{Cut}}{\Gamma_2(\Gamma_4(\Gamma_1,\Gamma_3);\Gamma_4(\Gamma_3,\Gamma_1)) \vdash H}\ \texttt{Cut}}{\Gamma_2(\Gamma_4(\Gamma_1,\Gamma_3)) \vdash H}\ CtrL$$

2. Straightforward, otherwise.

(WkL, U) : Straightforward. One contraction followed by one weakening to offset the effect of the contraction is simply pruned.

$$\frac{}{p \vdash p}\,id \qquad \frac{\mathbb{X} \vdash F \qquad F \vdash \mathbb{Y}}{\mathbb{X} \vdash \mathbb{Y}}\,\texttt{Cut'} \qquad \frac{}{\mathbb{1} \vdash \mathbb{Y}}\,\mathbb{1}L$$

$$\frac{}{\mathbb{X} \vdash \top}\,\top R \qquad \frac{}{\emptyset_m \vdash {}^{*}\top}\,{}^{*}\top R \qquad \frac{\emptyset_a \vdash \mathbb{Y}}{\top \vdash \mathbb{Y}}\,\top L$$

$$\frac{F ; G \vdash \mathbb{Y}}{F \wedge G \vdash \mathbb{Y}}\,\wedge L \qquad \frac{F \vdash \mathbb{Y} \qquad G \vdash \mathbb{Y}}{F \vee G \vdash \mathbb{Y}}\,\vee L \qquad \frac{\mathbb{X} \vdash F \qquad G \vdash \mathbb{Y}}{F \supset G \vdash \mathbb{X} \Rightarrow \mathbb{Y}}\,\supset L$$

$$\frac{\emptyset_m \vdash \mathbb{Y}}{{}^{*}\top \vdash \mathbb{Y}}\,{}^{*}\top L \qquad \frac{F , G \vdash \mathbb{Y}}{F * G \vdash \mathbb{Y}}\,* L \qquad \frac{\mathbb{X} \vdash F \qquad G \vdash \mathbb{Y}}{F -\!\!* G \vdash \mathbb{X} \multimap \mathbb{Y}}\,-\!\!* L$$

$$\frac{\mathbb{X} \vdash F \qquad \mathbb{X} \vdash G}{\mathbb{X} \vdash F \wedge G}\,\wedge R \qquad \frac{\mathbb{X} \vdash F ; G}{\mathbb{X} \vdash F \vee G}\,\vee R \qquad \frac{\mathbb{X} , F \vdash G}{\mathbb{X} \vdash F \supset G}\,\supset R$$

$$\frac{\mathbb{X}_1 \vdash F \qquad \mathbb{X}_2 \vdash G}{\mathbb{X}_1 , \mathbb{X}_2 \vdash F * G}\,* R \qquad \frac{\mathbb{X} , F \vdash G}{\mathbb{X} \vdash F -\!\!* G}\,-\!\!* R \qquad \frac{\mathbb{X}_1 \vdash \mathbb{Y}}{\mathbb{X}_1 ; \mathbb{X}_2 \vdash \mathbb{Y}}\,WkL$$

$$\frac{\mathbb{X} ; \mathbb{X} \vdash \mathbb{Y}}{\mathbb{X} \vdash \mathbb{Y}}\,CtrL \qquad \frac{\emptyset_a ; \mathbb{X} \vdash \mathbb{Y}}{\mathbb{X} \vdash \mathbb{Y}}\,EA_1L \qquad \frac{\emptyset_m , \mathbb{X} \vdash \mathbb{Y}}{\mathbb{X} \vdash \mathbb{Y}}\,EA_2L$$

$$\frac{\mathbb{X}_1 ; \mathbb{X}_2 \vdash \mathbb{Y}}{\mathbb{X}_1 \vdash \mathbb{X}_2 \Rightarrow \mathbb{Y}}\,DP_1 \qquad \frac{\mathbb{X}_1 , \mathbb{X}_2 \vdash \mathbb{Y}}{\mathbb{X}_1 \vdash \mathbb{X}_2 \multimap \mathbb{Y}}\,DP_2$$

Figure 2.1: `DL`$_{\texttt{BI}}$: a `BI` display calculus. An inference rule with a double-dotted line is bidirectional.

The rest: Straightforward or similar. □

2.1.2 `DL`$_{\texttt{BI}}$

Brotherston [2012] formulates `BI` display calculus as found in Figure 2.1. While the definition of a formula is carried over from Definition 24, postulated extra structural connectives ⇒ and $\multimap$ call for fresh definitions for what a `DL`$_{\texttt{BI}}$ structure or a `DL`$_{\texttt{BI}}$ sequent[1] is.

---

[1] Instead of a sequent, display calculus community tends to call it a consecution. I remain indifferent to the community-specific terminology in this thesis.

Definition 45 **($\mathtt{DL}_{\mathtt{BI}}$ structures)**
A $\mathtt{DL}_{\mathtt{BI}}$ negative structure $\mathbb{X}$ is defined by $\mathbb{X} := F \mid \emptyset_a \mid \emptyset_m \mid \mathbb{X};\mathbb{X} \mid \mathbb{X},\mathbb{X}$. A $\mathtt{DL}_{\mathtt{BI}}$ positive structure $\mathbb{Y}$ is defined by $\mathbb{Y} := F \mid \mathbb{X} \Rightarrow \mathbb{Y} \mid \mathbb{X} \multimap \mathbb{Y}$. The set of the $\mathtt{DL}_{\mathtt{BI}}$ negative structures is denoted by $\mathfrak{N}_{\mathtt{BI}}$, and that of the $\mathtt{DL}_{\mathtt{BI}}$ positive structures by $\mathfrak{P}_{\mathtt{BI}}$.

Lemma 10 **($\mathtt{DL}_{\mathtt{BI}}$ sequents)** *All the sequents appearing in `LBI` derivations are a set element of $\mathfrak{B}_{\mathtt{BI}} := \{\mathbb{X} \vdash \mathbb{Y} \mid \mathbb{X} \in \mathfrak{N}_{\mathtt{BI}} \wedge^{\dagger} \mathbb{Y} \in \mathfrak{P}_{\mathtt{BI}}\}$.*

The last two inference rules in Figure 2.1 are commonly termed display postulates which are the underlying proof-theoretical vehicles that set a display calculus apart from a sequent calculus.

Example 4 (A comparison of $\mathtt{DL}_{\mathtt{BI}}$ and `LBI` derivations) *A `LBI`-derivation and a $\mathtt{DL}_{\mathtt{BI}}$-derivation of $(p_1, p_1 -\!\!* p_2); p_2 \supset p_3 \vdash p_3$ (which is in both $\mathfrak{D}_{\mathtt{BI}}$ and $\mathfrak{B}_{\mathtt{BI}}$) is respectively:*

`LBI`:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p_1 \vdash p_1$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p_2 \vdash p_2$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p_3 \vdash p_3$}
\RightLabel{WkL}
\UnaryInfC{$p_2; p_3 \vdash p_3$}
\RightLabel{$\supset L$}
\BinaryInfC{$p_2; p_2 \supset p_3 \vdash p_3$}
\RightLabel{$-\!\!* L$}
\BinaryInfC{$(p_1, p_1 -\!\!* p_2); p_2 \supset p_3 \vdash p_3$}
\end{prooftree}
$$

$\mathtt{DL}_{\mathtt{BI}}$:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p_1 \vdash p_1$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p_2 \vdash p_2$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p_3 \vdash p_3$}
\RightLabel{$\supset L$}
\BinaryInfC{$p_2 \supset p_3 \vdash p_2 \Rightarrow p_3$}
\RightLabel{DP_1}
\UnaryInfC{$p_2; p_2 \supset p_3 \vdash p_3$}
\RightLabel{DP_1}
\UnaryInfC{$p_2 \vdash (p_2 \supset p_3) \Rightarrow p_3$}
\RightLabel{$-\!\!* L$}
\BinaryInfC{$p_1 -\!\!* p_2 \vdash p_1 \multimap ((p_2 \supset p_3) \Rightarrow p_3)$}
\RightLabel{DP_2}
\UnaryInfC{$p_1, p_1 -\!\!* p_2 \vdash (p_2 \supset p_3) \Rightarrow p_3$}
\RightLabel{DP_1}
\UnaryInfC{$(p_1, p_1 -\!\!* p_2); p_2 \supset p_3 \vdash p_3$}
\end{prooftree}
$$

It is a well-known fact that `Cut`' in a display calculus is admissible so long as it satisfies Belnap's conditions (*Cf.* Belnap [1982]). To spell out the conditions, readers are gently reminded of the fact that a proof system is a set of inference rules which are schemata. Take $\vee R$ in $\mathtt{DL}_{\mathtt{BI}}$ for example, it does not enforce that there be a unique structure "$\mathbb{X}$"

in the antecedent, and a unique "$F; G$" (in the premise) or a unique "$F \vee G$" (in the conclusion) in the consequent for the inference rule to be applicable during a backward derivation. In $\vee R$, all of the $\mathbb{X}$, F and G are but schemata which are to be instantiated by actual structures in $\mathfrak{N}_{\mathtt{BI}}$ and $\mathfrak{F}_{\mathtt{BI}}$. Similar for all the other inference rules.

Definition 46 (Belnap's conditions)

Preservation of formulas: *For every inference rule available, if it is not* `Cut`*', then a formula schema that appears in a premise sequent is a sub-formula schema of a formula schema that appears in the conclusion sequent.*

Shape-alikeness of parameters: [1] *For every inference rule, if the same schema occurs multiple-times, then it is instantiated by the same actual formula/structure.*

Non-proliferation of parameters: *For every inference rule and for every structural schema that occurs within the conclusion of the inference rule, it occurs only once there.*

Position-alikeness of parameters: *For every inference rule and for every positive structural schema that occurs in the inference rule, if it occurs in both premise(s) and conclusion, it does not occur as a negative structural schema; and for every inference rule and for every negative structural schema that occurs in the inference rule, if it occurs in both premise(s) and conclusion, it does not occur as a positive structural schema.*

Display of principal constituents: *For the principal (formula) of every inference rule, if any, if it occurs in the antecedent, then it is the only one constituent in the antecedent; and if it occurs in the consequent, then it is the only one constituent in the consequent.*

Closure under substitution for consequent parameters: *For every inference rule and for positive structural schemata in the inference rule, the inference rule is closed under simultaneous substitution of arbitrary (actual) structures into the positive structural schemata.*

---

[1] There is an inessential difference from the definition given by Belnap here.

Closure under substitution for antecedent parameters: *For every inference rule which may be partially or wholly instantiated with actual structure(s) and/or actual formula(s) such that*

1. *the conclusion sequent comes with some negative structure or some negative structural schema, say* $\mathbb{X}$, *and some consequent formula or some consequent formula schema as the principal, say* F *(which is a positive formula/formula schema), and*
2. *in the inference rule, sequents come with negative structure(s) and/or negative structural schema(ta) in which* F *occurs (A) as sub-formula(s) in case the* F *in the consequent of the conclusion sequent is a formula or (B) as sub-formula-schema(ta) in case the* F *in the consequent of the conclusion sequent is a formula schema,*

simultaneous substitution of $\mathbb{X}$ *into such occurrence(s) of* F *in sequents in the inference rule results in a partial, if not wholly, instantiation of the inference rule.*

Eliminability of matching principal constituents: *For every (finite or infinite) closed derivation tree constructable in a given proof system with a set of inference rules, if there are sequents in the schemata of* $\mathbb{X} \vdash F$ *and* $F \vdash \mathbb{Y}$ *such that (1) the* F *is the principal in the derivation, that (2) the* $\mathbb{X} \vdash F$ *is the conclusion of some inference rule* $\boldsymbol{Inf}_1$ *and that (3) the* $F \vdash \mathbb{Y}$ *is the conclusion of some inference rule* $\boldsymbol{Inf}_2$*, then one of the following must hold:*

1. $\mathbb{X}$ *is instantiated with* F *in the derivation.*
2. $\mathbb{Y}$ *is instantiated with* F *in the derivation.*
3. *it is possible to derive* $\mathbb{X} \vdash \mathbb{Y}$ *from the premise(s) of* $\boldsymbol{Inf}_1$ *and* $\boldsymbol{Inf}_2$ *by means of inference rules available in the given proof system (excluding* `Cut`*') and*

$$\frac{\mathbb{X}' \vdash F' \qquad F' \vdash \mathbb{Y}'}{\mathbb{X}' \vdash \mathbb{Y}'}\ \mathtt{Cut}'_{sub}$$

where $\mathbb{X}'$ *and* $\mathbb{Y}'$ *are some structure schemata, but* F' *may be instantiated only by a strict sub-formula of* F.

Lemma 11 (Satisfiability of Belnap’s conditions Brotherston [2012]) $\texttt{DL}_{\texttt{BI}}$ *satisfies Belnap’s conditions.*

Proposition 3 (Admissibility of Cut’ in $\texttt{DL}_{\texttt{BI}}$) *Any $D \in \mathfrak{B}_{\texttt{BI}}$ derivable in $\texttt{DL}_{\texttt{BI}}$ is also derivable in [$\texttt{DL}_{\texttt{BI}}$- Cut’] and vice versa.*

Proof. Follows directly from Lemma 11. ☐

One must note, however, that Cut’ in $\texttt{DL}_{\texttt{BI}}$ is not identical to Cut in LBI. Naturally, admissibility of Cut’ does not imply that of Cut. Although Brotherston [2012] indicates a method to attempt a proof that $\texttt{DL}_{\texttt{BI}}$ Cut’ admissibility is tantamount to LBI Cut admissibility, there is certain confusion concerning the difference between Cut and Cut’ in the cited reference.

To prove that Cut admissibility in LBI is implied from Cut’ admissibility in $\texttt{DL}_{\texttt{BI}}$, it must be proved that the translation be achievable not without introducing Cut’ but without introducing Cut. And the fundamental question is whether the display postulates - if there should be any necessity that they must be used in a derivation - can be shown Cut-free derivable. The demonstration, however, is impossible, because any display postulates in $\texttt{DL}_{\texttt{BI}}$, when they are introducing an extra structural connective $\Rightarrow$ or $\multimap$, are derivable in LBI[1] but not in [LBI- Cut]. What Brotherston [2012] terms a display-normalisation:

1. $\mathbb{X}_1 \vdash \mathbb{X}_2 \Rightarrow \mathbb{Y} \rightsquigarrow \mathbb{X}_1 ; \mathbb{X}_2 \vdash \mathbb{Y}$.
2. $\mathbb{X}_1 \vdash \mathbb{X}_2 \multimap \mathbb{Y} \rightsquigarrow \mathbb{X}_1 , \mathbb{X}_2 \vdash \mathbb{Y}$.

can be straightforwardly shown derivable in [LBI- Cut]. On the other hand, the following are not proved (but also cannot be so proved) derivable in $[\texttt{LBI} - \texttt{Cut}]$ in the cited reference.

1. $\mathbb{X}_1 ; \mathbb{X}_2 \vdash \mathbb{Y} \rightsquigarrow \mathbb{X}_1 \vdash \mathbb{X}_2 \Rightarrow \mathbb{Y}$.
2. $\mathbb{X}_1 , \mathbb{X}_2 \vdash \mathbb{Y} \rightsquigarrow \mathbb{X}_1 \vdash \mathbb{X}_2 \multimap \mathbb{Y}$.

---

[1] Under the following interpretation of structural connectives: $\Rightarrow$ as $\supset$, $\multimap$ as $-\!\!*$, “;” as $\wedge$, and “,” as $*$, as in Brotherston [2012].

Now the question is whether they are then admissible in $\mathtt{DL}_{\mathtt{BI}}$, in which case every $\mathtt{DL}_{\mathtt{BI}}$ derivation can be permuted into another $\mathtt{DL}_{\mathtt{BI}}$ derivation that does not require the problematic sequent transitions.

However, suppose for example that we have $p \wedge (p \supset q) \vdash q$ to derive:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,id \qquad \dfrac{}{q \vdash q}\,id}{D_2 : p \supset q \vdash p \Rightarrow q}\supset L}{D_1 : p; p \supset q \vdash q}\,DP_1}{D : p \wedge (p \supset q) \vdash q}\,\wedge L$$

But because the above $\mathtt{DL}_{\mathtt{BI}}$-derivation is the shortest possible to show that D is a `BI` theorem, and because the $\Rightarrow$ introduction must occur in the sequent transition of $D_1 \rightsquigarrow D_2$, it cannot be that the particular display postulate is admissible. But then, suppose some $\mathtt{DL}_{\mathtt{BI}}$-derivation:

$$\dfrac{\begin{matrix}\vdots\\ D_2 : \mathbb{X} \vdash (p_1; p_2) \Rightarrow q\end{matrix}}{D_1 : p_1; p_2; \mathbb{X} \vdash q}\,DP_1$$

`Cut`' elimination does not eliminate the occurrence of DP_1. The corresponding sequent calculus derivation with `LBI`:

$$\dfrac{\dfrac{}{\overline{p_1; p_2 \vdash p_1 \wedge p_2}} \qquad \dfrac{\begin{matrix}\vdots\\ D^* : \Gamma \vdash (p_1 \wedge p_2) \supset q\end{matrix} \qquad \dfrac{}{\overline{p_1 \wedge p_2; (p_1 \wedge p_2) \supset q \vdash q}}}{p_1 \wedge p_2; \Gamma \vdash q}\,\mathtt{Cut}}{p_1; p_2; \Gamma \vdash q}\,\mathtt{Cut}$$

then still comes with `Cut` instances which are not implied eliminable from `Cut`' elimination in $\mathtt{DL}_{\mathtt{BI}}$.[1] Here $\mathbb{X}$ corresponds to Γ, and D_2 corresponds to D^*.

The confusion may have been induced by certain similarity in appearance between `LBI` and $\mathtt{DL}_{\mathtt{BI}}$ in that both make use of left and right rules. It would not have arisen if the appearance of the $\mathtt{DL}_{\mathtt{BI}}$ inference rules had been slightly different. For example, suppose that we have a Hilbert-system for `BI`, say `HBI`, which comes with a MP. Had a proposition been this, "Suppose `HBI` is sound and complete with respect to the Kripke relational semantics. Since [`LBI` - `Cut`] is also sound and complete with respect to the same semantics, and since `Cut` is a rule of transitivity in `LBI`, the MP, which is also a

---

[1] In fact, already within the cited reference it holds that the display-normal right premise sequent of $\supset L$ (in the particular presentation of `LBI` in the cited reference) does not match with the right premise sequent of $\supset L$ in $\mathtt{DL}_{\mathtt{BI}}$, even with the equivalent (up to the display-normalisation) conclusion sequent for the two inference rules.

rule of transitivity in `HBI`[1], is admissible; that is, `HBI` is as expressive as [`HBI` - MP] as evidenced by the fact that we can demonstrate the expressiveness equivalence of [`LBI` - `Cut`] with `LBI` within `LBI`.", one would have immediately noticed a problem in the argument.

2.1.3 `TBI`

Galmiche et al. [2005] move away from sequent calculus into semantic tableaux to reformulate `BI` more semantically. Their calculus `TBI` is found in Figure 2.2. The inference rules are presented in a sequent-calculus style since there is no practical difference if a derivation tree grows downwards (Beth [1955]; Smullyan [1995]), or upwards in line with sequent calculus derivations. In `TBI`, every formula comes with a semantic label attached to it, denoted by $x\ (, y, z)$ (with or without a sub-script or a super-script). I go through a set of definitions for the calculus first.[2]

Definition 47 (Semantic labels) `TBI` *labelling structure is a 4-tuple* $(W', \circ, \dot{\epsilon}, \lhd)$ *with* W' *as a set of labels representing possible worlds in an underlying* `BI` *Kripke semantics, a binary function* $\circ : W' \times W' \to W'$*, a neutral element* $\dot{\epsilon}$ *and a pre-order* $\lhd$ *on elements of* W'*, satisfying, for all* $x, y, z \in W'$*:*

1. $x \circ \dot{\epsilon} = x$ *(neutrality).*
2. $(x \circ y) \circ z = x \circ (y \circ z)$ *(associativity).*
3. $x \circ y = y \circ x$ *(commutativity).*
4. $[x \lhd x]$ *(reflexivity).*
5. $[x \lhd y] \wedge^{\dagger} [y \lhd z] \to^{\dagger} [x \lhd z]$ *(transitivity).*
6. $[x \lhd y] \to^{\dagger} [(x \circ z) \lhd (y \circ z)]$ *(monotonicity).*

Definition 48 (A sub-label relation) *For any two semantic labels* x *and* y *in a* `TBI` *labelling structure,* $x \leq y$ *iff there exists some semantic label* z *such that* $x \circ z = y$*.*

---

[1] In the sense that F can be seen as $\top \supset F$.

[2] I use my own methodology to define notations; for that reason some may not necessarily appear or, if they do, coincide with those that are found in Galmiche et al. [2005].

$$\dfrac{\begin{array}{c} \mathsf{t}\,F : x \\ \mathsf{t}\,G : x \\ \mathsf{t}\,F \wedge G : x \\ K \end{array}}{\begin{array}{c} \mathsf{t}\,F \wedge G : x \\ K \end{array}}\ \mathsf{t}\wedge \qquad \dfrac{\begin{array}{cc} \mathsf{t}\,F : x & \mathsf{t}\,G : x \\ \mathsf{t}\,F \vee G : x & \mathsf{t}\,F \vee G : x \\ K & K \end{array}}{\begin{array}{c} \mathsf{t}\,F \vee G : x \\ K \end{array}}\ \mathsf{t}\vee \qquad \dfrac{\begin{array}{cc} \mathsf{f}\,F : x_2 & \mathsf{t}\,G : x_2 \\ x_1 \lhd x_2 & x_1 \lhd x_2 \\ \mathsf{t}\,F{\supset}G : x_1 & \mathsf{t}\,F{\supset}G : x_1 \\ K & K \end{array}}{\begin{array}{c} x_1 \lhd x_2 \\ \mathsf{t}\,F \supset G : x_1 \\ K \end{array}}\ \mathsf{t}\ \supset$$

$$\dfrac{\begin{array}{c} \mathsf{t}\,F : x'_1 \\ \mathsf{t}\,G : x'_2 \\ x'_1 \circ x'_2 \lhd x \\ x'_1 \circ x'_2 \lhd x'_1 \circ x'_2 \\ x'_1 \lhd x'_1 \\ x'_2 \lhd x'_2 \\ \mathsf{t}\,F * G : x \\ K \end{array}}{\begin{array}{c} \mathsf{t}\,F * G : x \\ K \end{array}}\ \mathsf{t}* \qquad \dfrac{\begin{array}{cc} \mathsf{f}\,F : x_2 & \mathsf{t}\,G : x_1 \circ x_2 \\ \mathsf{t}\,F {-\!\!*} G : x_1 & \mathsf{t}\,F {-\!\!*} G : x_1 \\ K & K \end{array}}{\begin{array}{c} \mathsf{t}\,F {-\!\!*} G : x_1 \\ K \end{array}}\ \mathsf{t}{-\!\!*} \qquad \begin{array}{c} \dot{\epsilon} \lhd x \\ \dfrac{\begin{array}{c} \mathsf{t}\,{}^{*}\!\top : x \\ K \end{array}}{\begin{array}{c} \mathsf{t}\,{}^{*}\!\top : x \\ K \end{array}} \end{array}$$

$$\dfrac{\begin{array}{cc} \mathsf{f}\,F : x & \mathsf{f}\,G : x \\ \mathsf{f}\,F \wedge G : x & \mathsf{f}\,F \wedge G : x \\ K & K \end{array}}{\begin{array}{c} \mathsf{f}\,F \wedge G : x \\ K \end{array}}\ \mathsf{f}\wedge \qquad \dfrac{\begin{array}{c} \mathsf{f}\,F : x \\ \mathsf{f}\,G : x \\ \mathsf{f}\,F \vee G : x \\ K \end{array}}{\begin{array}{c} \mathsf{f}\,F \vee G : x \\ K \end{array}}\ \mathsf{f}\vee \qquad \dfrac{\begin{array}{c} \mathsf{t}\,F : x' \\ \mathsf{f}\,G : x' \\ x \lhd x' \\ x' \lhd x' \\ \mathsf{f}\,F{\supset}G : x \\ K \end{array}}{\begin{array}{c} \mathsf{f}\,F \supset G : x \\ K \end{array}}\ \mathsf{f}\ \supset$$

$$\dfrac{\begin{array}{cc} \mathsf{f}\,F : x_2 & \mathsf{f}\,G : x_3 \\ x_2 \circ x_3 \lhd x_1 & x_2 \circ x_3 \lhd x_1 \\ \mathsf{f}\,F * G : x_1 & \mathsf{f}\,F * G : x_1 \\ K & K \end{array}}{\begin{array}{c} x_2 \circ x_3 \lhd x_1 \\ \mathsf{f}\,F * G : x_1 \\ K \end{array}}\ \mathsf{f}\,* \qquad \dfrac{\begin{array}{c} \mathsf{t}\,F : x'_1 \\ \mathsf{f}\,G : x \circ x'_1 \\ \mathsf{f}\,F {-\!\!*} G : x \\ x \circ x'_1 \lhd x \circ x'_1 \\ x'_1 \lhd x'_1 \\ K \end{array}}{\begin{array}{c} \mathsf{f}\,F {-\!\!*} G : x \\ K \end{array}}\ \mathsf{f}{-\!\!*}$$

Figure 2.2: `TBI`: a semantic tableaux for `BI` expressed in a sequent-style. Primed labels must be new labels distinct from all the others that are already in the conclusion `TBI` node.

More about the labelling structure that it is a partially defined labelling algebra is found in 3.1 in Galmiche et al. [2005].

Definition 49 (`TBI` **sub-node)** *A* `TBI` *sub-node Q is defined by:*

$Q := \mathsf{t}\, F : x \mid \mathsf{f}\, F : x \mid x \lhd y$

where $F \in \mathfrak{F}_{\mathtt{BI}}$ and the semantic labels x and y are an element of `TBI` *labelling structure (that is, an element of W').*

Definition 50 (`TBI` **node)** *Let "$Q_1, Q_2, \cdots, Q_n$" for some $n \geq 1$ denote a set comprising n* `TBI` *sub-nodes "Q_1", "Q_2",..., and "Q_n". Then a* `TBI` *node K is defined to be either empty or else defined by $K := Q \mid Q, K$.*

Every `TBI` inference rule then has one conclusion node and up to two premise nodes. In Figure 2.2, sub-nodes of each node are vertically placed for clarity.

Definition 51 (Labels in a `TBI` **node)** *Let K be a* `TBI` *node, then we denote by W'_K the set of all the semantic labels that appear within K.*

Definition 52 (Closable `TBI` **node Galmiche et al. [2005])** *A* `TBI` *node K is said to be closable iff at least one of the following four conditions holds for K:*

Ax: $\exists Q_1, Q_2 \in K\, \exists F \in \mathfrak{F}_{\mathtt{BI}}\, \exists x, y \in W'_K.[Q_1 = \mathsf{t}\, F : x] \wedge^{\dagger} [Q_2 = \mathsf{f}\, F : y] \wedge^{\dagger} [x \lhd y]$.[1]

$\top$: $\exists Q_1 \in K\, \exists x \in W'_K.[Q_1 = \mathsf{f}\, \top : x]$.

$\mathbb{1}$: $\exists Q_1, Q_2 \in K \exists F \in \mathfrak{F}_{\mathtt{BI}} \exists x, y, z \in W'_K.[Q_1 = \mathsf{f} F : x] \wedge^{\dagger} [Q_2 = \mathsf{t} \mathbb{1} : y] \wedge^{\dagger} [y \leq z] \wedge^{\dagger} [z \lhd x]$.

$^{*}\!\top$: $\exists Q_1 \in K\, \exists x \in W'_K.[Q_1 = \mathsf{f}\, {^{*}\!\top} : x] \wedge^{\dagger} [\dot{\epsilon} \lhd x]$.

Definition 53 (A `TBI` **derivation)** *A* `TBI` *derivation tree has as the root node two sub-nodes $\mathsf{f}\ F : \dot{\epsilon}$ and $\dot{\epsilon} \lhd \dot{\epsilon}$, and is said to be closable if all the leaves of the tree are a closable node.*

Galmiche et al. [2005] indicate a method to detect a counter-model for a given initial `TBI` node with two `TBI` sub-nodes $\mathsf{f}\, F : \dot{\epsilon}$ and $\dot{\epsilon} \lhd \dot{\epsilon}$.[2] They (implicitly) state that `TBI` is sound and complete with respect to the class of `BI` Kripke relational models.

---

[1] Note that $x \lhd y$ does only have to be inferrable in K: if, for example, $x \lhd z, z \lhd y \in K$, then it follows that $x \lhd y$. *Cf.* Definition 47.

[2] The knowledge is not needed in this section and is simply omitted here.

Proposition 4 (Soundness and completeness of `TBI` Galmiche et al. [2005])
For any $F \in \mathfrak{F}_{\texttt{BI}}$ and for any `TBI` node K with two `TBI` sub-nodes $\texttt{f}\ F : \dot{\epsilon}$ and $\dot{\epsilon} \lhd \dot{\epsilon}$, if there exists a closable derivation for K, then ${}^{*}\top \vdash F$ is `LBI`-derivable (soundness). For any `LBI`-derivable sequent ${}^{*}\top \vdash F$, there exists a closable `TBI` derivation tree for a `TBI` node K with two `TBI` sub-nodes $\texttt{f}\ F : \dot{\epsilon}$ and $\dot{\epsilon} \lhd \dot{\epsilon}$ (completeness).

They also state decidability of `BI` in Galmiche et al. [2005] and that of [`BI` - $\mathbb{1}$] in Galmiche and Méry [2003]. However, there are issues in their proof approaches.

To be able to conclude that `BI` (or [`BI`- $\mathbb{1}$]) is decidable, it must be shown that no derivation trees can grow infinitely without it turning out either to be closable or to be impossible to be closable. Galmiche and Méry [2003]; Galmiche et al. [2005] rely upon two techniques to state that `BI` is decidable: liberalisation of semantic tableaux rules and the concept of branch redundancy. The first rests upon a classical reasoning that, positing some entity that encompasses finite and infinite worlds, one could reason that it knows, before any construction of an initial `TBI` node: $\texttt{f}\ F : \dot{\epsilon}, \dot{\epsilon} \lhd \dot{\epsilon}$, whether F is a theorem or a non-theorem. It is implicit that the entity also knows the exact construction of a `TBI` derivation tree to prove or refute F. From its perspective, any derivation that eventually turns out closable in a finite or an infinite world has already turned out closable. Likewise, any derivation that eventually turns out not to be closable has already turned out as such. Any label assignment that may come into play in the course of the construction may then be carried out cleverly in some way to permit only a countable number of new labels to appear in the `TBI` derivation tree. Liberalisation of `TBI` rules attempts to do this by conjecturing the existence of least possible worlds in the underlying Kripke semantics that are absolutely necessary to prove F, which would then allow the entity to see the least labels corresponding to the least possible worlds in the construction of a closed derivation tree. But then each new label that $\texttt{f}\supset$, $\texttt{f}{-\!\!*}$ and $\texttt{t}{-\!\!*}$ introduce can be the least label for any particular principal (active formula) of the inference rule. It then follows that at most a countable number of labels are required in any `TBI` derivation process at least for closed derivation trees since `TBI` by design guarantees the subformula property.

To show that nothing that a finite world cannot conclude seeps into the argument, however, an effective identification of the leastness of labels is still not sufficient. Because it holds in general that $x \circ x \neq x$ for all $x \in W'$ such that $x \neq \dot{\epsilon}$, it must be also ascertained that the number of distinct labels by composition be bounded. Suppose the identification of the least labels can be indeed effectively achieved. Then the concept of branch redundancy provides an intuitive guideline for this matter: with now only a countable number of semantic labels at hand, even if a derivation branch grows infinitely, it must still become known, since `TBI` is sound and complete with respect to the underlying semantics, that a given formula is either a theorem or a non-theorem.

Let us first consier `BI` theorems for which there always exists a closed `TBI` derivation tree. We assume a very clever derivation tree construction (by the said entity) for any one of them such that it has the shortest derivation depth. From the presupposition, we observe that any such derivation tree contains no redundant derivation steps in any of its derivation tree branches: every derivation step in all the derivation tree branches is absolutely necessary. In particular, there is no possibility that there appear a sequence of derivation steps which makes no progress and which repeats itself. But then, if we let D_{bottom} denote the root sequent of the derivation tree and also let D_{top} denote a node in the derivation tree which is the conclusion of an axiom, then for all D_{top} in the derivation tree, we can plausibly define a well-order relation on $D_{bottom} \leadsto^* D_{top}$ such that it strictly decreases at each sequent transition. This is an easy case.

However, let us now consider a derivation tree where it must by necessity involve a repetition of previously taken steps in at least one of its derivation tree branches. Here we must be more wary. According to Galmiche and Méry [2003]; Galmiche et al. [2005], this case can be also detected successfully by terminating a `TBI` derivation tree construction upon a loop detection. The induction measure that they use is essentially the length of the loop since the induction measure in Galmiche et al. [2005] is some information that can be only found out when the loop is detected - which is really a sub-induction under the induction on the length of the loop. Firstly for finite `TBI` derivation trees, their approach certainly succeeds since the length of a loop cannot be infinitely long. Each such loop is in fact detectable. However, this induction measure somewhat falls short in the remaining case. Let us consider an infinite derivation branch tree which loops with an infinite period. Let us denote by D_{begin} the very beginning of the first loop (above the root of the derivation tree) and by D_{end} the end of the first loop. Then if we construct the derivation tree branch up to D_{end}, we will be still unable to tell that $D_{begin} \leadsto^+ D_{end}$ was a loop. To detect it as a loop, we must see that what follow D_{end}, say $D_{next\_begin} \leadsto^+ D_{next\_end}$, essentially correspond in a one-to-one manner to sequent transitions in $D_{begin} \leadsto^+ D_{end}$. Only then will we be able to terminate the derivation tree branch construction. Suppose, by means of showing contradiction, that the induction measure of the loop length provides an effective method to us for detection of the infinite loop. Then it must be that it is possible to get strictly closer to D_{next_end} at each sequent transition. In particular it must be that there is a possibility that the derivation tree branch construction goes past D_{end}. This, however, cannot be the case since the length of the loop is assumed infinite, and must therefore contradict the supposition. And here we see a fundamental problem in using the length of a loop as an inductive measure. It is not well-ordered: even if we prove all the cases up to the cycling period of k, we cannot apply induction hypothesis during the construction of a derivation branch which loops with a period of $k + 1$ because it cannot be that the loop of $k + 1$ *a priori* depends on loops of a shorter cycling period. Since neither Galmiche and Méry [2003] nor Galmiche et al. [2005] takes consideration over finiteness of all the loops, neither the decidability of $[\mathtt{BI} - \mathbb{1}]$ nor that of `BI` follows by the suggested proof approach. Whether `BI` or even [`BI` - $\mathbb{1}$] is decidable is still to be found out.

2.1.4 A forward `BI` sequent calculus

The formal systems introduced so far are mostly for backward proof searches given some conclusion `BI` sequent/formula. With `LBI` for instance, $F \in \mathfrak{F}_{\mathtt{BI}}$ is finitely identified as a theorem iff there exists an upward finite construction of a closed derivation tree for $\emptyset_a \vdash F \in \mathfrak{D}_{\mathtt{BI}}$ up to axioms.

A forward theorem proving Degtyarev [2001] on the other hand judges if a given formula is a theorem by presupposing a `Cut`-free proof system (and, with it, at most a finite set of axiom instances). If there exists a downward finite construction of a derivation tree up to the given proof obligation, then it will finitely turn out to be a theorem.

Of course upon a successful construction of a closed derivation tree by either of the two approaches, the distinction quickly diminishes, since we will then know a corresponding construction of a closed tree from the opposite direction. And if a closed derivation tree is not finitely constructed via forward theorem proving, neither is it possible to show a finite derivation tree construction via backward theorem proving. Therefore if there are merits in a study of forward theorem proving, they derive not from that it can identify more theorems than its backward counterparts but rather from that they may be potentially used to gain further insights about the nature of derivation tree construction process itself which is strongly linked to the behaviour of logic for which the formal systems have been developed. There is currently one sequent calculus for forward reasoning for a fragment of `BI` without units as Donnelly et al. [2004] present. Its extension to the full `BI` is yet to be seen.

2.2 Conclusion

In this chapter `BI` sequent-like proof systems were reviewed, some only briefly, some more critically. I showed the proof of `LBI` `Cut` admissibility, which was stated on occasions (*Cf.* Brotherston [2012]; Pym [2002]) but which was not accurately given. For this particular contribution, readers may choose a reference to Arisaka and Qin [2012]; however, while it supports a view that $\mathtt{DL}_{\mathtt{BI}}$ cut admissibility implies `LBI` cut admissibility, this thesis does not. On this matter, it is the view stated in this chapter that takes a precedence.

Chapter 3

Structural Interactions, Absorption of Structural Rules and Decidability in BI Sequent Calculus

The outline of this chapter is as follows:

1. Development of the concept of structural layers that is used in reasoning about `LBI` derivations in this thesis.
2. Development of a contraction-free `BI` sequent calculus α`LBI`, proving, through analysis on the syntactic phenomena of the base-logic interactions, admissibilities of `LBI` structural rules. It is also shown that [α`LBI` + `Cut`] is as expressive as α`LBI`, and that a cut admissibility holds in [α`LBI` + `Cut`].
3. A study on the significance of the structural units posited in `LBI`, deriving as the consequence a new `BI` sequent calculus `LBIZ` without those.
4. A purely syntactic study into `BI` decidability, to prove that a fragment of `BI` without the multiplicative implication and the multiplicative unit is decidable.

3.1 Reasoning BI as BI with structural layers

In earlier syntactic works, a fine distinction between additive/multiplicative structures is often encapsulated as a detail in coherent equivalence.

Definition 54 (Coherent equivalence) *$\bowtie$ is the equivalence relation on* `BI` *structures satisfying*

1. *associativity for the comma:* $\Gamma_1;(\Gamma_2;\Gamma_3) \bowtie (\Gamma_1;\Gamma_2);\Gamma_3$.
2. *associativity for the semi-colon:* $\Gamma_1,(\Gamma_2,\Gamma_3) \bowtie (\Gamma_1,\Gamma_2),\Gamma_3$.
3. *commutativity for the comma:* $\Gamma_1;\Gamma_2 \bowtie \Gamma_2;\Gamma_1$.
4. *commutativity for the semi-colon:* $\Gamma_1,\Gamma_2 \bowtie \Gamma_2,\Gamma_1$.
5. *structural equivalence around the additive structural unit:* $\Gamma \bowtie \Gamma;\emptyset_a$.
6. *structural equivalence around the multiplicative structural unit:* $\Gamma \bowtie \Gamma,\emptyset_m$.
7. *congruence:* $[\Gamma \bowtie \Gamma'] \rightarrow^{\dagger} [\Gamma_1(\Gamma) \bowtie \Gamma_1(\Gamma')]$.

However, an arbitrary choice of a representation of `BI` structures has a considerable downside of masking the semantically natural viewpoint about them, which is to view structures as nestings of additive and multiplicative *structural layers*.

Definition 55 (`BI` single structure) *A* `BI` *single structure α is defined by:*
$\alpha := F \mid \emptyset_m \mid \emptyset_a$.

Definition 56 (A `BI` structure in nested structural layers) *An antecedent structure Γ in nested structural layers is defined by:*

$$\begin{array}{rcl} \Gamma & := & \alpha \mid \mathcal{M} \mid \mathcal{A} \\ \mathcal{M} & := & \alpha, \mathcal{M}' \mid \mathcal{A}, \mathcal{M}' \\ \mathcal{M}' & := & \alpha \mid \mathcal{A} \mid \alpha, \mathcal{M}' \mid \mathcal{A}, \mathcal{M}' \\ \mathcal{A} & := & \alpha; \mathcal{A}' \mid \mathcal{M}; \mathcal{A}' \\ \mathcal{A}' & := & \alpha \mid \mathcal{M} \mid \alpha; \mathcal{A}' \mid \mathcal{M}; \mathcal{A}' \end{array}$$

Each of the $\mathcal{A}$ (resp. $\mathcal{M}$) substructures of Γ is termed an additive (resp. multiplicative) structural layer.

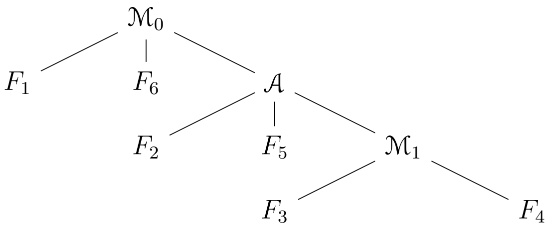

Figure 3.1: $F_1, ((F_3, F_4); F_2; F_5), F_6 \in \mathfrak{S}_{\mathtt{BI}}$ as represented in nested structural layers.

In a way, earlier works, by relying upon coherent equivalence, involuntarily relinquished a means of recognising the boundary between `BI` additives and `BI` multiplicatives (in which incidentally lies the distinct logical character of `BI`). An example of a `BI` structure in nested structural layers is found in Figure 3.1. There are two multiplicative structural layers: "$F_1, F_6, \mathcal{A}$" and "F_3, F_4"; and one additive structural layer "$F_2; F_5; \mathcal{M}_1$", with $\mathcal{A}$ denoting "$F_2; F_5; \mathcal{M}_1$" and $\mathcal{M}_1$ denoting "F_3, F_4". For any structure in which two structural layers nest, the structural layer holding the other structural layer within is described as the outer structural layer of the two, while that enclosed in the other is described as the inner structural layer.

3.2 α`LBI`: A Contraction-Free BI Sequent Calculus

In this section I present a new `BI` sequent calculus α`LBI` (Figure 3.2) in which no structural rules appear. Changes are made to the following `LBI` inference rules:

- $id_{\mathtt{LBI}}$: $id_{\alpha\mathtt{LBI}}$ replaces. Weakening and the following `LBI`-derivable rule:

  $$\frac{\Gamma(\Gamma_1) \vdash H}{\Gamma(\Gamma_1, (\emptyset_m; \Gamma_2)) \vdash H} \; EA_2$$

  are absorbed.

- ${}^{*}\!\top R_{\mathtt{LBI}}$: ${}^{*}\!\top R_{\alpha\mathtt{LBI}}$ replaces. Weakening and EA_2 are absorbed.

- $\supset L_{\mathtt{LBI}}$: $\supset L_{\alpha\mathtt{LBI}}$ replaces. Contraction, EA_2 and also weakening are absorbed.

- $-\!\!*L_{\mathtt{LBI}}$: $-\!\!*L_{1\,\alpha\mathtt{LBI}}$, $-\!\!*L_{2\,\alpha\mathtt{LBI}}$, $-\!\!*L_{3\,\alpha\mathtt{LBI}}$ and $-\!\!*L_{4\,\alpha\mathtt{LBI}}$ replace. Contraction is absorbed in all. $EqAnt_{2\,\mathtt{LBI}}$ is absorbed in $-\!\!*L_{3,4\,\alpha\mathtt{LBI}}$. EA_2 is absorbed in $-\!\!*L_{1,2,3\,\alpha\mathtt{LBI}}$. Weakening is absorbed "deeply" (to be explained shortly) in $-\!\!*L_1$.

$$\frac{}{\mathsf{E}(\Gamma; p) \vdash p}\; id \qquad \frac{}{\Gamma(\mathbb{1}) \vdash F}\; \mathbb{1}L \qquad \frac{}{\Gamma \vdash \top}\; \top R$$

$$\frac{}{\mathsf{E}(\Gamma; \emptyset_m) \vdash {}^{*}\!\top}\; {}^{*}\!\top R \qquad \frac{\Gamma(\emptyset_a) \vdash F}{\Gamma(\top) \vdash F}\; \top L \qquad \frac{\Gamma(\emptyset_m) \vdash F}{\Gamma({}^{*}\!\top) \vdash F}\; {}^{*}\!\top L$$

$$\frac{\Gamma(F; G) \vdash H}{\Gamma(F \wedge G) \vdash H}\; \wedge L \qquad \frac{\Gamma \vdash F \qquad \Gamma \vdash G}{\Gamma \vdash F \wedge G}\; \wedge R \qquad \frac{\Gamma(F) \vdash H \qquad \Gamma(G) \vdash H}{\Gamma(F \vee G) \vdash H}\; \vee L$$

$$\frac{\Gamma \vdash F_i}{\Gamma \vdash F_1 \vee F_2}\; \vee R \qquad \frac{\mathsf{E}(\Gamma_1; F \supset G) \vdash F \qquad \Gamma(G; \mathsf{E}(\Gamma_1; F \supset G)) \vdash H}{\Gamma(\mathsf{E}(\Gamma_1; F \supset G)) \vdash H}\; \supset L$$

$$\frac{\Gamma; F \vdash G}{\Gamma \vdash F \supset G}\; \supset R \qquad \frac{\Gamma, F \vdash G}{\Gamma \vdash F \mathrel{-\!\!*} G}\; \mathrel{-\!\!*} R \qquad \frac{\Gamma(F, G) \vdash H}{\Gamma(F * G) \vdash H}\; * L$$

$$\frac{Re_1 \vdash F_1 \qquad Re_2 \vdash F_2}{\Gamma' \vdash F_1 * F_2}\; * R_1 \qquad \frac{\emptyset_m \vdash F_1 \qquad \Gamma \vdash F_2}{\Gamma \vdash F_1 * F_2}\; * R_2$$

$$\frac{Re_1 \vdash F \qquad \Gamma((Re_2, G); (\Gamma', \mathsf{E}(\Gamma_1; F \mathrel{-\!\!*} G))) \vdash H}{\Gamma(\Gamma', \mathsf{E}(\Gamma_1; F \mathrel{-\!\!*} G)) \vdash H}\; \mathrel{-\!\!*} L_1$$

$$\frac{\Gamma' \vdash F \qquad \Gamma(G; (\Gamma', \mathsf{E}(\Gamma_1; F \mathrel{-\!\!*} G))) \vdash H}{\Gamma(\Gamma', \mathsf{E}(\Gamma_1; F \mathrel{-\!\!*} G)) \vdash H}\; \mathrel{-\!\!*} L_2$$

$$\frac{\emptyset_m \vdash F \qquad \Gamma((\Gamma', G); (\Gamma', \mathsf{E}(\Gamma_1; F \mathrel{-\!\!*} G)) \vdash H}{\Gamma(\Gamma', \mathsf{E}(\Gamma_1; F \mathrel{-\!\!*} G))) \vdash H}\; \mathrel{-\!\!*} L_3$$

$$\frac{\emptyset_m \vdash F \qquad \Gamma(G; F \mathrel{-\!\!*} G) \vdash H}{\Gamma(F \mathrel{-\!\!*} G) \vdash H}\; \mathrel{-\!\!*} L_4$$

Figure 3.2: α`LBI`: a `BI` sequent calculus with no explicit structural rules. $i \in \{1, 2\}$.

- $*R_{\texttt{LBI}}$: $*R_{1,2\,\alpha\texttt{LBI}}$ replace. Deep weakening absorption for $*R_{1\,\alpha\texttt{LBI}}$. Absorption of $EqAnt_{2\,\texttt{LBI}}$ for $*R_{2\,\alpha\texttt{LBI}}$.

In the rest of this section, formal definitions for

1. the ‘essence’ $\mathsf{E}(\Gamma)$ of Γ
2. the correspondence between Re_1/Re_2 and Γ' in $*R_{1\,\alpha\texttt{LBI}}$ and $\mathrel{-\!\!*}L_{1\,\alpha\texttt{LBI}}$

are provided and then the main properties of α`LBI`, *i.e.* admissibility of weakening, that of EA_2, that of both $EqAnt_{1\,\texttt{LBI}}$ and $EqAnt_{2\,\texttt{LBI}}$, that of contraction, and its equivalence to [`LBI- Cut`], are incrementally proved.

3.2.1 Essence of antecedent structures in interactions with the multiplicative unit

Co-existence of `IL` and `MILL` in `LBI` calls for new contraction-absorption techniques than those found in classics (Troelstra and Schwichtenberg [2000]). Possible interferences to one structural layer from others need specifically analysed.

To illustrate the technical difficulty, $EqAnt_{2\,\texttt{LBI}}$ for instance directly interacts with $WkL_{\texttt{LBI}}$. When $WkL_{\texttt{LBI}}$ is absorbed into the rest, the effect propagates to one direction of $EqAnt_{2\,\texttt{LBI}}$, resulting in:

$$\frac{\Gamma(\Gamma_1) \vdash H}{\Gamma(\Gamma_1, (\varnothing_m; \Gamma_2)) \vdash H}\, EA_2$$

Hence absorption of $WkL_{\texttt{LBI}}$ must involve study of $EqAnt_{2\,\texttt{LBI}}$ as well.

The solution I present for this particular issue is absorption of EA_2 together with $WkL_{\texttt{LBI}}$ into `LBI` logical inference rules. What is here termed the 'essence' of antecedent structures arises.

Definition 57 (Essence of structures)
Given a sequent $D : \Gamma(\Gamma_1) \vdash H$, $\mathsf{E}(\Gamma_1)$ *denotes a structure* Γ_a *for which the following holds:* $[\Gamma(\Gamma_a)) \vdash H] \rightsquigarrow^{*}_{EA_2} D$.

EA_2 is derivable in `LBI` with $EqAnt_{2\,\texttt{LBI}}$ and $WkL_{\texttt{LBI}}$. The following rules are enforced:

1. In a given derivation tree, the use of the notation $\mathsf{E}(\cdots)$ in multiple sequents in the derivation tree signifies the same `BI` structure.

2. $\mathsf{E}'(\Gamma)$ (or $\mathsf{E}_1(\Gamma)$ or any essence that differs from E by the presence of a sub-script, a super-script or both) in the same derivation tree does not have to be coincident with the `BI` structure that the $\mathsf{E}(\Gamma)$ denotes.

3. To prevent inundation of many super-scripts and sub-scripts, in the cases where no ambiguity is likely to arise such as in the following;

$$\frac{\Gamma(\mathsf{E}(\Gamma_1; F; G)) \vdash H}{\Gamma(\mathsf{E}(\Gamma_1; F \wedge G)) \vdash H}\, \wedge L$$

   the essence in the conclusion is assumed to be the same antecedent structure as the essence in the premise(s) save in what the inference rule modifies.

Example 5 *Given a* $\alpha\mathtt{LBI}$*-derivation:*

$$\dfrac{\dfrac{}{D_1 : F_1; ((\emptyset_m;\Gamma_1), F_1{\supset}F_2) \vdash F_1}\,id \qquad \dfrac{}{D_2 : F_2; F_1; ((\emptyset_m;\Gamma_1), F_1{\supset}F_2) \vdash F_2}\,id}{D : F_1; ((\emptyset_m;\Gamma_1), F_1{\supset}F_2) \vdash F_2}\supset L$$

the antecedent structures in D, D_1 *and* D_2 *can be viewed taking on the forms:* $\mathsf{E}(F_1; F_1{\supset}F_2)$*,* $\mathsf{E}(F_1; F_1{\supset}F_2)$*, and* $\mathsf{E}(F_2; F_1; F_1{\supset}F_2)$*.*

3.2.2 Correspondence between Re_1/Re_2 and Γ': deep weakening absorption

Correspondence between Re_1/Re_2 and Γ' in both $*R_{1\,\alpha\mathtt{LBI}}$ and $\mathrel{-\!*}L_{1\,\alpha\mathtt{LBI}}$ is defined through a binding to a corresponding $\mathtt{LBI}$-derivation.

Definition 58 (Re_1**/**Re_2 **in** $*R_1$**/**$\mathrel{-\!*}L_1$**)** *In* $\alpha\mathtt{LBI}$*, correspondence of premise/conclusion sequents in* $*R_1$ *and* $\mathrel{-\!*}L_1$ *are defined with respect to* $*R$*/*$\mathrel{-\!*}L$*/*WkL*/*$CtrL$*/*$EqAnt_2$*/*EA_2 *in* $\mathtt{LBI}$*:*

For $*R_{1\,\alpha\mathtt{LBI}}$**:** *Let* D_1 *be a sequent* $\Gamma' \vdash F * G$ *as the conclusion sequent of* $*R_{1\,\alpha\mathtt{LBI}}$*. Then the corresponding derivation of* $*R_{1\,\alpha\mathtt{LBI}}$ *within* $\mathtt{LBI}$ *is defined to be*

- $D_1 \rightsquigarrow^{*}_{WkL_{\mathtt{LBI}}} [D_1' : Re_1, Re_2 \vdash F * G]$
- $D_1' \rightsquigarrow_{*R_{\mathtt{LBI}}} [D_2 : Re_1 \vdash F]$
- $D_1' \rightsquigarrow_{*R_{\mathtt{LBI}}} [D_3 : Re_2 \vdash G]$

in which D_2 *and* D_3 *correspond to the premise sequents of* $*R_{1\,\alpha\mathtt{LBI}}$ *(with* D_1 *as its conclusion sequent).*

For $\mathrel{-\!*}L_{1\,\alpha\mathtt{LBI}}$**:** *Let* D_1 *be a sequent* $\Gamma(\Gamma', \mathsf{E}(\Gamma_1; F\mathrel{-\!*}G)) \vdash H$ *as the conclusion sequent of* $\mathrel{-\!*}L_{1\,\alpha\mathtt{LBI}}$*. Then the corresponding derivation of* $\mathrel{-\!*}L_{1\,\alpha\mathtt{LBI}}$ *within* $\mathtt{LBI}$ *is defined as below.* $\Gamma_a(-)$ *denotes* $\Gamma(-; (\Gamma', \mathsf{E}(\Gamma_1; F\mathrel{-\!*}G)))$ *and is used for simplification.*

- $D_1 \rightsquigarrow_{CtrL_{\mathtt{LBI}}} [D_1' : \Gamma_a(\Gamma', \mathsf{E}(\Gamma_1; F\mathrel{-\!*}G)) \vdash H]$
- $D_1' \rightsquigarrow^{*}_{EA_2} [D_1'' : \Gamma_a(\Gamma', (\Gamma_1; F\mathrel{-\!*}G)) \vdash H]$
- $D_1'' \rightsquigarrow_{WkL_{\mathtt{LBI}}} [D_1''' : \Gamma_a(\Gamma', F\mathrel{-\!*}G) \vdash H]$

- $D_1''' \quad \rightsquigarrow^*_{WkL_{\texttt{LBI}}} [D_1'''' : \Gamma_a(Re_1, Re_2, F{-\!\!*}G) \vdash H]$
- $D_1'''' \rightsquigarrow_{-\!\!*L_{\texttt{LBI}}} [D_2 : Re_1 \vdash F]$
- $D_1'''' \rightsquigarrow_{-\!\!*L_{\texttt{LBI}}} [D_3 : \Gamma_a(Re_2, G) \vdash H]$

in which D_2 and D_3 correspond to the premise sequents of $-\!\!*L_1\,_{\alpha\texttt{LBI}}$.

In effect, these rules deeply internalise general weakening ($WkL_{\texttt{LBI}}$) and, in case of $-\!\!*L_1$, also contraction ($CtrL_{\texttt{LBI}}$), which would be otherwise explicit in `LBI`. Since $WkL_{\texttt{LBI}}$ is general and can extend its reach to several additive structural layers of the antecedent structure, there naturally are many Re_1/Re_2 pairs to result through the internalised weakening process ($WkL_{\texttt{LBI}}$).

Proposition 5 below indicates that the use of weakening rules which only act for the outermost additive structural layer of Γ': WkL_1 for $*R_1$; or $WkL'_{1,2}$ for $-\!\!*L_1$, is not always sufficient.

$$\frac{\Gamma_1 \vdash H}{\Gamma_1; \Gamma_2 \vdash H}\, WkL_1 \quad \frac{\Gamma(\Gamma_1, F{-\!\!*}G) \vdash H}{\Gamma(\Gamma_1, (\Gamma_2; F{-\!\!*}G)) \vdash H}\, WkL'_1 \quad \frac{\Gamma(\Gamma_1, F{-\!\!*}G) \vdash H}{\Gamma((\Gamma_1; \Gamma_2), F{-\!\!*}G) \vdash H}\, WkL'_2$$

Proposition 5 *There are sequents $D : \Gamma \vdash F$ which are derivable in α`LBI` and `LBI` but not derivable in α`LBI`' which is identical to α`LBI` except for restriction on the internalised weakening for $*R_1$ to WkL_1 and for $-\!\!*L_1$ to $WkL'_{1,2}$.*

Proof. With $p_0; (p_1, ((p_2, p_3); p_4)) \vdash (p_5 \supset (p_1 * p_2)) * p_3$ for $*R_1$, and $p_1, ((p_2, p_3); p_5), (p_1 * p_2) -\!\!* (p_3 -\!\!* p_4) \vdash p_4$ for $-\!\!*L_1$. Details are left as an exercise. □

Similar `LBI`-derivations of other altered `LBI` inference rules are straightforward. Only Γ_1 in the conclusion sequent is discarded (in backward derivation) in $-\!\!*L_{2,3}$. For $-\!\!*L_4$, $[D : \Gamma(F{-\!\!*}G) \vdash H] \rightsquigarrow_{CtrL} [D' : \Gamma(F{-\!\!*}G; F{-\!\!*}G) \vdash H] \rightsquigarrow_{EqAnt_2}$ $[D'' : \Gamma((\emptyset_m, F{-\!\!*}G); F{-\!\!*}G) \vdash H]$ to take place first internally, followed by $-\!\!*L$.
For $\supset L$,

- $[D : \Gamma(\mathsf{E}(\Gamma_1; F{\supset}G)) \vdash H] \rightsquigarrow_{CtrL} [D' : \Gamma(\mathsf{E}(\Gamma_1; F{\supset}G); \mathsf{E}(\Gamma_1; F{\supset}G)) \vdash H]$
- $D' \rightsquigarrow^*_{EA_2} [D'' : \Gamma(\Gamma_1; F{\supset}G; \mathsf{E}(\Gamma_1; F{\supset}G)) \vdash H]$
- $D'' \rightsquigarrow_{WkL} [D''' : \Gamma(F{\supset}G; \mathsf{E}(\Gamma_1; F{\supset}G)) \vdash H]$

to take place firstly. Then, for the left premise sequent:

$D''' \rightsquigarrow_{\supset L} [D_1 : \mathsf{E}(\Gamma_1; F{\supset}G) \vdash F]$

and for the right premise sequent:

$D''' \rightsquigarrow_{\supset L} [D_2 : \Gamma(G; \mathsf{E}(\Gamma_1; F{\supset}G)) \vdash H]$.

3.2.3 Weakening admissibility and EA_2 admissibility

Admissibilities of weakening and EA_2 are both proved depth-preserving, which means, in case of weakening, that if a sequent $\Gamma(\Gamma_1) \vdash H$ is derivable with derivation depth of k, then $\Gamma(\Gamma_1; \Gamma_2) \vdash H$ is derivable with derivation depth of l such that $l \leq k$.

Proposition 6 (LBI3 weakening admissibility) *If a sequent $D : \Gamma(\Gamma_1) \vdash F$ is α`LBI`-derivable, then so is $D' : \Gamma(\Gamma_1; \Gamma_2) \vdash F$, preserving the derivation depth.*

Proof. By induction on derivation depth of $\Pi(D)$. If it is one, *i.e.* D is the conclusion sequent of an axiom, then so is D'. For inductive cases, assume that the current proposition holds for all the derivations of depth up to k. It must be now demonstrated that it still holds for derivations of depth $k+1$. Consider what the last inference rule is in $\Pi(D)$.

1. $\supset L$: $\Pi(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots \\ \mathsf{E}(\Gamma_1; F{\supset}G) \vdash F\end{matrix} \qquad \begin{matrix}\vdots \\ \Gamma(G; \mathsf{E}(\Gamma_1; F{\supset}G)) \vdash H\end{matrix}}{\Gamma(\mathsf{E}(\Gamma_1; F{\supset}G)) \vdash H} \supset L$$

   By induction hypothesis on both of the premises, $\mathsf{E}(\Gamma_1; \Gamma_2; F{\supset}G) \vdash F$ and $\Gamma(G; \mathsf{E}(\Gamma_1; \Gamma_2; F{\supset}G)) \vdash H$ are both α`LBI`-derivable.[1]
   Then so is $\Gamma(\mathsf{E}(\Gamma_1; \Gamma_2; F \supset G)) \vdash H$ via $\supset L$.

2. $-\!\!*L_1$: $\Pi(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots \\ Re_1 \vdash F\end{matrix} \qquad \begin{matrix}\vdots \\ \Gamma((Re_2, G); (\Gamma', \mathsf{E}(\Gamma_1; F{-\!\!*}G))) \vdash H\end{matrix}}{\Gamma(\Gamma', \mathsf{E}(\Gamma_1; F{-\!\!*}G)) \vdash H} -\!\!*L_1$$

---

[1] Γ_2 is assumed to appear at any convenient position to the present proof argument. This does not detract from the proof precision.

For this case (and also for $*R_1$), we must additionally take into account the internalised weakening.
Suppose that "$\Gamma'', \mathsf{E}'(\Gamma_1; F \text{-}\!\!* G)$" results from "$\Gamma', \mathsf{E}(\Gamma_1; F \text{-}\!\!* G)$" through a sequence of induction hypothesis applications at additive structural layers in "$\Gamma', \mathsf{E}(\Gamma_1; F \text{-}\!\!* G)$" on the right premise, then by induction hypothesis once more (on the right premise), $\Gamma((Re_2, G); (\Gamma'', \mathsf{E}'(\Gamma_1; F \text{-}\!\!* G)); \Gamma_2) \vdash H$.
Then $\Gamma((\Gamma'', \mathsf{E}'(\Gamma_1; F \text{-}\!\!* G)); \Gamma_2) \vdash H$ via $\text{-}\!\!* L_1$.

3. Other cases are simpler. □

Proposition 7 (Admissibility of EA_2) *If a sequent $D : \Gamma(\Gamma_1) \vdash F$ is α`LBI`-derivable, then so is $D' : \Gamma(\mathsf{E}(\Gamma_1)) \vdash F$, preserving the derivation depth.*

Proof. First and foremost, note that $\mathsf{E}(\Gamma_x)$ is some structure (*Cf.* Definition 57): there is no physical symbol E in the antecedent, since it is only for a notational convenience.

Proof is by induction on derivation depth of $\Pi(D)$. If it is one, *i.e.* D is the conclusion sequent of an axiom, then so is D'. Inductive cases are straightforward, and left as an exercise. □

3.2.4 Inversion lemma

The inversion lemma below is important in simplification of the subsequent discussion, as it signifies that a given sequent D_1 can be normalised into a simpler sequent D'_1, simpler in the sense that D'_1 has a smaller sum of the sizes of the formulas[1] found within D'_1 than D_1 does.

Lemma 12 (Inversion lemma for α`LBI`) *For the following sequent pairs, if the sequent on the left is α`LBI`-derivable at most with the derivation depth of k, then so is*

---

[1] Definition 43 for the definition of the formula size.

(are) the sequent(s) on the right.

$$\begin{array}{rl}
\Gamma(F \wedge G) \vdash H, & \Gamma(F; G) \vdash H \\
\Gamma(F_1 \vee F_2) \vdash H, & \textit{both}\ \Gamma(F_1) \vdash H\ \textit{and}\ \Gamma(F_2) \vdash H \\
\Gamma(F \supset G) \vdash H, & \Gamma(G) \vdash H \\
\Gamma(F * G) \vdash H, & \Gamma(F, G) \vdash H \\
\Gamma(\top) \vdash H, & \Gamma(\emptyset_a) \vdash H \\
\Gamma({}^{*}\top) \vdash H, & \Gamma(\emptyset_m) \vdash H \\
\Gamma(\Gamma_1; \emptyset_a) \vdash H, & \Gamma(\Gamma_1) \vdash H \\
\Gamma(\Gamma_1, \emptyset_m) \vdash H, & \Gamma(\Gamma_1) \vdash H \\
\Gamma \vdash F \wedge G, & \textit{both}\ \Gamma \vdash F\ \textit{and}\ \Gamma \vdash G \\
\Gamma \vdash F \supset G, & \Gamma; F \vdash G \\
\Gamma \vdash F \mathrel{-\!\!*} G, & \Gamma, F \vdash G
\end{array}$$

Proof. By induction on the derivation depth k.

1. For a α`LBI` sequent $\Gamma(F \wedge G) \vdash H$, the base case is when it is an axiom, and the proof is trivial. For inductive cases, assume that the statement holds true for all the derivation depths up to k, and show that it still holds true at $k + 1$. Consider what the last inference rule applied is.
   (a) $\top L$: The derivation ends in:

   $$\frac{\Gamma(F \wedge G)(\emptyset_a) \vdash F}{\Gamma(F \wedge G)(\top) \vdash F}\ \top L$$

   where the representation $\Gamma(\Gamma_1)(\Gamma_2)$ is an abbreviation of $(\Gamma(\Gamma_1))(\Gamma_2)$ which indicates that Γ_1 is not a subbunch of Γ_2 nor is Γ_2 a subbunch of Γ_1. By induction hypothesis, $\Gamma(F; G)(\emptyset_a) \vdash F$ (is α`LBI`-derivable). Then, $\Gamma(F; G)(\top) \vdash F$ (is α`LBI`-derivable) as required by (a forward application of) $\top L$.

   (b) ${}^{*}\top L$: Similar.

   (c) $\wedge L$: Similar, or trivial when the principal should coincide with $F \wedge G$.

   (d) $\vee L$: The derivation ends in:

$$\frac{\Gamma(F \wedge G)(F_1) \vdash H \qquad \Gamma(F \wedge G)(F_2) \vdash H}{\Gamma(F \wedge G)(F_1 \vee F_2) \vdash H} \vee L$$

By induction hypothesis, both $\Gamma(F; G)(F_1) \vdash H$ and $\Gamma(F; G)(F_2) \vdash H$. Then $\Gamma(F; G)(F_1 \vee F_2) \vdash H$ as required via $\vee L$.

(e) $\supset L$: The derivation ends in one of the following:

$$\frac{\mathsf{E}(\Gamma_1(F \wedge G); F_1 \supset G_1) \vdash F_1 \qquad \Gamma(G_1; \mathsf{E}(\Gamma_1(F \wedge G); F_1 \supset G_1)) \vdash H}{\Gamma(\mathsf{E}(\Gamma_1(F \wedge G); F_1 \supset G_1)) \vdash H} \supset L$$

$$\frac{\mathsf{E}(\Gamma_1'; F_1 \to G_1) \vdash F_1 \qquad \Gamma'(F \wedge G)(G_1; \mathsf{E}(\Gamma_1'; F_1 \supset G_1)) \vdash H}{\Gamma'(F \wedge G)(\mathsf{E}(\Gamma_1'; F_1 \supset G_1)) \vdash H} \supset L$$

By induction hypothesis, both $\mathsf{E}(\Gamma_1(F; G); F_1 \supset G_1) \vdash F_1$ and
$\Gamma(G_1; \mathsf{E}(\Gamma_1(F; G); F_1 \supset G_1)) \vdash H$ in case the former, or
$\Gamma'(F; G)(G_1; \mathsf{E}(\Gamma_1'; F_1 \supset G_1)) \vdash H$ in case the latter.
Then $\supset L$ (with the untouched left premise if the latter) produces the required result.

(f) $*L$: The derivation ends in:

$$\frac{\Gamma(F \wedge G)(F_1, G_1) \vdash H}{\Gamma(F \wedge G)(F_1 * G_1) \vdash H} *L$$

By induction hypothesis, $\Gamma(F; G)(F_1, G_1) \vdash H$. Then, $\Gamma(F; G)(F_1 * G_1) \vdash H$ as required via $*L$.

(g) $-\!\!*L_1$: The derivation ends in one of the following, depending on the location at which $F \wedge G$ appears. In the below inference steps, we assume that the particular formula $F \wedge G$ occurs in $Re_i(F \wedge G)$ ($i \in \{1, 2\}$) as the focused substructure, but not in Re_i.[1]

$$\frac{Re_1 \vdash F_1 \qquad \Gamma((Re_2, G_1); (\Gamma', \mathsf{E}(\Gamma_1(F \wedge G); F_1 -\!\!* G_1))) \vdash H}{\Gamma(\Gamma', \mathsf{E}(\Gamma_1(F \wedge G); F_1 -\!\!* G_1)) \vdash H}$$

---

[1] Note, however, that this does not preclude occurrences of $F \wedge G$ in case it occurs multiple times in the conclusion sequent.

$$\frac{Re_1 \vdash F_1 \qquad \Gamma((Re_2, G_1); (\Gamma'(F \wedge G), \mathsf{E}(\Gamma_1; F_1 -\!\!* G_1))) \vdash H}{\Gamma(\Gamma'(F \wedge G), \mathsf{E}(\Gamma_1; F_1 -\!\!* G_1)) \vdash H}$$

$$\frac{Re_1(F \wedge G) \vdash F_1 \qquad \Gamma((Re_2, G_1); (\Gamma'(F \wedge G), \mathsf{E}(\Gamma_1; F_1 -\!\!* G_1))) \vdash H}{\Gamma(\Gamma'(F \wedge G), \mathsf{E}(\Gamma_1; F_1 -\!\!* G_1)) \vdash H}$$

$$\frac{Re_1 \vdash F_1 \qquad \Gamma((Re_2(F \wedge G), G_1); (\Gamma'(F \wedge G), \mathsf{E}(\Gamma_1; F_1 -\!\!* G_1))) \vdash H}{\Gamma(\Gamma'(F \wedge G), \mathsf{E}(\Gamma_1; F_1 -\!\!* G_1)) \vdash H}$$

$$\frac{Re_1 \vdash F_1 \qquad \Gamma(F \wedge G)((Re_2, G_1); (\Gamma', \mathsf{E}(\Gamma_1; F_1 -\!\!* G_1))) \vdash H}{\Gamma(F \wedge G)(\Gamma', \mathsf{E}(\Gamma_1; F_1 -\!\!* G_1)) \vdash H}$$

$$\frac{Re_1 \vdash F_1 \qquad \Gamma((Re_2, G_1); (\Gamma_2(F \wedge G)(\Gamma', \mathsf{E}(\Gamma_1; F_1 -\!\!* G_1)))) \vdash H}{\Gamma(\Gamma_2(F \wedge G)(\Gamma', \mathsf{E}(\Gamma_1; F_1 -\!\!* G_1))) \vdash H}$$

For each, the required sequent results from induction hypothesis for the particular occurrences of $F \wedge G$ on both of the premises, and then $-\!\!*L_1$ by appropriately carrying out its internal weakening (forwardly) to recover Γ' (or $\Gamma'(F; G)$) from Re_1/Re_2 (*Cf.* Definition 58).

(h) $-\!\!*L_{2,3,4}$: Similar, but simpler.

(i) $\wedge R$: Similar to $\vee L$ in approach but simpler.

(j) $\vee R$: Similar.

(k) $\supset R$: The derivation ends in:

$$\frac{\Gamma(F \wedge G); F_1 \vdash G_1}{\Gamma(F \wedge G) \vdash F_1 \supset G_1} \supset R$$

By induction hypothesis, $\Gamma(F; G); F_1 \vdash G_1$. Then, $\Gamma(F; G) \vdash F_1 \supset G_1$ as required via $\supset R$.

(l) $*R_1$: The derivation ends in one of the following patterns:

$$\frac{Re_1 \vdash F_1 \qquad Re_2 \vdash G_1}{\Gamma'(F \wedge G) \vdash F_1 * G_1}$$

$$\frac{Re_1(F \wedge G) \vdash F_1 \qquad Re_2 \vdash G_1}{\Gamma'(F \wedge G) \vdash F_1 * G_1}$$

Trivial for the first case. For the second case, induction hypothesis on the left premise sequent produces $Re_1(F; G) \vdash F_1$, and then $*R_1$, appropriately carrying out its internal weakening to recover $\Gamma'(F; G)$ from $Re_1(F; G)$ and Re_2.

(m) $*R_2$: Trivial.

(n) $-\!\!*R$: Trivial.

2. For a α`LBI` sequent $\Gamma(F \vee G) \vdash H$: similar.

3. For a α`LBI` sequent $\Gamma(F * G) \vdash H$, the base case is when it is an axiom for which a proof is trivially given. For inductive cases, assume that it holds true for all the derivation depths up to k and show that the same still holds for the derivation depth of $k + 1$. Consider what the last inference rule is.

   (a) $*L$: Trivial if the principal coincides with $F * G$. Otherwise, the derivation looks like:

   $$\frac{\Gamma(F * G)(F_1, G_1) \vdash H}{\Gamma(F * G)(F_1 * G_1) \vdash H} *L$$

   By induction hypothesis, $\Gamma(F, G)(F_1, G_1) \vdash H$. Then, $\Gamma(F, G)(F_1 * G_1) \vdash H$ as desired via $*L$.

   (b) The rest: Similar to the previous cases.

4. For a α`LBI` sequent $D : \Gamma(\Gamma_1, \emptyset_m) \vdash H$, the base case is when it is the conclusion sequent of an axiom.

   (a) id: $D : \mathsf{E}(\Gamma'(\Gamma_1, \emptyset_m); p) \vdash p$. Then $D' : \mathsf{E}(\Gamma'(\Gamma_1); p) \vdash p$ is also an axiom.

   (b) $\mathbb{1}L$, $\top R$: straightforward.

   (c) $\top^* R$: similar to id case.

For inductive cases, assume that the statement holds true for all the derivation depths up to k, and show that it still holds true at $k+1$. Consider what the last inference rule applied is.

(a) $\top L$: The derivation ends in one of the following:

$$\dfrac{\Gamma(\Gamma_1, \emptyset_m)(\emptyset_a) \vdash H}{\Gamma(\Gamma_1, \emptyset_m)(\top) \vdash H}\ \top L \quad \dfrac{\Gamma(\Gamma_1(\emptyset_a), \emptyset_m) \vdash H}{\Gamma(\Gamma_1(\top), \emptyset_m) \vdash H}\ \top L$$

By induction hypothesis, $\Gamma(\Gamma_1)(\emptyset_a) \vdash H$ for the former, or $\Gamma(\Gamma_1(\emptyset_a)) \vdash H$ for the latter, is α`LBI`-derivable. Then so is $\Gamma(\Gamma_1)(\top) \vdash H$ or $\Gamma(\Gamma_1(\top)) \vdash H$ via $\top L$ as required.

(b) $\vee L$: The derivation ends in one of the following:

$$\dfrac{\Gamma(\Gamma_1, \emptyset_m)(F_1) \vdash H \qquad \Gamma(\Gamma_1, \emptyset_m)(F_2) \vdash H}{\Gamma(\Gamma_1, \emptyset_m)(F_1 \vee F_2) \vdash H}\ \vee L$$

$$\dfrac{\Gamma(\Gamma_1(F_1), \emptyset_m) \vdash H \qquad \Gamma(\Gamma_1(F_2), \emptyset_m) \vdash H}{\Gamma(\Gamma_1(F_1 \vee F_2), \emptyset_m) \vdash H}\ \vee L$$

For the former, $\Gamma(\Gamma_1)(F_1) \vdash H$ and $\Gamma(\Gamma_1)(F_2) \vdash H$ (induction hypothesis); then $\Gamma(\Gamma_1)(F_1 \vee F_2) \vdash H$ via $\vee L$ as required. For the latter, $\Gamma(\Gamma_1(F_1)) \vdash H$ and $\Gamma(\Gamma_1(F_2)) \vdash H$ (induction hypothesis); then $\Gamma(\Gamma_1(F_1 \vee F_2)) \vdash H$ via $\vee L$ as required.

(c) $\supset L$: The derivation ends in one of the following:

$$\dfrac{\mathsf{E}(\Gamma_1; F_1 \supset F_2) \vdash F_1 \qquad \Gamma(F_2; \mathsf{E}(\Gamma_1; F_1 \supset F_2))(\Gamma_2, \emptyset_m) \vdash H}{\Gamma(\mathsf{E}(\Gamma_1; F_1 \supset F_2))(\Gamma_2, \emptyset_m) \vdash H}\ \supset L$$

$$\dfrac{\mathsf{E}(\Gamma_1(\Gamma_2, \emptyset_m); F_1 \supset F_2) \vdash F_1 \qquad \Gamma(F_2; \mathsf{E}(\Gamma_1(\Gamma_2, \emptyset_m); F_1 \supset F_2)) \vdash H}{\Gamma(\mathsf{E}(\Gamma_1(\Gamma_2, \emptyset_m); F_1 \supset F_2)) \vdash H}\ \supset L$$

$$\dfrac{\mathsf{E}(\Gamma_1; F_1 \supset F_2) \vdash F_1 \qquad \Gamma(\Gamma_2(F_2; \mathsf{E}(\Gamma_1; F_1 \supset F_2)), \emptyset_m) \vdash H}{\Gamma(\Gamma_2(\mathsf{E}(\Gamma_1; F_1 \supset F_2)), \emptyset_m) \vdash H}\ \supset L$$

For the first case, $\Gamma(F_2;\mathsf{E}(\Gamma_1;F_1{\supset}F_2))(\Gamma_2)\vdash H$ (induction hypothesis); then $\Gamma(\mathsf{E}(\Gamma_1;F_1{\supset}F_2))(\Gamma_2)\vdash H$ via $\supset L$ as required.
For the second case, $\mathsf{E}(\Gamma_1(\Gamma_2);F_1{\supset}F_2)\vdash F_1$ and $\Gamma(\mathsf{E}(\Gamma_1(\Gamma_2);F_1{\supset}F_2))\vdash H$ (induction hypothesis); then $\Gamma(\mathsf{E}(\Gamma_1(\Gamma_2);F_1{\supset}F_2))\vdash H$ via $\supset L$ as required.
For the third case, induction hypothesis on the right premise sequent, then $\supset L$ to conclude.

(d) ${-\!\!*}L_1$: This case is non-trivial, and so I detail the proof. Firstly, we cover easier cases when the derivation ends in one of the following:

$$\dfrac{Re_1\vdash F \qquad \Gamma((Re_2,G);(\Gamma_2,\mathsf{E}(\Gamma_3(\Gamma_1,\emptyset_m);F{-\!\!*}G)))\vdash H}{\Gamma(\Gamma_2,\mathsf{E}(\Gamma_3(\Gamma_1,\emptyset_m);F{-\!\!*}G))\vdash H}\;{-\!\!*}L_1$$

$$\dfrac{Re_1\vdash F_1 \qquad \Gamma(\Gamma_1,\emptyset_m)((Re_2,G);(\Gamma_2,\mathsf{E}(\Gamma_3;F{-\!\!*}G)))\vdash H}{\Gamma(\Gamma_1,\emptyset_m)(\Gamma_2,\mathsf{E}(\Gamma_3;F{-\!\!*}G))\vdash H}\;{-\!\!*}L_1$$

$$\dfrac{Re_1\vdash F_1 \qquad \Gamma(\Gamma_1((Re_2,G);(\Gamma_2,\mathsf{E}(\Gamma_3;F{-\!\!*}G))),\emptyset_m)\vdash H}{\Gamma(\Gamma_1(\Gamma_2,\mathsf{E}(\Gamma_3;F{-\!\!*}G)),\emptyset_m)\vdash H}\;{-\!\!*}L_1$$

For each, induction hypothesis, if applicable, and ${-\!\!*}L_1$ conclude. Now consider more complex cases where "$\Gamma_1,\emptyset_m$" occurs in the conclusion sequent as $\Gamma(\Gamma_2(\Gamma_1,\emptyset_m),\mathsf{E}(\Gamma_3;F{-\!\!*}G))\vdash H$. Less involved cases are when the internalised weakening process either retains or discards the whole "$\Gamma_1,\emptyset_m$":

$$\dfrac{Re_1(\Gamma_1,\emptyset_m)\vdash F \qquad \Gamma((Re_2,G);(\Gamma_2(\Gamma_1,\emptyset_m),\mathsf{E}(\Gamma_3;F{-\!\!*}G)))\vdash H}{\Gamma(\Gamma_2(\Gamma_1,\emptyset_m),\mathsf{E}(\Gamma_3;F{-\!\!*}G))\vdash H}\;{-\!\!*}L_1$$

$$\dfrac{Re_1\vdash F \qquad \Gamma((Re_2(\Gamma_1,\emptyset_m),G);(\Gamma_2(\Gamma_1,\emptyset_m),\mathsf{E}(\Gamma_3;F{-\!\!*}G)))\vdash H}{\Gamma(\Gamma_2(\Gamma_1,\emptyset_m),\mathsf{E}(\Gamma_3;F{-\!\!*}G))\vdash H}\;{-\!\!*}L_1$$

$$\dfrac{Re_1\vdash F \qquad \Gamma((Re_2,G);(\Gamma_2(\Gamma_1,\emptyset_m),\mathsf{E}(\Gamma_3;F{-\!\!*}G)))\vdash H}{\Gamma(\Gamma_2(\Gamma_1,\emptyset_m),\mathsf{E}(\Gamma_3;F{-\!\!*}G))\vdash H}\;{-\!\!*}L_1$$

The first assumes that the specific "$\Gamma_1, \emptyset_m$" does not occur in Re_2; the second that it does not occur in Re_1; the third that it does not occur in Re_1 or in Re_2. Each of them is concluded via induction hypothesis and then $-\!\!*L_1$.

On the other hand, if "$\Gamma_1, \emptyset_m$" should be split among the two premises, then we must monitor the internalised derivation process (Definition 58) more closely. In case the internalised contraction, EA_2 and weakening on $[D : \Gamma(\Gamma_2(\Gamma_1, \emptyset_m), \mathsf{E}(\Gamma_3; F -\!\!* G)) \vdash H]$ lead to $[D_1 : \Gamma'(\Gamma'_1, \emptyset_m, F -\!\!* G) \vdash H]$ where

- $\Gamma'(-)$ abbreviates $\Gamma(-; (\Gamma_2(\Gamma_1, \emptyset_m), \mathsf{E}(\Gamma_3; F -\!\!* G)))$
- and the "$\emptyset_m$" in "$\Gamma'_1, \emptyset_m$" is the same (modulo contraction) "$\emptyset_m$" in "$\Gamma_1, \emptyset_m$" in D,

then we have the following transition in $\mathtt{LBI}$-space: $D \rightsquigarrow_{CtrL_{\mathtt{LBI}}} D' \rightsquigarrow^*_{EA_2} D'' \rightsquigarrow^*_{WkL_{\mathtt{LBI}}} [D_1 : \Gamma'(\Gamma'_1, \emptyset_m, F -\!\!* G) \vdash H]$. Consider possible scenarios for the last transitions in $\mathtt{LBI}$-space.

i. If $D_1 \rightsquigarrow_{-\!\!*L_{\mathtt{LBI}}} [D_2 : \Gamma'_1 \vdash F]$ and $D_1 \rightsquigarrow_{-\!\!*L_{\mathtt{LBI}}} [D_3 : \Gamma'(\emptyset_m, G) \vdash H]$: then induction hypothesis on D_3 (for both "$\emptyset_m, G$" and "$\Gamma_1, \emptyset_m$") and $-\!\!*L_2$ conclude.

ii. If $D_1 \rightsquigarrow_{-\!\!*L_{\mathtt{LBI}}} [D_2 : \emptyset_m \vdash F]$ and $D_1 \rightsquigarrow_{-\!\!*L_{\mathtt{LBI}}} [D_3 : \Gamma'(\Gamma'_1, G) \vdash H]$: then $D'_3 : \Gamma((\Gamma'_1, G); (\Gamma_2(\Gamma_1), \mathsf{E}(\Gamma_3; F -\!\!* G))) \vdash H$ is $\alpha\mathtt{LBI}$-derivable (induction hypothesis); $D''_3 : \Gamma((\Gamma_2(\Gamma_1), G); (\Gamma_2(\Gamma_1), \mathsf{E}(\Gamma_3; F -\!\!* G))) \vdash H$ is then also $\alpha\mathtt{LBI}$-derivable (Proposition 6)[1].
Then $\Gamma(\Gamma_2(\Gamma_1), \mathsf{E}(\Gamma_3; F -\!\!* G)) \vdash H$ as required via $-\!\!*L_3$.

iii. If (1) Γ'_1 is in the form: Γ'_2, Γ'_3 (2) $D_1 \rightsquigarrow_{-\!\!*L_{\mathtt{LBI}}} [D_2 : \Gamma'_2 \vdash F]$ and (3) $D_1 \rightsquigarrow_{-\!\!*L_{\mathtt{LBI}}} [D_3 : \Gamma'(\Gamma'_3, \emptyset_m, G) \vdash H]$: then induction hypothesis on D_3 and $-\!\!*L_1$ conclude.

iv. If (1) Γ'_1 is in the form: Γ'_2, Γ'_3 (2) $D_1 \rightsquigarrow_{-\!\!*L_{\mathtt{LBI}}} [D_2 : \Gamma'_2, \emptyset_m \vdash F]$ and (3) $D_1 \rightsquigarrow_{-\!\!*L_{\mathtt{LBI}}} [D_3 : \Gamma'(\Gamma'_3, G) \vdash H]$: then induction hypothesis on both of the premises and then $-\!\!*L_1$ conclude.

---

[1] The internalised derivation process from D into D_1 explains why this holds.

(e) $-\!\!*L_3$: The derivation ends in one of the following:

$$\cfrac{\emptyset_m \vdash F \qquad \Gamma((\Gamma_2(\Gamma_1, \emptyset_m), G); (\Gamma_2(\Gamma_1, \emptyset_m), \mathsf{E}(\Gamma_3; F -\!\!* G))) \vdash H}{\Gamma(\Gamma_2(\Gamma_1, \emptyset_m), \mathsf{E}(\Gamma_3; F -\!\!* G)) \vdash H} \; -\!\!*L_3$$

$$\cfrac{\emptyset_m \vdash F \qquad \Gamma((\Gamma_2, G); (\Gamma_2, \mathsf{E}(\Gamma_3(\Gamma_1, \emptyset_m); F -\!\!* G))) \vdash H}{\Gamma(\Gamma_2, \mathsf{E}(\Gamma_3(\Gamma_1, \emptyset_m); F -\!\!* G)) \vdash H} \; -\!\!*L_3$$

$$\cfrac{\emptyset_m \vdash F_1 \qquad \Gamma(\Gamma_1, \emptyset_m)((\Gamma_2, G); (\Gamma_2, \mathsf{E}(\Gamma_3; F -\!\!* G))) \vdash H}{\Gamma(\Gamma_1, \emptyset_m)(\Gamma_2, \mathsf{E}(\Gamma_3; F -\!\!* G)) \vdash H} \; -\!\!*L_3$$

$$\cfrac{\emptyset_m \vdash F_1 \qquad \Gamma(\Gamma_1((\Gamma_2, G); (\Gamma_2, \mathsf{E}(\Gamma_3; F -\!\!* G))), \emptyset_m) \vdash H}{\Gamma(\Gamma_1(\Gamma_2, \mathsf{E}(\Gamma_3; F -\!\!* G)), \emptyset_m) \vdash H} \; -\!\!*L_3$$

Trivial except for the first case by induction hypothesis and $-\!\!*L_3$. For the first case, again trivial if $\Gamma_2(\Gamma_1, \emptyset_m)$ is not "$\Gamma_1, \emptyset_m$"; otherwise, if it is "$\Gamma_1, \emptyset_m$", then by induction hypothesis on the right premise sequent, $\Gamma((\Gamma_2, G); (\Gamma_2, \mathsf{E}(\Gamma_3; F -\!\!* G))) \vdash H$ is $\alpha\mathtt{LBI}$-derivable. But, then, by eye inspection on $-\!\!*L_3$, it is immediate that $\Gamma(\Gamma_2, \mathsf{E}(\Gamma_3; F -\!\!* G)) \vdash H$ is also $\alpha\mathtt{LBI}$-derivable, as required.

(f) The rest: similar or straightforward.

5. The rest: similar or straightforward. □

3.2.5 Admissibility of $EqAnt_{1,2}$

Proposition 8 (Admissibility of $EqAnt_{1,2}$) *$EqAnt_1{}_{\mathtt{LBI}}$ and $EqAnt_2{}_{\mathtt{LBI}}$ are admissible in $\alpha\mathtt{LBI}$. Depth preservation holds.*

Proof. Follows from (1) inversion lemma showing depth-preserving admissibility of

$$\cfrac{\Gamma(\Gamma_1; \emptyset_a) \vdash H}{\Gamma(\Gamma_1) \vdash H} \quad \cfrac{\Gamma(\Gamma_1, \emptyset_m) \vdash H}{\Gamma(\Gamma_1) \vdash H}$$

(2) Proposition 6 showing depth-preserving admissibility of

$$\frac{\Gamma(\Gamma_1) \vdash H}{\Gamma(\Gamma_1; \emptyset_a) \vdash H}$$

and (3) Proposition 7 showing depth-preserving admissibility of

$$\frac{\Gamma(\Gamma_1) \vdash H}{\Gamma(\Gamma_1, \emptyset_m) \vdash H}$$

□

3.2.6 Preparation for contraction admissibility in $*R_1$/$-\!\!*L_1$

I dedicate one subsection here to fortify ourselves with a further observation about the generation process of Re_1/Re_2, preparing for the main proof of contraction admissibility. Following Proposition 6 and Proposition 8, an observation is made concerning the internalised weakening ($WkL_{\mathtt{LBI}}$) within $*R_1$ and $-\!\!*L_1$: there is no need to consider an arbitrary $WkL_{\mathtt{LBI}}$ application in the process.

Lemma 13 (Sufficiency of incremental weakening) *In an application of $*R_1$ (in backward derivation) on a αLBI-derivable sequent $D : \Gamma' \vdash F * G$, if there exists a αLBI-derivable pair of D_1 and D_2 such that $D \rightsquigarrow_{*R_1} D_1$ and $D \rightsquigarrow_{*R_1} D_2$, then there exists a αLBI-derivable pair of D'_1 and D'_2 such that $D \rightsquigarrow_{*R'_1} D'_1$ and $D \rightsquigarrow_{*R'_1} D'_2$ where $*R'_1$ is defined here to be $*R$ except that its internalised weakening is carried out only with WkL_1 and WkL_2 as stated below:*

$$\frac{\Gamma_1 \vdash H}{\Gamma_1; \Gamma_2 \vdash H}\, WkL_1 \quad \frac{\Gamma_1, \Gamma_2 \vdash H}{\Gamma_1, (\Gamma_2; \Gamma_3) \vdash H}\, WkL_2$$

Similarly, in an application of $-\!\!*L_1$ (in backward derivation) on a sequent $D : \Gamma(\Gamma', \mathsf{E}(\Gamma_1; F -\!\!* G))) \vdash H$, it suffices to apply the following restricted weakening rules in the internalised weakening process:

$$\frac{\Gamma(\Gamma_2, F -\!\!* G) \vdash H}{\Gamma(\Gamma_2, (\Gamma_1; F -\!\!* G)) \vdash H}\, WkL'_1 \quad \frac{\Gamma(\Gamma_2, F -\!\!* G) \vdash H}{\Gamma((\Gamma_2; \Gamma_3), F -\!\!* G) \vdash H}\, WkL'_2$$

$$\frac{\Gamma(\Gamma_2, \Gamma_3, F -\!\!* G) \vdash H}{\Gamma(\Gamma_2, (\Gamma_3; \Gamma_4), F -\!\!* G) \vdash H}\, WkL'_3$$

Proof.

$*R_1$: Under the assumption made, there exists a α`LBI`-derivable pair of $D_1 : Re_1 \vdash F$ and $D_2 : Re_2 \vdash G$ from the conclusion sequent $D : \Gamma' \vdash F * G$ such that $D \rightsquigarrow_{*R_1} D_1$ and $D \rightsquigarrow_{*R_1} D_2$. Internally (*Cf.* Definition 58) Re_1/Re_2 results from a finite number of $WkL_{\texttt{LBI}}$ applications on D as follows: $D \rightsquigarrow^*_{WkL_{\texttt{LBI}}} [D' : Re_1, Re_2 \vdash F{*}G]$. In D', notice that the outermost structural layer of the antecedent structure is multiplicative. If Γ' in D was an additive structural layer, *i.e.* Γ' denoting $\alpha_1; \ldots; \alpha_m; \mathcal{M}_1; \ldots; \mathcal{M}_n$ for $m + n \geq 2$, $m \geq 0$ and $n \geq 1$, then a finite number of $WkL_{\texttt{LBI}}$ applications must have taken place at this additive structural layer (which is the outermost structural layer in Γ') such that (in backward derivation) all but one multiplicative structural layer $\mathcal{M}_k$, $1 \leq k \leq n$ were discarded in the transition. But this process is also achieved via WkL_1. Once the outermost structural layer is multiplicative, it is either the case that some Re'_1/Re'_2 pair can be formed on the antecedent part for D'_1 and D'_2 such that $Re'_1 \vdash F$ and $Re'_2 \vdash G$ are both α`LBI`-derivable, or not. We are done if it can be formed. Otherwise, the current outermost multiplicative structural layer holds $\mathcal{A}$(s) as its constituent(s) whose $\mathcal{M}$ constituent (again only one of them) must be connected at the current outermost multiplicative structural layer, which is achieved through WkL_2. This incremental process eventually produces the Re'_1/Re'_2 pair on the antecedent part, *provided that a situation that satisfies all the below conditions does not arise.*

- for all $D^* : Re^*_1, Re^*_2 \vdash F * G$ such that $D \rightsquigarrow^*_{\{WkL_1, WkL_2\}} D^*$ as the internal weakening process within $*R_1$, not both $D^*_1 : Re^*_1 \vdash F$ and $D^*_2 : Re^*_2 \vdash G$ are α`LBI`-derivable.
- there exists $D^{**} : Re^{**}_1, Re^{**}_2 \vdash F * G$ such that $D^* \rightsquigarrow^*_{WkL_{\texttt{LBI}}} D^{**}$ (as the internal weakening process within $*R_1$), and that both $D^{**}_1 : Re^{**}_1 \vdash F$ and $D^{**}_2 : Re^{**}_2 \vdash G$ are α`LBI`-derivable.

Suppose, by way of showing contradiction, that there exists a α`LBI`-derivation in which both conditions above satisfy. Then it cannot be the case that $D^* \rightsquigarrow^*_{\{WkL_1, WkL_2\}} D^{**}$ for the obvious reason that otherwise we would have D^{**} as D^*, an immediate contradiction to the supposition. Therefore it must be

the case that the Re_i^{**} (for $i \in \{1, 2\}$) looks like $\alpha_1, \alpha_2, \cdots, \alpha_m, \mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n$ for $m + n \geq 2, m \geq 0, n \geq 1$ such that it does not coincide with any possible Re_i^*. However, such a condition perforce implies by the definition of $*R_1$ that there exists a possible $D_1^* : Re_1^* \vdash F$ (resp. D_2^*) that looks like: $\alpha_1, \alpha_2, \cdots, \alpha_m, \mathcal{A}_1^*, \mathcal{A}_2^*, \cdots, \mathcal{A}_n^* \vdash F$ (similarly for D_2^*) such that, in `LBI`-space: $[D_1^* : \alpha_1, \alpha_2, \cdots, \alpha_m, \mathcal{A}_1^*, \mathcal{A}_2^*, \cdots, \mathcal{A}_n^* \vdash F] \rightsquigarrow_{WkL}^*$
$[D_1^{**} : \alpha_1, \alpha_2, \cdots, \alpha_m, \mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n \vdash F]$ and similarly for $D_2^* \rightsquigarrow_{WkL}^* D_2^{**}$. But then Proposition 6 dictates that α`LBI`-derivability of D_1^{**} (resp. D_2^{**}) implies α`LBI`-derivability of D_1^* (resp. D_2^*), a direct contradiction to the supposition.

$-\!\!*L_1$: Similar. The starting point for the implicit weakening in these rules is Γ' in the conclusion sequent $D : \Gamma(\Gamma', \mathsf{E}(\Gamma_1; F -\!\!* G))) \vdash H$. An application of WkL_1' is mandatory (*Cf.* Definition 58) in case the principal is (or will be) in an additive structural layer connected to Γ_1.

□

Corollary 1 (Maximal Re_1/Re_2) *For a α`LBI`-derivable sequent $D : \Gamma' \vdash F * G$, if there exists a pair of α`LBI`-derivable sequents $D_1' : Re_1' \vdash F$ and $D_2' : Re_2' \vdash G$ such that $D \rightsquigarrow_{*R_1} D_1'$ and $D \rightsquigarrow_{*R_1} D_2'$, then there exists a pair of α`LBI`-derivable sequents $D_1 : Re_1 \vdash F$ and $D_2 : Re_2 \vdash G$ such that all the following conditions satisfy.*

- *$D \rightsquigarrow_{*R_1} D_1$ (resp. $D \rightsquigarrow_{*R_1} D_2$) with incremental weakening (Lemma 13).*
- *D_1 (resp. D_2) is a sequent that results from Proposition 6 on D_1' (resp. D_2')*[1].
- *there exists no $D_1^* : Re_1^* \vdash F$ (resp. $D_2^* : Re_2^* \vdash G$) such that all the following conditions satisfy.*
  - *D_1^* (resp. D_2^*) is a sequent that results from Proposition 6 on D_1 (resp. D_2).*
  - *$D_1^* \not\bowtie D_1$ (resp. $D_2^* \not\bowtie D_2$).*
  - *$D \rightsquigarrow_{*R_1} D_1^*$ (resp. $D \rightsquigarrow_{*R_1} D_2^*$).*

---

[1] That is, there is a transition $D_i \rightsquigarrow_{WkL_{\text{LBI}}}^* D_i'$ in `LBI`-space. An application of Proposition 6 on D_i' is reflexive if it only introduces "$\emptyset_a$"s due to Proposition 8.

Such a Re_1/Re_2 pair is called a maximal Re_1/Re_2 pair. Likewise, with an abbreviation $\Gamma_a(-)$ denoting $\Gamma(-;(\Gamma', \mathsf{E}(\Gamma_1; F{-\!\!*}G)))$, for a α`LBI`-derivable sequent $D : \Gamma(\Gamma', \mathsf{E}(\Gamma_1; F{-\!\!*}G)) \vdash H$, if there exists a pair of α`LBI`-derivable sequents $D_1' : Re_1' \vdash F$ and $D_2' : \Gamma_a(Re_2', G) \vdash H$ such that $D \rightsquigarrow_{-\!\!*L_1} D_1'$ and $D \rightsquigarrow_{-\!\!*L_1} D_2'$, then there exists a pair of α`LBI`-derivable sequents $D_1 : Re_1 \vdash F$ and $D_2 : \Gamma_a(Re_2, G) \vdash H$ such that the following conditions all satisfy.

- *$D \rightsquigarrow_{-\!\!*L_1} D_1$ (resp. $D \rightsquigarrow_{-\!\!*L_1} D_2$) with incremental weakening (Lemma 13).*
- *D_1 (resp. D_2) is a sequent that results from Proposition 6 on D_1' (resp. D_2').*
- *there exists no $D_1^* : Re_1^* \vdash F$ (resp. $D_2^* : \Gamma_a(Re_2^*, G) \vdash H$) such that the following conditions all satisfy.*
  - *D_1^* (resp. D_2^*) is a sequent that results from Proposition 6 on D_1 (resp. D_2).*
  - *$D_1^* \not\bowtie D_1$ (resp. $D_2^* \not\bowtie D_2$).*
  - *$D \rightsquigarrow_{-\!\!*L_1} D_1^*$ (resp. $D \rightsquigarrow_{-\!\!*L_1} D_2^*$).*

It is inferrable from Corollary 1 that neither $*R_2$, $-\!\!*L_2$ nor $-\!\!*L_3$ needs embed an internalised weakening. In the rest, I assume only some maximal Re_1/Re_2 pair for $*R_1$ and $-\!\!*L_1$.

3.2.7 Admissibility of contraction in α`LBI`

Contraction admissibility in α`LBI` follows.

Theorem 6 (Contraction admissibility in α`LBI`)
If $D : \Gamma(\Gamma_a; \Gamma_a) \vdash F$ is α`LBI`-derivable, then so is $D' : \Gamma(\Gamma_a) \vdash F$, preserving the derivation depth.

Proof. By induction on the derivation depth of $\Pi(D)$. The base cases are when it is 1, *i.e.* when D is the conclusion sequent of an axiom. To consider which axiom has been applied, if it is $\top R$, then it is trivial to show that if $\Gamma(\Gamma_a; \Gamma_a) \vdash \top$ is α`LBI`-derivable, then so is $\Gamma(\Gamma_a) \vdash \top$. Also for $\mathbb{1}L$, a single occurrence of $\mathbb{1}$ on the antecedent part of D suffices for the $\mathbb{1}L$ application, and the current theorem is trivially provable in this case, too. For both id and ${}^*\!\top R$, $\Pi(D)$ looks like:

$$\frac{}{\Gamma; \alpha \vdash F}$$

where (α, F) is (p, p) for id, and $(\emptyset_m, {}^{*}\top)$ for ${}^{*}\top R$. As the antecedent is either a single structure or its outermost structural layer, *i.e.* $\Gamma; \alpha$, is additive, irrespective of where Γ_a in D is, if D is α`LBI`-derivable, then so is D'.

For inductive cases, suppose that the current theorem has been proved for any derivation depth of up to k, it must be then demonstrated that it still holds for the derivation depth of $k+1$. Consider what the α`LBI` inference rule applied last is, and, in case of a left inference rule, consider where the active structure Γ_b of the inference rule is in $\Gamma(\Gamma_a; \Gamma_a)$.

1. $\top L$, and Γ_b is $\top$: if it does not appear in Γ_a, induction hypothesis on the premise sequent concludes. Otherwise, $\Pi(D)$ looks like:

   $$\begin{array}{c}\vdots \\ \dfrac{D_1 : \Gamma(\Gamma'_a(\emptyset_a); \Gamma'_a(\top)) \vdash H}{D : \Gamma(\Gamma'_a(\top); \Gamma'_a(\top)) \vdash H}\,\top L\end{array}$$

   where $\Gamma'_a(\top)$ represents Γ_a (assumed similarly for all the remaining cases). From α`LBI` inversion lemma, if D_1 is derivable, so is $D'_1 : \Gamma(\Gamma'_a(\emptyset_a); \Gamma'_a(\emptyset_a)) \vdash H$. By induction hypothesis on D'_1, $D''_1 : \Gamma(\Gamma'_a(\emptyset_a)) \vdash H$ is also derivable. Then a forward (assumed similarly for all the remaining cases) application of $\top L$ on D''_1, *i.e.* D''_1 as the premise sequent, deriving the conclusion sequent via $\top L$ at derivation depth $k+1$, concludes.

2. ${}^{*}\top L$, and Γ_b is ${}^{*}\top$: similar to the case $\top$.

3. $\wedge L$, and Γ_b is $F_1 \wedge F_2$: if it does not appear in Γ_a, induction hypothesis on the premise sequent. Otherwise, $\Pi(D)$ looks like:

   $$\begin{array}{c}\vdots \\ \dfrac{D_1 : \Gamma(\Gamma'_a(F_1; F_2); \Gamma'_a(F_1 \wedge F_2)) \vdash H}{D : \Gamma(\Gamma'_a(F_1 \wedge F_2); \Gamma'_a(F_1 \wedge F_2)) \vdash H}\,\wedge L\end{array}$$

   $D'_1 : \Gamma(\Gamma'_a(F_1; F_2); \Gamma'_a(F_1; F_2)) \vdash H$ is α`LBI`-derivable (inversion lemma); $D''_1 : \Gamma(\Gamma'_a(F_1; F_2)) \vdash H$ is also α`LBI`-derivable (induction hypothesis); then $\wedge L$ on D''_1 concludes.

4. $\supset L$, and Γ_b is $\Gamma'; F \supset G$: if it does not appear in Γ_a, then the induction hypothesis on both of the premises concludes. If it is entirely in Γ_a, then $\Pi(D)$ looks either like:

$$\dfrac{\begin{matrix}\vdots & \vdots\\ D_1 : \mathsf{E}(\Gamma'; F \supset G) \vdash F & D_2\end{matrix}}{D : \Gamma(\Gamma'_a(\mathsf{E}(\Gamma'; F \supset G)); \Gamma'_a(\mathsf{E}(\Gamma'; F \supset G))) \vdash H} \supset L$$

where $D_2 : \Gamma(\Gamma'_a(G; \mathsf{E}(\Gamma'; F{\supset}G)); \Gamma'_a(\mathsf{E}(\Gamma'; F{\supset}G))) \vdash H$, or, in case Γ_a is $\mathsf{E}(\Gamma'_a); F{\supset}G$, like:

$$\dfrac{\begin{matrix}\vdots & \vdots\\ D_1 : \mathsf{E}(\Gamma'_a); F{\supset}G; \mathsf{E}(\Gamma'_a); F{\supset}G \vdash F & D_2\end{matrix}}{D : \Gamma(\mathsf{E}(\Gamma'_a); F{\supset}G; \mathsf{E}(\Gamma'_a); F{\supset}G) \vdash H} \supset L$$

where $D_2 : \Gamma(G; \mathsf{E}(\Gamma'_a); F{\supset}G; \mathsf{E}(\Gamma'_a); F{\supset}G) \vdash H$. In the former, $D'_2 : \Gamma(\Gamma'_a(G; \mathsf{E}(\Gamma'; F{\supset}G)); \Gamma'_a(G; \mathsf{E}(\Gamma'; F{\supset}G))) \vdash H$ (weakening admissibility); $D''_2 : \Gamma(\Gamma'_a(G; \mathsf{E}(\Gamma'; F{\supset}G))) \vdash H$ (induction hypothesis); then $\supset L$ on D_1 and D''_2 concludes. In the latter, induction hypothesis on D_1 and on D_2 followed by $\supset L$ conclude. Finally, if only a substructure of Γ_b is in Γ_a with the rest spilling out of Γ_a, then similar to the latter case.

5. $*R_1$: $\Pi(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots & \vdots\\ D_1 : Re_1 \vdash F_1 & D_2 : Re_2 \vdash F_2\end{matrix}}{D : \Gamma(\Gamma_a; \Gamma_a) \vdash F_1 * F_2} *R_1$$

We show that the internalised weakening process to generate a maximal Re_1/Re_2 pair must either weaken away one Γ_a completely or preserve $\Gamma_a; \Gamma_a$ as a substructure of Re_1 (or Re_2). But due to the formulation of the pair (*c.f.* Corollary 1), such must be the case. If $\Gamma_a; \Gamma_a$ is preserved in Re_1, then induction hypothesis on D_1 concludes; otherwise, it is trivial to see that only a single Γ_a (if any) needs to be present in D.

6. $-\!\!*L_1$, and Γ_b is $\Gamma', \mathsf{E}(\Gamma_1; F{-\!\!*}G)$: if Γ_b is not in Γ_a, then induction hypothesis on the right premise sequent concludes. If it is in Γ_a, $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ D_1 : Re_1 \vdash F & D_2\end{array}}{D : \Gamma(\Gamma'_a(\Gamma', \mathsf{E}(\Gamma_1; F{-\!\!*}G)); \Gamma'_a(\Gamma', \mathsf{E}(\Gamma_1; F{-\!\!*}G))) \vdash H}\,{-\!\!*}L_1$$

where D_2 is:

$$\Gamma(\Gamma'_a((Re_2, G); (\Gamma', \mathsf{E}(\Gamma_1; F{-\!\!*}G))); \Gamma'_a(\Gamma', \mathsf{E}(\Gamma_1; F{-\!\!*}G))) \vdash H$$

$D'_2 : \Gamma(\Gamma'_a((Re_2, G); (\Gamma', \mathsf{E}(\Gamma_1; F{-\!\!*}G))); \Gamma'_a((Re_2, G); (\Gamma', \mathsf{E}(\Gamma_1; F{-\!\!*}G)))) \vdash H$ via Proposition 6 is also α`LBI`-derivable. $D''_2 : \Gamma(\Gamma'_a((Re_2, G); (\Gamma', \mathsf{E}(\Gamma_1; F{-\!\!*}G)))) \vdash H$ via induction hypothesis. Then ${-\!\!*}L_1$ on D_1 and D''_2 concludes. If, on the other hand, Γ_b is in Γ_a, then it is (or will be after "EA_2"s) either in Γ_1 or in Γ'. But if it is in Γ_1, then it must be weakened away, and if it is in Γ', similar to the $*R_1$ case.

7. Other cases are similar to one of the cases already examined. □

I now justify the absorption of structural contraction in $\supset L$ and ${-\!\!*}L_{1,2,3}$.

Proposition 9 (Non-admissible structural contraction) *There exist sequents which are derivable in [*`LBI - Cut`*] but not derivable in [*`LBI - Cut`*] without structural contraction.*

Proof. For ${-\!\!*}L_{1,2,3}$, use a sequent $\top{-\!\!*}p_1, \top{-\!\!*}(p_1 \supset p_2) \vdash p_2$ and assume that every propositional variable is distinct. Then without contraction, there are several derivations of which two sensible ones are shown below (the rest similar).

1. $$\dfrac{\dfrac{}{\top{-\!\!*}(p_1 \supset p_2) \vdash \top}\,\top R \qquad p_1 \vdash p_2}{D : \top{-\!\!*}p_1, \top{-\!\!*}(p_1 \supset p_2) \vdash p_2}\,{-\!\!*}L$$

2. $$\dfrac{\dfrac{}{\top{-\!\!*}p_1 \vdash \top}\,\top R \qquad \dfrac{\dfrac{\emptyset_a \vdash p_1 \qquad \dfrac{}{p_2 \vdash p_2}\,id}{\emptyset_a; p_1 \supset p_2 \vdash p_2}\,\supset L}{p_1 \supset p_2 \vdash p_2}\,EqAnt_1L}{D : \top{-\!\!*}p_1, \top{-\!\!*}(p_1 \supset p_2) \vdash p_2}\,{-\!\!*}L$$

In both of the derivation trees above, one branch is open. Moreover, such holds true when only formula-level contraction is permitted in `LBI`. The sequent D cannot be derived under the given restriction. In the presence of structural contraction, however, another construction is possible:

$$\dfrac{\dfrac{\Pi(D_1) \qquad \Pi(D_2)}{(\top {-\!\!*} p_1, \top {-\!\!*} (p_1 \supset p_2)); (\top {-\!\!*} p_1, \top {-\!\!*} (p_1 \supset p_2)) \vdash p_2}\, {-\!\!*}L}{D : \top {-\!\!*} p_1, \top {-\!\!*} (p_1 \supset p_2) \vdash p_2}\, CtrL$$

where $\Pi(D_1)$ and $\Pi(D_2)$ are:

$\Pi(D_1)$**:**

$$\dfrac{}{\top {-\!\!*} (p_1 \supset p_2) \vdash \top}\, \top R$$

$\Pi(D_2)$**:**

$$\dfrac{\dfrac{}{\top {-\!\!*} p_1 \vdash \top}\, \top R \qquad \dfrac{\dfrac{}{p_1 \vdash p_1}\, id \qquad \dfrac{\dfrac{}{p_2 \vdash p_2}\, id}{p_1; p_2 \vdash p_2}\, WkL}{p_1; p_1 \supset p_2 \vdash p_2}\, \supset L}{p_1; (\top {-\!\!*} (p_1 \supset p_2)) \vdash p_2}\, {-\!\!*}L$$

where all the derivation tree branches are closed upward.

For $\supset L$, with $(\emptyset_m; p_1), (\emptyset_m; p_1 \supset p_2) \vdash p_2$. Without structural contraction we have (only two sensible ones are shown; the rest similar):

1.

$$\dfrac{\emptyset_m \vdash p_1 \qquad \dfrac{\dfrac{\dfrac{}{p_2 \vdash p_2}\, id}{\emptyset_m; p_2 \vdash p_2}\, WkL}{(\emptyset_m; p_1), (\emptyset_m; p_2) \vdash p_2}\, EA_2}{D : (\emptyset_m; p_1), (\emptyset_m; p_1 \supset p_2) \vdash p_2}\, \supset L$$

2.

$$\dfrac{\dfrac{p_1 \vdash p_2}{\emptyset_m; p_1 \vdash p_2}\, WkL}{D : (\emptyset_m; p_1), (\emptyset_m; p_1 \supset p_2) \vdash p_2}\, EA_2$$

In the presence of structural contraction, there is a closed derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p_1 \vdash p_1}\,id}{\emptyset_m; p_1; \emptyset_m \vdash p_1}\,WkL \qquad \dfrac{\dfrac{}{p_2 \vdash p_2}\,id}{\emptyset_m; p_1; \emptyset_m; p_2 \vdash p_2}\,WkL}{\emptyset_m; p_1; \emptyset_m; p_1 \supset p_2 \vdash p_2}\,\supset L}{((\emptyset_m; p_1), (\emptyset_m; p_1 \supset p_2)); ((\emptyset_m; p_1), (\emptyset_m; p_1 \supset p_2)) \vdash p_2}\,EA_2}{D : (\emptyset_m; p_1), (\emptyset_m; p_1 \supset p_2) \vdash p_2}\,CtrL$$

□

3.2.8 Equivalence of α`LBI` to `LBI`

The following equivalence theorem of α`LBI` to `LBI` concludes this section.

Theorem 7 (Equivalence between α`LBI` and `LBI`) *$D : \Gamma \vdash F$ is α`LBI`-derivable iff it is [`LBI- Cut`]-derivable (iff it is `LBI`-derivable).*

Proof. Into the *only if* direction, assume that D is α`LBI`-derivable, and then show that there is a [`LBI- Cut`]-derivation for each α`LBI` derivation. But this is obvious because each α`LBI` inference rule is derivable in `LBI`: $*R_{1,2\,\alpha\mathtt{LBI}}$, $-\!\!*L_{1,2,3,4\,\alpha\mathtt{LBI}}$, $\supset L_{\alpha\mathtt{LBI}}$, $id_{\alpha\mathtt{LBI}}$ and ${}^{*}\!\top R_{\alpha\mathtt{LBI}}$ as stated in 3.2.2; all the other α`LBI` rules are identical to `LBI`'s.[1]

Into the *if* direction, assume that D is [`LBI- Cut`]-derivable, and then show that there is a corresponding α`LBI`-derivation to each [`LBI- Cut`] derivation by induction on the derivation depth of $\Pi_{\mathtt{LBI}}(D)$ ($\Pi_{\mathtt{LBI}}(D)$ denotes a [`LBI- Cut`]-derivation of D).

If it is 1, *i.e.* if D is the conclusion sequent of an axiom, I note that $\mathbb{1}L_{\mathtt{LBI}}$ is identical to $\mathbb{1}L_{\alpha\mathtt{LBI}}$; $id_{\mathtt{LBI}}$ and ${}^{*}\!\top R_{\mathtt{LBI}}$ via $id_{\alpha\mathtt{LBI}}$ and resp. ${}^{*}\!\top R_{\alpha\mathtt{LBI}}$ with Proposition 6 and Proposition 7; and $\top R_{\mathtt{LBI}}$ as identical to $\top R_{\alpha\mathtt{LBI}}$. For inductive cases, assume that the *if* direction holds true up to the [`LBI- Cut`]-derivation depth of k, then it must be demonstrated that it still holds true for the [`LBI- Cut`]-derivation depth of $k + 1$. Consider what the `LBI` rule applied last is:

1. $\supset L_{\mathtt{LBI}}$: $\Pi_{\mathtt{LBI}}(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots \\ D_1 : \Gamma_1 \vdash F\end{matrix} \qquad \begin{matrix}\vdots \\ D_2 : \Gamma(\Gamma_1; G) \vdash H\end{matrix}}{D : \Gamma(\Gamma_1; F \supset G) \vdash H}\,\supset L_{\mathtt{LBI}}$$

---

[1] Note again that EA_2 is [`LBI- Cut`]-derivable with $WkL_{\mathtt{LBI}}$ and $EqAnt_{2\,\mathtt{LBI}}$.

By induction hypothesis, both D_1 and D_2 are also αLBI-derivable. Proposition 6 on D_1 in αLBI-space results in $D_1' : \Gamma_1; F \supset G \vdash F$, and on D_2 results in $D_2' : \Gamma(\Gamma_1; G; F \supset G) \vdash H$. Then an application of $\supset L_{\alpha\texttt{LBI}}$ on D_1' and D_2 concludes in αLBI-space.

2. $-\!\!*L_{\texttt{LBI}}$: $\Pi_{\texttt{LBI}}(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots \\ D_1 : \Gamma_1 \vdash F\end{matrix} \qquad \begin{matrix}\vdots \\ D_2 : \Gamma(G) \vdash H\end{matrix}}{D : \Gamma(\Gamma_1, F -\!\!* G) \vdash H} -\!\!*L_{\texttt{LBI}}$$

By induction hypothesis, D_1 and D_2 are also αLBI-derivable.

   (a) If $\Gamma(G)$ is G, *i.e.* if the antecedent part of D_2 is a single structure (G), then Proposition 6 on D_2 results in $D_2' : G; (\Gamma_1, F -\!\!* G) \vdash H$ in αLBI-space. Then $-\!\!*L_{2\,\alpha\texttt{LBI}}$ on D_1 and D_2' leads to $D' : \Gamma_1, F -\!\!* G \vdash H$ as required. Instead of D_2', $D_2^* : G; F -\!\!* G \vdash H$ in case Γ_1 is $\emptyset_m$, and $-\!\!*L_{4\,\alpha\texttt{LBI}}$ instead of $-\!\!*L_{2\,\alpha\texttt{LBI}}$.

   (b) If $\Gamma(G)$ is $\Gamma'(\Gamma'', G)$, then Proposition 6 on D_2 leads to $D_2' : \Gamma'((\Gamma'', G); (\Gamma'', \Gamma_1, F -\!\!* G)) \vdash H$. Then $-\!\!*L_{1\,\alpha\texttt{LBI}}$ on D_1 and D_2' leads to $D' : \Gamma'(\Gamma'', \Gamma_1, F -\!\!* G) \vdash H$ as required. Instead of D_2', $D_2^* : \Gamma'((\Gamma'', G); (\Gamma'', F -\!\!* G)) \vdash H$ in case Γ_1 is $\emptyset_m$, and $-\!\!*L_{3\,\alpha\texttt{LBI}}$ instead of $-\!\!*L_{1\,\alpha\texttt{LBI}}$.

   (c) Finally, if $\Gamma(G)$ is $\Gamma'(\Gamma''; G) \vdash H$, then Proposition 6 on D_2 leads to $D_2' : \Gamma'(\Gamma''; G; (\Gamma_1, F -\!\!* G)) \vdash H$. Then $-\!\!*L_{2\,\alpha\texttt{LBI}}$ on D_1 and D_2' leads to $D' : \Gamma'(\Gamma''; (\Gamma_1, F -\!\!* G)) \vdash H$ as required. Instead of D_2', $D_2^* : \Gamma'(\Gamma''; G; F -\!\!* G) \vdash H$ in case Γ_1 is $\emptyset_m$. Then $-\!\!*L_{4\,\alpha\texttt{LBI}}$ instead of $-\!\!*L_{2\,\alpha\texttt{LBI}}$.

3. $WkL_{\texttt{LBI}}$: Proposition 6.

4. $CtrL_{\texttt{LBI}}$: Theorem 6.

5. $EqAnt_{1\,\texttt{LBI}}$: Proposition 8.

6. $EqAnt_{2\,\texttt{LBI}}$: Proposition 8.

7. The rest: straightforward. □

3.3 α`LBI` Cut Elimination

I now prove admissibility of `Cut` in α`LBI + Cut` directly. Just as in the case of intuitionistic logic, cut admissibility proof for a contraction-free `BI` sequent calculus is simpler than that for `LBI` (which is found in Chapter 2 of this thesis). Since it has been already proved that weakening admissibility holds preserving derivation depth, we may simplify permutation via a context sharing cut, $\texttt{Cut}_{CS}$, which is easily verified derivable in α`LBI + Cut`:

$$\dfrac{\Gamma_1 \vdash F \qquad \Gamma_2(F;\Gamma_1) \vdash H}{\Gamma_2(\Gamma_1) \vdash H}\,\texttt{Cut}_{CS}$$

where Γ_1 is shared across the premises.

Theorem 8 (Cut admissibility in α`LBI`) *There is a cut elimination procedure to prove admissibility of* `Cut` *directly within* α`LBI + Cut`.

Proof. I show the procedure by induction on cut rank and a sub-induction on cut level, making use of $\texttt{Cut}_{CS}$. In this proof, (U, V) denotes, for some α`LBI` inference rules U and V, that one of the premises has been just derived with U and the other with V. In the pairs of derivations, the first is the derivation tree to be permuted and the second the permuted derivation tree.

(id, id):

1. $$\dfrac{\dfrac{}{\mathsf{E}(\Gamma_1;p) \vdash p}\,id \qquad \dfrac{}{\mathsf{E}'(\Gamma_2;p) \vdash p}\,id}{\mathsf{E}''(\Gamma_2;\mathsf{E}(\Gamma_1;p)) \vdash p}\,\texttt{Cut}$$

   $\Rightarrow$

   $$\dfrac{}{\mathsf{E}''(\Gamma_2;\mathsf{E}(\Gamma_1;p)) \vdash p}\,id$$

   Of course, for the above permutation to be correct, we must be able to demonstrate the fact that the antecedent structure of the conclusion sequent of the permuted derivation tree is $\mathsf{E}'''(\Gamma_2;\Gamma_1;p)$ such that $[\mathsf{E}'''(\Gamma_2;\Gamma_1;p)] \equiv [\mathsf{E}''(\Gamma_2;\mathsf{E}(\Gamma_1;p))]$ (equivalence up to associativity and commutativity of binary structural connectives). But note that it only takes a finite number of (backward) EA_2 applications (*Cf.* Proposition 7) on $\Gamma_2; \mathsf{E}(\Gamma_1; p) \vdash p$ to upward derive $\Gamma_2; \Gamma_1; p \vdash p$. The implication is that, since $\Gamma_2; \mathsf{E}(\Gamma_1; p) \vdash p$ results upward from $\mathsf{E}''(\Gamma_2; \mathsf{E}(\Gamma_1; p)) \vdash p$ also in a finite number of backward EA_2 applications, the antecedent structure must be in the form: $\mathsf{E}'''(\Gamma_2; \Gamma_1; p)$.

2.

$$\dfrac{\dfrac{}{\mathsf{E}(\Gamma_1; p) \vdash p}\,id \qquad \dfrac{}{\mathsf{E}'(\Gamma_2(p); q) \vdash q}\,id}{\mathsf{E}''(\Gamma_2(\mathsf{E}(\Gamma_1; p)); q) \vdash q}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{}{\mathsf{E}''(\Gamma_2(\mathsf{E}(\Gamma_1; p)); q) \vdash q}\,id$$

Other patterns for which one of the premises is an axiom sequent are straightforward.

For the rest, if the cut formula is principal only for one of the premise sequents, then we follow the routine (*Cf.* Troelstra and Schwichtenberg [2000]) to permute up the other premise sequent for which it is the principal. For example, in case we have the derivation pattern below:

$$\dfrac{\dfrac{D_1 \qquad D_2}{D_5 : \Gamma_1(H_1 \vee H_2) \vdash F_1 \supset F_2}\,\vee L \qquad \dfrac{D_3 : \mathsf{E}(\Gamma_3; F_1 \supset F_2) \vdash F_1 \qquad D_4 : \Gamma_2(F_2; \mathsf{E}(\Gamma_3; F_1 \supset F_2)) \vdash H}{D_6 : \Gamma_2(\mathsf{E}(\Gamma_3; F_1 \supset F_2)) \vdash H}\,\supset L}{\Gamma_2(\mathsf{E}'(\Gamma_3; \Gamma_1(H_1 \vee H_2))) \vdash H}\,\texttt{Cut}$$

where $D_1 : \Gamma_1(H_1) \vdash F_1 \supset F_2$ and $D_2 : \Gamma_1(H_2) \vdash F_1 \supset F_2$, the cut formula $F_1 \supset F_2$ is not the principal on the left premise. In this case we simply apply `Cut` on the pairs: (D_1, D_6) and (D_2, D_6), to conclude:

$$\dfrac{\dfrac{D_1 \qquad D_6}{\Gamma_2(\mathsf{E}''(\Gamma_3; \Gamma_1(H_1))) \vdash H}\,\texttt{Cut} \qquad \dfrac{D_2 \qquad D_6}{\Gamma_2(\mathsf{E}'''(\Gamma_3; \Gamma_1(H_2))) \vdash H}\,\texttt{Cut}}{\Gamma_2(\mathsf{E}'(\Gamma_3; \Gamma_1(H_1 \vee H_2))) \vdash H}\,\vee L$$

Of course for this particular permutation to be correct, we must be able to demonstrate in the permuted derivation tree that $\mathsf{E}'(\Gamma_3; \Gamma_1(H_1 \vee H_2)) \equiv \mathsf{E}''''(\Gamma_3) \star \Gamma_1(H_1 \vee H_2)$ with $\star$ either a semi-colon or a comma (equivalence up to associativity and commutativity of binary structural connectives), that $\mathsf{E}''(\Gamma_3; \Gamma_1(H_1)) \equiv \mathsf{E}''''(\Gamma_3) \star \Gamma_1(H_1)$, and that

$\mathsf{E}'''(\Gamma_3; \Gamma_1(H_2)) \equiv \mathsf{E}''''(\Gamma_3) \star \Gamma_1(H_2)$. But this is vacuous since the cut formula which is replaced with the structure $\Gamma_1(H_1)$ or $\Gamma_1(H_2)$ is a formula.

Cases that remain are those for which both premises of the cut instance have the cut formula as the principal. We go through each to conclude the proof.

($\wedge L, \wedge R$):

$$\dfrac{\dfrac{D_1 : \Gamma_1 \vdash F_1 \qquad D_2 : \Gamma_1 \vdash F_2}{\Gamma_1 \vdash F_1 \wedge F_2}\,\wedge R \qquad \dfrac{D_3 : \Gamma_2(F_1; F_2) \vdash H}{\Gamma_2(F_1 \wedge F_2) \vdash H}\,\wedge L}{\Gamma_2(\Gamma_1) \vdash H}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{D_2 \qquad \dfrac{D_1 \qquad D_3}{\Gamma_2(\Gamma_1; F_2) \vdash H}\,\texttt{Cut}}{\Gamma_2(\Gamma_1) \vdash H}\,\texttt{Cut}_{CS}$$

($\vee L, \vee R$):

$$\dfrac{\dfrac{D_1 : \Gamma_1 \vdash F_i \quad (i \in \{1, 2\})}{\Gamma_1 \vdash F_1 \vee F_2}\,\vee R \qquad \dfrac{D_2 : \Gamma_2(F_1) \vdash H \qquad D_3 : \Gamma_2(F_2) \vdash H}{\Gamma_2(F_1 \vee F_2) \vdash H}\,\vee L}{\Gamma_2(\Gamma_1) \vdash H}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{D_1 \qquad D_{(2 \; or \; 3)}}{\Gamma_2(\Gamma_1) \vdash H}\,\texttt{Cut}$$

The right premise sequent is D_2 if $i = 1$; or D_3 if $i = 2$.

($\supset L, \supset R$):

$$\dfrac{\dfrac{D_1 : \Gamma_1; F_1 \vdash F_2}{D_4 : \Gamma_1 \vdash F_1 \supset F_2}\,\supset R \qquad \dfrac{D_2 : \mathsf{E}(\Gamma_3; F_1 \supset F_2) \vdash F_1 \qquad D_3 : \Gamma_2(F_2; \mathsf{E}(\Gamma_3; F_1 \supset F_2)) \vdash H}{\Gamma_2(\mathsf{E}(\Gamma_3; F_1 \supset F_2)) \vdash H}\,\supset L}{\Gamma_2(\mathsf{E}(\Gamma_3; \Gamma_1)) \vdash H}\,\texttt{Cut}$$

$\Rightarrow$

$$\begin{array}{c}
\dfrac{\dfrac{\dfrac{D_4 \qquad D_2}{\mathsf{E}(\Gamma_3;\Gamma_1)\vdash F_1}\,\texttt{Cut} \qquad D_1}{\Gamma_1;\mathsf{E}(\Gamma_3;\Gamma_1)\vdash F_2}\,\texttt{Cut} \qquad \dfrac{D_4 \qquad D_3}{\Gamma_2(F_2;\mathsf{E}(\Gamma_3;\Gamma_1))\vdash H}\,\texttt{Cut}}{\Gamma_2(\Gamma_1;\mathsf{E}(\Gamma_3;\Gamma_1))\vdash H}\,\texttt{Cut}_{CS} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ \text{Proposition 6} \\
\Gamma_2(\Gamma_3;\Gamma_1;\mathsf{E}(\Gamma_3;\Gamma_1))\vdash H \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ \text{Proposition 7} \\
\Gamma_2(\mathsf{E}(\Gamma_3;\Gamma_1);\mathsf{E}(\Gamma_3;\Gamma_1))\vdash H \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ \text{Theorem 6} \\
\Gamma_2(\mathsf{E}(\Gamma_3;\Gamma_1))\vdash H
\end{array}$$

($*L, *R_1$):

$$\dfrac{\dfrac{D_1 : Re_1 \vdash F_1 \qquad D_2 : Re_2 \vdash F_2}{\Gamma_1 \vdash F_1 * F_2}\,*R_1 \qquad \dfrac{D_3 : \Gamma_2(F_1, F_2)\vdash H}{\Gamma_2(F_1 * F_2)\vdash H}\,*L}{\Gamma_2(\Gamma_1)\vdash H}\,\texttt{Cut}$$

$\Rightarrow$

$$\begin{array}{c}
\dfrac{D_2 \qquad \dfrac{D_1 \qquad D_3}{\Gamma_2(Re_1, F_2)\vdash H}\,\texttt{Cut}}{\Gamma_2(Re_1, Re_2)\vdash H}\,\texttt{Cut} \\
\cdots\cdots\cdots\cdots\cdots \ \text{Proposition 6} \\
\Gamma_2(\Gamma_1)\vdash H
\end{array}$$

($*L, *R_2$):

$$\dfrac{\dfrac{D_1 : \emptyset_m \vdash F_1 \qquad D_2 : \Gamma_1 \vdash F_2}{\Gamma_1 \vdash F_1 * F_2}\,*R_1 \qquad \dfrac{D_3 : \Gamma_2(F_1, F_2)\vdash H}{\Gamma_2(F_1 * F_2)\vdash H}\,*L}{\Gamma_2(\Gamma_1)\vdash H}\,\texttt{Cut}$$

$\Rightarrow$

$$\begin{array}{c}
\dfrac{D_2 \qquad \dfrac{D_1 \qquad D_3}{\Gamma_2(\emptyset_m, F_2)\vdash H}\,\texttt{Cut}}{\Gamma_2(\emptyset_m, \Gamma_1)\vdash H}\,\texttt{Cut} \\
\cdots\cdots\cdots\cdots\cdots \ \text{Proposition 8} \\
\Gamma_2(\Gamma_1)\vdash H
\end{array}$$

($\mathrel{-\!\!*}L_1, \mathrel{-\!\!*}R$):

$$\dfrac{\dfrac{D_1 : \Gamma_1, F_1 \vdash F_2}{D_4 : \Gamma_1 \vdash F_1 \mathrel{-\!\!*} F_2}\,\mathrel{-\!\!*}R \qquad \dfrac{D_2 : Re_1 \vdash F_1 \qquad D_3 : \Gamma_2((Re_2, F_2);\mathsf{E}(\Gamma', (\Gamma_3; F_1 \mathrel{-\!\!*} F_2)))\vdash H}{\Gamma_2(\mathsf{E}(\Gamma', (\Gamma_3; F_1 \mathrel{-\!\!*} F_2)))\vdash H}\,\mathrel{-\!\!*}L_1}{\Gamma_2(\mathsf{E}(\Gamma', (\Gamma_3; \Gamma_1)))\vdash H}\,\texttt{Cut}$$

⇒

$$\dfrac{D_2 \qquad \dfrac{D_1 \qquad \dfrac{D_4 \qquad D_3}{\Gamma_2((Re_2, F_2); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H}\,\texttt{Cut}}{\Gamma_2((Re_2, \Gamma_1, F_1); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H}\,\texttt{Cut}}{\Gamma_2((Re_2, \Gamma_1, Re_1); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H}\,\texttt{Cut}$$

$$\begin{array}{cl} \Gamma_2((Re_2, \Gamma_1, Re_1); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \text{Proposition 6} \\ \Gamma_2((\Gamma', (\Gamma_3; \Gamma_1)); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \text{Proposition 7} \\ \Gamma_2((\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1)); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \text{Theorem 6} \\ \Gamma_2(\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1)) \vdash H & \end{array}$$

($-\!\!*L_2$, $-\!\!*R$):

$$\dfrac{\dfrac{D_1 : \Gamma_1, F_1 \vdash F_2}{D_4 : \Gamma_1 \vdash F_1 -\!\!* F_2}\,-\!\!*R \qquad \dfrac{D_2 : \Gamma' \vdash F_1 \qquad D_3 : \Gamma_2(F_2; (\Gamma', \mathsf{E}(\Gamma_3; F_1 -\!\!* F_2))) \vdash H}{\Gamma_2(\Gamma', \mathsf{E}(\Gamma_3; F_1 -\!\!* F_2)) \vdash H}\,-\!\!*L_2}{\Gamma_2(\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H}\,\texttt{Cut}$$

⇒

$$\dfrac{D_2 \qquad \dfrac{D_1 \qquad \dfrac{D_4 \qquad D_3}{\Gamma_2(F_2; (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H}\,\texttt{Cut}}{\Gamma_2((\Gamma_1, F_1); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H}\,\texttt{Cut}}{\Gamma_2((\Gamma', \Gamma_1); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H}\,\texttt{Cut}$$

$$\begin{array}{cl} \Gamma_2((\Gamma', \Gamma_1); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \text{Proposition 6} \\ \Gamma_2((\Gamma', (\Gamma_3; \Gamma_1)); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \text{Proposition 7} \\ \Gamma_2((\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1)); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \text{Theorem 6} \\ \Gamma_2(\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1)) \vdash H & \end{array}$$

($-\!\!*L_3$, $-\!\!*R$):

$$\dfrac{\dfrac{D_1 : \Gamma_1, F_1 \vdash F_2}{D_4 : \Gamma_1 \vdash F_1 -\!\!* F_2}\,-\!\!*R \qquad \dfrac{D_2 : \emptyset_m \vdash F_1 \qquad D_3 : \Gamma_2((\Gamma', F_2); (\Gamma', \mathsf{E}(\Gamma_3; F_1 -\!\!* F_2))) \vdash H}{\Gamma_2(\Gamma', \mathsf{E}(\Gamma_3; F_1 -\!\!* F_2)) \vdash H}\,-\!\!*L_2}{\Gamma_2(\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1)) \vdash H}\,\texttt{Cut}$$

⇒

$$\begin{array}{c}
\dfrac{D_2 \qquad \dfrac{D_1 \qquad \dfrac{D_4 \qquad D_3}{\Gamma_2((\Gamma', F_2); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H}\,\mathtt{Cut}}{\Gamma_2((\Gamma', \Gamma_1, F_1); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H}\,\mathtt{Cut}}{\Gamma_2((\Gamma', \Gamma_1, \emptyset_m); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H}\,\mathtt{Cut} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{ Proposition 8} \\
\Gamma_2((\Gamma', \Gamma_1); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{ Proposition 6} \\
\Gamma_2((\Gamma', (\Gamma_3; \Gamma_1)); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{ Proposition 7} \\
\Gamma_2((\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1)); (\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1))) \vdash H \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{ Theorem 6} \\
\Gamma_2(\Gamma', \mathsf{E}(\Gamma_3; \Gamma_1)) \vdash H
\end{array}$$

($-\!\!*L_4$, $-\!\!*R$):

$$\dfrac{\dfrac{D_1 : \Gamma_1, F_1 \vdash F_2}{D_4 : \Gamma_1 \vdash F_1 -\!\!* F_2}\,-\!\!*R \qquad \dfrac{D_2 : \emptyset_m \vdash F_1 \qquad D_3 : \Gamma_2(F_2; F_1 -\!\!* F_2) \vdash H}{\Gamma_2(F_1 -\!\!* F_2) \vdash H}\,-\!\!*L_2}{\Gamma_2(\Gamma_1) \vdash H}\,\mathtt{Cut}$$

$\Rightarrow$

$$\begin{array}{c}
\dfrac{D_2 \qquad \dfrac{D_1 \qquad \dfrac{D_4 \qquad D_3}{\Gamma_2(F_2; \Gamma_1) \vdash H}\,\mathtt{Cut}}{\Gamma_2((\Gamma_1, F_1); \Gamma_1) \vdash H}\,\mathtt{Cut}}{\Gamma_2((\Gamma_1, \emptyset_m); \Gamma_1) \vdash H}\,\mathtt{Cut} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{ Proposition 8} \\
\Gamma_2(\Gamma_1; \Gamma_1) \vdash H \\
\cdots\cdots\cdots\cdots\cdots\cdots \text{ Theorem 6} \\
\Gamma_2(\Gamma_1) \vdash H
\end{array}$$

□

Proposition 10 (Analyticity of α`LBI`) α`LBI` *is analytic.*

Proof. It suffices to demonstrate that the number of Re_1/Re_2 pairs that $*R_1$ and $-\!\!*L_1$ can generate onto the premise sequents is finitely bounded since, even if a $\emptyset_m$ should be introduced on the left premise via a $-\!\!*L_3$ or a $-\!\!*L_4$, subsequent applications of the either of the α`LBI` inference rules would only result in introducing the same $\emptyset_m$. But due to the generation process of the pair in each α`LBI` inference rule, it must be the case. □

$$\frac{}{\mathsf{E}(\Gamma; p) \vdash p}\,id \qquad \frac{}{\Gamma(\mathbb{1}) \vdash F}\,\mathbb{1}L \qquad \frac{}{\Gamma \vdash \top}\,\top R \quad \frac{}{\mathsf{E}(\Gamma; {}^{*}\!\top) \vdash {}^{*}\!\top}\,{}^{*}\!\top R$$

$$\frac{\Gamma(F; G) \vdash H}{\Gamma(F \wedge G) \vdash H}\,\wedge L \qquad \frac{\Gamma \vdash F \qquad \Gamma \vdash G}{\Gamma \vdash F \wedge G}\,\wedge R$$

$$\frac{\Gamma(F) \vdash H \qquad \Gamma(G) \vdash H}{\Gamma(F \vee G) \vdash H}\,\vee L \qquad \frac{\Gamma \vdash F_i}{\Gamma \vdash F_1 \vee F_2}\,\vee R$$

$$\frac{\mathsf{E}(\Gamma_1; F \supset G) \vdash F \qquad \Gamma(G; \mathsf{E}(\Gamma_1; F \supset G)) \vdash H}{\Gamma(\mathsf{E}(\Gamma_1; F \supset G)) \vdash H} \supset L \qquad \frac{\Gamma; F \vdash G}{\Gamma \vdash F \supset G} \supset R$$

$$\frac{\Gamma(F, G) \vdash H}{\Gamma(F * G) \vdash H}\,{*L} \qquad \frac{Re_1 \vdash F_1 \qquad Re_2 \vdash F_2}{\Gamma_a, \Gamma_b \vdash F_1 * F_2}\,{*R}$$

$$\frac{Re_1 \vdash F \qquad \Gamma((Re_2, G); (\Gamma_a, \Gamma_b, \mathsf{E}(\Gamma_1; F \mathrel{-\!\!*} G))) \vdash H}{\Gamma(\Gamma_a, \Gamma_b, \mathsf{E}(\Gamma_1; F \mathrel{-\!\!*} G)) \vdash H} \mathrel{-\!\!*} L \qquad \frac{\Gamma, F \vdash G}{\Gamma \vdash F \mathrel{-\!\!*} G} \mathrel{-\!\!*} R$$

Figure 3.3: `LBIZ`: a `BI` sequent calculus with zero occurrence of explicit structural rules and structural units. $i \in \{1, 2\}$.

3.4 Departure from Coherent Equivalence

In this section we study emptiness of an antecedent structure within $\mathfrak{D}_{\texttt{BI}}$, and develop a new presentation for α`LBI`. As we just saw, α`LBI` comes with several inference rules for the left multiplicative implication and the right multiplicative conjunction, which is necessary under the present assumption of ours (*Cf.* Definition 54) that structural counterparts of the zero-place logical operators be quantifiable. It holds in both `LBI` and α`LBI` derivations that Γ be different from "$\Gamma; \emptyset_a$" which is also different from "$\Gamma, \emptyset_m$", *i.e.* both $\emptyset_a$ and $\emptyset_m$ have an entity. To identify where the many inference rules originate, however, it is precisely in the fact that the coherent equivalence within a `BI` proof system expresses the structural equivalence. By departing from it those many rules for the mentioned two connectives are merged into one, as preferred.

3.4.1 Demerit of coherent equivalence

Coherent equivalence is an equivalence relation up to associativity, commutativity and the structural equivalence $\Gamma = \Gamma, \emptyset_m = \Gamma; \emptyset_a$. Earlier works on `BI` proof theory (apart from one semantic calculus by Galmiche et al. [2005]; it is, however, unsound[1]) appear to all incorporate it into their respective proof system. Though associativity and commutativity can harmlessly reside within a BI proof system as:

$$\frac{\Gamma_2 \vdash F \quad (\Gamma_1 \equiv \Gamma_2)}{\Gamma_1 \vdash F}\,\text{Exchange}$$

where $\Gamma_1 \equiv \Gamma_2$ denotes the equivalence of Γ_1 and Γ_2 up to associativity and commutativity, the structural equivalences ($EqAnt_{1,2\,\mathtt{LBI}}$; *Cf.* Figure 1.7) cannot be so innocuous. Unlike with Exchange above which only permutes structures, they permit an arbitrary introduction of new structural units at any structural layers of an antecedent structure. Further, to achieve such manipulations, there must be posited "structural" units - a fair amount of notational cost, obscuring the intrinsic nature of the system.

3.4.2 Emergence of LBIZ

Thanks to the earlier analysis in the development of αLBI, however, we know that neither $\Gamma = \Gamma; \emptyset_a$ nor $\Gamma = \Gamma, \emptyset_m$ needs given any autonomy as structural rules in a BI sequent calculus. The cumbersome variation of inference rules for the left multiplicative implication and the right multiplicative conjunction can be thus unified into a single inference rule, obviating, in so doing, also structural units themselves. The new BI sequent calculus LBIZ is found in Figure 3.3 which brings BI sequent calculus in line with other logics' (*Cf.* Definition 13 and Definition 23).

Definition 59 (Sequent calculus convention in LBIZ) *For an antecedent structure in the form: "$\Gamma_1; \Gamma_2$", its emptiness is identified with $\top$,* i.e. *"$\Gamma_1; \Gamma_2$" is identified with Γ_1 (resp. with Γ_2) in case Γ_2 (resp. Γ_1) is empty. Likewise, for an antecedent structure in the form: "Γ_1, Γ_2", its emptiness is identified with $^*\top$,* i.e. *"Γ_1, Γ_2" is identified with Γ_1 (resp. Γ_2) in case Γ_2 (resp. Γ_1) is empty.*

Under the convention, $*R2_{\alpha\mathtt{LBI}}$ for instance does not need defined in separation, since the condition for the (backward) inference rule application is precisely when Γ_a, Γ_b is identified with Γ_a (or Γ_b). To accommodate the absence of $\emptyset_m$ in LBIZ, we also slightly modify the EA_2 inference rule as relevant in the essence.

$$\frac{\Gamma(\Gamma_1) \vdash H}{\Gamma(\Gamma_1, (^*\top; \Gamma_2)) \vdash H}\,EA_2$$

---

[1] *Cf.* Chapter 2 of this thesis.

$$\frac{\Gamma(p; G) \vdash H}{\Gamma(p; p \supset G) \vdash H} \supset L_p \qquad \frac{\Gamma(F_1 \supset G; F_2 \supset G) \vdash H}{\Gamma((F_1 \vee F_2) \supset G) \vdash H} \supset L_\vee$$

$$\frac{\Gamma(F_1 \supset (F_2 \supset G)) \vdash H}{\Gamma((F_1 \wedge F_2) \supset G) \vdash H} \supset L_\wedge \qquad \frac{\Gamma(G) \vdash H}{\Gamma(\top \supset G) \vdash H} \supset L_\top$$

$$\frac{\Gamma_1; F_2 \supset G \vdash F_1 \supset F_2 \qquad \Gamma(\Gamma_1; G) \vdash H}{\Gamma(\Gamma_1; (F_1 \supset F_2) \supset G) \vdash H} \supset L_\supset \qquad \frac{\Gamma_1 \vdash F_1 * F_2 \qquad \Gamma(\Gamma_1; G) \vdash H}{\Gamma(\Gamma_1; (F_1 * F_2) \supset G) \vdash H} \supset L_*$$

Figure 3.4: A set of $\supset L$ rules. No implicit contraction occurs for all.

It is trivial to see that LBIZ is otherwise equivalent to αLBI, and that all the previous results go through.

3.5 On BI Decidability: A Syntactic Observation

In this section, I consider BI decidability from a syntactic perspective, based on LBIZ. Though LBIZ is contraction-free in the sense that an explicit structural rule of contraction does not appear within, the termination property is not immediately apparent once $\supset L$ and/or $-\!\!*L$ appear in a derivation. This is because contraction, though only implicit, does occur within the two inference rules. Towards the conclusion of the BI decision problem, I here initiate the research by showing the decidability of [BI - multiplicative implication - multiplicative unit], which is at the time of this thesis writing the largest BI fragment that is demonstrably provable to be decided.

Definition 60 (LBIZ$_1$) LBIZ$_1$ *comprises the following* LBIZ *inference rules:*

Axioms: $id \quad \mathbb{1}L \quad \top R$

Other logical rules: $\wedge L \quad \wedge R \quad \vee L \quad \vee R \quad \supset L \quad \supset R \quad * L \quad * R$

In line with the restriction, we assume the availability of only those connectives in LBIZ$_1$ to all the sequents appearing in a LBIZ$_1$ derivation, and term the fragment BI$_{\texttt{base}}$.

3.5.1 Implicit contraction elimination in LBIZ$_1$

My intention is to prove that replacement of $\supset L_{\texttt{LBIZ}_1}$ with those in Figure 3.4 results neither in a loss nor in a gain of expressiveness, to render LBIZ$_1$ contraction-free even implicitly.

3.5.1.1 Preparation

First of all, we make an observation that $\supset L$ in $\texttt{LBIZ}_1$ does not have to be as general as $\supset L$ in $\texttt{LBIZ}$ due to the absence of the multiplicative unit. It suffices to have:

$$\frac{\Gamma_1; F{\supset}G \vdash F \qquad \Gamma(\Gamma_1; G) \vdash H}{\Gamma(\Gamma_1; F{\supset}G) \vdash H} \supset L_{\texttt{LBIZ}_1}$$

Lemma 14 (Inversion) *Along with* $\texttt{LBIZ}$ *inversion which inherits* $\alpha\texttt{LBI}$ *inversion, the following holds in* $\texttt{LBIZ}_1$*: if* $\Gamma(F{\supset}G) \vdash H$ *is* $\texttt{LBIZ}_1$*-derivable at most with the derivation depth of* k*, then so is* $\Gamma(G) \vdash H$.

Proof. For a $\alpha\texttt{LBI}$ sequent $\Gamma(F \supset G) \vdash H$, the base case is when it is an axiom, in which case the proof is trivial. For inductive cases, assume that it holds true for the derivation depths up to k, and show that it still holds true at the derivation depth of $k+1$. Consider what the last inference rule is.

1. $\supset L_{\texttt{LBIZ}_1}$: If the principal coincides with the $F \supset G$, then it is trivial via $\supset L_{\texttt{LBIZ}_1}$. Otherwise, the derivation ends in either of the two below:

$$\frac{\Gamma'(F \supset G); F_1 \supset G_1 \vdash F_1 \qquad \Gamma_2(\Gamma'(F \supset G); G_1) \vdash H}{\Gamma_2(\Gamma'(F \supset G); F_1 \supset G_1) \vdash H} \supset L_{\texttt{LBIZ}_1}$$

$$\frac{\Gamma_2; F_1 \supset G_1 \vdash F_1 \qquad \Gamma(F \supset G)(\Gamma_2; G_1) \vdash H}{\Gamma(F \supset G)(\Gamma_2; F_1 \supset G_1) \vdash H} \supset L_{\texttt{LBIZ}_1}$$

   Induction hypothesis on both of the premises in case the former, or on the right premise, and then $\supset L_{\texttt{LBIZ}_1}$ to conclude.

2. The rest: Similar to the consideration taken in the proof of the $\alpha\texttt{LBI}$ inversion lemma. □

Lemma 15 ($\mathtt{LBIZ}_1$ weakening and contraction admissibility) *Both weakening and contraction ($CtrL_{\mathtt{LBI}}$ and $WkL_{\mathtt{LBI}}$) are admissible in* $\mathtt{LBIZ}_1$.

Proof. Trivial for weakening admissibility (*Cf.* Proposition 6). Theorem 6 for contraction admissibility apart from the case of left additive implication which is handled with Lemma 14. □

Proposition 11 (Equivalence) *For a given* $\mathtt{LBIZ}_1$ *sequent* $D : \Gamma \vdash H$*, it is* $\mathtt{LBIZ}_1$*-derivable (with* $\supset L_{\mathtt{LBIZ}_1}$*) iff it is [*$\mathtt{LBIZ}$ *-* $-\!\!*L$ *-* $-\!\!*R$ *-* $^*\!\top L$ *-* $^*\!\top R$*]-derivable.*

Proof. There is only one that differs in the absence of the multiplicative unit and the multiplicative implication between $\mathtt{LBIZ}_1$ and the subset of $\mathtt{LBIZ}$, namely $\supset L$. Hence we only need prove that $\supset L$ in the one is derivable in the other. Proof is by induction on derivation depth into both directions. Into the *if* direction, we need to show that $\supset L_{\mathtt{LBIZ}}$ with the restriction is derivable in $\mathtt{LBIZ}_1$. By induction hypothesis, we have both $D_1 : \Gamma_1; F \supset G \vdash F$ and $D_2 : \Gamma(G; \Gamma_1; F \supset G) \vdash H$ derivable in $\mathtt{LBIZ}_1$. By $\supset L_{\mathtt{LBIZ}_1}$ on D_1 and D_2, we then have $D' : \Gamma(\Gamma_1; F \supset G; F \supset G) \vdash H$ derivable in $\mathtt{LBIZ}_1$. A conclusion is via Lemma 15. Similar for the *only if* direction via the admissible weakening and contraction in the restricted $\mathtt{LBIZ}$ to $\mathtt{BI}_{\mathtt{base}}$. □

With this, we simply assume $\supset L_{\mathtt{LBIZ}_1}$ as the left additive implication rule in $\mathtt{LBIZ}_1$, dropping the subscript hereafter.

For the main result to follow, two more concepts are needed: (1) sequent weights; and (3) irreducible $\mathtt{LBIZ}_1$ sequents. Readers may find it useful to refer back to 1.1.2 of this thesis, which is a pre-requisite for the current discussion.

Definition 61 (Sequent weights) *Given a sequent* $D : \Gamma \vdash H$*, its weight is defined to be the sum of the formula weight of all the formulas in* D*. The formula weight of a formula* F*,* $\mathtt{f\_weight}(F)$*, is defined as follows:*

- $\mathtt{f\_weight}(F) = 2$ *if* $F \in \{\top, \mathbb{1}, p\}$.
- $\mathtt{f\_weight}(F) = \mathtt{f\_weight}(F_1)(1 + \mathtt{f\_weight}(F_2))$ *if* $F \in \{F_1 \wedge F_2, F_1 * F_2\}$.
- $\mathtt{f\_weight}(F) = 1 + \mathtt{f\_weight}(F_1) + \mathtt{f\_weight}(F_2)$ *if* $F = F_1 \vee F_2$.

- $\texttt{f\_weight}(F) = 1 + \texttt{f\_weight}(F_1)\texttt{f\_weight}(F_2)$ *if* $F = F_1 \supset F_2$.

Definition 62 (Irreducible $\texttt{LBIZ}_1$ sequents)
An antecedent structure Γ *in* $\texttt{LBIZ}_1$ *is said to be irreducible if it contains as its sub-structure none of the following:*

1. $p; p \supset G$
2. $\top \supset G$
3. $\mathbb{1}$
4. $H_1 \wedge H_2$
5. $H_1 \vee H_2$
6. $H_1 * H_2$.

A $\texttt{LBIZ}_1$ *sequent* $D : \Gamma \vdash F$ *is said to be irreducible if* Γ *is irreducible.*

Lemma 16 (Normalisation) *Any* $\texttt{LBIZ}_1$ *sequent* D *which is not irreducible can be reduced into a set of irreducible sequents such that* D *be derivable iff they are.*

Proof. Basically follows from $\texttt{LBIZ}$ inversion which inherits the $\alpha\texttt{LBI}$ inversion lemma, and Lemma 14. A sequent with a $\mathbb{1}$ in the antecedent part is immediately inconsistent[1] and derivable. □

3.5.1.2 Implicit contraction elimination for $\supset L_p$, and $\supset L_{*1}$

I now show that any $\supset L_{\texttt{LBIZ}_1}$ application on "$p \supset G$" can be deferred until "$p; p \supset G$" appears as a sub-structure in the antecedent part. Such also is the case for $(F_1 * F_2) \supset F_3$ under a set of conditions.

---

[1] Here, by a sequent $\Gamma \vdash F$ being inconsistent, I mean that $\Gamma \vdash \mathbb{1}$ is derivable.

Lemma 17 *Any* $\texttt{LBIZ}_1$*-derivable irreducible sequent* $D : \Gamma \vdash H$ *has a closed derivation in which the principal of the last rule applied is neither* $p{\supset}G$ *(on the antecedent part of* D*), nor* $(F_1 * F_2){\supset}G$ *if not all of the following conditions satisfy:*

- $[D : \Gamma(\Gamma_1; (F_1 * F_2){\supset}G) \vdash H] \rightsquigarrow_{\supset L} [D_1 : \Gamma_1; (F_1 * F_2){\supset}G \vdash F_1 * F_2]$
- $D_1 \rightsquigarrow_{*R} [D_2 : Re_1 \vdash F_1]$
- $D_1 \rightsquigarrow_{*R} [D_3 : Re_2 \vdash F_2]$
- D_2 *and* D_3 *(and hence also* D_1*) are both* $\texttt{LBIZ}_1$*-derivable.*

Proof. By contradiction. As in Dyckhoff [1992] (*Cf.* Chapter 1 of this thesis), we assume that inference rules to apply in the leftmost branch were cleverly chosen so that the derivation length between D and the conclusion sequent of an axiom in the leftmost branch is shortest.[1] Suppose, by way of showing contradiction, that there cannot exist any other shorter derivations of D than the ones ending in $\supset L$ with the principal of a formula in the form either $p{\supset}G$, or $(F_1 * F_2){\supset}G$ (under the condition that not all the four conditions satisfy). Then $\Pi(D)$, a derivation of D, looks like:

$$\dfrac{\dfrac{\begin{matrix}\vdots & \vdots\\ D_3 & D_4\end{matrix}}{D_1 : \Gamma_1; F_{77}{\supset}G \vdash F_{77}}\,\textbf{Inf} \qquad \begin{matrix}\vdots\\ D_2 : \Gamma(\Gamma_1; G) \vdash H\end{matrix}}{D : \Gamma(\Gamma_1; F_{77}{\supset}G) \vdash H}\,\to L$$

where F_{77} is p if the principal is "$p{\supset}G$"; or is "$F_1 * F_2$" if it is "$(F_1 * F_2){\supset}G$". As D is irreducible, so is D_1 which, therefore, cannot be the conclusion sequent of an axiom. If F_{77} is p, then the consequent formula of D_1 can be active only for an axiom. Likewise, due to the given condition, if F_{77} is $F_1 * F_2$ in D_1, its consequent part cannot be active for **Inf**. Therefore, **Inf** is known to be $\supset L$. Moreover, as the leftmost branch is supposed shortest, the principal for **Inf** must be from among those constituents residing in the same additive structural layer as the $F_{77}{\supset}G$. Furthermore, that the leftmost branch is shortest has to dictate that the principal for **Inf** is in neither of the following forms: "$p_i{\supset}G_i$", or "$(F_{j1} * F_{j2}){\supset}G_j$" for some propositional variable p_i, some $F_{j1} * F_{j2}$ (satisfying the same condition as stated) and some formula G_i, or G_j.

These points taken into account, D, D_1, D_2, D_3 and D_4 are actually seen taking the following forms for some other formula F:

---

[1] This, incidentally, is a classical proof. I leave a constructive proof open.

- $D : \Gamma(\Gamma_1; F \supset G'; F_{77} \supset G) \vdash H$
- $D_1 : \Gamma_1; F \supset G'; F_{77} \supset G \vdash F_{77}$
- $D_2 : \Gamma(\Gamma_1; F \supset G'; G) \vdash H$
- $D_3 : \Gamma_1; F \supset G'; F_{77} \supset G \vdash F$
- $D_4 : \Gamma_1; G'; F_{77} \supset G \vdash F_{77}$

But, then, this perforce implies the existence of an alternative derivation $\Pi'(D)$ which results by permuting $\Pi(D)$:

$$\dfrac{\begin{matrix}\vdots \\ D_3\end{matrix} \qquad \dfrac{\begin{matrix}\vdots \\ D_4\end{matrix} \qquad \begin{matrix}\vdots \\ D_2' : \Gamma(\Gamma_1; G'; G) \vdash H\end{matrix}}{\Gamma(\Gamma_1; G'; F_{77} \supset G) \vdash H} \supset L}{D} \supset L$$

D_2' can be shown derivable from D_2 via Lemma 14. A direct contradiction to the supposition has been drawn, for the leftmost branch in $\Pi'(D)$ is shorter. □

From Lemma 17 follows an observation.

Lemma 18 *In* LBIZ$_1$, $\supset L_*'$ *as below is admissible.*

$$\dfrac{D_1 : \Gamma_1; (F_1 * F_2) \supset G \vdash F_1 * F_2 \qquad D_2 : \Gamma(\Gamma_1; G) \vdash H}{D : \Gamma(\Gamma_1; (F_1 * F_2) \supset G) \vdash H} \supset L_*'$$

Proof. Any application of $\supset L$ with $(F_1 * F_2) \supset G$ as its principal can be deferred until all the four conditions hoisted in Lemma 17 are satisfied. Under the assumption, there exists a pair of sequent transitions via $*R$ from the left premise sequent D_1 of the $\supset L_*'$ into D_2 and D_3 such that (1) $D_1 \leadsto_{*R} D_2$; (2) $D_1 \leadsto_{*R} D_3$; and (3) both D_2 and D_3 are LBIZ$_1$-derivable. Then, because in D_1 the outermost structural layer of the antecedent for which $(F_1 * F_2) \supset G$ is a constituent is not a multiplicative structural layer, nor can it be $(F_1 * F_2) \supset G$ (otherwise D_1 is not LBIZ$_1$-derivable), it must be an additive structural layer, and moreover, it must be such that there exists at least one multiplicative structural layer as its constituent (because the four conditions in Lemma 17 are assumed satisfied). By the way a maximal Re_1/Re_2 pair is formed, it cannot be the case that two constituents of the outermost additive structural layer be retained simultaneously. And so there could be only one from among the $\mathcal{M}$ constituents which is to remain after a sequence of the internalised weakening so that the result be a multiplicative structural layer to appear at the outermost structural layer. But $(F_1 * F_2) \supset F_3$ is not a multiplicative structural layer. □

Proposition 12 *Replacement of* $\supset L_{\texttt{LBIZ}_1}$ *with those in Figure 3.4 is sound and complete.*

Proof. One direction: to assume inference rules in Figure 3.4 and to show corresponding derivations with $\supset L_{\texttt{LBIZ}_1}$, is trivial. Into the other direction, proof is by induction on sequent weight. We consider what the actual instance F is in the principal $F \supset G$, and turn to Lemma 17 and Lemma 18, for $\supset L_{*1}$, and $\supset L_p$. $\supset L_\top$ is straightforward. If $F = \mathbb{1}$, $F \supset G$ is a useless construct in the antecedent. *Cf.* Lemma 5 for the other cases. □

A decidability result follows.

Theorem 9 (Decidability of `BI`$_{\texttt{base}}$) `BI`$_{\texttt{base}}$ *is decidable.*

Proof. For all the `LBIZ`$_1$ inference rules, the sequent weight defined strictly decreases from any conclusion sequent into premise sequent(s). Furthermore, every sequent to appear during a derivation is finite (and so the weight of any sequent to appear during a derivation is also finite). □

3.6 Conclusion

Here is a summary of the contributions in the present chapter.

1. Delivery of α`LBI` as a structural-rule-free `BI` sequent calculus.

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2. A direct proof of admissibility of `Cut` within [α`LBI`+ `Cut`].

3. Development of a sequent calculus convention to align `BI` proof theory with other logics' in the manner emptiness of a structure in a sequent is treated. Many varieties in the two α`LBI` multiplicative inference rules, the left multiplicative implication and the right multiplicative conjunction, were unified into `LBIZ`.

4. A purely syntactic proof of $\texttt{BI}_{\texttt{base}}$ decidability, which, at the time of this writing, is the largest decidable fragment of the logic `BI` which actually comes with a proof.[1]

This chapter was motivated broadly by the two objectives: the one, the analysis of interactions between structural inference rules and logical inference rules in `BI` sequent calculi; the other, derivation of a purely syntactic decidability result for $\texttt{BI}_{\texttt{base}}$. Both concluded successfully.

To begin with, a new `BI` sequent calculus α`LBI` was presented, which concluded a long standing open research problem of absorption of the `LBI` structural rules into the logical rules, in particular of contraction and of the structural equivalences hindering scalable backward proof searches. The goal was attained through analysis of the way they behave in `LBI` derivations. Weakening, contraction and the structural equivalence around the units, *i.e.* all the `LBI` structural rules, were all found depth-preserving admissible in α`LBI`. To the best of my knowledge, none of them were closely analysed in earlier work, let alone the simultaneous solution.

Though fairly remote, a work by Donnelly et al. [2004] is related closest nonetheless for weakening absorption which succeeded in absorption of the effect into the other inference rules within their forward sequent calculus for a unit-less subset of `BI`. The approach of theirs, however, comes with certain shortage in that the said effect of weakening is absorbed not only into logical inference rules, but spills out also into another structural rule of contraction that lingers on. Defeated to an extent is their intention of structural weakening elimination, because, as a matter of fact, it still occurs through the new structural rule, though now bearing a different label. Another issue to

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[1] If one is permitted to restrict the occurrences of the multiplicative unit or those of the multiplicative implication only to those subformula positions which do not need incur structural contraction, one may artificially derive a larger decidable fragment. This restriction itself, however, would impose an expensive restriction on what form a `BI` formula can fit in, and therefore impracticable.

ensue from coupling the two structural inference rules is amplification of the difficulty in analysing the behaviour of contraction.

For the remaining two `LBI` structural inference rules, there existed no known sensible results. They manifest as sources of non-termination in earlier `BI` proof systems such as `LBI` and $\texttt{DL}_{\texttt{BI}}$ (*Cf.* Chapter 1 and Chapter 2); it is very hard to actually prove that an invalid `BI` formula is underivable in those systems.

Of these, contraction absorption seems to have remained a particularly hard problem, and the reasons that I consider added to the complexity are the following:

1. that practically any structure, not just formulas, may duplicate in `BI` sequent calculus.
2. that a sound understanding of interactions between weakening and the structural equivalence is a prerequisite for a successful contraction analysis.
3. that a sound understanding of both the interactions between additive and multiplicative structural layers and the behaviour of weakening are the prerequisites for a successful contraction analysis.

To pierce the layered complexity, this chapter presented two key ideas: the essence of structures, and deep weakening absorption which led to the discovery of the concept of maximal Re_1/Re_2 pairs. The former provided a satisfactory clue to the second difficulty cited above, and the latter to the third difficulty. Interactions between the structural rules were analysed and, as a result, the effect of contraction was for the first time fully decoupled from that of weakening and the structural equivalence, leading to a concise and natural proof of the contraction admissibility as exhibited. A direct cut elimination procedure was then laid down for [α`LBI`+ `Cut`].

The technical inquiry into the nature of `BI` proof systems was farther forged ahead with evaluation of the significance of the coherent equivalence - a common wisdom, as has been, in `BI` proof theory. Owing largely to the α`LBI` delivery, an insight nonetheless dawned on: the notational complexity around the `BI` units in `LBI` germinates from incorporation of the structural equivalence (which is one of the conditions of the coherent equivalence) into the proof system. By shedding off the extrinsic legacy, the core of α`LBI` was successfully extracted into `LBIZ` with zero structural rules and structural units.

It was then used to prove decidability of $\mathtt{BI}_{\mathtt{base}}$ (a fragment of `BI` without the multiplicative unit and the multiplicative implication), purely syntactically. The viewpoint of a structure as a nesting of structural layers provided a clue (*Cf.* Lemma 18) to extend the Dyckhoff's method (*Cf.* Chapter 1) beyond propositional intuitionistic logic, to constitute a proof of $\mathtt{BI}_{\mathtt{base}}$ decidability. To the best of my knowledge, it is the largest `BI` fragment that has ever been given an actual decidability proof. A $\mathtt{BI}_{\mathtt{base}}$ decision procedure is found in Chapter 5 as a side contribution of the chapter.

Chapter 4

Defining BBI Sequent Calculi

The outline of this chapter is as follows:

1. Development of adequate conventions for a `BBI` sequent calculus, and analysis of `BBI` semantics and its implication on syntax.
2. Formalisation of a `BBI` sequent calculus $\mathtt{LBBI}_p$ with proofs of its soundness and completeness.
3. Development of another `BBI` sequent calculus $\alpha\mathtt{LBBI}_p$, which results from absorbing structural rules of $\mathtt{LBBI}_p$.
4. Adaptation of the sequent calculi to separation logic.
5. Identification of a conservative cut eliminable fragment of $[\alpha\mathtt{LBBI}_p + \mathtt{Cut}]$.
6. Comparisons of the `BBI` sequent calculi with earlier `BBI` proof systems.

4.1 Preparations with Fundamental Notations

A `BBI` sequent calculus requires a new syntactic convention that captures a logical combination of classical logic and a variant of multiplicative intuitionistic linear logic with the non-intuitionistic $^*\top$ (*Cf.* Chapter 1). Since there are no `BBI` sequent calculi known at present, I begin by providing intuition, introducing only so many fundamental notations as sufficient to get us started on this topic. To reflect the effect of the law of the excluded middle, we shall strictly distinguish negative structures from positive ones.

Definition 63 *A negative* `BBI` *structure* $\mathsf{\Gamma}$ *is defined by:*

$$\begin{array}{rcl} \mathsf{\Gamma} & := & F \mid \mathsf{M} \mid \mathsf{A} \\ \mathsf{M} & := & F, \mathsf{M}' \mid \mathsf{A}, \mathsf{M}' \\ \mathsf{M}' & := & F \mid \mathsf{A} \mid F, \mathsf{M}' \mid \mathsf{A}, \mathsf{M}' \\ \mathsf{A} & := & F; \mathsf{A}' \mid \mathsf{M}; \mathsf{A}' \\ \mathsf{A}' & := & F \mid \mathsf{M} \mid F; \mathsf{A}' \mid \mathsf{M}; \mathsf{A}' \end{array}$$

Each of the A *(resp.* M*) sub-structures of* $\mathsf{\Gamma}$ *is termed a negative additive (resp. multiplicative) structural layer.*

Definition 64 *A positive* `BBI` *structure* Δ *is defined by:* $\Delta := F \mid \Delta; \Delta$.

The set of $\mathsf{\Gamma}$ that the above grammar generates is denoted by $\mathfrak{N}$ and that of Δ by $\mathfrak{P}$.

Property 6 (Associativity and commutativity)
The following properties hold within $\mathfrak{N}$ *and* $\mathfrak{P}$*:*

1. $(\mathsf{\Gamma}_1, \mathsf{\Gamma}_2), \mathsf{\Gamma}_3 = \mathsf{\Gamma}_1, (\mathsf{\Gamma}_2, \mathsf{\Gamma}_3)$.
2. $(\mathsf{\Gamma}_1; \mathsf{\Gamma}_2); \mathsf{\Gamma}_3 = \mathsf{\Gamma}_1; (\mathsf{\Gamma}_2; \mathsf{\Gamma}_3)$.
3. $\mathsf{\Gamma}_1, \mathsf{\Gamma}_2 = \mathsf{\Gamma}_2, \mathsf{\Gamma}_1$.
4. $\mathsf{\Gamma}_1; \mathsf{\Gamma}_2 = \mathsf{\Gamma}_2; \mathsf{\Gamma}_1$.
5. $(\Delta_1; \Delta_2); \Delta_3 = \Delta_1; (\Delta_2; \Delta_3)$.
6. $\Delta_1; \Delta_2 = \Delta_2; \Delta_1$.

Negative structures are represented in nested structural layers as in `BI` proof theory (*Cf.* Chapter 3). Positive structures are represented in list.

4.1.1 Exponent-less $\mathtt{LBBI}_p$ sequents

To cement the basic, we for now limit our attention to the sequents in the following set $\mathfrak{C} := \{\Gamma \vdash \Delta \mid [\Gamma \in \mathfrak{N}] \wedge^\dagger [\Delta \in \mathfrak{P}]\}$. The notation "$\Gamma(\Gamma_1)$" is used (just as in `BI` proof theory *Cf.* Chapter 1) to specify which part of an antecedent structure is currently being accessed via an inference rule, stating that Γ_1 occurs as a sub-structure of $\Gamma(\Gamma_1)$. That is, informally, $\Gamma(-)$ represents a negative structure with a "hole" which is filled with Γ_1 like $\Gamma(\Gamma_1)$. For a formal definition (albeit for $\mathfrak{S}_{\mathtt{BI}}$) of a context, readers are referred back to Chapter 1.

For the correspondence between structures and formulas, I define the following interpretation of (exponent-less) sequents.

Definition 65 (Interpretation of a positive structure) *Interpretation of a positive structure is a function* $\bar{\cdot} : \mathfrak{P} \to \mathfrak{F}_{\mathtt{BBI}}$ *recursively defined as follows:*

- $\overline{F} \to F$.
- $\overline{\Delta_1; \Delta_2} \to \overline{\Delta_1} \vee \overline{\Delta_2}$.

Definition 66 (Interpretation of a negative structure) *Interpretation of a negative structure is a function* $\underline{\cdot}_\partial : \mathfrak{N} \to \mathfrak{F}_{\mathtt{BBI}}$ *recursively defined as follows:*

- $\underline{F}_\partial \to F$.
- $\underline{\Gamma_1; \Gamma_2}_\partial \to (\underline{\Gamma_1}_\partial \wedge \underline{\Gamma_2}_\partial)$.
- $\underline{\Gamma_1, \Gamma_2}_\partial \to (\underline{\Gamma_1}_\partial * \underline{\Gamma_2}_\partial)$.

4.2 $\mathtt{LBBI}_p$: BBI Sequent Calculus

In this section, three major technical difficulties concerning the development of a `BBI` sequent calculus are discussed. The first one arises from the fact that negation normal form of a `BBI` formula (via De Morgan and other laws in Boolean algebra for reduction) is not always knowable unlike classical logic. The second one comes from collapsing of `BBI` multiplicative conjuncts. And the third one from the non-intuitionistic $^*\top$. A

`BBI` sequent calculus $\texttt{LBBI}_p$ is developed by heeding these. For the exact order of the materials to appear in this section, a discussion on the partial negation normal form and a development of sequent calculus conventions precede the others. The collapsing of multiplicative conjuncts and the impact of the non-intuitionistic multiplicative unit will then be analysed with reference to corresponding $\texttt{LBBI}_p$ inference rules. The role of the falsity within `BBI` sequent calculi will be also mentioned.

4.2.1 On the partial negation normalisation

Let us suppose:

1. We have a sequent $D : (F_1; F_2 \supset F_3), F_4 \vdash F_5$ such that $D \in \mathfrak{C}$.
2. It holds that $\forall W \in \texttt{ND}\, \forall m \in W.[m \models (F_1 \wedge (F_2 \supset F_3)) * F_4] \rightarrow^{\dagger} [m \models F_5]$ (*Cf.* Definition 65 and Definition 66).
3. We have a sequent calculus which is sound and complete with respect to `BBI` Kripke non-deterministic semantics (*Cf.* Chapter 1).

Then there should exist a closed derivation tree for D constructable in the supposed proof system.

Now suppose we know that it is $F_2 \supset F_3$ that becomes the principal. Also recall the inference rule $\supset L$ in classical logic sequent calculus such as `G1c` (*Cf.* Chapter 1):

$$\frac{\Psi \vdash A_1; \Phi \qquad \Psi; A_2 \vdash \Phi}{\Psi; A_1 \supset A_2 \vdash \Phi} \supset L_{\texttt{G1c}}$$

for some $A_1, A_2 \in \mathfrak{F}_{\texttt{CL}}$ and some $\Psi, \Phi \in \mathfrak{S}_{\texttt{CL}}$. By sheer syntactic speculation on $\supset L_{\texttt{G1c}}$, then, we could have the following backward derivation of D with $F_2 \supset F_3$ as the principal:

$$\frac{D_2 : F_1, F_4 \vdash F_5; F_2 \qquad (F_1; F_3), F_4 \vdash F_3}{D : (F_1; F_2 \supset F_3), F_4 \vdash F_5}$$

which transfers F_2 onto the consequent (looked from conclusion to premises). This, however, does not reflect the `BBI` base-logic interactions property onto syntax. To elucidate, by the set of current suppositions, D_2 should be universally valid, *i.e.* $\forall W \in \texttt{ND}\, \forall m \in W.[m \models F_1 * F_4] \rightarrow^{\dagger} [m \models F_5 \vee F_2]$. But an error is immediately noticed by recalling that $[m \models F \vee G] \leftrightarrow^{\dagger} [m \models F] \vee^{\dagger} [m \models G]$: suppose that some possible world m forces $F_1 * F_4$, then D_2 says that the same possible world is used for judgement of F_5

and F_2, which cannot be generally appropriate. More informally, the problem that this bottom-up derivation step has is that F_2 is somehow connected additively to F_5, or to "F_1, F_4" interchangeably, whereas it is "F_1" to which it should be additively connected.

Solution To respect the property of multiplicative capturing of additive components, therefore, an additive implication must be resolved at the additive structural layer of whom it is a constituent. However, recall that classical logic exhibits symmetry. Instead of $\supset L_{\texttt{G1c}}$, a one-sided inference rule works just as well:

$$\frac{\Psi; A_1 \supset \mathbb{1} \vdash \mathbb{1} \qquad \Psi; A_2 \vdash \mathbb{1}}{\Psi; A_1 \supset A_2 \vdash \mathbb{1}} \supset L'_{\texttt{G1c}}$$

We may therefore consider an alternative derivation of D:

$$\frac{D_3 : ((F_1; F_2 \supset \mathbb{1}), F_4); F_5 \supset \mathbb{1} \vdash \mathbb{1} \qquad ((F_1; F_3), F_4); F_5 \supset \mathbb{1} \vdash \mathbb{1}}{D : ((F_1; F_2 \supset F_3), F_4); F_5 \supset \mathbb{1} \vdash \mathbb{1}}$$

as semantics dictates (*Cf.* Chapter 1). A solution to the multiplicative capturing of additive components was successfully given. Since we have

- $[(F_1 \wedge F_2) \supset \mathbb{1}] \simeq [(F_1 \supset \mathbb{1}) \vee (F_2 \supset \mathbb{1})]$.[1]
- $[(F_1 \vee F_2) \supset \mathbb{1}] \simeq [(F_1 \supset \mathbb{1}) \wedge (F_2 \supset \mathbb{1})]$.
- $[(F_1 \supset F_2) \supset \mathbb{1}] \simeq [F_1 \wedge (F_2 \supset \mathbb{1})]$.
- $[\mathbb{1} \supset \mathbb{1}] \simeq \top$.
- $[\top \supset \mathbb{1}] \simeq \mathbb{1}$.

we can define a set of inference rules to deal with formulas in the form: $F \supset \mathbb{1}$, in case F is in one of the forms: $\{\top, \mathbb{1}, F_1 \wedge F_2, F_1 \vee F_2, F_1 \supset F_2\}$.

Syntactic issue We must, however, consider the possibility that F_2 be in one of the forms: $\{{}^{*}\top, F_1 * F_2, F_1 \mathrel{-\!\!*} F_2\}$, for which no pseudo De Morgan equivalence is (yet) defined. The above solution of ours then do not automatically reduce a formula in the form: $G \supset \mathbb{1}$ into its (pseudo) negation normal form. Though somehow extrinsic, it is also non-aesthetic to have to state those formulas with the tailing "$\supset \mathbb{1}$".

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[1] Equivalence via De Morgan and duality.

4.2.1.1 Refinement of $\texttt{LBBI}_p$ sequents

To handle the slight syntactic inconvenience, we now extend the earlier notion of sequents in order to have a tidier syntactic representation.

Definition 67 (Negative structure with exponents) *By* $\mathbb{\Gamma}$ *we denote a* $\texttt{LBBI}_p$ *negative structure with exponents which is defined by:*

$$\begin{array}{rcl}
\mathbb{\Gamma} & := & F^{\Delta} \mid \mathcal{M}^{\Delta} \mid \mathcal{A}^{\Delta} \\
\mathcal{M} & := & F^{\Delta}, \mathcal{M}'^{\Delta} \mid \mathcal{A}^{\Delta}, \mathcal{M}'^{\Delta} \\
\mathcal{M}' & := & F \mid \mathcal{A} \mid F^{\Delta}, \mathcal{M}'^{\Delta} \mid \mathcal{A}^{\Delta}, \mathcal{M}'^{\Delta} \\
\mathcal{A} & := & F^{\mathbb{1}}; \mathcal{A}'^{\mathbb{1}} \mid \mathcal{M}^{\mathbb{1}}; \mathcal{A}'^{\mathbb{1}} \\
\mathcal{A}' & := & F \mid \mathcal{M} \mid F^{\mathbb{1}}; \mathcal{A}'^{\mathbb{1}} \mid \mathcal{M}^{\mathbb{1}}; \mathcal{A}'^{\mathbb{1}}
\end{array}$$

All the sub-structures $\mathbb{\Gamma}_1$ *of* $\mathbb{\Gamma}$ *are associated with some positive structure* $\Delta \in \mathfrak{P}$ *which is termed the exponent (of the sub-structure it is associated to). The set of* $\mathbb{\Gamma}$ *is denoted by* $\mathfrak{A}$*. Also we define a function* $\log : \mathfrak{A} \to \mathfrak{P}$ *such that* $\log \mathbb{\Gamma}$ *is the exponent associated to* $\mathbb{\Gamma}$*.*

Just as in Definition 63, each of the $\mathcal{A}$ (resp. $\mathcal{M}$) sub-structures is termed a (negative) additive (resp. multiplicative) structural layer. For presentational convenience, we also use the equivalent notation, Γ^{Δ}, to explicitly state $\log \mathbb{\Gamma} (= \Delta)$ of $\mathbb{\Gamma}$.

Example 6
For a negative structure with exponents: $(F_1^{\Delta_1}, F_2^{\Delta_2})^{\Delta_3}$ *, the following hold:*

- $\log F_1^{\Delta_1} = \Delta_1$.
- $\log F_2^{\Delta_2} = \Delta_2$.
- $\log (F_1^{\Delta_1}, F_2^{\Delta_2})^{\Delta_3} = \Delta_3$.

Property 7 (Associativity and commutativity) *The following associativity and commutativity hold within* $\mathfrak{A}$*:*

1. $(\mathbb{\Gamma}_1, \mathbb{\Gamma}_2), \mathbb{\Gamma}_3 = \mathbb{\Gamma}_1, (\mathbb{\Gamma}_2, \mathbb{\Gamma}_3)$.
2. $(\mathbb{\Gamma}_1; \mathbb{\Gamma}_2); \mathbb{\Gamma}_3 = \mathbb{\Gamma}_1; (\mathbb{\Gamma}_2; \mathbb{\Gamma}_3)$.

3. $\mathbb{\Gamma}_1, \mathbb{\Gamma}_2 = \mathbb{\Gamma}_2, \mathbb{\Gamma}_1$.

4. $\mathbb{\Gamma}_1; \mathbb{\Gamma}_2 = \mathbb{\Gamma}_2; \mathbb{\Gamma}_1$.

Definition of $\texttt{LBBI}_p$ sequents now follows.

Definition 68 **($\texttt{LBBI}_p$ sequents)** *A* $\texttt{LBBI}_p$ *sequent is defined to be a set element of* $\mathfrak{D}_{\texttt{BBI}} := \{\mathbb{\Gamma} \vdash \Delta \mid [\mathbb{\Gamma} \in \mathfrak{A}] \wedge^{\dagger} [\Delta \in \mathfrak{P}]\}$.

4.2.1.2 Interpretation of exponents and relation between $\mathfrak{E}$ and $\mathfrak{D}_{\texttt{BBI}}$

It is clear that $\mathfrak{E}$ and $\mathfrak{D}_{\texttt{BBI}}$ are essentially the same by the following interpretation of $\mathbb{\Gamma} \in \mathfrak{A}$.

Definition 69 (Interpretation of an exponentiated negative structure) *Interpretation of an exponentiated negative structure is a function* $\underline{\ } : \mathfrak{A} \to \mathfrak{F}_{\texttt{BBI}}$ *recursively defined as follows:*

- $\underline{F^{\mathbb{1}}} \to F$.
- *If* $\overline{\Delta} \neq \mathbb{1}$, *then* $\underline{F^{\Delta}} \to (\underline{F^{\mathbb{1}}} \wedge (\overline{\Delta} \supset \mathbb{1}))$.
- $\underline{(\Gamma_1^{\mathbb{1}}; \Gamma_2^{\mathbb{1}})^{\mathbb{1}}} \to (\underline{\Gamma_1^{\mathbb{1}}} \wedge \underline{\Gamma_2^{\mathbb{1}}})$.
- $\underline{(\Gamma_1^{\mathbb{1}}, \Gamma_2^{\mathbb{1}})^{\mathbb{1}}} \to (\underline{\Gamma_1^{\mathbb{1}}} * \underline{\Gamma_2^{\mathbb{1}}})$.
- *If* $\overline{\Delta_1} \neq \mathbb{1}$, *then* $(\underline{(\Gamma_1^{\Delta_1}, \Gamma_2^{\mathbb{1}})^{\mathbb{1}}}) \to ((\underline{\Gamma_1^{\mathbb{1}}} \wedge (\overline{\Delta_1} \supset \mathbb{1})) * \underline{\Gamma_2^{\mathbb{1}}})$.
- *If* $\overline{\Delta_1} \neq \mathbb{1}$ *and if* $\overline{\Delta_2} \neq \mathbb{1}$, *then* $\underline{(\Gamma_1^{\Delta_1}, \Gamma_2^{\Delta_2})^{\mathbb{1}}} \to ((\underline{\Gamma_1^{\mathbb{1}}} \wedge (\overline{\Delta_1} \supset \mathbb{1})) * (\underline{\Gamma_2^{\mathbb{1}}} \wedge (\overline{\Delta_2} \supset \mathbb{1})))$.
- *If* $\overline{\Delta} \neq \mathbb{1}$, *then* $\underline{\Gamma^{\Delta}} \to \underline{\Gamma^{\mathbb{1}}} \wedge (\overline{\Delta} \supset \mathbb{1})$.

We have the following result concerning the relation between $\mathfrak{E}$ and $\mathfrak{D}_{\texttt{BBI}}$.

Lemma 19 (Isomorphism) $\mathfrak{D}_{\texttt{BBI}}$ *and* $\mathfrak{E}$ *are isomorphic.*

Proof. Obvious by Definition 65, Definition 66, Definition 69, De Morgan and duality. □

I end this sub-sub-section 4.2.1.2 with a concluding remark on the derivation of $D : (F_1; F_2 \supset F_3), F_4 \vdash F_5$, which looked like:

$$\frac{D_3 : ((F_1; F_2 \supset \mathbb{1}), F_4); F_5 \supset \mathbb{1} \vdash \mathbb{1} \qquad ((F_1; F_3), F_4); F_5 \supset \mathbb{1} \vdash \mathbb{1}}{D : ((F_1; F_2 \supset F_3), F_4); F_5 \supset \mathbb{1} \vdash \mathbb{1}}$$

By mapping each sequent in $\mathfrak{C}$ in the derivation into $\mathfrak{D}_{\texttt{BBI}}$, we gain:

$$\frac{D_3 : (F_1^{F_2}, F_4)^{F_5} \vdash \mathbb{1} \qquad D' : ((F_1; F_3), F_4)^{F_5} \vdash \mathbb{1}}{D : ((F_1; F_2 \supset F_3), F_4)^{F_5} \vdash \mathbb{1}}$$

4.2.1.3 Conventions for `BBI` sequent calculi

Sequent calculus conventions are now formally introduced for $\texttt{LBBI}_p$.

Definition 70 (BBI equivalences) *"$\leftrightharpoons_{ant}$" is the equivalence relation on exponentiated negative structures satisfying:*

1. *If $\Delta_1 \equiv \Delta_2$ (up to assoc. and commut.), then $\Gamma^{\Delta_1} \leftrightharpoons_{ant} \Gamma^{\Delta_2}$.*
2. *$\mathbb{\Gamma}_1 \equiv \mathbb{\Gamma}_2$ (up to assoc. and commut.; Cf. Property 7).*
3. *If $\mathbb{\Gamma}_1 \leftrightharpoons_{ant} \mathbb{\Gamma}_2$ and $\mathbb{\Gamma}(\mathbb{\Gamma}_1) \in \mathfrak{D}_{\texttt{BBI}}$, then $\mathbb{\Gamma}(\mathbb{\Gamma}_1) \leftrightharpoons_{ant} \mathbb{\Gamma}(\mathbb{\Gamma}_2)$.*

"$\asymp_{pos}$" is the equivalence relation on positive structures satisfying:

1. *$\Delta_1 \equiv \Delta_2$ (up to assoc. and commut.).*
2. *$\Delta; \mathbb{1} \asymp_{pos} \Delta$.*

"$\asymp_{ant}$" is the equivalence relation on exponentiated negative structures satisfying:

1. *$\mathbb{\Gamma}_1 \equiv \mathbb{\Gamma}_2$ (up to assoc. and commut. as in Property 7).*
2. *If $\Delta_1 \asymp_{pos} \Delta_2$, then $\Gamma^{\Delta_1} \asymp_{ant} \Gamma^{\Delta_2}$.*
3. *$(\Gamma^{\mathbb{1}}; \top^{\mathbb{1}})^{\Delta} \asymp_{ant} \Gamma^{\Delta}$.*
4. *$(\Gamma_1^{\Delta_1}, {}^{*}\!\top^{\mathbb{1}})^{\Delta_2} \asymp_{ant} \Gamma^{(\Delta_1;\Delta_2)}$.*
5. *If $\mathbb{\Gamma}_1 \asymp_{ant} \mathbb{\Gamma}_2$ and $\mathbb{\Gamma}(\mathbb{\Gamma}_1) \in \mathfrak{D}_{\texttt{BBI}}$, then $\mathbb{\Gamma}(\mathbb{\Gamma}_1) \asymp_{ant} \mathbb{\Gamma}(\mathbb{\Gamma}_2)$.*

For a close syntax-semantic correspondence, it is more useful to be able to focus on an antecedent sub-structure which is at least as large as an additive structural layer (provided there is any structural layer) rather than some of its constituents. This is because all the constituents of an additive structural layer are judged by the same possible world in the underlying `BBI` Kripke non-deterministic semantics.

Definition 71 (Layer focus) *Given an antecedent structure* $\mathbb{\Gamma}(\mathbb{\Gamma}_1)$ *of which* $\mathbb{\Gamma}_1$ *is a sub-structure, we denote by* $\mathtt{return}_A(\mathbb{\Gamma}(\mathbb{\Gamma}_1))$ *the following:*

- *it is* $\mathbb{\Gamma}_2$ *with* $\mathbb{\Gamma}_2 \leftrightharpoons_{ant} (\mathbb{\Gamma}_1; \mathbb{\Gamma}_3)^{\Delta}$ *if there exists a structure* $\mathbb{\Gamma}_3$ *such that:*
  (a) $\mathbb{\Gamma}_3$ *is a sub-structure of* $\mathbb{\Gamma}(\mathbb{\Gamma}_1)$*.*
  (b) there exists no sub-structure $\mathbb{\Gamma}_4$ *of* $\mathbb{\Gamma}(\mathbb{\Gamma}_1)$ *such that* $(\mathbb{\Gamma}_2; \mathbb{\Gamma}_4)^{\Delta'}$ *is a sub-structure of* $\mathbb{\Gamma}(\mathbb{\Gamma}_1)$*.*
  (c) (the $\mathbb{\Gamma}_1$ *that occurs in* $\mathbb{\Gamma}_2$ *is the same as the focused sub-structure* $\mathbb{\Gamma}_1$ *of* $\mathbb{\Gamma}(\mathbb{\Gamma}_1)$*.)*
- *it is* $\mathbb{\Gamma}_1$*, otherwise.*

Definition 72 (Sequent calculus convention) *Notational conventions for* `BBI` *sequent calculi are set forth as follows:*

1. Γ *abbreviates* $\Gamma^{\mathbb{1}}$.
2. $\mathbb{\Gamma}\{\Gamma_1^{\Delta_1}\} \vdash \{\Delta_2\}$ *abbreviates* $\mathbb{\Gamma}(\Gamma_1^{(\Delta_1;\Delta_2)}) \vdash \mathbb{1}$ *where* $\mathtt{return}_A(\mathbb{\Gamma}(\Gamma_1^{(\Delta_1;\Delta_2)})) = \Gamma_1^{(\Delta_1;\Delta_2)}$.
3. *In a* $\mathtt{LBBI}_p$ *sequent, emptiness of an exponentiated negative structure in "*$\mathbb{\Gamma}_1; \mathbb{\Gamma}_2$*" is identified with* $\top$, i.e. *"*$\mathbb{\Gamma}_1; \mathbb{\Gamma}_2$*" is identified with* $\mathbb{\Gamma}_2$ *(resp.* $\mathbb{\Gamma}_1$*) in case* $\mathbb{\Gamma}_1$ *(resp.* $\mathbb{\Gamma}_2$*) is empty.*
4. *In a* $\mathtt{LBBI}_p$ *sequent, emptiness of an exponentiated negative structure in "*$\mathbb{\Gamma}_1, \mathbb{\Gamma}_2$*" is identified with* $^{*}\!\top$, i.e. *"*$\mathbb{\Gamma}_1, \mathbb{\Gamma}_2$*" is identified with* $\mathbb{\Gamma}_2$ *(resp.* $\mathbb{\Gamma}_1$*) in case* $\mathbb{\Gamma}_1$ *(resp.* $\mathbb{\Gamma}_2$*) is empty.*
5. *In a* $\mathtt{LBBI}_p$ *sequent, emptiness of a positive structure in "*$\Delta_1; \Delta_2$*" is identified with* $\mathbb{1}$, i.e. *"*$\Delta_1; \Delta_2$*" is identified with* Δ_2 *(resp.* Δ_1*) in case* Δ_1 *(resp.* Δ_2*) is empty.*

$\mathtt{LBBI}_p$ is found in Figure 4.1.

4.2.2 On the collapse of multiplicative conjuncts

Under the `BBI` Kripke non-deterministic semantics, collapsing of multiplicative conjuncts (which is nominally considered a special *distribution* in the rest) is permitted. I first introduce the notion of the relative structural distance.

$$\dfrac{}{F \vdash F}\,\text{Ax} \qquad \dfrac{\Gamma_1^{\Delta_1} \vdash \mathbb{1}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1\}}\,\mathbb{1}_{ps} \qquad \dfrac{\mathbb{\Gamma}\{\Gamma_1; F_1; F_2\} \vdash \{\Delta_1\}}{\mathbb{\Gamma}\{\Gamma_1; F_1 \wedge F_2\} \vdash \{\Delta_1\}}\,\wedge L$$

$$\dfrac{\mathbb{\Gamma}\{\Gamma_1; F_1\} \vdash \{\Delta_1\} \qquad \mathbb{\Gamma}\{\Gamma_1; F_2\} \vdash \{\Delta_1\}}{\mathbb{\Gamma}\{\Gamma_1; F_1 \vee F_2\} \vdash \{\Delta_1\}}\,\vee L \qquad \dfrac{\mathbb{\Gamma}\{\Gamma_1; (F_1, F_2)\} \vdash \{\Delta_1\}}{\mathbb{\Gamma}\{\Gamma_1; F_1 * F_2\} \vdash \{\Delta_1\}}\,*L$$

$$\dfrac{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1\} \qquad \mathbb{\Gamma}\{\Gamma_1; F_2\} \vdash \{\Delta_1\}}{\mathbb{\Gamma}\{\Gamma_1; F_1 \supset F_2\} \vdash \{\Delta_1\}}\,\supset L \qquad \dfrac{\mathbb{\Gamma}\{\Gamma_1; G\} \vdash \{\Delta_1\} \quad (F \in \Xi)}{\mathbb{\Gamma}\{\Gamma_1; F \mathrel{-\!\!*} G\} \vdash \{\Delta_1\}}\,\mathrel{-\!\!*} L_{{}^{*}\top}$$

$$\dfrac{\Gamma_1^{\Delta_1} \vdash F \quad (\Xi \cap F = \emptyset) \qquad \mathbb{\Gamma}\{\Gamma_2^{\Delta_2}, G\} \vdash \{\Delta_3\}}{\mathbb{\Gamma}\{\Gamma_1^{\Delta_1}, \Gamma_2^{\Delta_2}, F \mathrel{-\!\!*} G\} \vdash \{\Delta_3\}}\,\mathrel{-\!\!*} L_I$$

$$\dfrac{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1\} \qquad \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_2\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 \wedge F_2\}}\,\wedge R \qquad \dfrac{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1; F_2\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 \vee F_2\}}\,\vee R$$

$$\dfrac{{}^{*}\top \vdash F_1 \qquad \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; G_1\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 * G_1\}}\,*R_{{}^{*}\top} \qquad \dfrac{\Gamma_1^{\Delta_1} \vdash F_1 \qquad \Gamma_2^{\Delta_2} \vdash G_1}{\Gamma_1^{\Delta_1}, \Gamma_2^{\Delta_2} \vdash F_1 * G_1}\,*R_I$$

$$\dfrac{\Gamma_1^{\Delta_1}, F_1 \vdash F_2 \quad (\Xi \cap F_1 = \emptyset)}{\Gamma_1^{\Delta_1} \vdash F_1 \mathrel{-\!\!*} F_2}\,\mathrel{-\!\!*} R_I \qquad \dfrac{\mathbb{\Gamma}\{\Gamma_1; F_1\} \vdash \{\Delta_1; F_2\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 \supset F_2\}}\,\supset R$$

$$\dfrac{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; G\} \quad (F \in \Xi)}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F \mathrel{-\!\!*} G\}}\,\mathrel{-\!\!*} R_{{}^{*}\top} \qquad \dfrac{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1\}}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1\}}\,\textit{Wk L}$$

$$\dfrac{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F\}}\,\textit{Wk R} \qquad \dfrac{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2; \Gamma_2\} \vdash \{\Delta_1\}}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1\}}\,\textit{Ctr L}$$

$$\dfrac{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F; F\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F\}}\,\textit{Ctr R} \qquad \dfrac{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; \mathsf{S}_\wedge(F_1 \times F_2) * \mathsf{S}_\vee(G_1 \times G_2)\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 * G_1; F_2 * G_2\}}\,\text{dR}$$

$$\dfrac{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; \Delta_2\}}{\mathbb{\Gamma}\{\Gamma_1^{\Delta_1}, ({}^{*}\top; \Gamma_2)^{\Delta_3}\} \vdash \{\Delta_2\}}\,{}^{*}\top WkL \qquad \dfrac{\mathbb{\Gamma}\{({}^{*}\top; \Gamma_1)^{\Delta_2}, ({}^{*}\top; \Gamma_1)^{\Delta_2}\} \vdash \{\Delta_1\}}{\mathbb{\Gamma}\{{}^{*}\top; \Gamma_1\} \vdash \{\Delta_1; \Delta_2\}}\,{}^{*}\top CtrL$$

$$\dfrac{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F\} \qquad \mathbb{\Gamma}\{\Gamma_2; F\} \vdash \{\Delta_2\}}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2\}}\,\texttt{Cut}$$

Ξ is a set of `BBI` formulas such that $\forall F \in \Xi.[F \vdash {}^{*}\top] \wedge^{\dagger} [{}^{*}\top \vdash F]$.

Figure 4.1: $\texttt{LBBI}_p$: a sequent calculus for `BBI`. Definition 72 for calculus conventions.

Definition 73 (Relative structural distance) *Relative structural distance between a structure* $\mathbb{\Gamma}(\mathbb{\Gamma}_1)$ *and its focused sub-structure* $\mathbb{\Gamma}_1$*, denoted by* $\texttt{str\_dist}(\mathbb{\Gamma}(-))$*, is defined as follows:*

- *it is 0 if* “$\mathbb{\Gamma}(-) \leftrightharpoons_{ant} -$”.
- *it is* `str_dist`$(\Gamma_2(-)) + 1$ *if the set of conditions below are satisfied:*
  1. *there exists some* Γ'_1 *and some* Δ *such that* $\mathbb{\Gamma}(-) \leftrightharpoons_{ant} (\Gamma_2(-); \Gamma'_1)^{\Delta}$.
  2. *there exists no pair of* Γ''_1 *and* $\Gamma_3(-)$ *such that* $\Gamma_2(-) \leftrightharpoons_{ant} \Gamma_3(-); \Gamma''_1$.
- *it is* `str_dist`$(\mathbb{\Gamma}_2(-)) + 1$ *if the set of conditions below are satisfied:*
  1. *there exists some* $\mathbb{\Gamma}'_1$ *and some* Δ *such that* $\mathbb{\Gamma}(-) \leftrightharpoons_{ant} (\mathbb{\Gamma}_2(-), \mathbb{\Gamma}'_1)^{\Delta}$.
  2. *there exists no pair of* $\mathbb{\Gamma}''_1$ *and* $\mathbb{\Gamma}_3(-)$ *such that* $\mathbb{\Gamma}_2(-) \leftrightharpoons_{ant} \mathbb{\Gamma}_3(-), \mathbb{\Gamma}''_1$.

The special distributivity holds between the multiplicative conjunction and the additive disjunction.

Lemma 20 *Let* $dR_{another}$ *denote the following rule:*

$$\frac{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; (F_1 \wedge F_2) * (G_1 \vee G_2)\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 * G_1; F_2 * G_2\}}\ dR_{another}$$

Then it is locally sound with respect to the `BBI` *Kripke non-deterministic semantics,* i.e. *it holds that if* $\underline{\mathbb{\Gamma}(\Gamma_1^{(\Delta_1;(F_1 \wedge F_2)*(G_1 \vee G_2))})} \supset \mathbb{1}$ *is universally valid, then so is* $\underline{\mathbb{\Gamma}(\Gamma_1^{(\Delta_1;F_1*G_1;F_2*G_2)})} \supset \mathbb{1}$.

Proof. In Appendix A. □

To illustrate how $dR_{another}$ acts in a `BBI` sequent calculus, let us replace dR with $dR_{another}$ in `LBBI`$_p$. Let us call the alternative proof system `LBBI`$_p$’.

Example 7 *Let* $D \in \mathfrak{D}_{\texttt{BBI}}$ *be* $p, q \vdash (p * (r \supset q)); (p * r)$. *Then it is* `LBBI`$_p$*’-derivable:*

$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\,Ax \qquad \dfrac{}{p \vdash p}\,Ax}{p \vdash p \wedge p}\,\wedge R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{r \vdash r}\,Ax}{q; r \vdash s; r}\,\{Wk\,L,\ Wk\,R\}}{q \vdash r \supset q; r}\,\supset R}{q \vdash (r \supset s) \vee r}\,\vee R}{\dfrac{p, q \vdash (p \wedge p) * ((r \supset s) \vee r)}{p, q \vdash (p * (r \supset s)); (p * r)}\,dR_{another}}\,*R_I$$

Also, let $D' \in \mathfrak{D}_{\texttt{BBI}}$ *be* $(\top^{(p*(q\supset\mathbb{1}));((p\supset\mathbb{1})*(r\supset\mathbb{1}))}, p)^{(q\vee r)*p} \vdash \mathbb{1}$*:*

$$\begin{prooftree}
\AxiomC{}
\doubleLine
\UnaryInfC{${}^{*}\!\top \vdash p \vee (p \supset \mathbb{1})$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$p \vdash p$}
\AxiomC{}
\doubleLine
\UnaryInfC{$q \vdash q; r$}
\RightLabel{$\vee R$}
\UnaryInfC{$q \vdash q \vee r$}
\RightLabel{$*R_I$}
\BinaryInfC{$(\{q\}, p)^{(q \vee r) * p} \vdash \{\mathbb{1}\}$}
\RightLabel{$\supset R$}
\UnaryInfC{$(\{\top\}, p)^{(q \vee r) * p} \vdash \{q \supset \mathbb{1}\}$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$p \vdash p$}
\AxiomC{}
\doubleLine
\UnaryInfC{$r \vdash q; r$}
\RightLabel{$\vee R$}
\UnaryInfC{$r \vdash q \vee r$}
\RightLabel{$*R_I$}
\BinaryInfC{$(\{r\}, p)^{(q \vee r) * p} \vdash \{\mathbb{1}\}$}
\RightLabel{$\supset R$}
\UnaryInfC{$(\{\top\}, p)^{(q \vee r) * p} \vdash \{r \supset \mathbb{1}\}$}
\RightLabel{$\wedge R$}
\BinaryInfC{$(\{\top\}, p)^{(q \vee r) * p} \vdash \{(q \supset \mathbb{1}) \wedge (r \supset \mathbb{1})\}$}
\RightLabel{$*R_{{}^{*}\!\top}$}
\BinaryInfC{$(\{\top\}, p)^{(q \vee r) * p} \vdash \{(p \vee (p \supset \mathbb{1})) * ((q \supset \mathbb{1}) \wedge (r \supset \mathbb{1}))\}$}
\RightLabel{$dR_{another}$}
\UnaryInfC{$(\{\top\}, p)^{(q \vee r) * p} \vdash \{(p * (q \supset \mathbb{1})); ((p \supset \mathbb{1}) * (r \supset \mathbb{1}))\}$}
\end{prooftree}$$

(The second derivation can be shortened by brining the derivation step of $*R_I$ below $*R_{{}^{*}\!\top}$ and dR; this particular derivation tree is simply to illustrate the use of $*R_{{}^{*}\!\top}$ when it does not take as the principal a formula in the exponent of the outermost antecedent structure.)

`LBBI`$_p$ achieves nearly the same effect with dR as `LBBI`$_p$' with $dR_{another}$. In fact, for soundness and completeness, it does not really matter which is used. However, `LBBI`$_p$' is unfortunately non-analytic even without `Cut` due to the interactions between $dR_{another}$ and $CtrR$.

Proposition 13 (Non-analyticity of `LBBI`$_p$')

$dR_{another}$ and $CtrR$ lead to non-analyticity, i.e. *infinite introduction of distinct new constructs, in* `LBBI`$_p$*'.*

Proof. Let $D \in \mathfrak{D}_{\mathtt{BBI}}$ be $H \vdash F_1 * G_1$. Then there exists a derivation tree such as the following (arbitrarily chosen for the sake of illustration; note that associativity and commutativity as defined earlier are freely available) in which the introduction of new constructs is unbounded:

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$H \vdash F_1 * G_1; (F_1 \vee G_1) * (G_1 \wedge F_1); ((F_1 \vee G_1) \wedge F_1) * ((G_1 \wedge F_1) \vee G_1)$}
\RightLabel{$dR_{another}$}
\UnaryInfC{$H \vdash F_1 * G_1; (F_1 \vee G_1) * (G_1 \wedge F_1); (F_1 \vee G_1) * (G_1 \wedge F_1); F_1 * G_1$}
\doubleLine
\RightLabel{$CtrR$}
\UnaryInfC{$H \vdash F_1 * G_1; (F_1 \vee G_1) * (G_1 \wedge F_1)$}
\RightLabel{$dR_{another}$}
\UnaryInfC{$H \vdash F_1 * G_1; F_1 * G_1; F_1 * G_1$}
\RightLabel{$CtrR$}
\UnaryInfC{$H \vdash F_1 * G_1; F_1 * G_1$}
\end{prooftree}$$

□

The given example derivation tree in Proposition 13 introduces (looked backward) a new construct $(F_1 \vee G_1) * (G_1 \wedge F_1)$ in the first $dR_{another}$ application and another new construct $((F_1 \vee G_1) \wedge F_1) * ((G_1 \wedge F_1) \vee G_1)$ in the second $dR_{another}$ application. Since $CtrR$ and $dR_{another}$ are available in [`LBBI`$_p$' - `Cut`], this infinite process cannot be prevented from occurring.

4.2.2.1 Ensuring analyticity with dR through synthesising operators

We nonetheless notice with a knowledge of laws in Boolean algebra that satisfiability of $(F_1 \lor G_1) \land F_1$ (resp. $(G_1 \land F_1) \lor G_1$) coincides with that of F_1 (resp. G_1). Hence if the process of synthesising two multiplicative conjuncts is systematically carried out, then starting with a finite number of multiplicative conjuncts in an exponent, there could result only a finite number of multiplicative conjuncts even in the presence of $CtrR$. We achieve this by internally processing the premise sequent of $dR_{another}$ farther, drawing ideas from the Boolean algebra, resulting in dR as we see in Figure 4.1. For this purpose, first we define `BBI` literals and an equivalence relation $\doteq$.

Definition 74 (`BBI` literals) *A `BBI` literal $f, (g, h)$ is defined by the following grammar: $f := p \mid {}^*\top \mid F \supset F \mid F * F \mid F -\!\!* F$.*

Definition 75 (Equivalence relation $\doteq$) *We define a binary relation $\doteq$ as one that satisfies the following:*

1. $f \land (g \land h) \doteq (f \land g) \land h$ *(associativity 1).*
2. $f \lor (g \lor h) \doteq (f \lor g) \lor h$ *(associativity 2).*
3. $f \land g \doteq g \land f$ *(commutativity 1).*
4. $f \lor g \doteq g \lor f$ *(commutativity 2).*
5. $f \land (g \lor h) \doteq (f \land g) \lor (f \land h)$ *(distributivity 1).*
6. $f \lor (g \land h) \doteq (f \lor g) \land (f \lor h)$ *(distributivity 2).*
7. $f \land f \doteq f$ *(identity 1).*
8. $f \lor f \doteq f$ *(identity 2).*
9. $f \lor (f \land g) \doteq f$ *(absorption 1).*
10. $f \land (f \lor g) \doteq f$ *(absorption 2).*
11. $\mathbb{1} \lor f \doteq f$.
12. $\mathbb{1} \land f \doteq \mathbb{1}$.

13. $\top \vee f \doteq \top$.

14. $\top \wedge f \doteq f$.

Based on the above knowledge, we let $dR_{\texttt{LBBI}_p}$:

$$\frac{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; \mathsf{S}_\wedge(F_1 \times F_2) * \mathsf{S}_\vee(G_1 \times G_2)\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 * G_1; F_2 * G_2\}} \, dR$$

perform certain normalisation operations on the premise.

Definition 76 (Synthesis operators)
The conjunctive synthesis operator $\mathsf{S}_\wedge : \mathfrak{F}_{\texttt{BBI}} \times \mathfrak{F}_{\texttt{BBI}} \rightarrow \mathfrak{F}_{\texttt{BBI}}$ *is a function that takes two* `BBI` *formulas* F *and* G *and returns a* `BBI` *conjunctive normal form of* $F \wedge G$ *such that (1) it be the least modulo* $\doteq$ *and that (2) all the* `BBI` *literals in the* `BBI` *conjunctive normal form occur as a sub-formula of at least either* F *or* G*.*

The disjunctive synthesis operator $\mathsf{S}_\vee : \mathfrak{F}_{\texttt{BBI}} \times \mathfrak{F}_{\texttt{BBI}} \rightarrow \mathfrak{F}_{\texttt{BBI}}$ *is a function that takes two* `BBI` *formulas* F *and* G *and returns a* `BBI` *disjunctive normal form of* $F \vee G$ *such that (1) it be the least modulo* $\doteq$ *and that (2) all the* `BBI` *literals in the* `BBI` *disjunctive normal form occur as a sub-formula of at least either* F *or* G*.*

The normalisation ensures a nice property.

Proposition 14 (Analyticity) *dR and $CtrR$ do not lead to non-analyticity in* $\texttt{LBBI}_p$*.*

Proof. With a finite number of `BBI` literals to synthesise through dR and $CtrR$, no clause of either the `BBI` disjunctive or the `BBI` conjunctive normal form can contain an infinite number of `BBI` literals. The number of clauses is necessarily finite in the `BBI` disjunctive/conjunctive normal form. □

Moreover, the operations can be syntactically carried out, by comparing the formulas to be synthesised, say F, with all the possible `BBI` conjunctive (disjunctive) normal forms, say $G_1, \cdots, G_k$ (k is necessarily finite), to test if there is a match. The process can be conducted in some sequent calculus for classical logic.

4.2.3 On direct and indirect cut formulas

The availability of dR has an implication on `Cut`. Suppose that we have a $\texttt{LBBI}_p$-derivation:

$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\text{Ax} \qquad \dfrac{\dfrac{\overline{\overline{{}^{*}\!\top ; q \vdash q ; \mathbb{1}}}}{{}^{*}\!\top \vdash q ; q \supset \mathbb{1}}\supset R}{{}^{*}\!\top \vdash q \vee (q \supset \mathbb{1})}\vee R}{D_3 : p \vdash p * (q \vee (q \supset \mathbb{1}))}*R_{*\top}}{D_1 : p \vdash p * q ; p * (q \supset \mathbb{1})}dR \qquad \begin{matrix} \vdots \\ D_2 : p * (q \supset \mathbb{1}) \vdash \mathbb{1} \end{matrix}}{p \vdash p * q}\,\texttt{Cut}$$

The `Cut` instance introduces a cut formula of $p * (q \supset \mathbb{1})$ (a direct cut formula) on D_1 and D_2. If we carefully examine the derivation, however, there is actually another cut formula indirectly introduced through a synthesis of the direct cut formula with a formula which existed prior to the backward `Cut` application. In the above derivation, it is $p * (q \vee (q \supset \mathbb{1}))$ as appearing in D_3. It is a cut formula since it could not have existed if not for the `Cut` application introducing the direct cut formula. Hence, comparing the left branch with D_2 to which only the direct cut formula is introduced, we notice that the indirect cut formula has been forgotten being introduced into D_2, resulting in an intention mismatch on the part of `Cut` between the two derivation branches of its. This is not in keeping with our intention (*i.e.* with our understanding about `Cut`) and certainly not desirable for cut admissibility analysis since the intention mismatch would inflict on a successful elimination of `Cut`. In a more general scenario than the very simple example given above, it is rather difficult to know how many indirect cut formulas a direct formula would introduce over the course of a derivation. Practically, there are two conceivable solutions to the intention mismatch: one that modifies the direct cut formula on the right premise of a `Cut` instance in case it should be, in the left derivation branch, used to introduce an indirect cut formula - a try-and-amend approach. The second that I adopt in this thesis is to not permit introduction of indirect cut formulas, since any such formulas can be directly introduced, precluding any potential intention mismatches. Under this strategy, the above derivation becomes:

$$\dfrac{\begin{matrix} \vdots \\ D_1' : p \vdash p * q ; H \end{matrix} \qquad \begin{matrix} \vdots \\ D_2' : H \vdash \mathbb{1} \end{matrix}}{p \vdash p * q}\,\texttt{Cut}$$

where the direct cut formula is $H \equiv (p * (q \supset \mathbb{1})) \vee (p * (q \vee (q \supset \mathbb{1})))$ (or $H = (p * (q \supset \mathbb{1})) \vee (p * (q \vee (q \supset \mathbb{1})))$ without a loss of generality).

4.2.4 Significance of the multiplicative unit

Significance of the multiplicative unit in `BBI` proof theory has not been paid so much attention. Unlike a general consensus that the multiplicative base logic of `BBI` is the same as that of `BI`, there is a crucial difference in that $^{*}\top$ in `BBI` does not behave intuitionistically but Boolean.

For a closer syntax-semantics correspondence, I regard $^{*}\top$ explicitly as a Boolean component, which leads to dedicated inference rules around $^{*}\top$. To sharply recognise the `BBI` formulas that are semantically indistinguishable from $^{*}\top$, `LBBI`$_p$ defines a set Ξ which holds certain class of `BBI` formulas. They are termed `BBI` multiplicative theorems.

Definition 77 (Multiplicative theorems) *A multiplicative theorem is defined to be a* `BBI` *formula* F *iff* $\forall W \in \mathtt{ND}.([\epsilon \models F] \wedge^{\dagger} (\forall m \in W.[m \neq \epsilon] \rightarrow^{\dagger} \neg^{\dagger}[m \models F]))$.

Definition 78 (Collector) *The* `LBBI`$_p$*-collector* Ξ *is a set of* `BBI` *formulas such that for all* F *that it holds, both* $F \vdash {}^{*}\top$ *and* ${}^{*}\top \vdash F$ *are* `LBBI`$_p$*-derivable.*

Intuitively, Ξ can hold all the multiplicative theorems in an event where `LBBI`$_p$ is complete with respect to the `BBI` Kripke non-deterministic semantics.

We now go through relevant `LBBI`$_p$ inference rules. First, although contraction is generally understood to be unavailable in a multiplicative context, the Boolean $^{*}\top$ semantics strongly determines satisfiability of other formulas additively connected to it (F is said to be additively connected to $^{*}\top$ iff it is a constituent of the same additive structural layer as $^{*}\top$ is) with respect to ϵ, and so we have $^{*}\top CtrL$. Second, although the multiplicative implication $F_1 \mathrel{-\!\!*} F_2$ is generally understood to be intuitionistic, the Boolean semantics for $^{*}\top$ makes an exception: if F_1 is in Ξ, then the behaviour of F_1 is no longer distinguishable from $^{*}\top$. We therefore have $\mathrel{-\!\!*}R_{^{*}\top}$ and $\mathrel{-\!\!*}L_{^{*}\top}$ which do not act intuitionistically. Additionally, we have another inference rule $*R_{^{*}\top}$ for multiplicative conjuncts.

By having syntax that closely matches with the underlying semantics, we for instance have a very natural derivation of $^{*}\top \wedge F \supset F * F$, eliciting no surprise:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F \vdash F}\,\mathrm{Ax}}{\top;{}^{*}\top;F \vdash F}\,WkL \qquad \dfrac{\dfrac{}{F \vdash F}\,\mathrm{Ax}}{\top;{}^{*}\top;F \vdash F}\,WkL}{(\top;{}^{*}\top;F),(\top;{}^{*}\top;F) \vdash F * F}\,*R_I}{\top;{}^{*}\top;F \vdash F * F}\,{}^{*}\top CtrL}{\top;{}^{*}\top \wedge F \vdash F * F}\,\wedge L}{\top \vdash ({}^{*}\top \wedge F)\supset(F * F)}\supset R$$

4.2.5 On the treatment of falsity

The role that the falsity "$\mathbb{1}$" undertakes in `BBI` sequent calculus must be studied closely. It holds in `BBI` that if $F \supset \mathbb{1}$ is universally valid, then neither F nor $F \wedge G$, nor $F * G$ is satisfiable. An observation follows.

Lemma 21 (Bottom lemma) *If* $\forall W \in \mathtt{ND}\, \forall m \in W.\neg^{\dagger}[m \models \mathbb{\Gamma}_1]$, *then* $\forall W \in \mathtt{ND}\, \forall m \in W.[m \models \underline{\mathbb{\Gamma}(\mathbb{\Gamma}_1)} \supset \mathbb{1}]$ *for an arbitrary* $\mathbb{\Gamma}(-)$ *(but such that* $\mathbb{\Gamma}(\mathbb{\Gamma}_1) \in \mathfrak{A}$*).*

Proof. In Appendix B. □

$\mathbb{1}_{ps}$ (Figure 4.1) results based on Lemma 21. We now also know that the following `Cut` is derivable with $\mathbb{1}_{ps}$ in $\mathtt{LBBI}_p$:

$$\frac{D_1 : \mathbb{\Gamma}'\{\Gamma_1\} \vdash \{F; \Delta_1\} \qquad \mathbb{\Gamma}\{\Gamma_2; F\} \vdash \{\Delta_2\}}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2\}}\ \mathtt{Cut}$$

where $\mathbb{\Gamma}\{\Gamma_1\} \vdash \{F; \Delta_1\} \rightsquigarrow_{\mathbb{1}_{ps}} D_1$.

I call a special case where $\mathbb{\Gamma}'(-) = -$ as the intuitionistic `Cut` that can exhibit certain multiplicity:

$$\frac{\Gamma'^{\Delta'} \vdash F \qquad \mathbb{\Gamma}((\Gamma_1; F)^{\Delta_1})_A \ldots ((\Gamma_k; F)^{\Delta_k})_A \vdash \mathbb{1}}{\mathbb{\Gamma}((\Gamma_1; \Gamma')^{(\Delta_1;\Delta')})_A \ldots ((\Gamma_k; \Gamma')^{(\Delta_1;\Delta')})_A \vdash \mathbb{1}}\ \mathtt{Cut}^*$$

where (1) $\mathtt{return}_A(\mathbb{\Gamma}(\Gamma_a^{\Delta_a})_A) = \Gamma_a^{\Delta_a}$ for some $\Gamma_a^{\Delta_a}$ and (2) $\mathbb{\Gamma}(\Gamma_1)(\Gamma_2)\ldots(\Gamma_k)$ (for some $\Gamma_1, \ldots, \Gamma_k$) abbreviates $(\ldots((\mathbb{\Gamma}(\Gamma_1))(\Gamma_2))\ldots)(\Gamma_k)$.

Lemma 22 $\mathtt{Cut}^*$ *is derivable in* $\mathtt{LBBI}_p$*.*

Proof. Straightforward. □

4.3 Main Properties of LBBI_p

In this section main properties of $\mathtt{LBBI}_p$: soundness and completeness, are proved. For the soundness proof, the result below around $\vee$ (and dually $\wedge$) is made use of.

Lemma 23 (Equivalence for $\vee$)
$\forall W \in \mathtt{ND}.(\forall m \in W.[m \models \underline{\mathbb{\Gamma}(F_1)}] \vee^{\dagger} [m \models \underline{\mathbb{\Gamma}(F_2)}]) \leftrightarrow^{\dagger} (\forall m \in W.[m \models \underline{\mathbb{\Gamma}(F_1 \vee F_2)}])$.

Proof. In Appendix C. □

Theorem 10 (Soundness of LBBI$_p$) *If* $\Gamma^{\Delta} \vdash \mathbb{1}$ *is* LBBI$_p$*-derivable, then* $\underline{\Gamma^{\Delta}} \supset \mathbb{1}$ *(or* $\underline{\Gamma} \supset \overline{\Delta}$*) is universally valid.*

Proof. By induction on the derivation depth. Mostly straightforward. Note that $[m \models \underline{\Gamma^{\Delta}} \supset \mathbb{1}] \leftrightarrow^{\dagger} ([m \models \underline{\Gamma}] \rightarrow^{\dagger} [m \models \overline{\Delta}])$, as expected. Trivial for Ax. For inductive cases, assume that the current theorem holds true for all the LBBI$_p$ derivations of derivation depth up to k, and show that it still holds true for LBBI$_p$ derivations of derivation depth $k+1$. Consider what the last inference rule applied is.

$\mathbb{1}_{ps}$**:** by induction hypothesis, we have $\underline{\Gamma_1^{\Delta_1}} \supset \mathbb{1}$ as a universally valid BBI formula. Then immediate via Bottom lemma.

$\wedge L$**:** ";" in antecedent is the structural proxy of $\wedge$. Vacuous.

$\vee L$**:** Lemma 23.

$* L$**:** "," in the antecedent part is the structural proxy of $*$. Vacuous.

$\supset L$**:** similar to "$\vee L$". Lemma 23.

$-\!\!* L_{*\top}$**:** trivial by the definition of a multiplicative theorem.

$-\!\!* L_I$**:** no essential difference from BI $-\!\!* L$ (*Cf.* literature Brotherston [2009]; Pym [2002]). Straightforward.

$\wedge R$**:** similar to $\vee L$. Note $[(F_1 \wedge F_2) \supset \mathbb{1}] \simeq [(F_1 \supset \mathbb{1}) \vee (F_2 \supset \mathbb{1})]$ (equivalence via De Morgan). Lemma 23.

$\vee R$**:** ";" in a positive structure is the structural proxy of "$\vee$". Vacuous.

$* R_{*\top}$**:** formally a sub-induction on the relative structural distance `str_dist`$(\mathbb{\Gamma}(-))$. The straightforward proof is nonetheless omitted except I note that ϵ satisfies F_1 due to induction hypothesis on the left premise.

$* R_I$**:** no essential difference from BI $* R$.

$\supset R$: a sub-induction on relative structural distance. Straightforward by definition of the Kripke semantics for “$\supset$” for the base case. For inductive cases, Lemma 23.

$-\!\!*R_I$: By induction hypothesis, we have:
$\forall W \in \texttt{ND}\, \forall m \in W.(\exists m_1, m_2 \in W.[m \in m_1 \circ m_2] \wedge^{\dagger} [m_1 \models \underline{\mathbb{\Gamma}}] \wedge^{\dagger} [m_2 \models F]) \rightarrow^{\dagger} [m \models G]$.
We must now show that $\forall W \in \texttt{ND}\, \forall m', m_1, m_2 \in W.[m' \models \mathbb{\Gamma}] \rightarrow^{\dagger} ([m_1 \models F] \rightarrow^{\dagger} ([m_2 \in m' \circ m_1] \rightarrow^{\dagger} [m_2 \models G]))$.
Here assume arbitrary $m_a, m_b, m_c \in W$ such that $[m_a \models \mathbb{\Gamma}] \wedge^{\dagger} [m_b \models F] \wedge^{\dagger} [m_c \in m_a \circ m_b]$. If the assumption is not self-contradictory in itself, we must be able to show that $[m_c \models G]$ from the hypotheses, which holds true trivially.

$-\!\!*R_{{}^*\!\top}$: Trivial by the definition of a multiplicative theorem.

WkL: straightforward.

WkR: straightforward.

$CtrL$: straightforward.

$CtrR$: straightforward.

dR: Lemma 20 and Definition 76.

${}^*\!\top WkL$: straightforward.

${}^*\!\top CtrL$: a sub-induction on the relative structural distance $\texttt{str\_dist}(\mathbb{\Gamma}(-))$. For base cases of the sub-induction, $\forall W \in \texttt{ND}\, \forall m \in W$:

1. if $\exists m_1, m_2.[m \in m_1 \circ m_2] \wedge^{\dagger} [m_1 \models \underline{({}^*\!\top; \Gamma_1)^{\Delta_2}}] \wedge^{\dagger} [m_2 \models \underline{({}^*\!\top; \Gamma_1)^{\Delta_2}}]$, then it must hold that $m_1 = m_2 = \epsilon = m$ and, by induction hypothesis of the main induction, also that $[m \models \overline{\Delta_1}]$.
2. otherwise, $(\neg^{\dagger}(\exists m_1, m_2.[m \in m_1 \circ m_2] \wedge^{\dagger} [m_1 \models \underline{({}^*\!\top; \Gamma_1)^{\Delta_2}}] \wedge^{\dagger} [m_2 \models \underline{({}^*\!\top; \Gamma_1)^{\Delta_2}}])) \rightarrow^{\dagger} ([m_1 \neq \epsilon] \wedge^{\dagger} [m_2 \neq \epsilon])$. Then m cannot be ϵ. Then m cannot force $\underline{({}^*\!\top; \Gamma_1)^{\Delta_2}}$.

Then straightforward. Proofs for inductive cases follow the approaches in Lemma 23, though simpler.

`Cut`**:** a sub-induction on $\texttt{str\_dist}(\mathbb{\Gamma}(-))$. Base case is trivial. For inductive cases,

1. if $\mathbb{\Gamma}(-) \leftrightharpoons_{ant} (\Gamma'(-); \Gamma'_1)^{\Delta'}$ such that $\texttt{str\_dist}(\Gamma'(-)) = l$ and that $\texttt{str\_dist}(\mathbb{\Gamma}(-)) = l + 1$, then $\forall W \in \texttt{ND}$:
   (a) if $\forall m \in W.\neg^{\dagger}[m \models \underline{\Gamma'_1}]$, then by Bottom lemma, we have $\forall m \in W.\neg^{\dagger}[m \models \underline{(\Gamma'((\Gamma_1; \Gamma_2)^{\Delta_1;\Delta_2}); \Gamma'_1)^{\Delta'}}]$.
   (b) otherwise, by induction hypothesis of the sub-induction and that of the main induction, we have either $\neg^{\dagger}[m \models \underline{(\Gamma'(\Gamma_1; \Gamma_2)^{\Delta_1;\Delta_2}; \Gamma'_1)^{\Delta'}}]$, or $[m \models \overline{\Delta'}]$ as required.
2. if $\mathbb{\Gamma}(-) \leftrightharpoons_{ant} (\mathbb{\Gamma}'(-), \mathbb{\Gamma}'_1)^{\Delta'}$ such that $\texttt{str\_dist}(\mathbb{\Gamma}'(-)) = l$ and that $\texttt{str\_dist}(\mathbb{\Gamma}(-)) = l + 1$: also the proof approaches as we saw in Lemma 23 work.

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Theorem 11 (Completeness of LBBI$_p$) *Any universally valid* `BBI` *formula* $F \in \mathfrak{F}_{\texttt{BBI}}$ *is derivable in* `LBBI`$_p$, i.e. $\top^F \vdash \mathbb{1}$ *(or* $\top \vdash F$*).*

Proof. It suffices to show that each Hilbert inference rule (*Cf.* Figure 1.8) is derivable in `LBBI`$_p$. Proof is by induction on the derivation depth in the `BBI` Hilbert system.

$F \supset (G \supset F)$**:**

$$\cfrac{\cfrac{\cfrac{\cfrac{}{F \vdash F}\,Ax}{\top; F; G \vdash F}\,WkL}{\top; F \vdash G \supset F}\,\supset R}{\top \vdash F \supset (G \supset F)}\,\supset R$$

$(F \supset (G \supset H)) \supset ((F \supset G) \supset (F \supset H))$**:** Also straightforward.

$F \supset F \vee G$**:** Also straightforward.

$(F \supset H) \supset ((G \supset H) \supset (F \vee G \supset H))$**:**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F \vdash F}\,Ax}{F \vdash F; H}\,WkR \quad \dfrac{\dfrac{}{H \vdash H}\,Ax}{F; H \vdash H}\,WkL}{F; F{\supset}H \vdash H}\supset L}{\top; F; F{\supset}H; G{\supset}H \vdash H}\,WkL \quad \dfrac{\dfrac{\dfrac{\dfrac{}{G \vdash G}\,\mathrm{Ax}}{G \vdash G; H}\,WkR \quad \dfrac{\dfrac{}{H \vdash H}\,\mathrm{Ax}}{G; H \vdash H}\,WkL}{G; G{\supset}H \vdash H}\supset L}{\top; G; F{\supset}H; G{\supset}H \vdash H}\,WkL}{\top; F \vee G; F{\supset}H; G{\supset}H \vdash H}\vee L}{\top; F{\supset}H; G{\supset}H \vdash F \vee G{\supset}H}\supset R}{\top; F{\supset}H \vdash (G{\supset}H){\supset}(F \vee G{\supset}H)}\supset R}{\top \vdash (F{\supset}H){\supset}((G{\supset}H){\supset}(F \vee G{\supset}H))}\supset R$$

$F \wedge G \supset F$**:** straightforward.

$F{\supset}(G{\supset}F \wedge G)$**:** straightforward.

$\mathbb{1}{\supset}F$**:** straightforward.

$F{\supset}\top$**:** straightforward.

$((F{\supset}\mathbb{1}){\supset}\mathbb{1}) \supset F$**:** straightforward.

$F{\supset}({}^{*}\!\top * F)$**:**

$$\dfrac{\dfrac{\dfrac{}{{}^{*}\!\top \vdash {}^{*}\!\top}\,\mathrm{Ax} \quad \dfrac{\dfrac{}{F \vdash F}\,\mathrm{Ax}}{\top; F \vdash F}\,WkL}{\top; F \vdash {}^{*}\!\top * F}\,{*R_{{}^{*}\!\top}}}{\top \vdash F{\supset}({}^{*}\!\top * F)}\supset R$$

$({}^{*}\!\top * F){\supset}F$**:**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F \vdash F}\,Ax}{{}^{*}\!\top, F \vdash F}\,{}^{*}\!\top WkL}{{}^{*}\!\top * F \vdash F}\,{*L}}{\top; {}^{*}\!\top * F \vdash F}\,WkL}{\top \vdash {}^{*}\!\top * F{\supset}F}\supset R$$

MP:

$$\dfrac{\top \vdash F \qquad \dfrac{\top \vdash F{\supset}G \qquad \dfrac{\dfrac{\dfrac{}{F \vdash F}\,Ax}{F \vdash F; G}\,WkR \quad \dfrac{\dfrac{}{G \vdash G}\,Ax}{F; G \vdash G}\,WkL}{F; F{\supset}G \vdash G}\supset L}{F \vdash G}\,\mathtt{Cut}}{\top \vdash G}\,\mathtt{Cut}$$

$*$**:**

$$\dfrac{\dfrac{\dfrac{\dfrac{D_1 : F_1 \vdash G_1 \qquad D_2 : F_2 \vdash G_2}{F_1, F_2 \vdash G_1 * G_2}\,{*R_I}}{F_1 * F_2 \vdash G_1 * G_2}\,{*L}}{\top; F_1 * F_2 \vdash G_1 * G_2}\,{WkL}}{\top \vdash (F_1 * F_2) \supset (G_1 * G_2)}\,{\supset R}$$

Cf. MP for the remaining derivation tree construction for both D_1 and D_2.

$-\!\!*_1$:

1. G is not in Ξ:

$$\dfrac{\dfrac{\dfrac{\dfrac{\top \vdash F \supset (G -\!\!* H) \qquad \dfrac{\dfrac{\dfrac{}{F \vdash F}\,\text{Ax}}{(\{F\}, G)^H \vdash \{F\}}\,\mathbb{1}_{ps} \qquad \dfrac{\dfrac{\dfrac{}{G \vdash G}\,\text{Ax} \qquad \dfrac{}{H \vdash H}\,Ax}{G -\!\!* H, G \vdash H}\,{-\!\!*L_I}}{(G -\!\!* H; F), G \vdash H}\,WkL}{(F \supset (G -\!\!* H); F), G \vdash H}\,{\supset L}}{F, G \vdash H}\,\texttt{Cut}}{F * G \vdash H}\,{*L}}{\top; F * G \vdash H}\,WkL}{\top \vdash F * G \supset H}\,{\supset R}$$

2. G is in Ξ.

$$\dfrac{\dfrac{\dfrac{\dfrac{\top \vdash F \supset (G -\!\!* H) \qquad \dfrac{\dfrac{\dfrac{}{F \vdash F}\,\text{Ax}}{(\{F\}, G)^H \vdash \{F\}}\,\mathbb{1}_{ps} \qquad \dfrac{\dfrac{\dfrac{\overline{\overline{G \vdash {}^{*}\!\top}} \qquad \dfrac{\dfrac{}{H \vdash H}\,\text{Ax}}{{}^{*}\!\top, H \vdash H}\,{}^{*}\!\top WkL}{H, G \vdash H}\,\texttt{Cut}}{G -\!\!* H, G \vdash H}\,{-\!\!*L_{{}^{*}\!\top}}}{(G -\!\!* H; F), G \vdash H}\,WkL}{(F \supset (G -\!\!* H); F), G \vdash H}}{F, G \vdash H}\,\texttt{Cut}}{F * G \vdash H}\,{*L}}{\top; F * G \vdash H}\,WkL}{\top \vdash F * G \supset H}\,{\supset R}$$

$-\!\!*_2$:

1. G is not in Ξ:

$$\dfrac{\dfrac{\dfrac{\dfrac{\top \vdash F * G \supset H \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{F \vdash F}\,Ax \qquad \dfrac{}{G \vdash G}\,Ax}{F, G \vdash F * G}\,{*R_I}}{F, G \vdash F * G; H}\,WkR \qquad \dfrac{\dfrac{}{H \vdash H}\,Ax}{H; (F, G) \vdash H}\,WkL}{F * G \supset H; (F, G) \vdash H}\,{\supset L}}{F, G \vdash H}\,\texttt{Cut}}{F \vdash G -\!\!* H}\,{-\!\!*R_I}}{\top; F \vdash G -\!\!* H}\,WkL}{\top \vdash F \supset (G -\!\!* H)}\,{\supset R}$$

2. G is in Ξ:

$$\dfrac{}{\mathbb{\Gamma}\{\mathbb{E}((\Gamma_1; p)^{(p;\Delta)})\} \vdash \{\mathbb{1}\}}\ id \qquad\qquad \dfrac{}{\mathbb{\Gamma}\{\mathbb{E}((\Gamma_1; {}^{*}\!\top)^{({}^{*}\!\top;\Delta_1)})\} \vdash \{\mathbb{1}\}}\ {}^{*}\!\top R$$

$$\dfrac{}{\mathbb{\Gamma}\{\Gamma_1; \mathbb{1}\} \vdash \{\Delta_1\}}\ \mathbb{1}L \qquad\qquad \dfrac{}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\top; \Delta_1\}}\ \top R$$

$$\dfrac{Re_1^{\Delta_1} \vdash \mathsf{S}^{+}(F_1 * G_1; \ldots; F_k * G_k) \qquad Re_2^{\Delta_2} \vdash \mathsf{S}^{-}(F_1 * G_1; \ldots; F_k * G_k)}{\mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(\Delta';F_1 * G_1;\ldots;F_k * G_k)})\} \vdash \{\mathbb{1}\}}\ {*}R_I$$

$$\dfrac{{}^{*}\!\top \vdash \mathsf{S}^{+}(F_1 * G_1; \ldots; F_k * G_k) \qquad \mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(\Delta_1;F_1 * G_1;\ldots;F_k * G_k)})\} \vdash \{\mathsf{S}^{-}(F_1 * G_1; \ldots; F_k * G_k)\}}{\mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(\Delta_1;F_1 * G_1;\ldots;F_k * G_k)})\} \vdash \{\mathbb{1}\}}\ {*}R_{{}^{*}\!\top}$$

$$\dfrac{\Gamma_1^{(\Delta_1;F_1 \mathrel{-\!\!*} F_2)}, F_1 \vdash F_2 \quad (\Xi \cap F_1 = \emptyset)}{\mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(\Delta_1;F_1 \mathrel{-\!\!*} F_2)})\} \vdash \{\mathbb{1}\}}\ {-\!\!*}R_I$$

$$\dfrac{Re_1^{\Delta_1} \vdash F \quad (\Xi \cap F = \emptyset) \qquad \mathbb{\Gamma}\{(Re_2^{\Delta_2}, G); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_2; F \mathrel{-\!\!*} G)^{\Delta_3}))\} \vdash \{\Delta'\}}{\mathbb{\Gamma}\{\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_2; F \mathrel{-\!\!*} G)^{\Delta_3})\} \vdash \{\Delta'\}}\ {-\!\!*}L_I$$

$$\dfrac{\mathbb{\Gamma}\{({}^{*}\!\top; \Gamma_1)^{\Delta}, ({}^{*}\!\top; \Gamma_1)^{\Delta}\} \vdash \{\Delta\}}{\mathbb{\Gamma}\{{}^{*}\!\top; \Gamma_1\} \vdash \{\Delta\}}\ {}^{*}\!\top CtrL$$

Figure 4.2: A set of α`LBBI`$_p$ inference rules to replace corresponding `LBBI`$_p$ inference rules with.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\phantom{{}^{*}\!\top \vdash G}}}{{}^{*}\!\top \vdash G} \qquad \dfrac{}{F \vdash F}\ \mathrm{Ax}}{F \vdash F * G}\ {*}R_C}{F \vdash F * G; H}\ WkR \qquad \dfrac{\dfrac{}{H \vdash H}\ \mathrm{Ax}}{H; F \vdash H}\ WkL}{}\ \supset L \\ \top \vdash F * G \supset H \qquad\qquad F * G \supset H; F \vdash H}{F \vdash H}\ \mathtt{Cut}}{F \vdash G \mathrel{-\!\!*} H}\ {-\!\!*}R_{{}^{*}\!\top}$$

$$\dfrac{\dfrac{F \vdash G \mathrel{-\!\!*} H}{\top; F \vdash G \mathrel{-\!\!*} H}\ WkL}{\top \vdash F \supset (G \mathrel{-\!\!*} H)}\ \supset R$$

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4.4 A Variant of `LBBI`$_p$: Absorption of Structural Rules

In this section, a variant of `LBBI`$_p$: α`LBBI`$_p$, is presented by adapting the earlier methodology within `BI` proof theory. This sequent calculus is both weakening-free and distribution-free in the sense of not permitting their phenomena to occur in any structural rules within α`LBBI`$_p$. It is also partially contraction-free in that the phenomenon of $CtrL$ does not occur in any structural rules.

$\alpha\mathtt{LBBI}_p$ derives from modifying the following inference rules in $\mathtt{LBBI}_p$.

- $Ax_{\,\mathtt{LBBI}_p}$: split into four rules: $id_{\alpha\mathtt{LBBI}_p}$, $\mathbb{1}L_{\alpha\mathtt{LBBI}_p}$, $\top R_{\alpha\mathtt{LBBI}_p}$ and $^*\!\top R_{\alpha\mathtt{LBBI}_p}$. $\mathbb{1}_{ps\,\mathtt{LBBI}_p}$, $WkL_{\mathtt{LBBI}_p}$ and $WkR_{\mathtt{LBBI}_p}$ are absorbed into these. In addition, $^*\!\top WkL$ is absorbed into $id_{\alpha\mathtt{LBBI}_p}$ and $^*\!\top R_{\alpha\mathtt{LBBI}_p}$.
- $*R_{I\,\mathtt{LBBI}_p}$: $*R_{I\,\alpha\mathtt{LBBI}_p}$ replaces. $\mathbb{1}_{ps\,\mathtt{LBBI}_p}$, $dR_{\mathtt{LBBI}_p}$, $WkL_{\mathtt{LBBI}_p}$, $WkR_{\mathtt{LBBI}_p}$, $^*\!\top WkL_{\mathtt{LBBI}_p}$, and $CtrR_{\mathtt{LBBI}_p}$ are absorbed.
- $*R_{^*\!\top\,\mathtt{LBBI}_p}$: $*R_{^*\!\top\,\alpha\mathtt{LBBI}_p}$ replaces. $dR_{\mathtt{LBBI}_p}$, $^*\!\top WkL_{\mathtt{LBBI}_p}$, $WkL_{\mathtt{LBBI}_p}$, $WkR_{\mathtt{LBBI}_p}$ and $CtrR_{\mathtt{LBBI}_p}$ are absorbed.
- $-\!\!*R_{I\,\mathtt{LBBI}_p}$: $-\!\!*R_{I\,\alpha\mathtt{LBBI}_p}$ replaces. $\mathbb{1}_{ps\,\mathtt{LBBI}_p}$, $CtrR_{\mathtt{LBBI}_p}$ and $^*\!\top WkL_{\mathtt{LBBI}_p}$ are absorbed.
- $-\!\!*L_{I\,\mathtt{LBBI}_p}$: $-\!\!*L_{I\,\alpha\mathtt{LBBI}_p}$ replaces. $WkL_{\mathtt{LBBI}_p}$, $WkR_{\mathtt{LBBI}_p}$, $^*\!\top WkL_{\mathtt{LBBI}_p}$ and $CtrL_{\mathtt{LBBI}_p}$ are absorbed.
- $^*\!\top CtrL_{\mathtt{LBBI}_p}$: $^*\!\top CtrL_{\alpha\mathtt{LBBI}_p}$ replaces. $CtrR_{\mathtt{LBBI}_p}$ is absorbed.

These rules are found in Figure 4.2.

Definition 79 ($\alpha\mathtt{LBBI}_p$) *$\alpha\mathtt{LBBI}_p$ comprises the following inference rules.*

Axioms: $\underline{id}$ $\underline{\top R}$ $\underline{\mathbb{1}L}$ $\underline{^*\!\top R}$

Other logical rules: $\wedge L$ $\wedge R$ $\vee L$ $\vee R$ $\supset L$ $\supset R$ $*\,L$

$\underline{*R_I}$ $\underline{*R_{^*\!\top}}$ $\underline{-\!\!*R_I}$ $-\!\!*R_{^*\!\top}$ $\underline{-\!\!*L_I}$ $-\!\!*L_{^*\!\top}$

Structural rule: $\underline{^*\!\top CtrL}$

each of which is identical to a corresponding inference rule in $\mathtt{LBBI}_p$ unless otherwise stated earlier (underlined in this definition for clarity). Collector Ξ holds a set of $\mathtt{BBI}$ *multiplicative theorems which are $\alpha\mathtt{LBBI}_p$-derivable.*

The rest of this section exhibits main properties of $\alpha\mathtt{LBBI}_p$: admissibility of $\mathbb{1}_{ps\,\mathtt{LBBI}_p}$; that of dR; that of $^*\!\top WkL_{\mathtt{LBBI}_p}$, $WkL_{\mathtt{LBBI}_p}$ and $WkR_{\mathtt{LBBI}_p}$; $\alpha\mathtt{LBBI}_p$ inversion lemma; admissibility of $CtrL_{\mathtt{LBBI}_p}$ and $CtrR_{\mathtt{LBBI}_p}$[1]; and the equivalence of $\alpha\mathtt{LBBI}_p$ to [$\mathtt{LBBI}_p$-$\mathtt{Cut}$]. The presentation style is kept closely aligned to that in Chapter 3 for ease of

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[1] Note for $CtrR_{\mathtt{LBBI}_p}$ that its phenomenon still occurs in the structural rule $^*\!\top CtrL$.

comparisons between `LBIZ` and $\alpha\texttt{LBBI}_p$. My hope is that such juxtapositions be an aid for further study into the intrinsic characters of structural interactions (*i.e.* syntactic phenomena of the base logic interactions) within those specific combined logics.

The essence of `BBI` structures; synthesis operators $\texttt{S}^{+}(\cdots)$ and $\texttt{S}^{-}(\cdots)$; and the correspondence between $Re^{\Delta_1}/Re^{\Delta_2}$ in the premise sequent(s) and negative structures in the conclusion sequent are first defined.

4.4.1 Essence, synthesis and $Re_1^{\Delta_1}/Re_2^{\Delta_2}$ pair

4.4.1.1 Essence of negative structures

As in `BI` proof theory (*Cf.* Chapter 3), ${}^{*}\top WkL$ is not totally disjoint from WkL. The concept of the 'essence' introduced back then is adjusted here for `BBI`.

Definition 80 (Essence of structures) *An essence of a negative structure* $\mathbb{\Gamma}_a \in \mathfrak{A}$ *is a negative structure* $\mathbb{E}(\mathbb{\Gamma}_a) \in \mathfrak{A}$*, satisfying the following for some context* $\mathbb{\Gamma}(-)$*:*

- $[D : \mathbb{\Gamma}(\mathbb{E}(\mathbb{\Gamma}_a)) \vdash \mathbb{1}] \rightsquigarrow^{*}_{{}^{*}\top WkL} [D' : \mathbb{\Gamma}(\mathbb{\Gamma}_a) \vdash \mathbb{1}]$
- *Both D and D' are an element of* $\mathfrak{D}_{\texttt{BBI}}$.

4.4.1.2 On the synthesis operators and the $Re_1^{\Delta_1}/Re_2^{\Delta_2}$ pair

As dR is no longer available in $\alpha\texttt{LBBI}_p$, what it achieves must be made possible within $\alpha\texttt{LBBI}_p$ logical inference rules. Further, it is noticeable that not only dR but $CtrR$ is unavailable in the calculus. The $\texttt{LBBI}_p$ synthesis operators need adjusted adequately, taking into account the effects of $CtrR$ and dR upon them.

Definition 81 ($\alpha\texttt{LBBI}_p$ synthesis operators)
Let D denote $\mathbb{\Gamma}\{\Gamma_1^{\Delta_1}\} \vdash \{F_1 * G_1; \cdots ; F_k * G_k\}$ *for* $k \geq 1$ *such that* $D \in \mathfrak{D}_{\texttt{BBI}}$. *Let* $D \rightsquigarrow_{syn} D'$ *denote the following transition on D:*

- $D \rightsquigarrow^{*}_{\{CtrR_{\texttt{LBBI}_p}, dR_{\texttt{LBBI}_p}\}} [D'' : \mathbb{\Gamma}\{\Gamma_1^{\Delta_1}\} \vdash \{H_1 * H_2; \Delta_x\}]$
- $D'' \rightsquigarrow^{*}_{WkR_{\texttt{LBBI}_p}} [D' : \mathbb{\Gamma}\{\Gamma_1^{\Delta_1}\} \vdash \{H_1 * H_2\}]$

such that $CtrR_{\texttt{LBBI}_p}$ *in* $D \rightsquigarrow^{*} D''$ *applies only to those* $F_i * G_i$ $(1 \leq i \leq k)$ *and/or any formula in the form* $F_x * G_x$ *that* $dR_{\texttt{LBBI}_p}$ *in the same transition may produce, that (2)*

$dR_{\texttt{LBBI}_p}$ takes place on those formulas, and that (3) all the $F_i * G_i$ are synthesised at least once. Then the pair $(\mathsf{S}^+(F_1 * G_1; \cdots; F_k * G_k), \mathsf{S}^-(F_1 * G_1; \cdots; F_k * G_k))$ is defined to be either (H_1, H_2) or (H_2, H_1).

The $Re_1^{\Delta_1}/Re_2^{\Delta_2}$ pair is also defined through a binding of $*R_I$ and $-\!\!*L_I$ to a corresponding $\texttt{LBBI}_p$-derivation.

Definition 82 **(**$Re_1^{\Delta_1}/Re_2^{\Delta_2}$ **in** $*R_I/-\!\!*L_I$**)** *In $\alpha\texttt{LBBI}_p$, correspondence of premise and conclusion sequents in $*R_I$ and $-\!\!*L_I$ are defined with respect to $*R_I/-\!\!*L_I/{}^{*}\!\top WkL$ $/WkL/WkR/CtrL/CtrR/dR/\mathbb{1}_{ps}$ in $\texttt{LBBI}_p$:*

For $*R_{I\alpha\texttt{LBBI}_p}$**:**

Let $D_1 : \mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(\Delta';F_1*G_1;\cdots;F_k*G_k)})\} \vdash \{\mathbb{1}\}$ be the conclusion sequent of the inference rule. Let F_x denote $F_1 * G_1; \cdots; F_k * G_k$. Then the corresponding derivation of $*R_{I\alpha\texttt{LBBI}_p}$ within $\texttt{LBBI}_p$ is defined to be

- $D \rightsquigarrow^{*}_{{}^{*}\!\top WkL_{\texttt{LBBI}_p}} [D_1' : \mathbb{\Gamma}\{\Gamma_1^{\Delta'}\} \vdash \{F_x)\}]$
- $D_1' \rightsquigarrow^{*}_{\mathbb{1}_{ps}\,\texttt{LBBI}_p} [D_1'' : \Gamma_1^{\Delta'} \vdash F_x]$
- $D_1'' \rightsquigarrow_{syn} [D_1''' : \Gamma_1^{\Delta'} \vdash \mathsf{S}^+(F_x) * \mathsf{S}^-(F_x)]$
- $D_1''' \rightsquigarrow^{*}_{\{WkR_{\texttt{LBBI}_p}, WkL_{\texttt{LBBI}_p}\}} [D_1'''' : Re_1^{\Delta_1}, Re_2^{\Delta_2} \vdash \mathsf{S}^+(F_x) * \mathsf{S}^-(F_x)]$
- $D_1'''' \rightsquigarrow_{*R_{\texttt{LBBI}_p}} [D_2 : Re_1^{\Delta_1} \vdash \mathsf{S}^+(F_x)]$
- $D_1'''' \rightsquigarrow_{*R_{\texttt{LBBI}_p}} [D_3 : Re_2^{\Delta_2} \vdash \mathsf{S}^-(F_x)]$

D_2 and D_3 correspond to the premise sequents of $*R_{I\alpha\texttt{LBBI}_p}$ (with D_1 as its conclusion sequent).

For $-\!\!*L_{I\;\alpha\texttt{LBBI}_p}$**:**

Let $D_1 : \mathbb{\Gamma}\{\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_2; F -\!\!* G)^{\Delta_3})\} \vdash \{\Delta'\}$ as the conclusion sequent of the inference rule. Then the corresponding derivation of $-\!\!*L_{I\;\alpha\texttt{LBBI}_p}$ within $\texttt{LBBI}_p$ is defined as below.

- $D_1 \rightsquigarrow_{CtrL_{\texttt{LBBI}_p}} [D_1' : \mathbb{\Gamma}\{(\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_2; F -\!\!* G)^{\Delta_3}));$
  $(\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_2; F -\!\!* G)^{\Delta_3}))\} \vdash \{\Delta'\}]$
- $D_1' \rightsquigarrow^{*}_{{}^{*}\!\top WkL} [D_1'' : \mathbb{\Gamma}\{(\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, (\Gamma_2; F -\!\!* G)^{\Delta_3});$
  $(\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_2; F -\!\!* G)^{\Delta_3}))\} \vdash \{\Delta'\}]$

- $D_1'' \rightsquigarrow^*_{\{WkR_{\texttt{LBBI}_p}, WkL_{\texttt{LBBI}_p}\}} [D_1''' : ℾ\{(Re_1^{\Delta_1}, Re_2^{\Delta_2}, F \mathbin{-\!\!*} G);$
  $(\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_2; F \mathbin{-\!\!*} G)^{\Delta_3}))\} \vdash \{\Delta'\}]$
- $D_1''' \rightsquigarrow_{\mathbin{-\!\!*} L_{I\ \texttt{LBBI}_p}} [D_2 : Re_1^{\Delta_1} \vdash F]$
- $D_1''' \rightsquigarrow_{\mathbin{-\!\!*} L_{I\ \texttt{LBBI}_p}} [D_3 : ℾ\{(Re_2^{\Delta_2}, G);(\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}(\Gamma_2; F \mathbin{-\!\!*} G))\} \vdash \{\Delta'\}]$

D_2 and D_3 correspond to the premise sequents of $\mathbin{-\!\!*} L_{I\ \alpha\texttt{LBBI}_p}$ (with D_1 as its conclusion sequent).

As we saw in Chapter 3, these rules internalise $\texttt{LBBI}_p$ backward derivation steps. Derivations of other new inference rules within $\texttt{LBBI}_p$ are similar and straightforward except possibly for $*R_{{*}\!\top\ \alpha\texttt{LBBI}_p}$ whose $\texttt{LBBI}_p$-binding is:

- Let F_x denote $F_1 * G_1; \cdots ; F_k * G_k$, let D denote the conclusion sequent of the inference rule, *i.e.* $D : ℾ\{\mathbb{E}(\Gamma_1^{(\Delta_1;F_x)})\} \vdash \{\mathbb{1}\}$. Also let Δ_y denote $\log \mathbb{E}(\Gamma_1^{(\Delta_1;F_x)})$.
- $D \rightsquigarrow_{CtrL} [D' : ℾ\{\mathsf{E}(\Gamma_1^{(\Delta_1;F_x)}); \mathsf{E}(\Gamma_1^{(\Delta_1;F_x)})\} \vdash \{\Delta_y\}$
- $D' \rightsquigarrow^*_{{*}\!\top WkL} [D'' : ℾ\{\Gamma_1; \mathsf{E}(\Gamma_1^{(\Delta_1;F_x)})\} \vdash \{\Delta_1; F_x; \Delta_y\}]$
- $D'' \rightsquigarrow_{syn} [D''' : ℾ\{\Gamma_1; \mathsf{E}(\Gamma_1^{(\Delta_1;F_x)})\} \vdash \{\Delta_1; \mathsf{S}^+(F_x) * \mathsf{S}^-(F_x); \Delta_y\}]$
- $D''' \rightsquigarrow^*_{\{WkL, WkR\}} [D'''' : ℾ\{\mathbb{E}(\Gamma_1^{(\Delta_1;F_x)})\} \vdash \{\mathsf{S}^+(F_x) * \mathsf{S}^-(F_x)\}]$
- $D'''' \rightsquigarrow_{*R_{{*}\!\top}} [D_2 : {*}\!\top \vdash \mathsf{S}^+(F_x)]$
- $D'''' \rightsquigarrow_{*R_{{*}\!\top}} [D_3 : ℾ\{\mathbb{E}(\Gamma_1^{(\Delta_1;F_x)})\} \vdash \{\mathsf{S}^-(F_x)\}]$

where D_2 and D_3 correspond to the premise sequents of the $\alpha\texttt{LBBI}_p$ inference rule.

The essence of structures does not appear in additive inference rules (axioms excluded) unlike `BI` (*Cf.* `LBIZ` in Chapter 3).

4.4.2 Main results

Main results about $\alpha\texttt{LBBI}_p$ are proved in the following order: admissibility of $\mathbb{1}_{ps}$; that of WkL, WkR and ${*}\!\top WkL$; $\alpha\texttt{LBBI}_p$ inversion lemma; admissibility of $CtrL$, $CtrR$ and dR; equivalence of $\alpha\texttt{LBBI}_p$ to [$\texttt{LBBI}_p$- `Cut`]. Most of the results (in this subsection) are dependent on earlier results (in this subsection). Admissibility results are depth-preserving.

4.4.2.1 Admissibility of $\mathbb{1}_{ps}$

The effect of $\mathbb{1}_{ps\,\texttt{LBBI}_p}$ is taken into account in the set of $\alpha\texttt{LBBI}_p$ inference rules.

Proposition 15 (Admissibility of $\mathbb{1}_{ps}$) *If a sequent $D : \Gamma^{\Delta} \vdash \mathbb{1}$ is $\alpha\texttt{LBBI}_p$-derivable, then so is $\mathbb{\Gamma}\{\Gamma\} \vdash \{\Delta\}$, preserving the derivation depth.*

Proof. By induction on derivation depth of $\Pi(D)$. For the base cases, we note that $id, \mathbb{1}L, \top R$ and $^*\!\top R$ all absorb the effect of $\mathbb{1}_{ps}$ within. But then inductive cases are trivial. □

4.4.2.2 Admissibility of weakening

Proposition 16 (Admissibility of $^*\!\top WkL$)
If a sequent $D : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1\}$ is $\alpha\texttt{LBBI}_p$-derivable, then so is $D' : \mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{\Delta_1})\} \vdash \{\mathbb{1}\}$, preserving the derivation depth.

Proof. In Appendix D. □

Proposition 17 (Admissibility of additive weakening)
If a sequent $D : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1\}$ is $\alpha\texttt{LBBI}_p$-derivable, then so is $D' : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2\}$, preserving the derivation depth.

Proof. In Appendix E. □

4.4.2.3 Inversion lemma

The following depth-preserving inversion lemma holds in $\alpha\texttt{LBBI}_p$.

Lemma 24 (Inversion lemma for $\alpha\texttt{LBBI}_p$) *For the following sequent pairs, if the sequent shown on the left is $\alpha\texttt{LBBI}_p$-derivable at most with the derivation depth of k,* *then the sequent(s) shown on the right are also $\alpha\mathtt{LBBI}_p$-derivable, preserving derivation depth.*

$$
\begin{aligned}
\mathbb{\Gamma}(F \wedge G) \vdash \mathbb{1} \quad &, \quad \mathbb{\Gamma}(F; G) \vdash \mathbb{1} && (4.1)\\
\mathbb{\Gamma}(F_1 \vee F_2) \vdash \mathbb{1} \quad &, \quad \textit{both}\, \mathbb{\Gamma}(F_1) \vdash \mathbb{1} && \\
&\quad\;\; \textit{and}\, \mathbb{\Gamma}(F_2) \vdash \mathbb{1} && (4.2)\\
\mathbb{\Gamma}\{\Gamma_1; F \supset G\} \vdash \{\Delta\} \quad &, \quad \textit{both}\, \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta; F\} && \\
&\quad\;\; \textit{and}\, \mathbb{\Gamma}\{\Gamma_1; G\} \vdash \{\Delta\} && (4.3)\\
\mathbb{\Gamma}(F * G) \vdash \mathbb{1} \quad &, \quad \mathbb{\Gamma}(F, G) \vdash \mathbb{1} && (4.4)\\
\mathbb{\Gamma}(F \mathrel{-\!\!*} G) \vdash \mathbb{1}(F \in \Xi) \quad &, \quad \mathbb{\Gamma}(G) \vdash \mathbb{1}(F \in \Xi) && (4.5)\\
\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta; F \wedge G\} \quad &, \quad \textit{both}\, \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta; F\} && \\
&\quad\;\; \textit{and}\, \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta; G\} && (4.6)\\
\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta; F \vee G\} \quad &, \quad \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta; F; G\} && (4.7)\\
\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta; F \supset G\} \quad &, \quad \mathbb{\Gamma}\{\Gamma_1; F\} \vdash \{\Delta; G\} && (4.8)\\
\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta; F \mathrel{-\!\!*} G\}(F \in \Xi) \quad &, \quad \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta; G\}(F \in \Xi) && (4.9)
\end{aligned}
$$

Proof. In Appendix F. □

4.4.2.4 Admissibility of contraction

Definition 83 (Incremental weakening for $*R_I/\mathrel{-\!\!*}L_I$) *Incremental internal weakening for $*R_I$ makes use only of the following weakening rule in the internalised weakening process:*

$$\frac{\mathbb{\Gamma}_1, \Gamma_2 \vdash F_1 * F_2}{\mathbb{\Gamma}_1, (\Gamma_2; \Gamma_3)^{\Delta_1} \vdash F_1 * F_2}\, Wk$$

Similarly, incremental internal weakening for $\mathrel{-\!\!*}L_I$ makes use only of the following weakening rules in the internal weakening process:

$$\frac{\mathbb{\Gamma}\{\Gamma_1^{\Delta_1}, F \mathrel{-\!\!*} G\} \vdash \{\Delta'\}}{\mathbb{\Gamma}\{\Gamma_1^{\Delta_1}, (\Gamma_2; F \mathrel{-\!\!*} G)^{\Delta_3}\} \vdash \{\Delta'\}}\, Wk_1'$$

$$\frac{\mathbb{\Gamma}\{\mathbb{\Gamma}_1, \Gamma_2, F{-\!\!*}G\} \vdash \{\Delta'\}}{\mathbb{\Gamma}\{\mathbb{\Gamma}_1, (\Gamma_2; \Gamma_3)^{\Delta_2}, F{-\!\!*}G\} \vdash \{\Delta'\}} Wk'_2$$

Lemma 25 (Maximal $Re_1^{\Delta_1}/Re_2^{\Delta_2}$) *Let F_x denote $(F_1 * G_1; \cdots; F_k * G_k)$ for some $k \geq 1$, then for a $\alpha\texttt{LBBI}_p$-derivable sequent $D : \mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(\Delta_1;F_x)})\} \vdash \{\mathbb{1}\}$, if there exists a pair of $\alpha\texttt{LBBI}_p$-derivable sequents $D'_1 : Re_{1'}^{\Delta'_1} \vdash \mathsf{S}^+(F_x)$ and $D'_2 : Re_{2'}^{\Delta'_2} \vdash \mathsf{S}^-(F_x)$ such that $D \rightsquigarrow_{*R_I} D'_1$ and $D \rightsquigarrow_{*R_I} D'_2$, then there exists a pair of $\alpha\texttt{LBBI}_p$-derivable sequents $D_1 : Re_1^{\Delta_1} \vdash \mathsf{S}^+(F_x)$ and $D_2 : Re_2^{\Delta_2} \vdash \mathsf{S}^-(F_x)$ such that all the following conditions satisfy.*

- *$D \rightsquigarrow_{*R_I} D_1$ (resp. $D \rightsquigarrow_{*R_I} D_2$) with incremental weakening (Definition 83).*
- *D_1 (resp. D_2) is a sequent that results from Proposition 17 on D'_1 (resp. D'_2).*[1]
- *there exists no $D_1^\star : Re_{1^\star}^{\Delta_1^\star} \vdash \mathsf{S}^+(F_x)$ (resp. $D_2^\star : Re_{2^\star}^{\Delta_2^\star} \vdash \mathsf{S}^-(F_x)$) such that all the following conditions satisfy.*
    - *$D_1^\star$ (resp. $D_2^\star$) is a sequent that results from Proposition 17 on D_1 (resp. D_2).*
    - *$Re_1^{\Delta_1} \not\succsim_{ant} Re_{1^\star}^{\Delta_1^\star}$ (resp. $Re_2^{\Delta_2} \not\succsim_{ant} Re_{2^\star}^{\Delta_2^\star}$).*
    - *$D \rightsquigarrow_{*R_I} D_1^\star$ (resp. $D \rightsquigarrow_{*R_I} D_2^\star$).*

Such a $Re_1^{\Delta_1}/Re_2^{\Delta_2}$ pair is called a maximal $Re_1^{\Delta_1}/Re_2^{\Delta_2}$ pair. Likewise, with Γ' denoting $(\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_2; F{-\!\!*}G)^{\Delta_3}))^{\mathbb{1}}$, if there exists a pair of $\alpha\texttt{LBBI}_p$-derivable sequents $D'_1 : Re_{1'}^{\Delta'_1} \vdash F$ and $D'_2 : \mathbb{\Gamma}\{(Re_{2'}^{\Delta'_2}, G); \Gamma'\}) \vdash \{\Delta'\}$ such that (1) F is not in Ξ, that (2) $D \rightsquigarrow_{-\!*L_I} D'_1$ and that (3) $D \rightsquigarrow_{-\!*L_I} D'_2$, then there exists a pair of $\alpha\texttt{LBBI}_p$-derivable sequents $D_1 : Re_1^{\Delta_1} \vdash F$ and $D_2 : \mathbb{\Gamma}\{(Re_2^{\Delta_2}, G); \Gamma'\} \vdash \{\Delta'\}$ such that the following conditions all satisfy.

- *$D \rightsquigarrow_{-\!*L_I} D_1$ (resp. $D \rightsquigarrow_{-\!*L_I} D_2$) with incremental internal weakening.*
- *D_1 (resp. D_2) is a sequent that results from Proposition 17 on D'_1 (resp. D'_2).*

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[1] That is, there is a transition $D_i \rightsquigarrow^* D'_i$ in $\texttt{LBBI}_p$-space with $\texttt{LBBI}_p$-additive-weakening applications.

- *there exists no* $D_1^\star : Re_{1^\star}^{\Delta_1^\star} \vdash F$ *(resp.* $D_2^\star : \Gamma\{(Re_{2^\star}^{\Delta_2^\star}, G); \Gamma'\} \vdash \{\Delta'\}$*) such that the following conditions all satisfy.*
  - $D_1^\star$ *(resp.* $D_2^\star$*) is a sequent that results from Proposition 17 on* D_1 *(resp.* D_2*).*
  - $Re_{1^\star}^{\Delta_1^\star} \not\preceq_{ant} Re_1^{\Delta_1}$ *(resp.* $\Gamma(((Re_{2^\star}^{\Delta_2^\star}, G); \Gamma')^{\Delta'}) \not\preceq_{ant} \Gamma(((Re_2^{\Delta_2}, G); \Gamma')^{\Delta'})$*).*
  - $D \rightsquigarrow_{-*L} D_1^\star$ *(resp.* $D \rightsquigarrow_{-*L} D_2^\star$*).*

Proof. In Appendix G. □

Proposition 18 (α`LBBI`$_p$ **contraction admissibility)** *If a sequent* $D : \Gamma\{\Gamma_1; \Gamma_2; \Gamma_2\} \vdash \{\Delta_1; H; H\}$ *is* α`LBBI`$_p$*-derivable, then so is* $D' : \Gamma\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; H\}$*, preserving the derivation depth.*[1]

Proof. In Appendix H. □

Proposition 19 (Admissibility of dR**)** *If a sequent* $\Gamma\{\Gamma_1\} \vdash \{\Delta_1; \mathsf{S}_\wedge(F_1 \times F_2) * \mathsf{S}_\vee(G_1 \times G_2)\}$ *is* α`LBBI`$_p$*-derivable, then so is* $\Gamma\{\Gamma_1\} \vdash \{\Delta_1; F_1 * G_1; F_2 * G_2\}$*, preserving the derivation depth.*

Proof. By induction on derivation depth of $\Pi(D)$. Trivial for the base cases. Also trivial for all the left inference rules by induction hypothesis on the premise sequents. For the right inference rules, neither $F_1 * G_1$ nor $F_2 * G_2$ can become the principal for any inference rules other than $*R_I$ or $*R_{*\top}$. But these rules already absorb the effect within. Induction hypothesis for the rest. □

---

[1] It may appear more rigorous if the statement is divided into two cases, *i.e.* into the left additive contraction and the right additive contraction (with the proof via a simultaneous induction). However, as the depth-preserving additive weakening admissibility has been proved, they can be handled at once without a loss of generality.

4.4.2.5 Equivalence of α`LBBI`$_p$ to `LBBI`$_p$ - `Cut`

For the proof, we make use of the formula depth along with derivation depth.

Definition 84 (Formula depth) *The depth of a formula F, $\texttt{f\_depth}(F)$, is defined as follows. It is 1 if $F = \{\mathbb{1}, \top, {}^{*}\!\top, p\}$; is $max(\texttt{f\_depth}(F_1), \texttt{f\_depth}(F_2)) + 1$ if $F = F_1 \cdot F_2$ for $\cdot \in \{\wedge, \vee, \supset, *, -\!\!*\}$.*

Proposition 20 (Equivalence of α`LBBI`$_p$ with `LBBI`$_p$- `Cut`)
$D : \Gamma \vdash \mathbb{1}$ is α`LBBI`$_p$-derivable with the collector $\Xi_{\alpha\texttt{LBBI}_p}$ containing all the `BBI` multiplicative theorems derivable in α`LBBI`$_p$ iff it is [`LBBI`$_p$- `Cut`]-derivable with the collector $\Xi_{\texttt{LBBI}_p}$ containing all the `BBI` multiplicative theorems derivable in `LBBI`$_p$- `Cut`.

Proof. In Appendix I. □

4.5 Adaptation to Separation Logic

Thanks to a theoretical result by Larchey-Wendling and Galmiche [2012], adaptation of `BBI` sequent calculi into separation logic sequent calculi come into an easy reach. One characteristic semantic difference exists in the condition of disjointness in the composition of elements of `Heap`, however; in the heap semantics, a composition of two elements is defined only if there is no overlap between them. To capture the condition, I define the following axiom:

$$\frac{}{Ex_1 \mapsto Ex_2, Ex_1 \mapsto Ex_3 \vdash \Delta}\ \mathbb{1}_{overlap}$$

The soundness of $\mathbb{1}_{overlap}$ is immediate. Then a sound separation logic sequent calculus results by adding it to `LBBI`$_p$ such that (1) the resultant calculus takes separation logic formulas instead of `BBI` formulas and that (2) structures to appear in the calculus comprise separation logic formulas instead of `BBI` formulas.

Similar holds true for α`LBBI`$_p$ in that to use it for separation logic, it suffices to add to it the following inference rule:

$$\frac{}{\mathbb{\Gamma}(\mathbb{E}((Ex_1 \mapsto Ex_2; \Gamma_1)^{\Delta_1}), \mathbb{E}((Ex_1 \mapsto Ex_3; \Gamma_2)^{\Delta_2})) \vdash \mathbb{1}}\ \mathbb{1}_{overlap}$$

in addition to the same two changes in the case of $\mathtt{LBBI}_p$. Soundness of the resultant separation logic calculus is again immediate due to the relation between `BBI` semantics and separation logic semantics (*Cf.* Chapter 1).

4.6 On Cut Eliminations

Whether `Cut` is admissible in [$\alpha\mathtt{LBBI}_p$+ `Cut`] (which is equivalent to whether it is admissible in $\mathtt{LBBI}_p$) is, however, not very straightforward, particularly around $-\!\!*R_I$. Nonetheless, cut elimination succeeds for a conservative fragment of [$\alpha\mathtt{LBBI}_p$+ `Cut`]: $[\alpha\mathtt{LBBI}_p + \mathtt{Cut}]^-$, with the following changes:

1. $-\!\!*R_{*\top}$: use instead:

$$\frac{\Gamma_1^{\Delta_1}, F_1 \vdash F_2}{\Gamma_1 \vdash F_1 -\!\!* F_2; \Delta_1}\ -\!\!*R_I$$

2. $*R_{*\top}$: use instead:

$$\frac{{}^{*}\!\top \vdash F_1 \qquad \mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(\Delta_1;G_1)})\} \vdash \{G_1\}}{\mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(\Delta_1;F_1 * G_1)})\} \vdash \{\mathbb{1}\}}\ *R_{*\top}$$

3. $*R_I$: use instead:

$$\frac{Re_1^{\Delta_1} \vdash F_1 \qquad Re_2^{\Delta_2} \vdash G_1}{\mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(\Delta_1;F_1 * G_1)})\} \vdash \{\mathbb{1}\}}\ *R_I$$

4. The following collector Ξ is used for $[\alpha\mathtt{LBBI}_p + \mathtt{Cut}]^-$: Ξ holds `BBI` multiplicative theorems that are $[\alpha\mathtt{LBBI}_p + \mathtt{Cut}]^-$ derivable.

Proposition 21 (Cut elimination procedure) `Cut` *is admissible in* $[\alpha\mathtt{LBBI}_p + \mathtt{Cut}]^-$.

Proof. Found in Appendix J. □

$$\dfrac{}{p \vdash p}\,id \qquad \dfrac{\mathbb{X} \vdash F \quad F \vdash \mathbb{Y}}{\mathbb{X} \vdash \mathbb{Y}}\,\texttt{Cut'} \qquad \dfrac{}{\mathbb{1} \vdash \mathbb{Y}}\,\mathbb{1}L \qquad \dfrac{}{\mathbb{X} \vdash \top}\,\top R$$

$$\dfrac{}{\emptyset_m \vdash {}^{*}\!\top}\,{}^{*}\!\top R \qquad \dfrac{\emptyset_a \vdash \mathbb{Y}}{\top \vdash \mathbb{Y}}\,\top L \qquad \dfrac{\sharp F \vdash \mathbb{Y}}{\neg F \vdash \mathbb{Y}}\,\neg L \qquad \dfrac{F ; G \vdash \mathbb{Y}}{F \wedge G \vdash \mathbb{Y}}\,\wedge L$$

$$\dfrac{F \vdash \mathbb{Y} \quad G \vdash \mathbb{Y}}{F \vee G \vdash \mathbb{Y}}\,\vee L \qquad \dfrac{\mathbb{X} \vdash F \quad G \vdash \mathbb{Y}}{F \supset G \vdash \sharp \mathbb{X} ; \mathbb{Y}}\,\supset L \qquad \dfrac{\emptyset_m \vdash \mathbb{Y}}{{}^{*}\!\top \vdash \mathbb{Y}}\,{}^{*}\!\top L \qquad \dfrac{F , G \vdash \mathbb{Y}}{F * G \vdash \mathbb{Y}}\,*L$$

$$\dfrac{\mathbb{X} \vdash F \quad G \vdash \mathbb{Y}}{F \mathrel{-\!\!*} G \vdash \mathbb{X} \mathrel{-\!\!\circ} \mathbb{Y}}\,\mathrel{-\!\!*}L \qquad \dfrac{\mathbb{X} \vdash \sharp F}{\mathbb{X} \vdash \neg F}\,\neg R \qquad \dfrac{\mathbb{X} \vdash F \quad \mathbb{X} \vdash G}{\mathbb{X} \vdash F \wedge G}\,\wedge R \qquad \dfrac{\mathbb{X} \vdash F ; G}{\mathbb{X} \vdash F \vee G}\,\vee R$$

$$\dfrac{\mathbb{X} ; F \vdash G}{\mathbb{X} \vdash F \supset G}\,\supset R \qquad \dfrac{\mathbb{X}_1 \vdash F \quad \mathbb{X}_2 \vdash G}{\mathbb{X}_1 , \mathbb{X}_2 \vdash F * G}\,*R \qquad \dfrac{\mathbb{X} , F \vdash G}{\mathbb{X} \vdash F \mathrel{-\!\!*} G}\,\mathrel{-\!\!*}R \qquad \dfrac{\mathbb{X}_1 \vdash \mathbb{Y}}{\mathbb{X}_1 ; \mathbb{X}_2 \vdash \mathbb{Y}}\,WkL$$

$$\dfrac{\mathbb{X} \vdash \mathbb{Y}_1}{\mathbb{X} \vdash \mathbb{Y}_1 ; \mathbb{Y}_2}\,WkR \qquad \dfrac{\mathbb{X} ; \mathbb{X} \vdash \mathbb{Y}}{\mathbb{X} \vdash \mathbb{Y}}\,CtrL \qquad \dfrac{\mathbb{X} \vdash \mathbb{Y} ; \mathbb{Y}}{\mathbb{X} \vdash \mathbb{Y}}\,CtrR \qquad \begin{array}{c} \emptyset_a ; \mathbb{X} \vdash \mathbb{Y} \\ \hdashline \mathbb{X} \vdash \mathbb{Y} \end{array}\,EA_1L$$

$$\begin{array}{c} \mathbb{X} \vdash \mathbb{Y} ; \emptyset_a \\ \hdashline \mathbb{X} \vdash \mathbb{Y} \end{array}\,EA_1R \qquad \begin{array}{c} \emptyset_m , \mathbb{X} \vdash \mathbb{Y} \\ \hdashline \mathbb{X} \vdash \mathbb{Y} \end{array}\,EA_2L \qquad \begin{array}{c} \mathbb{X}_1 ; \mathbb{X}_2 \vdash \mathbb{Y} \\ \hdashline \mathbb{X}_1 \vdash \sharp \mathbb{X}_2 ; \mathbb{Y} \end{array}\,DP_1 \qquad \begin{array}{c} \mathbb{X}_1 ; \sharp\sharp \mathbb{X}_2 \vdash \mathbb{Y} \\ \hdashline \mathbb{X}_1 ; \mathbb{X}_2 \vdash \mathbb{Y} \end{array}\,DP_2$$

$$\begin{array}{c} \mathbb{X} \vdash \mathbb{Y}_1 ; \mathbb{Y}_2 \\ \hdashline \mathbb{X} ; \sharp \mathbb{Y}_1 \vdash \mathbb{Y}_2 \end{array}\,DP_3 \qquad \begin{array}{c} \mathbb{X} \vdash \mathbb{Y}_1 ; \sharp\sharp \mathbb{Y}_2 \\ \hdashline \mathbb{X} \vdash \mathbb{Y}_1 ; \mathbb{Y}_2 \end{array}\,DP_4 \qquad \begin{array}{c} \mathbb{X}_1 , \mathbb{X}_2 \vdash \mathbb{Y} \\ \hdashline \mathbb{X}_1 \vdash \mathbb{X}_2 \mathrel{-\!\!\circ} \mathbb{Y} \end{array}\,DP_5$$

Figure 4.3: $\mathtt{DL_{BBI}}$: a display calculus for `BBI` by Brotherston [2012]. DP_1, DP_2, DP_3, DP_4 and DP_5 are commonly termed display postulates.

4.7 Conclusion and Related Work

Here is a summary of contributions in this chapter:

1. Development of `BBI` sequent calculus conventions and `BBI` sequent calculi through analysis on the implication of the underlying semantics onto syntax.
2. Identification of a conservative `Cut`-eliminable class of `BBI` sequent calculi with the first detailed cut elimination procedure known so far as a foothold for further research into this direction.

4.7.1 Related work

Known results related closest are ones by Brotherston [2012]; Park et al. [2013]. Brotherston introduced a `BBI` display calculus $\mathtt{DL_{BBI}}$, as shown in Figure 4.3.

Definition 85 **($\mathtt{DL_{BBI}}$ structures)** *A* $\mathtt{DL_{BBI}}$ *negative structure* $\mathbb{X}$ *and a* $\mathtt{DL_{BBI}}$ *positive structure* $\mathbb{Y}$ *are mutually defined by the following grammars:*

$\mathbb{X} := F \mid \varnothing_a \mid \varnothing_m \mid \mathbb{X};\mathbb{X} \mid \mathbb{X},\mathbb{X} \mid \sharp\mathbb{Y}.$

$\mathbb{Y} := F \mid \varnothing_a \mid \mathbb{Y};\mathbb{Y} \mid \sharp\mathbb{X} \mid \mathbb{X}\multimap\mathbb{Y}.$

The set of $\mathtt{DL_{BBI}}$ *negative structures is denoted by* $\mathfrak{N}_{\mathtt{DL_{BBI}}}$*, whereas that of* $\mathtt{DL_{BBI}}$ *positive structures by* $\mathfrak{P}_{\mathtt{DL_{BBI}}}$.

Definition 86 **($\mathtt{DL_{BBI}}$ sequents)** *The set of* $\mathtt{DL_{BBI}}$ *sequents is defined by:*

$\mathfrak{D}_{\mathtt{DL_{BBI}}} := \{\mathbb{X} \vdash \mathbb{Y} \mid [\mathbb{X} \in \mathfrak{N}_{\mathtt{DL_{BBI}}}] \wedge^{\dagger} [\mathbb{Y} \in \mathfrak{P}_{\mathtt{DL_{BBI}}}]\}.$

$\neg$ and $\sharp$ are both primitive in $\mathtt{DL_{BBI}}$, but I treat $\neg F$ as an abbreviation of $F \supset \mathbb{1}$, and $\sharp\mathbb{X}$ (resp. $\sharp\mathbb{Y}$) as an abbreviation of $\mathbb{X} \Rightarrow \mathbb{1}$ (resp. $\mathbb{Y} \Rightarrow \mathbb{1}$) where $\Rightarrow \mathbb{1}$ is what represents $\sharp$. `Cut'` is admissible in $\mathtt{DL_{BBI}}$ (*Cf.* Brotherston [2012]). Introduction of both $\sharp$ and $\multimap$ in a backward proof search must be allowed, however. To show that an upward introduction of $\sharp$ is necessary, $p; p \supset q \vdash q$ may be used. To show that an upward introduction of $\multimap$ is necessary, $p, p{-\!\!*}q \vdash q$ may be used.

For a more scalable proof search, some de-displaying seems to have proved useful (*Cf.* Goré et al. [2009]). Park et al. [2013] borrowed the idea of nested sequents (*Cf.* Kashima [1994]) and envisaged an optimisation of $\mathtt{DL_{BBI}}$ into $\mathtt{S_{BBI}}$ by the following main modifications:

1. The presence of $\multimap$ on the consequent is not permitted in $\mathtt{S_{BBI}}$, which can only appear on the antecedent. The following $\mathtt{DL_{BBI}}$ derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathbb{X}_1; \sharp(\mathbb{X}_2\multimap(F_1;\cdots;F_k)) \vdash \varnothing_a}{\mathbb{X}_1 \vdash \sharp\sharp(\mathbb{X}_2\multimap(F_1;\cdots;F_k)); \varnothing_a}\,DP_1}{\mathbb{X}_1 \vdash \sharp\sharp(\mathbb{X}_2\multimap(F_1;\cdots;F_k))}\,EA_1R}{\mathbb{X}_1 \vdash \mathbb{X}_2\multimap(F_1;\cdots;F_k)}\,DP_4}{\mathbb{X}_1, \mathbb{X}_2 \vdash F_1;\cdots;F_k}\,DP_5$$

are compiled as one step derivation in $\mathtt{S_{BBI}}$:

$$\dfrac{\mathbb{X}_1; (\mathbb{X}_2 \vdash \varnothing_a) \ll \varnothing_a \vdash F_1;\cdots;F_k \gg\vdash \varnothing_a}{\mathbb{X}_1, \mathbb{X}_2 \vdash F_1;\cdots;F_k}\,TC_S$$

One characteristic of the calculus $\mathtt{S_{BBI}}$ is that nesting of sequents within a sequent is permitted, as clear from the above derivation step. $(\mathbb{X}_2 \vdash \varnothing_a) \ll \varnothing_a \vdash F_1;\cdots;F_k \gg$ corresponds to $\sharp(\mathbb{X}_2\multimap(F_1;\cdots;F_k))$ in the first derivation.

2. No explicit use of DL_{1-4} occurs in a $\mathtt{S_{BBI}}$ derivation.

Since several $\mathtt{DL_{BBI}}$ derivation steps are identified redundant (as shown above) within $\mathtt{S_{BBI}}$, “proof searches in $\mathtt{S_{BBI}}$ are always simpler than in $\mathtt{DL_{BBI}}$ (except in trivial cases)” (a quotation from Park et al. [2013]).

To make an observation on cut admissibility and analyticity, they state that $\mathtt{S_{BBI}}$ is sound and complete with respect to the underlying `BBI` semantics, and that `Cut`” (a rule of transitivity in $\mathtt{S_{BBI}}$) in the proof system is admissible. A few remarks could be useful, nonetheless. First for analyticity, introductions of a $\mathtt{S_{BBI}}$ structural connective internalising both $\multimap$ and $\sharp$ are still permitted to occur in $\mathtt{S_{BBI}}$, which, in conjunction with available contractions, leads to an infinite introduction of new distinct constructs. Again the new structural connective $(\cdots) \ll \cdots \gg$ can be shown neither admissible nor implied eliminable from admissibility of `Cut`”. The use of multiplicative-implication-like display-like postulates to simulate not only the multiplicative implication but also the semantic peculiarity of the multiplicative conjunction and of the multiplicative unit makes the identification of maximally analytical `BBI` fragment a taxing process.

Proposition 22 (Couplings of multiplicative connectives in $\mathtt{DL_{BBI}}$ and $\mathtt{S_{BBI}}$) *The structural connective for the multiplicative implication available within* $\mathtt{DL_{BBI}}$ *and* $\mathtt{S_{BBI}}$ *is expressive enough to simulate the semantics of the multiplicative conjunction and that of the multiplicative unit.*

Proof. For the multiplicative conjunction, use the following $\mathtt{DL_{BBI}}$ sequent Park et al. [2013]: $p \vdash (p * q); (p * \neg q)$. Its $\mathtt{DL_{BBI}}$-derivation is:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$q \vdash q$}
\RightLabel{WkL}
\UnaryInfC{$\varnothing_m; q \vdash q$}
\doubleLine
\RightLabel{$\{DP_1, DP_3\}$}
\UnaryInfC{$\varnothing_m; \sharp q \vdash \sharp q$}
\RightLabel{$\neg R$}
\UnaryInfC{$\varnothing_m; \sharp q \vdash \neg q$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p \vdash p$}
\RightLabel{$*R$}
\BinaryInfC{$p, (\varnothing_m; \sharp q) \vdash p * \neg q$}
\RightLabel{WkR}
\UnaryInfC{$p, (\varnothing_m; \sharp q) \vdash (p * q); (p * \neg q)$}
\doubleLine
\RightLabel{$\{DP_1, DP_3, DP_4, DP_5\}$}
\UnaryInfC{$\varnothing_m; \sharp(p \multimap ((p * q); (p * \neg q))) \vdash q$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p \vdash p$}
\RightLabel{$*R$}
\BinaryInfC{$p, (\varnothing_m; \sharp(p \multimap ((p * q); (p * \neg q)))) \vdash p * q$}
\RightLabel{WkR}
\UnaryInfC{$p, (\varnothing_m; \sharp(p \multimap ((p * q); (p * \neg q)))) \vdash (p * q); (p * \neg q)$}
\doubleLine
\RightLabel{$\{DP_1, DP_4, DP_5\}$}
\UnaryInfC{$\varnothing_m \vdash p \multimap ((p * q); (p * \neg q))$}
\RightLabel{DP_5}
\UnaryInfC{$\varnothing_m, p \vdash (p * q); (p * \neg q)$}
\RightLabel{EA_2L}
\UnaryInfC{$p \vdash (p * q); (p * \neg q)$}
\end{prooftree}
$$

The use of DP_5 is necessary to derive the given sequent, even though there appears no $-\!\!*$ within the sequent.

For the multiplicative unit, use the following sequent: $\varnothing_m; p \vdash p * p$. Its $\mathtt{DL}_{\mathtt{BBI}}$-derivation is as follows:

$$
\begin{prooftree}
\AxiomC{}
\doubleLine
\UnaryInfC{$\varnothing_m; p \vdash p$}
\AxiomC{}
\doubleLine
\UnaryInfC{$\sharp p \vdash \sharp p$}
\doubleLine
\RightLabel{$\{WkL, WkR, EA_2L\}$}
\UnaryInfC{$(\varnothing_m; \sharp p), (\varnothing_m; p) \vdash p * p; \sharp p$}
\doubleLine
\RightLabel{$DP_{1,3,4,5}$}
\UnaryInfC{$\varnothing_m; \sharp((\varnothing_m; p) \multimap (p * p; \sharp p)) \vdash p$}
\doubleLine
\RightLabel{$\{*R, WkR\}$}
\BinaryInfC{$(\varnothing_m; \sharp((\varnothing_m; p) \multimap (p * p; \sharp p))), (\varnothing_m; p) \vdash p * p; \sharp p$}
\doubleLine
\RightLabel{$\{DP_{1,4,5}, CtrR\}$}
\UnaryInfC{$\varnothing_m, (\varnothing_m; p) \vdash p * p; \sharp p$}
\RightLabel{EA_2L}
\UnaryInfC{$\varnothing_m; p \vdash p * p; \sharp p$}
\doubleLine
\RightLabel{$\{DP_2, DP_3, CtrL\}$}
\UnaryInfC{$\varnothing_m; p \vdash p * p$}
\end{prooftree}
$$

The use of DP_5 is again necessary to derive the given sequent despite the absense of $-\!\!*$ in the sequent. □

For the $\mathtt{S}_{\mathtt{BBI}}$ cut elimination which is stated holding in Park et al. [2013], it may be indeed holding; its proof may not, however, follow from (1) equivalence of $\mathtt{S}_{\mathtt{BBI}}$ to $\mathtt{DL}_{\mathtt{BBI}}$ and (2) `Cut`' admissibility in $\mathtt{DL}_{\mathtt{BBI}}$: $\mathtt{S}_{\mathtt{BBI}}$ does not satisfy Belnap's conditions (*Cf.* Belnap [1982]) and `Cut`" is not the same as `Cut`' in $\mathtt{DL}_{\mathtt{BBI}}$.

In Park et al. [2013], it is also shown how to absorb contraction inference rules in the calculus into the other logical inference rules to derive a variant $\mathtt{CS}_{\mathtt{BBI}}$ for a more efficient proof search than possible within $\mathtt{S}_{\mathtt{BBI}}$. The absorption spills out into structural rules, nonetheless, and elimination of structural contraction remains a future work.

4.7.2 Conclusion

In this chapter, a groundwork was taken towards a delivery of a `BBI` sequent calculus. Great cares were taken to ensure a closer semantic-syntax correspondence than previously envisaged such as by Brotherston [2012]; Park et al. [2013]. A sound and analytic `BBI` proof system $\alpha\mathtt{LBBI}_p$ was derived, which aimed for a complementary study to theirs towards a mutual goal of an efficient `BBI` theorem proving. Instead of starting analysis on the highly expressive mechanism with display postulates, taming their expressiveness wherever redundancies are witnessed (Park et al. [2013]), I began a search for analytic fragments of `BBI` proof systems, carrying out a semantic examination primarily on the following: (1) a partial distributivity between the additive-multiplicative base-logics and (2) the significance of the non-intuitionistic multiplicative unit in Logic `BBI` which is more commonly considered as a part of the multiplicative intuitionistic linear logic (which behaves intuitionistically). If we consider the work by Park et al. [2013] as a top-down approach, the approach taken in this chapter is then bottom-up. For (1), it was reported that a multiplicative conjunct $(F_1 * F_2)$ may exhibit certain coupling effect with other multiplicative conjuncts. This phenomenon was treated in the special distributivity rule dR. Consideration over its implication upon analyticity and `Cut` was then taken. One advantage of $\texttt{LBBI}_p$ over $\texttt{DL}_{\texttt{BBI}}$ and $\texttt{S}_{\texttt{BBI}}$ draws from the absence of certain inconvenience in Brotherston [2012]; Park et al. [2013] where the semantic peculiarity of the multiplicative conjunction and of the multiplicative unit has to be simulated with the multiplicative implication ($-\!\!\circ$, $(\cdots) \ll \cdots \gg$; *Cf.* Proposition 22). $\texttt{LBBI}_p$ will be able to handle the fragment of `BBI` without the multiplicative implication by simply dropping the inference rules for the connective, to contrast. For the multiplicative unit, $\texttt{LBBI}_p$ strictly treats it as a Boolean component with dedicated inference rules. Further studies into semantics-syntax correspondences, I here believe, should unfold a more precise picture about how the multiplicative unit behaves within `BBI` sequent calculus. From $\texttt{LBBI}_p$, a variant $\alpha\texttt{LBBI}_p$ was derived that absorbed the effects of $\texttt{LBBI}_p$ structural rules (partial absorption for contraction). Admissibility of `Cut` was then stated for a conservative $\alpha\texttt{LBBI}_p$ with a cut elimination procedure. It is hoped that these `BBI` sequent calculi become the starting point for a set of work to follow into bottom-up `BBI` sequent calculus derivation, just as $\texttt{DL}_{\texttt{BBI}}$ was for $\texttt{S}_{\texttt{BBI}}$. The technique that Ciabattoni et al. [2008] showed in order to extract structural inference rules from `Cut` may be one useful approach to consider.

Chapter 5

Phased Sequent Calculus

Complex analysis is often anticipated for expressive logics, as we saw in the previous chapters. For instance, in Chapter 3 as part of `LBIZ` development, deep absorption of `LBI` weakening into `LBI` logical inference rules was proposed, which did what was required of in the proof of contraction admissibility. Nevertheless, the Re_1/Re_2 pair is somewhat mystified, since it may not be any substructure but rather a collection of some substructures that occur in the conclusion sequent. Further, the internalisation of a number of `LBI` operations within a `LBIZ` logical rule does not fail to obscure the manner by which base logic interactions take place, nor does it bespeak the rationale behind in a perspicuous manner. But the farther we step away from those logics, the more apparent becomes the following more fundamental inquiry: "First of all, how were intuitionistic logic and multiplicative intuitionistic linear logic combined?" `LBIZ` retains almost no trace of the origin of the logical combination. From the inference rules of `LBIZ`, it cannot be made entirely certain, either, if `BI` is the combined logic of the base logics or a combined logic of the two. In this chapter, I consider a general problem of formulation of a combined logic within sequent calculus, which gives rise to the concept of phased sequent calculus as one that allows development of a manner in which base-logics interact, of the origin of the logical combination itself.

5.1 Phased Sequent Calculus: Sequent Calculus to Derive Logical Combinations

Phased sequent calculus is a sequent calculus that externally observes a set of base logics. Put in another way, it is a sequent calculus as a meta sequent calculus. Base logics of a combined logic form base components in phased sequent calculus of the combined logic. Those base components each have their own derivability relation so that, if we have a combined logic of two base logics, one of the base components has a derivability relation $\vdash_1$, while the other has a derivability relation $\vdash_2$. This first consideration is akin to the one found in Schechter [2011] or else in Cruz-fillipe and Sernadas [2005] to physically isolate base logics. This achieves the intended effect of mine where none of the base components can be conscious of any others. In fact there is a physical separation between any two of them. Apart from the base components, there can - though does not have to - be meta-base-component components called communication components. Each communication component, defined over two base components, can *observe* both components in a particular manner, and in the particular observation is a particular logical combination. I say "particular" because exactly what the observation is remains unknown until it is determined. This achieves the intended effect of mine that base logics combine when they are recognised at once somehow. Therefore, unlike orthodox sequent calculi, the phased sequent calculus consists of two basic building-blocks: two or more base components and zero or more communication components. As a fundamental principle of the formalism, a base component does not spontaneously communicate with other base components (as stated above). Suppose that we are conscious of two propositional logics[1] $\mathfrak{L}_A = \mathfrak{L}(\mathcal{P}, \mathcal{C}_A, \mathbf{Inf}_A)$ and $\mathfrak{L}_B = \mathfrak{L}(\mathcal{P}, \mathcal{C}_B, \mathbf{Inf}_B)$ with propositional variables $\mathcal{P}$, logical connectives $\mathcal{C}_A$ and respectively $\mathcal{C}_B$, and a set of inference rules $\mathbf{Inf}_A$ and respectively $\mathbf{Inf}_B$. A phased sequent calculus for a combined logic $\mathfrak{L}_C$ of $\mathfrak{L}_A$ and $\mathfrak{L}_B$ with two base components representative of $\mathfrak{L}_A$ and $\mathfrak{L}_B$ but without a communication component will, as another principle of the formalism, recognise all the formulas built from propositional variables and $\mathcal{C}_A \bigcup \mathcal{C}_B$. Inferences on formulas themselves, however, can take place only within a base component, which implies, in the absence of communication facilitated between

---

[1] It does not matter if a logic is propositional or otherwise, nor does it matter if the number of logics to be combined is two or greater; but I only talk about two propositional logics here to provide intuition.

the two, that, while a set of formulas in $\mathfrak{L}_C$ becomes generally larger than that recognised in $\mathfrak{L}_A$ or in $\mathfrak{L}_B$, what $\mathfrak{L}_A$ (resp. $\mathfrak{L}_B$) may handle are only those connectives that it recognises. This is defined to be the most basic combined logic derivable from $\mathfrak{L}_A$ and $\mathfrak{L}_B$ within the phased sequent calculus that respects the two said fundamental principles, which we represent as $\mathfrak{L}_C = \mathfrak{L}(\mathcal{P}, \mathcal{C}_A \bigcup \mathcal{C}_B, \textbf{Inf}_A, \textbf{Inf}_B, \emptyset)$, the third and the fourth defining available inference rules to each base component. This maps to a psychological state of ours where we do not systematically know links between $\mathfrak{L}_A$ and $\mathfrak{L}_B$, though somehow conscious of them. As we direct more attention to combination itself, however, $\mathfrak{L}_C$ may gradually mould into an agreeable logic to us with fair properties, into a stand-alone logic, when an interaction principle, *i.e.* the meaning of 'combination' of $\mathfrak{L}_A$ and $\mathfrak{L}_B$, is agreed upon that thenceforth acts as a bridge over $\mathfrak{L}_A$ and $\mathfrak{L}_B$. In phased sequent calculus, it is built as a set of inference rules: $\textbf{Inf}_{A \leftrightarrow B}$. A more general definition of $\mathfrak{L}_C$ now derives: $\mathfrak{L}_C = \mathfrak{L}(\mathcal{P}, \mathcal{C}_A \bigcup \mathcal{C}_B, \textbf{Inf}_A, \textbf{Inf}_B, \textbf{Inf}_{A \leftrightarrow B})$.

We could also take the following viewpoint about the phased sequent calculus: base components represent small worlds that are self-functional but unaware of the world outside themselves, whereas communication components act as mediators (or interpreters) on behalf of the small worlds. By the strength of mediation the mediators exercise, the small worlds become more glued or more detached (with respect to certain criteria that allow us to measure the detachedness).

The phased sequent calculus permits us to freely set those mediators as the roots of distinct logical characteristics of a combined logic, *i.e.* the semantics of 'combination' itself.[1]

Let us consider our specific example of combining $\texttt{BI}_{\texttt{base}}$ additive sub-logic and $\texttt{BI}_{\texttt{base}}$ multiplicative sub-logic in order to enforce intuition of the previous paragraphs. A general (in our sense of respecting the two properties) representation of a combined logic of $\texttt{BI}_{\texttt{base}}$ additive sub-logic and $\texttt{BI}_{\texttt{base}}$ multiplicative sub-logic is: $\mathfrak{L}(\mathcal{P}, \{\top, \mathbb{1}, \wedge, \vee, \supset, *\}, \textbf{Inf}_{BIm}, \textbf{Inf}_{BIa}, \textbf{Infs})$ where:

---

[1] In case of a dissent to the use of the term 'semantics', I reckon there in fact is conceivable a correspondence of some semantics to derivations a set of inference rules may construct. By considering derivations within a proof system, I reckon we are studying the semantic characteristics at a higher-level than at the level of individual models. Note, however, that phased sequent calculus allows some interesting enterprise: provided we first know semantics, it can view any logic $\mathfrak{L}$ as a combined logic of a restricted $\mathfrak{L}$ with a set of particular models and a restricted $\mathfrak{L}$ with a set of all the other conceivable models which are connected through mediation of an appropriate mediator.

- $\mathbf{Inf}_{BIm} = \{id_1, *L, *R, \texttt{Cut}_1\}$
- $\mathbf{Int}_{BIa} = \{id_2, \mathbb{1}L, \top R, \wedge L, \vee L, \supset_{\{p,\top,\wedge,\vee,\supset,*\}} L, \wedge R, \vee R, \supset R, \texttt{Cut}_2\}$

in which id_1 and id_2 are respectively:

$$\frac{}{p \vdash p}\,id_1 \qquad\qquad \frac{}{\Gamma; p \vdash p}\,id_2$$

and both $\texttt{Cut}_1$ and $\texttt{Cut}_2$ are appropriate inference rules that reflect transitivity ($\texttt{Cut}_{\texttt{LBIZ}_1}$ in this specific case of ours), while all the rest of inference rules are exactly from $\texttt{LBIZ}_1$. Now, let us suppose **Infs** to be empty. This represents an immature form of combination of the two that are more fairly said to have been confounded (perhaps amid confusion on the part of one who works on the two logics) than calculatingly combined. The resultant combined logic, whose name shall remain anonymous, does not lack in any inconvenience. It recognises all the $\texttt{BI}_{\texttt{base}}$ formulas; and yet it exhibits knowledge either of $\{id_1, *L, *R, \texttt{Cut}_1\}$ (**Inf**$_1$) or of $\{id_2, \mathbb{1}L, \top R, \wedge L, \vee L, \supset_{\ldots} L, \wedge R, \vee R, \supset R, \texttt{Cut}_2\}$ (**Inf**$_2$) but not of both; a somewhat fashionable state of a logic. Let us suppose a formula: $(((p_1 \wedge p_2) * p_3) \wedge p_4) \supset (p_1 * p_3)$. If it so happens that the anonymous combined logic that we just conjured up exhibits knowledge of **Inf**$_1$, we are immediately struck by this feeling that we be quite unable to do anything with the given formula. It may, equally probably, exhibit knowledge of **Inf**$_2$ on the other hand, in which case may be drawn the following derivation:

$$\dfrac{\dfrac{((p_1 \wedge p_2) * p_3); p_4 \vdash p_1 * p_3}{((p_1 \wedge p_2) * p_3) \wedge p_4 \vdash p_1 * p_3}\,\wedge L}{\vdash (((p_1 \wedge p_2) * p_3) \wedge p_4) \supset (p_1 * p_3)}\,\supset R$$

although it does not present much prospect from there on.

It seems, in the context of phased sequent calculus, that what is not knowable in a small world becomes an unknown predicate to it: the given formula would contain the following unknown predicate $??_1((((p_1 \wedge p_2) * p_3) \wedge p_4) \supset (p_1 * p_3))$ if the combined logic exhibits knowledge of **Inf**$_1$, or else it would the following unknown predicates $??_2((p_1 \wedge p_2) * p_3)$ and $??_3(p_1 * p_3)$. These predicates cannot be knowable in a respective small world (either the case), and reasoning comes to an abrupt halt upon exhausting the other derivation options.[1]

---

[1] *Nonetheless, they still exist in the anonymous logic as what ordinarily appear to be pointless existences!*

By positing a mediator with an adequate strength of mediation, however, the clumsy motley can turn into a logic which may be perceived to be more interesting to a wider audience. With an adequate mediation, the combination becomes $\texttt{BI}_{\texttt{base}}$ in which the supposed formula is recognised as a theorem. But the mediator in our present specific case is none other than **Infs**, which is a formalisation of what it means by a logic combination of the base logics. Phased sequent calculus is then a fairly general proof-theoretical framework that allows the sense of logical combination to be studied: it assumes no premeditated logical combination (apart from the two mentioned conditions), which is determined not in the way that we see agreeable to some understanding that we may entertain about what it should mean by 'combination', but in accordance with the definition of communication components (the mediators), which may on some occasion afford us precisely what to us is an ideal combination or may simply not.

5.2 PBI: A Specific Calculus for $\texttt{BI}_{\texttt{base}}$

A thorough investigation into the mechanism of 'combination' given multiple logics within the proposed formalism is, however, too large to be even marginally exhausted in a single chapter of a thesis. Such being my present foreboding, it is kept, for theory, an introductory note to phased sequent calculus with a specific example of $\texttt{BI}_{\texttt{base}}$, and for application, a supplementary material to Chapter 3 of this thesis providing a decision procedure for the $\texttt{BI}$ fragment.

To begin, a set of definitions for $\texttt{BI}_{\texttt{base}}$ are (re-)constructed. The definition for $\texttt{BI}_{\texttt{base}}$ formulas is the same, but listed here as a reminder.

Definition 87 ($\texttt{BI}_{\texttt{base}}$ formulas) *A* $\texttt{BI}_{\texttt{base}}$ *formula* F $(, G, H)$ *is defined by:*

$F := p \mid \top \mid \mathbb{1} \mid F \wedge F \mid F \vee F \mid F \supset F \mid F * F.$

By $\mathfrak{F}_{\texttt{BI}_{\texttt{base}}}$ *we denote the set of the* $\texttt{BI}_{\texttt{base}}$ *formulas that this grammar generates.*

Definition 88 (Structures) *A* $\texttt{BI}_{\texttt{base}}$ *structure* Γ *is defined by:*

$\Gamma := \mathcal{A} \mid \mathcal{M} \qquad \mathcal{M} := F \mid \underbrace{\mathcal{A}_1, \cdots, \mathcal{A}_m}_{m \geq 2} \qquad \mathcal{A} := F \mid \underbrace{\mathcal{M}_1; \cdots; \mathcal{M}_n}_{n \geq 2}$

By $\mathfrak{S}_{\texttt{BI}_{\texttt{base}}}$ *we denote the set of the structures that this grammar generates. Each* $\mathcal{A}$ *(resp.* $\mathcal{M}$*) is termed an additive (resp. a multiplicative) structural layer.*

Associativity and commutativity hold within $\mathfrak{F}_{\texttt{BI}_{\texttt{base}}}$ and $\mathfrak{S}_{\texttt{BI}_{\texttt{base}}}$ as noted in 1.1.4 (for `BI`), but re-listed here also for a reminder:

Property 8 (Associativity and commutativity)

1. $F_1 \wedge (F_2 \wedge F_3) = (F_1 \wedge F_2) \wedge F_3$.
2. $F_1 \vee (F_2 \vee F_3) = (F_1 \vee F_2) \vee F_3$.
3. $F_1 * (F_2 * F_3) = (F_1 * F_2) * F_3$.
4. $F_1 \wedge F_2 = F_2 \wedge F_1$.
5. $F_1 \vee F_2 = F_2 \vee F_1$.
6. $F_1 * F_2 = F_2 * F_1$.
7. $\Gamma_1; (\Gamma_2; \Gamma_3) = (\Gamma_1; \Gamma_2); \Gamma_3$.
8. $\Gamma_1, (\Gamma_2, \Gamma_3) = (\Gamma_1, \Gamma_2), \Gamma_3$.
9. $\Gamma_1; \Gamma_2 = \Gamma_2; \Gamma_1$.
10. $\Gamma_1, \Gamma_2 = \Gamma_2, \Gamma_1$.

In defining contexts, note that a phased sequent calculus poses a mediator (or mediators) observing base-logic interactions which the small worlds (or base components) themselves cannot observe. The following definition reflects the viewpoint of our specific `BI`$_{\texttt{base}}$ mediator.

Definition 89 (Depth-aware contexts and structures) *SL_1 and SL_2 contexts of degree $i \geq 0$ are defined inductively:*

- ${}^0\Omega_1(-) := -_{\mathcal{M}}$.
- ${}^1\Omega_2(-) := -_{\mathcal{A}}, \Gamma$ *(for some $\Gamma \in \mathfrak{S}_{\texttt{BI}_{\texttt{base}}}$).*
- ${}^{(2i+2)}\Omega_1(-) := ({}^{(2i)}\Omega_1(-); \Gamma_1), \Gamma$ *(for some $\Gamma_1, \Gamma \in \mathfrak{S}_{\texttt{BI}_{\texttt{base}}}$; assumed similarly in the rest) such that Γ_1 is not empty.*

- ${}^{(2i+3)}\Omega_2(-) := ({}^{(2i+1)}\Omega_2(-); \Gamma_1), \Gamma$. *such that* Γ_1 *is not empty.*
- ${}^{0}\Omega_2(-) := -_{\mathcal{A}}$.
- ${}^{1}\Omega_1(-) := -_{\mathcal{M}}; \Gamma$ *such that* Γ *is not empty.*
- ${}^{(2i+2)}\Omega_2(-) := ({}^{(2i)}\Omega_2(-), \Gamma_1); \Gamma$ *such that* Γ *is not empty.*
- ${}^{(2i+3)}\Omega_1(-) := ({}^{(2i+1)}\Omega_1(-), \Gamma_1); \Gamma$ *such that* Γ *is not empty.*

where a $-_{\mathcal{M}}$ *or a* $-_{\mathcal{A}}$ *denotes that only a* $\mathcal{M}$ *and respectively a* $\mathcal{A}$ *may replace the hole.* ${}^{i}\Omega_j$ *for* $i \geq 0$ *and* $j \in \{1, 2\}$ *then denotes a* $\texttt{BI}_{\texttt{base}}$ *structure with an associated* SL_j *context of degree* i *such that there exists at least one focusable sub-strucure. Given some* ${}^{i}\Omega_j(\Gamma)$*, we say that* Γ *is focused.*

Example 8

1. *If* ${}^{0}\Omega_1 = (((p_1; p_2), F_3); p_4), F_5$*, then the entire "*$(((p_1; p_2), F_3); p_4), F_5$*" can be focused, but not any others.*
2. *there cannot be any* ${}^{0}\Omega_2$ *such that* ${}^{0}\Omega_2 = (((p_1; p_2), F_3); p_4), F_5$.
3. *there cannot be any* ${}^{1}\Omega_1$ *such that* ${}^{1}\Omega_1 = (((p_1; p_2), F_3); p_4), F_5$.
4. *If* ${}^{1}\Omega_2 = (((p_1; p_2), F_3); p_4), F_5$*, then the entire "*$((p_1; p_2), F_3); p_4$*" and the entire "*$F_5$*" can be focused, but not any others.*
5. *If* ${}^{2}\Omega_1 = (((p_1; p_2), F_3); p_4), F_5$*, then the entire "*$(p_1; p_2), F_3$*" can be focused, but not any others.*
6. *there cannot be any* ${}^{2}\Omega_2$ *such that* ${}^{2}\Omega_2 = (((p_1; p_2), F_3); p_4), F_5$.
7. *there cannot be any* ${}^{3}\Omega_1$ *such that* ${}^{3}\Omega_1 = (((p_1; p_2), F_3); p_4), F_5$.
8. *If* ${}^{3}\Omega_2 = (((p_1; p_2), F_3); p_4), F_5$*, then the entire "*$p_1; p_2$*" and the entire "*$F_3$*" can be focused, but not any others.*
9. *If* ${}^{4}\Omega_1 = (((p_1; p_2), F_3); p_4), F_5$*, then the entire "*$p_1$*" and the entire "*$p_2$*" can be focused, but not any others.*

10. *there cannot be any* ${}^{i}\Omega_2$ *with* $i \geq 4$ *such that* ${}^{i}\Omega_2 = (((p_1;p_2),F_3);p_4),F_5$.

11. *there cannot be any* ${}^{i}\Omega_1$ *with* $i \geq 5$ *such that* ${}^{i}\Omega_1 = (((p_1;p_2),F_3);p_4),F_5$.

Lemma 26 *For all* $\Gamma \in \mathfrak{S}_{\mathtt{BI}_{\mathtt{base}}}$*, if there exists some* ${}^{(2i)}\Omega_j$ *for* $i \geq 0$ *and* $j \in \{1,2\}$ *such that* ${}^{(2i)}\Omega_j \equiv \Gamma$ *(up to assoc. and commut. as in Property 8), there exists no* ${}^{(2i+1)}\Omega_j$ *such that* ${}^{(2i+1)}\Omega_j \equiv \Gamma$.

Definition 90 (`PBI` **Sequents**)
The set $\mathfrak{D}_{SL_1}$ *of* SL_1 *sequents is defined by:*
$\mathfrak{D}_{SL_1} := \{{}^{i}\Omega_1 \vdash_1 F \mid [i \geq 0] \wedge^{\dagger} [{}^{i}\Omega_1 \in \mathfrak{S}_{\mathtt{BI}_{\mathtt{base}}}] \wedge^{\dagger} [F \in \mathfrak{F}_{\mathtt{BI}_{\mathtt{base}}}]\}$.
The set $\mathfrak{D}_{SL_2}$ *of* SL_2 *sequents is defined by:*
$\mathfrak{D}_{SL_2} := \{{}^{i}\Omega_2 \vdash_2 F \mid [i \geq 0] \wedge^{\dagger} [{}^{i}\Omega_2 \in \mathfrak{S}_{\mathtt{BI}_{\mathtt{base}}}] \wedge^{\dagger} [F \in \mathfrak{F}_{\mathtt{BI}_{\mathtt{base}}}]\}$.
The set $\mathfrak{D}_{\mathtt{PBI}}$ *of* `PBI` *sequents is then defined by* $\mathfrak{D}_{\mathtt{PBI}} := \mathfrak{D}_{SL_1} \bigcup \mathfrak{D}_{SL_2}$.

It trivially holds that $\mathfrak{D}_{SL_1} \bigcap \mathfrak{D}_{SL_2} = \emptyset$. `PBI` is found in Figure 5.1.

Definition 91 (`PBI`) `PBI` *comprises two non-communication components and one communication component (a mediator).*

- *SL_1 rules (base):*
  $SL_1id \quad *L \quad *R$
- *Interaction rules (mediator):*
  $\mathtt{Transfer}\downarrow \quad *\mathtt{Lock}\downarrow \quad \vee\mathtt{Lock}\downarrow \quad \mathbb{1}\mathtt{Peel}\downarrow \quad \mathtt{Revert}\downarrow$
  $\mathtt{Transfer}\uparrow \quad *\mathtt{Lock}\uparrow \quad \vee\mathtt{Lock}\uparrow \quad \mathbb{1}\mathtt{Peel}\uparrow \quad \mathtt{Revert}\uparrow$
- *SL_2 rules (base):*
  $SL_2id \quad \mathbb{1}L \quad \top R \quad \wedge L \quad \vee L \quad \supset L_{\{p,\top,\wedge,\vee,\supset,*\}} \quad \wedge R \quad \vee R \quad \supset R$

I denote the set of interaction rules by ***Inf***$_I$*. Those in [· · ·] in* `PBI` *inference rules are side conditions (and not premise sequents) that must satisfy when the particular inference rules apply.*

All the logical inference rules in `PBI`, apart from the difference in the nature of contexts, behave in the same way as those corresponding ones in $\mathtt{BI}_{\mathtt{base}}$. Significance of interaction rules are detailed in 5.3.

$$\frac{}{{}^{0}\Omega_1(p) \vdash_1 p}\; SL_1 id \qquad \frac{{}^{k}\Omega_1(\langle\Gamma\rangle, F, G) \vdash_1 H}{{}^{k}\Omega_1(\langle\Gamma\rangle, F * G) \vdash_1 H}\; *L$$

$$\frac{{}^{j_1}\Omega_1 \vdash_1 H_a \qquad {}^{j_2}\Omega_1 \vdash_1 H_b \quad [\Gamma_a = {}^{j_1}\Omega_1 \text{ and } \Gamma_b = {}^{j_2}\Omega_1]}{{}^{0}\Omega_1(\Gamma_a, \Gamma_b) \vdash_1 H_a * H_b}\; *R$$

$$\frac{{}^{j}\Omega_1 \vdash_1 H \quad [{}^{j}\Omega_1 = {}^{0}\Omega_2]}{{}^{0}\Omega_2 \vdash_2 H}\; \texttt{Transfer}\uparrow \qquad \frac{{}^{j}\Omega_2 \vdash_2 H \quad [{}^{j}\Omega_2 = {}^{0}\Omega_1]}{{}^{0}\Omega_1 \vdash_1 H}\; \texttt{Transfer}\downarrow$$

$$\frac{{}^{k}\Omega_1 \vdash_1 H \quad [{}^{k}\Omega_1 = {}^{(k+1)}\Omega_2]}{{}^{(k+1)}\Omega_2 \vdash_2 H}\; \texttt{Revert}\uparrow \qquad \frac{{}^{k}\Omega_2 \vdash_2 H \quad [{}^{k}\Omega_2 = {}^{(k+1)}\Omega_1]}{{}^{(k+1)}\Omega_1 \vdash_1 H}\; \texttt{Revert}\downarrow$$

$$\frac{{}^{0}\Omega_1(\Gamma_a) \vdash_1 \mathbb{1}}{{}^{0}\Omega_2(\Gamma_a; \Gamma_b) \vdash_2 H}\; \mathbb{1}\texttt{Peel}\uparrow \qquad \frac{{}^{0}\Omega_2(\Gamma_a) \vdash_2 \mathbb{1}}{{}^{0}\Omega_1(\Gamma_a, \Gamma_b) \vdash_1 H}\; \mathbb{1}\texttt{Peel}\downarrow$$

$$\frac{{}^{(k+1)}\Omega_1 \vdash_1 H_a \vee H_b \quad [{}^{(k+1)}\Omega_1 = {}^{k}\Omega_2]}{{}^{k}\Omega_2 \vdash_2 H_a \vee H_b}\; \vee\texttt{Lock}\uparrow$$

$$\frac{{}^{(k+2)}\Omega_2 \vdash_2 H_a \vee H_b \quad [{}^{(k+2)}\Omega_2 = {}^{(k+1)}\Omega_1]}{{}^{(k+1)}\Omega_1 \vdash_1 H_a \vee H_b}\; \vee\texttt{Lock}\downarrow$$

$$\frac{{}^{0}\Omega_1 \vdash_1 H_a * H_b \quad [{}^{0}\Omega_1 = \textit{proj}({}^{j}\Omega_2)]}{{}^{j}\Omega_2 \vdash_2 H_a * H_b}\; *\texttt{Lock}\uparrow \qquad \frac{{}^{j_2}\Omega_2 \vdash_2 H_l * H_r \quad [{}^{j_2}\Omega_2 = {}^{j_1}\Omega_1]}{{}^{j_1}\Omega_1 \vdash_1 H_l * H_r}\; *\texttt{Lock}\downarrow$$

$$\frac{}{{}^{0}\Omega_2(\Gamma; p) \vdash_2 p}\; SL_2 id \qquad \frac{}{{}^{j}\Omega_2 \vdash_2 \top}\; \top R \qquad \frac{}{{}^{k}\Omega_2(\Gamma; \mathbb{1}) \vdash_2 H}\; \mathbb{1}L$$

$$\frac{{}^{k}\Omega_2(\Gamma_a; F; G) \vdash_2 H}{{}^{k}\Omega_2(\Gamma_a; F \wedge G) \vdash_2 H}\; \wedge L \qquad \frac{{}^{k}\Omega_2(\Gamma_a; F) \vdash_2 H \qquad {}^{k}\Omega_2(\Gamma_a; G) \vdash_2 H}{{}^{k}\Omega_2(\Gamma_a; F \vee G) \vdash_2 H}\; \vee L$$

$$\frac{{}^{k}\Omega_2(\Gamma_a; p; F) \vdash_2 H}{{}^{k}\Omega_2(\Gamma_a; p; p \supset F) \vdash_2 H}\; \supset L_p \qquad \frac{{}^{k}\Omega_2(\Gamma_a; G) \vdash_2 H}{{}^{k}\Omega_2(\Gamma_a; \top \supset G) \vdash_2 H}\; \supset L_\top$$

$$\frac{{}^{k}\Omega_2(\Gamma_a; F_a \supset (F_b \supset F_c)) \vdash_2 H}{{}^{k}\Omega_2(\Gamma_a; (F_a \wedge F_b) \supset F_c) \vdash_2 H}\; \supset L_\wedge \qquad \frac{{}^{k}\Omega_2(\Gamma_a; (F_a \supset F_c); (F_b \supset F_c)) \vdash_2 H}{{}^{k}\Omega_2(\Gamma_a; (F_a \vee F_b) \supset F_c) \vdash_2 H}\; \supset L_\vee$$

$$\frac{{}^{0}\Omega_2(\Gamma_a; F_b \supset F_c) \vdash_2 F_a \supset F_b \qquad {}^{k}\Omega_2(\Gamma_a; F_c) \vdash_2 H}{{}^{k}\Omega_2(\Gamma_a; (F_a \supset F_b) \supset F_c) \vdash_2 H}\; \supset L_\supset$$

$$\frac{{}^{j}\Omega_2 \vdash_2 F_a * F_b \quad [{}^{j}\Omega_2 = \Gamma_a] \qquad {}^{k}\Omega_2(\Gamma_a; F_c) \vdash_2 H}{{}^{k}\Omega_2(\Gamma_a; (F_a * F_b) \supset F_c) \vdash_2 H}\; \supset L_*$$

$$\frac{{}^{0}\Omega_2(\Gamma; F_a) \vdash_2 F_b \quad [{}^{j}\Omega_2 = \Gamma]}{{}^{j}\Omega_2 \vdash_2 F_a \supset F_b}\; \supset R \qquad \frac{{}^{j}\Omega_2 \vdash_2 F_a \qquad {}^{j}\Omega_2 \vdash_2 F_b}{{}^{j}\Omega_2 \vdash_2 F_a \wedge F_b}\; \wedge R$$

$$\frac{{}^{j}\Omega_2 \vdash_2 F_x \quad [x \in \{a, b\}]}{{}^{j}\Omega_2 \vdash_2 F_a \vee F_b}\; \vee R$$

Figure 5.1: `PBI`: a phased sequent calculus defining $\texttt{BI}_{\texttt{base}}$. $j \in \{0, 1\}$ with or without a sub-script. $k \geq 0$.

Definition 92 (`PBI` **conventions)**

1. *Given a* `PBI` *sequent with some antecedent (sub-) structure in the form: "*$\Gamma_1; \Gamma_2$*", Γ_1 or Γ_2 may be empty. Its emptiness is identified with a $\top$.*[1]

2. *Given a* `PBI` *sequent, for an antecedent structure in the form: "*Γ_1, Γ_2*", neither Γ_1 nor Γ_2 can be empty. The notation: "$\langle\Gamma\rangle$", is used to explicitly denote that Γ which is enclosed in "$\langle\cdots\rangle$" may be empty.*

Example 9 *$*L$ is (backward) applicable to a* `PBI` *sequent $F * G \vdash_1 H$ even if there are no other surrounding structures. $*R$ does not (backward) apply to a* `PBI` *sequent $D : F_a \vdash H_a * H_b$ since there is only one formula in the antecedent.*

The function *proj* applies in the process of a backward derivation step of $*$`Lock` $\uparrow$. In the context of a `PBI` derivation, it acts as an incremental weakening as we saw in Chapter 3.

Definition 93 (Projection) *Let $\mathfrak{M}$ denote either a formula in the form: $H_1 * H_2$ or a structure in the form: "Γ_1, Γ_2". Let proj$_{unit} : \mathfrak{S}_{SL_2} \rightharpoonup \mathfrak{S}_{SL_1}$ be a partial function defined by: proj$_{unit}({}^0\Omega_2(\mathfrak{M}; \Gamma_2)) = {}^0\Omega_1(\mathfrak{M})$. Then a partial function proj $: \mathfrak{S}_{SL_2} \rightharpoonup \mathfrak{S}_{SL_1}$ is defined as follows:*

- *proj*$({}^0\Omega_2(\Gamma_1; \Gamma_2)) =$ *proj*$_{unit}({}^0\Omega_2(\Gamma_1; \Gamma_2))$.
- *proj*$({}^1\Omega_2) = {}^0\Omega_1(\Gamma)$ *where Γ is a* `BI`$_{\texttt{base}}$ *structure that derives from applying proj$_{unit}$ to one or more focusable sub-structures of ${}^1\Omega_2$ (assume that each of them is a SL_2 structure with the 0-th context degree).*

The following example shows how *proj* works.

Example 10 *Let $p_1 * p_2, (p_3; (p_4, p_5))$ be a SL_2 antecedent structure of the context degree 1. Then the output of proj$((p_1 * p_2, (p_3; (p_4, p_5))))$ is either of the below;*

- $p_1 * p_2, (p_3; (p_4, p_5))$ *(when proj$_{unit}$ applies on $p_1 * p_2$)*

---

[1] This identification is only relevant in the left premise sequent of $\supset L_*$ when Γ_a is empty in the conclusion sequent.

- $p_1 {*} p_2, p_4, p_5$ *(when it applies on* $p_3; (p_4, p_5)$ *and when it applies both on* $p_3; (p_4, p_5)$ *and on* $p_1 * p_2$*)*

each of which is a SL_1 *antecedent structure of the 0-th context.*

And the following example clarifies why it is a partial function.

Example 11 *Let* $p_1 \supset p_2, p_3 \supset p_4$ *be another* SL_2 *antecedent structure of the context degree 1. Then proj*$(p_1 \supset p_2, p_3 \supset p_4)$ *is undefined. Let* $p_1 \supset p_2, p_3 \supset p_4 \vdash_2 p_5 * p_6$ *be a* SL_2 *sequent with the context degree of 1. Then* $*$`Lock` $\uparrow$ *cannot apply on the sequent since the side condition cannot be satisfied.*

The expressiveness of `PBI` is to be subsequently shown equivalent to $\texttt{LBIZ}_1$'s. A characterisation of semantics more faithful to the underlying principle of the phased sequent calculus is a fruitful future work.

5.3 Base-Logic Interactions within `PBI`

The set of `PBI` sequents is a union of $\mathfrak{D}_{SL_1}$ and $\mathfrak{D}_{SL_2}$ with an empty intersection (*Cf.* Definition 90). In the context of a backward derivation tree construction, *i.e.* backward theorem-proving, then, we know *a priori* that no SL_1 (resp. SL_2) rules apply to any $D \in \mathfrak{D}_{SL_2}$ (resp. $D \in \mathfrak{D}_{SL_1}$). I term by *phase* what imposes such a condition on `PBI` sequents, to emphasise the syntactic viewpoint about a small world (or a base component). A phase can be of SL_2 or of SL_1. All the SL_1 (resp. SL_2) sequents are in a SL_1 (resp. SL_2) phase. Interactions between the two phases, *i.e.* phase switches, cannot be achieved within the base components, which must be facilitated by the mediator. A `PBI` phase switch (looked from conclusion to premise(s)) from a SL_1 phase into a SL_2 phase is induced via `Transfer` $\downarrow$, `Revert` $\downarrow$, $\mathbb{1}$`Peel` $\downarrow$, $*$`Lock` $\downarrow$ and $\vee$`Lock` $\downarrow$, while that from a SL_2 phase into a SL_1 phase is induced via `Transfer` $\uparrow$, `Revert` $\uparrow$, $\mathbb{1}$`Peel` $\uparrow$, $*$`Lock` $\uparrow$ and $\vee$`Lock` $\uparrow$.

5.3.1 Interactions as a set of transitions

Since a mediator is an interpretation superimposed on the two phases, it is possible to entirely characterise its actions in a state diagram, as shown in Figure 5.2, abstracting

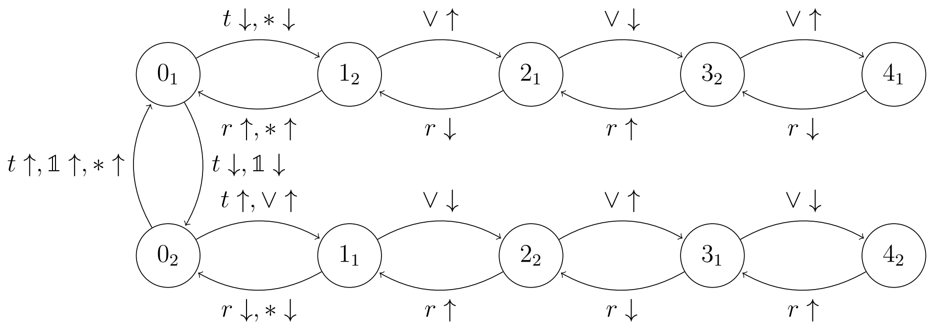

Figure 5.2: A state diagram of `PBI` base-logic interactions characterised by `PBI` interaction rules, shown up to the 4th contexts (to continue farther, if possible at all, in the same pattern thereupon). State labels indicate the context-degree and their sub-scripts either of the small worlds SL_1 which 1 represents or SL_2 which 2 does. Following short-hand labels for `PBI` interaction rules are used in the state diagram: `Transfer` $\downharpoonleft \mapsto t \downharpoonleft$, `Revert` $\downharpoonleft \mapsto r \downharpoonleft$, $\mathbb{1}$`Peel` $\downharpoonleft \mapsto \mathbb{1} \downharpoonleft$, $\vee$`Lock` $\downharpoonleft \mapsto \vee \downharpoonleft$, $*$`Lock` $\downharpoonleft \mapsto * \downharpoonleft$, `Transfer` $\upharpoonleft \mapsto t \upharpoonleft$, `Revert` $\upharpoonleft \mapsto r \upharpoonleft$, $\mathbb{1}$`Peel` $\upharpoonleft \mapsto \mathbb{1} \upharpoonleft$, $\vee$`Lock` $\upharpoonleft \mapsto \vee \upharpoonleft$, $*$`Lock` $\upharpoonleft \mapsto * \upharpoonleft$.

away logical inference steps to take place in each phase. The implication is that, when we intend to derive a combined logic, a particular manner of logical combination *can be proof-theoretically developed* and analysed by working on the abstract model. This can aid, given some base logics, an adequate derivation of a combined logic befitting a particular application.

5.3.2 Types of `PBI` phase switches

From the perspective of a logical combination engineering, it is useful that we classify interaction rules by types and grasp the senses of logical combination in effect.

5.3.2.1 Phase switches by transfer

Phase switches by transfer: `Transfer` $\upharpoonleft$, `Transfer` $\downharpoonleft$, `Revert` $\upharpoonleft$, `Revert` $\downharpoonleft$, represent natural transitions from one phase to another, adjusting the context-degree adequately. `Transfer` $\upharpoonleft$, `Transfer` $\downharpoonleft$ never decrease the context-degree, whereas `Revert` $\upharpoonleft$, `Revert` $\downharpoonleft$ always do.

Example 12 *An example with* $(p_1 \vee p_2) * p_3 \vdash_1 (p_1 \vee p_2) * p_3$*; logical connectives accessible in a sequent are highlighted in red:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p_1 \vdash_2 p_1}\,SL_2id}{p_1 \vdash_2 p_1 \vee p_2}\,\vee R \quad \dfrac{\dfrac{}{p_2 \vdash_2 p_2}\,SL_2id}{p_2 \vdash_2 p_1 \vee p_2}\,\vee R}{p_1 \vee p_2 \vdash_2 p_1 \vee p_2}\,\vee L}{p_1 \vee p_2 \vdash_1 p_1 \vee p_2}\,\mathtt{Transfer}\downarrow \quad \dfrac{}{p_3 \vdash_1 p_3}\,SL_1id}{p_1 \vee p_2, p_3 \vdash_1 (p_1 \vee p_2) * p_3}\,*R}{(p_1 \vee p_2) * p_3 \vdash_1 (p_1 \vee p_2) * p_3}\,*L}{(p_1 \vee p_2) * p_3 \vdash_2 (p_1 \vee p_2) * p_3}\,\mathtt{Transfer}\uparrow$$

Another example with $p_1; (p_2, p_3) \vdash_1 p_1$*:*

$$\dfrac{\dfrac{}{p_1; (p_2, p_3) \vdash_2 p_1}\,SL_1id}{p_1; (p_2, p_3) \vdash_1 p_1}\,\mathtt{Revert}\downarrow$$

5.3.2.2 Phase switches by peeling

A phase switch can also be induced via a peeling operation which discards, *i.e.* peels, structural layers of a sequent in the current phase. A peeling operation of the mediator intends to identify formula(s)/structure(s) which the mediator supposes is buried in a structure inaccessible in the current phase with the current context-degree. In `PBI`, there are two peeling rules: $\mathbb{1}\mathtt{Peel}\downarrow$ and $\mathbb{1}\mathtt{Peel}\uparrow$ with the intention of locating $\mathbb{1}$ which may be hidden in an inner structural layer.

Example 13 *An example of peelings with* $p_1, (p_2 \wedge (p_3 * (p_4 \wedge \mathbb{1}))) \vdash_1 p_2$*; those logical connectives accessible are highlighted in red:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p_4; \mathbb{1} \vdash_2 \mathbb{1}}\,\mathbb{1}L}{p_4 \wedge \mathbb{1} \vdash_2 \mathbb{1}}\,\wedge L}{p_3, (p_4 \wedge \mathbb{1}) \vdash_1 \mathbb{1}}\,\mathbb{1}\mathtt{Peel}\downarrow}{p_3 * (p_4 \wedge \mathbb{1}) \vdash_1 \mathbb{1}}\,*L}{p_2; (p_3 * (p_4 \wedge \mathbb{1})) \vdash_2 \mathbb{1}}\,\mathbb{1}\mathtt{Peel}\uparrow}{p_2 \wedge (p_3 * (p_4 \wedge \mathbb{1})) \vdash_2 \mathbb{1}}\,\wedge L}{p_1, (p_2 \wedge (p_3 * (p_4 \wedge \mathbb{1}))) \vdash_1 p_2}\,\mathbb{1}\mathtt{Peel}\downarrow$$

5.3.2.3 Phase switches by locking

Lockings are a mode of communication that facilitates a greater flexibility in the way phases interact. Within `PBI`, they are the most complex phase switches. While the intention of a locking is - to the extent that it tries to collect sufficient information by digging deeper into inner structural layers - similar to that by peeling, there is also a difference in that it does not in general reveal an inner structural layer as the outermost one, which would happen if with peeling. Instead, a phase shift by locking lets the intention of the mediator (to collect sufficient information for some *locked* formula that requires them) carried over to the next phase, along with adjustment of the context-degree.

Example 14 *An example of* `PBI` *derivation of* $p_1, ((p_2 * p_3); p_4) \vdash_1 (p_5 \supset (p_1 * p_2)) * p_3$ *involving* $*\mathtt{Lock}\uparrow$ *and* $*\mathtt{Lock}\downarrow$*; in the below derivation those logical connectives accessible are highlighted in red:*

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{SL_1id}
\UnaryInfC{$p_1 \vdash_1 p_1$}
\AxiomC{}
\RightLabel{SL_1id}
\UnaryInfC{$p_2 \vdash_1 p_2$}
\RightLabel{$*R$}
\BinaryInfC{$p_1, p_2 \vdash_1 p_1 {\color{red}*} p_2$}
\RightLabel{$*\mathtt{Lock}\uparrow$}
\UnaryInfC{$p_5; (p_1, p_2) \vdash_2 p_1 * p_2$}
\RightLabel{$*\mathtt{Lock}\downarrow$}
\UnaryInfC{$p_5; (p_1, p_2) \vdash_1 p_1 {\color{red}*} p_2$}
\RightLabel{$\mathtt{Transfer}\uparrow$}
\UnaryInfC{$p_5; (p_1, p_2) \vdash_2 p_1 * p_2$}
\RightLabel{$\supset R$}
\UnaryInfC{$p_1, p_2 \vdash_2 p_5 {\color{red}\supset} (p_1 * p_2)$}
\RightLabel{$\mathtt{Transfer}\downarrow$}
\UnaryInfC{$p_1, p_2 \vdash_1 p_5 \supset (p_1 * p_2)$}
\AxiomC{}
\RightLabel{SL_1id}
\UnaryInfC{$p_3 \vdash_1 p_3$}
\RightLabel{$*R$}
\BinaryInfC{$p_1, p_2, p_3 \vdash_1 (p_5 \supset (p_1 * p_2)) {\color{red}*} p_3$}
\RightLabel{$*L$}
\UnaryInfC{$p_1, p_2 {\color{red}*} p_3 \vdash_1 (p_5 \supset (p_1 * p_2)) {\color{red}*} p_3$}
\RightLabel{$*\mathtt{Lock}\uparrow$}
\UnaryInfC{$p_1, (p_2 * p_3; p_4) \vdash_2 (p_5 \supset (p_1 * p_2)) * p_3$}
\RightLabel{$*\mathtt{Lock}\downarrow$}
\UnaryInfC{$p_1, (p_2 * p_3; p_4) \vdash_1 (p_5 \supset (p_1 * p_2)) {\color{red}*} p_3$}
\end{prooftree}
$$

As the above derivation illustrates, $*\mathtt{Lock}\uparrow\downarrow$ *dig out multiplicative components in order to apply* $*R$*.*

Another example of `PBI` *derivation of* $p_1; ((p_2 * p_3) \vee p_4) * p_5 \vdash_2 (p_2 * p_3 * p_5) \vee (p_4 * p_5)$ *involving* $\vee\mathtt{Lock}\uparrow$ *and* $\vee\mathtt{Lock}\downarrow$*; those logical connectives accessible are highlighted in red:*

$$
\begin{prooftree}
\AxiomC{}
\doubleLine
\UnaryInfC{$p_2 {\color{red}*} p_3 \vdash_1 p_2 {\color{red}*} p_3$}
\AxiomC{}
\RightLabel{SL_1id}
\UnaryInfC{$p_5 \vdash_1 p_5$}
\RightLabel{$*R$}
\BinaryInfC{$p_2 {\color{red}*} p_3, p_5 \vdash_1 p_2 {\color{red}*} p_3 {\color{red}*} p_5$}
\RightLabel{$*\mathtt{Lock}\uparrow$}
\UnaryInfC{$p_1; ((p_2 * p_3), p_5) \vdash_2 p_2 * p_3 * p_5$}
\RightLabel{$*\mathtt{Lock}\downarrow$}
\UnaryInfC{$p_1; ((p_2 * p_3), p_5) \vdash_1 p_2 {\color{red}*} p_3 {\color{red}*} p_5$}
\RightLabel{$\mathtt{Transfer}\uparrow$}
\UnaryInfC{$p_1; ((p_2 * p_3), p_5) \vdash_2 p_2 * p_3 * p_5$}
\RightLabel{$\vee R$}
\UnaryInfC{$p_1; ((p_2 * p_3), p_5) \vdash_2 (p_2 * p_3 * p_5) {\color{red}\vee} (p_4 * p_5)$}
\RightLabel{$\mathtt{Revert}\downarrow$}
\UnaryInfC{$p_1; ((p_2 * p_3), p_5) \vdash_1 (p_2 * p_3 * p_5) \vee (p_4 * p_5)$}
\RightLabel{$\mathtt{Revert}\uparrow$}
\UnaryInfC{$p_1; ((p_2 * p_3), p_5) \vdash_2 (p_2 * p_3 * p_5) \vee (p_4 * p_5)$}
\AxiomC{}
\RightLabel{SL_1id}
\UnaryInfC{$p_4 \vdash_1 p_4$}
\AxiomC{}
\RightLabel{SL_1id}
\UnaryInfC{$p_5 \vdash_1 p_5$}
\RightLabel{$*R$}
\BinaryInfC{$p_4, p_5 \vdash_1 p_4 {\color{red}*} p_5$}
\RightLabel{$*\mathtt{Lock}\uparrow$}
\UnaryInfC{$p_1; (p_4, p_5) \vdash_2 p_4 * p_5$}
\RightLabel{$*\mathtt{Lock}\downarrow$}
\UnaryInfC{$p_1; (p_4, p_5) \vdash_1 p_4 {\color{red}*} p_5$}
\RightLabel{$\mathtt{Transfer}\uparrow$}
\UnaryInfC{$p_1; (p_4, p_5) \vdash_2 p_4 * p_5$}
\RightLabel{$\vee R$}
\UnaryInfC{$p_1; (p_4, p_5) \vdash_2 (p_2 * p_3 * p_5) {\color{red}\vee} (p_4 * p_5)$}
\RightLabel{$\mathtt{Revert}\downarrow$}
\UnaryInfC{$p_1; (p_4, p_5) \vdash_1 (p_2 * p_3 * p_5) \vee (p_4 * p_5)$}
\RightLabel{$\mathtt{Revert}\uparrow$}
\UnaryInfC{$p_1; (p_4, p_5) \vdash_2 (p_2 * p_3 * p_5) \vee (p_4 * p_5)$}
\RightLabel{$\vee L$}
\BinaryInfC{$p_1; (((p_2 * p_3) {\color{red}\vee} p_4), p_5) \vdash_2 (p_2 * p_3 * p_5) \vee (p_4 * p_5)$}
\RightLabel{$\vee\mathtt{Lock}\downarrow$}
\UnaryInfC{$p_1; (((p_2 * p_3) \vee p_4), p_5) \vdash_1 (p_2 * p_3 * p_5) \vee (p_4 * p_5)$}
\RightLabel{$*L$}
\UnaryInfC{$p_1; ((p_2 * p_3) \vee p_4) {\color{red}*} p_5 \vdash_1 (p_2 * p_3 * p_5) \vee (p_4 * p_5)$}
\RightLabel{$\vee\mathtt{Lock}\uparrow$}
\UnaryInfC{$p_1; ((p_2 * p_3) \vee p_4) * p_5 \vdash_2 (p_2 * p_3 * p_5) {\color{red}\vee} (p_4 * p_5)$}
\end{prooftree}
$$

As the above derivation illustrates, $\vee$`Lock` $\uparrow\downarrow$ *let the sense of intuitionistic* $\vee$ *to extend to all the structural layers.*

In the above example only one disjunction in the antecedent part was required to be first handled before the $\vee R$. In general, however, there could be more than one disjunctions in the antecedent that must be solved. The mediator may need fine-tune the context-degree via `Revert` $\uparrow$ so that all the required $\vee L$ for the $\vee R$ can take place without an omission.

5.4 Equivalence of `PBI` with `LBIZ`$_1$

I now show that `PBI` is as expressive as `LBIZ`$_1$. One direction is trivial with the following bottom lemma.

Lemma 27 (Bottom lemma) *Let* t *be some positive integer and let* $\Gamma(\mathbb{1}) \vdash F$ *denote some* `LBIZ`$_1$*-derivable* `BI`$_{\texttt{base}}$ *sequent. Then*

- $\Gamma(\mathbb{1}) \vdash G$ *is also* `LBIZ`$_1$*-derivable for an arbitrary* $G \in \mathfrak{F}_{\texttt{BI}_{\texttt{base}}}$.
- *If* $\Gamma(\mathbb{1}) \equiv \Gamma_1(\mathbb{1}); \cdots ; \Gamma_t$*, then* $\Gamma_1(\mathbb{1}); \cdots ; \Gamma_s \vdash F$ *for any* $0 \leq s \leq t$ *is also* `LBIZ`$_1$*-derivable.*
- *If* $\Gamma(\mathbb{1}) \equiv \Gamma_1(\mathbb{1}), \cdots , \Gamma_t$*, then* $\Gamma_1(\mathbb{1}), \cdots , \Gamma_s \vdash F$ *for any* $0 \leq s \leq t$ *is also* `LBIZ`$_1$*-derivable.*
- $\Gamma(\mathbb{1}); \Gamma_1; \cdots ; \Gamma_t \vdash F$ *is also* `LBIZ`$_1$*-derivable.*
- $\Gamma(\mathbb{1}), \Gamma_1, \cdots , \Gamma_t \vdash F$ *is also* `LBIZ`$_1$*-derivable.*

Proof. Straightforward. □

5.4.1 $\mathtt{LBIZ}_1$ is not less expressive than $\mathtt{PBI}$

Proposition 23 *If* ${}^k\Omega_i \vdash_i F \in \mathfrak{D}_{\mathtt{PBI}}$ *with* $k \geq 0$ *and* $i \in \{1, 2\}$ *is derivable in* $\mathtt{PBI}$*, then* $\Gamma \vdash F(\in \mathfrak{D}_{\mathtt{BI}_{\mathtt{base}}})$ *with* $\Gamma = {}^k\Omega_i$ *is* $\mathtt{LBIZ}_1$*-derivable.*

Proof. It suffices to show that each $\mathtt{PBI}$ inference rule is derivable in $\mathtt{LBIZ}_1$. Trivial for all the non-interaction rules. Consider now the remaining inference rules:

1. $\mathtt{Transfer}\uparrow, \mathtt{Transfer}\downarrow, \mathtt{Revert}\uparrow, \mathtt{Revert}\downarrow$: implicit in $\mathtt{LBIZ}_1$.
2. $\mathbb{1}\mathtt{Peel}\uparrow, \mathbb{1}\mathtt{Peel}\downarrow$: Lemma 27.
3. $\vee\mathtt{Lock}\uparrow, \vee\mathtt{Lock}\downarrow$: implicit in $\mathtt{LBIZ}_1$.
4. $*\mathtt{Lock}\uparrow, *\mathtt{Lock}\downarrow$: $*\mathtt{Lock}\downarrow$ is implicit in $\mathtt{LBIZ}_1$. To the projection in $*\mathtt{Lock}\uparrow$, weakening (which is admissible in $\mathtt{LBIZ}_1$) corresponds. □

5.4.2 $\mathtt{PBI}$ is not less expressive than $\mathtt{LBIZ}_1$

Some observations about $\mathtt{LBIZ}_1$ derivations precede the main result.

5.4.2.1 Preparations

The following sub-structural layer relation is used in the rest.

Definition 94 (Sub-structural layer relation) *We define a relation* $\preceq$ *on two structures* $s \in \mathfrak{S}_{\mathtt{BI}_{\mathtt{base}}}$ *and* $t \in \mathfrak{S}_{\mathtt{BI}_{\mathtt{base}}}$ *such that:* $s \preceq t$ *iff* s *is a sub-structural layer of* t*. By* $s \prec t$ *we denote that* s *is a strict sub-structural layer of* t*.*

Example 15 *Let* $p_1; (p_2, (p_3; p_4))$ *be a* $\mathtt{LBIZ}_1$ *structure. Then we have the following to hold:*

1. $p_1 \prec (p_1; (p_2, (p_3; p_4)))$.
2. $(p_2, (p_3; p_4)) \prec (p_1; (p_2, (p_3; p_4)))$.
3. $p_2 \prec (p_1; (p_2, (p_3; p_4)))$.

4. $(p_3; p_4) \prec (p_1; (p_2, (p_3; p_4)))$.

5. $p_3 \prec (p_1; (p_2, (p_3; p_4)))$.

6. $p_4 \prec (p_1; (p_2, (p_3; p_4)))$.

7. $p_2 \prec (p_2, (p_3; p_4))$.

8. $(p_3; p_4) \prec (p_2, (p_3; p_4))$.

9. $p_3 \prec (p_2, (p_3; p_4))$.

10. $p_4 \prec (p_2, (p_3; p_4))$.

11. $p_3 \prec (p_3; p_4)$.

12. $p_4 \prec (p_3; p_4)$.

To reason about structural layers in the course of a backward derivation, meta-notations as references to structural layers and their transitions are additionally defined.

Definition 95 (Names) *We define by* $\mathfrak{T} := \{\tau_1, \tau_2, \cdots, \tau_a, \tau_b, \cdots\}$ *the set of names.*

Definition 96 (Named structures/sequents)
A structure $\Gamma \in \mathfrak{S}_{\texttt{BI}_{\texttt{base}}}$ *is said to be named iff, for every structural layer* s *such that* $s \preceq \Gamma$*, there exists a distinct name* τ_s *that refers to* s *(*i.e. *a name used always refers to one structural layer). A sequent* $D \in \mathfrak{D}_{\texttt{BI}_{\texttt{base}}}$ *is said to be named iff the antecedent part of* D *is named.*

I conveniently use the standard terminologies in computer science: $^\star\tau_s$ to denote s and $\&s$ to denote τ_s if τ_s is the name that refers to s.

Definition 97 (Transitions on named sequents) *Let* $D : \Gamma \vdash F$ *be a named* $\texttt{LBIZ}_1$ *sequent. Let* s_D *denote a structural layer within a particular sequent* D*. Let* $\texttt{fresh} : \mathfrak{S}_{\texttt{BI}_{\texttt{base}}} \times \mathfrak{T} \rightarrow \{\boldsymbol{T}, \boldsymbol{F}\}$ *be a predicate such that, for some named* $\texttt{BI}_{\texttt{base}}$ *structure* Γ *and some name* τ*,* $\texttt{fresh}(\Gamma, \tau)$ *iff* τ *is not a name already used for some structural layer in* Γ*. Then we define transitions on the names of* D *in* $D \leadsto_{\boldsymbol{Inf}} D'$ *(*D' *is also a named sequent) as follows for each non-axiom* ***Inf*** *available in* $\texttt{LBIZ}_1$*:*

---

1. *$\wedge R, \vee R$: the antecedent of D' is the same named* $\texttt{BI}_{\texttt{base}}$ *structure as that of D.*

2. *$\supset R$: D and D' look like* $\Gamma \vdash F_1 \supset F_2$ *and respectively* $\Gamma; F_1 \vdash F_2$.

   (a) *if Γ is a $\mathcal{M}$ with at least two constituents, then $\Gamma; F_1$ is a named* $\texttt{BI}_{\texttt{base}}$ *structure such that Γ in D' is the same named* $\texttt{BI}_{\texttt{base}}$ *structure as in D, that* $\texttt{fresh}(\Gamma, \tau)$, *and that* $^\star\tau = (\Gamma; F)_{D'}$.

   (b) *otherwise, if Γ is a $\mathcal{A}$, then for all $s_{D'} \preceq (\Gamma; F_1)_{D'}$:*

      i. *if* $[s_{D'} = (\Gamma_1; F_1)_{D'}]$, *then* $[\& s_{D'} = \& \Gamma_D]$.

      ii. *otherwise, if $s_{D'}$ is the particular F_1 in D', then* $\texttt{fresh}(\Gamma_D, \tau)$ *and* $[\& {F_1}_{D'} = \tau]$.

      iii. *otherwise,* $[s_{D'} = (^\star(\& s_{D'}))_D]$ *(That is to say, informally, that apart from the outermost structural layer in the antecedent of D', all the names are simply carried over from Γ_D to $\Gamma_{D'}$.)*

      such that, for all two distinct names τ_1 and τ_2 shared between D and D' if $^\star\tau_{1D} \prec {^\star\tau_{2D}}$, *then* $^\star\tau_{1D'} \prec {^\star\tau_{2D'}}$. *(That is to say, informally, that all the structural layers in D' are equivalent up to commut. and assoc. to those in D.)*

3. *$*R$: For the internalised weakening Wk_1 and Wk_2:*

$$\frac{\Gamma_1 \vdash F}{\Gamma_1; \Gamma_2 \vdash F}\, WkL_1 \quad \frac{\Gamma_1, \Gamma_2 \vdash F}{\Gamma_1, (\Gamma_2; \Gamma_3) \vdash F}\, WkL_2$$

   we have;

   - $[D : \Gamma_1; \Gamma_2 \vdash F] \rightsquigarrow_{Wk_1} [D' : \Gamma_1 \vdash F]$*: all the names in D are preserved in D' except that, if Γ_2 is not empty, then (1) the name τ_a such that* $\tau_a = \&(\Gamma_1; \Gamma_2)_D$ *and (2) all the names used for ${\Gamma_2}_D$, are absent in D'.*
   - $[D : \Gamma_1, (\Gamma_2; \Gamma_3) \vdash F] \rightsquigarrow_{Wk_2} [D' : \Gamma_1, \Gamma_2 \vdash F]$*: all the names in D are preserved in D' except that, if Γ_3 is not empty, then (1) the name τ_a such that* $\tau_a = \&(\Gamma_2; \Gamma_3)_D$ *and (2) all the names used for ${\Gamma_3}_D$, are absent in D'.*

   With the above preparation, suppose that the internalised weakening produces $D_a : \Gamma_1, \Gamma_2 \vdash F_1 * F_2$, *then D' looks like $\Gamma_i \vdash F_i$ for $i \in \{1, 2\}$. Then the antecedent part of D' is the same named structure Γ_i as in D_a.*

4. *$\wedge$L: D and D' look like $\Gamma(F \wedge G) \vdash H$ and respectively $\Gamma(F; G) \vdash H$. Let τ_1 and τ_2 be two names such that $[\tau_1 \neq \tau_2] \wedge^{\dagger} \texttt{fresh}(\Gamma(F \wedge G), \tau_1) \wedge^{\dagger} \texttt{fresh}(\Gamma(F \wedge G), \tau_2)$. Then for all $s_{D'} \preceq \Gamma(F; G)_{D'}$:*
   1. *(a) if $s_{D'}$ is the focused F, then $[\& s_{D'} = \tau_1]$.*
   2. *(b) otherwise, if $s_{D'}$ is the focused G, then $[\& s_{D'} = \tau_2]$.*
   3. *(c) otherwise, $[s_{D'} = ((^{\star}(\& s_{D'}))_D)_{rep}]$ where t_{rep} is the structural layer that results from replacing, in a structural layer t ($\preceq \Gamma(F \wedge G)$), the particular occurrence of $F \wedge G$ (if any) as the focused part of $\Gamma(F \wedge G)$ with $F; G$.*

   such that, for every two distinct names τ_a and τ_b shared between D and D', if $^{\star}\tau_{aD} \prec {^{\star}\tau_{bD}}$, then $^{\star}\tau_{aD'} \prec {^{\star}\tau_{bD'}}$.

5. *$\vee$L: D and D' look like $\Gamma(F_1 \vee F_2) \vdash H$ and respectively $\Gamma(F_i) \vdash H$ for $i \in \{1, 2\}$. Then for all $s_{D'} \preceq \Gamma(F_i)$, $[s_{D'} = ((^{\star}(\& s_{D'}))_D)_{rep}]$ where t_{rep} is the structural layer that results from replacing, in a structural layer t, the particular occurence of $F_1 \vee F_2$ (if any) as the focused part of $\Gamma(F_1 \vee F_2)$ with F_i.*

6. *$*$L: D and D' look like $\Gamma(F * G) \vdash H$ and respectively $\Gamma(F, G) \vdash H$. Let τ_1 and τ_2 be two names such that $[\tau_1 \neq \tau_2] \wedge^{\dagger} \texttt{fresh}(\Gamma(F * G), \tau_1) \wedge^{\dagger} \texttt{fresh}(\Gamma(F * G), \tau_2)$. Then for all $s_{D'} \preceq \Gamma(F, G)$:*
   1. *(a) if $s_{D'}$ is the focused F in $\Gamma(F, G)$, then $\& s_{D'} = \tau_1$.*
   2. *(b) otherwise, if $s_{D'}$ is the focused G in $\Gamma(F, G)$, then $\& s_{D'} = \tau_2$.*
   3. *(c) otherwise, $[s_{D'} = ((^{\star}(\& s_{D'}))_D)_{rep}]$ where t_{rep} is the structural layer which results from replacing, in a structural layer t, the particular $F * G$ (if any) as occurring in the focused part of $\Gamma(F * G)$.*

   such that, for every two distinct names τ_a and τ_b shared between D and D', if $^{\star}\tau_{aD} \prec {^{\star}\tau_{bD}}$, then $^{\star}\tau_{aD'} \prec {^{\star}\tau_{bD'}}$.

7. *$\supset L_{\{p, \top, \wedge\}}$: D looks like $\Gamma(F) \vdash H$ and D' looks like $\Gamma(F') \vdash H$ where F and F' for each inference rule are:*
   1. *(a) $p; p \supset G$ and $p; G$ ($\supset L_p$).*

(b) $\top \supset G$ *and* $G \quad (\supset L_{\top})$.

(c) $(F_1 \wedge F_2) \supset G$ *and* $F_1 \supset (F_2 \supset G) \quad (\supset L_{\wedge})$.

Then for all $s_{D'} \preceq \Gamma(F')$*:*

(a) *if* $s_{D'}$ *is the focused* G*, then* $\& s_{D'}$ *is* $\&(p \supset G)_D$, $\&(\top \supset G)_D$ *or* $\&((F_1 \wedge F_2) \supset G)_D$ *(for the focused formula) depending on which inference rule,*

(b) *otherwise,* $[s_{D'} = ((\star(\& s_{D'}))_D)_{rep}]$ *where* t_{rep} *is the structural layer that results from replacing, in a structural layer* t*, the particular* $p \supset G$, $\top \supset G$ *or* $(F_1 \wedge F_2) \supset G$*) (if any) as occurring in the focused part of* D *with* G*,* G*, or* $F_1 \supset (F_2 \supset G)$

such that, for every two distinct names τ_a *and* τ_b *shared between* D *and* D'*, if* $\star\tau_{a\,D} \prec \star\tau_{b\,D}$*, then* $\star\tau_{a\,D'} \prec \star\tau_{b\,D'}$*.*

8. $\supset L_{\vee}$*:* D *looks like* $\Gamma((F_1 \vee F_2) \supset G) \vdash H$ *and* D' *like* $\Gamma(F_1 \supset G; F_2 \supset G) \vdash H$*. Let* τ_1 *and* τ_2 *be two names such that* $[\tau_1 \neq \tau_2] \wedge^{\dagger} \texttt{fresh}(\Gamma((F_1 \vee F_2) \supset G), \tau_1) \wedge^{\dagger} \texttt{fresh}(\Gamma((F_1 \vee F_2) \supset G), \tau_2)$*. Then for all* $s_{D'} \preceq \Gamma(F_1 \supset G; F_2 \supset G)_{D'}$*:*

   (a) *if* $s_{D'}$ *is the focused* $F_i \supset G$ *for* $i \in \{1, 2\}$*, then* $[\& s_{D'} = \tau_i]$*,*

   (b) *otherwise,* $[s_{D'} = ((\star(\& s_{D'}))_D)_{rep}]$ *where* t_{rep} *is the structural layer that results from replacing, in a structural layer* t*, the particular* $(F_1 \vee F_2) \supset G$ *(if any) as occurring in the focused part of* D *with* $F_1 \supset G; F_2 \supset G$*,*

   such that, for every two distinct names τ_a *and* τ_b *shared between* D *and* D'*, if* $\star\tau_{a\,D} \prec \star\tau_{b\,D}$*, then* $\star\tau_{a\,D'} \prec \star\tau_{b\,D'}$*.*

9. $\supset L_{\supset}$*:* D *looks like* $\Gamma(\Gamma_1; (F_1 \supset F_2) \supset G) \vdash H$*.* D' *looks like* $\Gamma(\Gamma_1; G) \vdash H$ *if it is the right premise sequent of the inference rule, or like* $\Gamma_1; F_2 \supset G \vdash F_1 \supset F_2$ *if it is the left premise sequent of the inference rule. Assume without loss of generality that* $\Gamma_1; (F_1 \supset F_2) \supset G$ *is an additive structural layer in* $\Gamma(\Gamma_1; (F_1 \supset F_2) \supset G)$*, then for all* $s_{D'} \preceq \Gamma(\Gamma_1; G)_{D'}$ *(if the right premise) or for all* $s_{D'} \preceq (\Gamma_1; F_2 \supset G)_{D'}$ *(if the left premise):*

   (a) *if $s_{D'}$ is the focused G (if the right premise) or is the focused $F_2 \supset G$ (if the left premise), then $[\& s_{D'} = \&((F_1 \supset F_2) \supset G)_D]$ for the focused $(F_1 \supset F_2) \supset G$ in D.*

   (b) *otherwise, $[s_{D'} = ((^\star(\& s_{D'}))_D)_{rep}]$ where t_{rep} is the structural layer that results from replacing, in a structural layer t, the particular $(F_1 \supset F_2) \supset G$ (if any) as occurring in the focused part of D with G (if the right premise) or with $F_2 \supset G$ (if the left premise);*

such that, for every two distinct names τ_a and τ_b shared between D and D', if ${}^\star\tau_{aD} \prec {}^\star\tau_{bD}$, then ${}^\star\tau_{aD'} \prec {}^\star\tau_{bD'}$.

10. *$\supset L_*$: D looks like $\Gamma(\Gamma_1; (F_1 * F_2) \supset G) \vdash H$. D' looks like $\Gamma(\Gamma_1; G) \vdash H$ if it is the right premise sequent of the inference rule, or like $\Gamma_1 \vdash F_1 * F_2$ if it is the left premise sequent of the inference rule. Assume without loss of generality that the focused $\Gamma_1; (F_1 * F_2) \supset G$ is an additive structural layer in D, then for all $s_{D'} \preceq \Gamma(\Gamma_1; G)_{D'}$ (if the right premise) or for all $s_{D'} \preceq \Gamma_{1\,D'}$ (if the left premise):*

   (a) *if $s_{D'}$ is the focused G (if the right premise), then $[\& s_{D'} = \&((F_1 * F_2) \supset G)_D]$ for the focused $(F_1 * F_2) \supset G$ in D,*

   (b) *otherwise, $[s_{D'} = ((^\star(\& s_{D'}))_D)_{rep}]$ where t_{rep} is the structural layer that results from replacing, in a structural layer t, the particular $(F_1 * F_2) \supset G$ (if any) as occurring in the focused part of D with G (if the right premise) or with emptiness (if the left premise);*

such that, for every distinct names τ_a and τ_b shared between D and D', if ${}^\star\tau_{aD} \prec {}^\star\tau_{bD}$, then ${}^\star\tau_{aD'} \prec {}^\star\tau_{bD'}$.

5.4.2.2 $\texttt{LBIZ}_1$ permutation properties

Lemma 28 (Permutation of additive inference rules) *Let $D : \Gamma \vdash F$ be a named $\texttt{LBIZ}_1$ sequent. Then, for any transition $D \rightsquigarrow^{+}_{\textit{\textbf{Infs}}} D'$ with a set of inference rules **Infs** that includes all the left inference rules but no right inference rules, and for any two additive structural layers ${}^\star\tau_1$ and ${}^\star\tau_2$ such that ${}^\star\tau_1 \prec {}^\star\tau_2 \preceq \Gamma$, it holds that if an additive inference rule **Inf** $\in$ **Infs** applies at ${}^\star\tau_1$, there exists a transition $D \rightsquigarrow^{*}_{\textit{\textbf{Infs}}}$*

$D_a \rightsquigarrow_{\textit{Inf}} D_b \rightsquigarrow^*_{\textit{Infs}} D'$ such that no additive inference rules apply at ${}^\star\tau_2$ in the transition $D_b \rightsquigarrow^*_{\textit{Infs}} D'$.

Proof. Assume, without loss of generality, that a name, once introduced, will never be introduced anew later in a transition. First prove for a specific case where ${}^\star\tau_2$ is the outermost additive structural layer in Γ, *i.e.* either ${}^\star\tau_2 = \Gamma$ or there exists a structural layer u such that $[u = \Gamma] \wedge^\dagger (\forall u' \preceq \Gamma.[{}^\star\tau_2 \prec u'] \rightarrow^\dagger [u' = u])$. Suppose that no additive inference rules apply at ${}^\star\tau_2$, then the condition vacuously holds. Suppose that no additive inference rules apply at ${}^\star\tau_2$ in $D_b \rightsquigarrow^*_{\textbf{Infs}} D'$, then again vacuous. Otherwise, we would originally have the following transition: $D \rightsquigarrow^*_{\textbf{Infs}} D_c \rightsquigarrow_{\textbf{Inf}} D_d \rightsquigarrow^*_{\textbf{Infs}} D_e \rightsquigarrow_{\textbf{Inf'}} D_f \rightsquigarrow^*_{\textbf{Infs}} D'$ such that **Inf'** is the first additive left inference rule that applies at ${}^\star\tau_2$ in $D_d \rightsquigarrow^*_{\textbf{Infs}} D_e \rightsquigarrow_{\textbf{Inf'}} D_f \rightsquigarrow^*_{\textbf{Infs}} D'$. Proof is then by induction on the number of additive inference rules that apply at ${}^\star\tau_2$ in the transition: $D_e \rightsquigarrow_{\textbf{Inf'}} D_f \rightsquigarrow^*_{\textbf{Infs}} D'$. Base case was just exhibited (*i.e.* no such **Inf'**). For inductive cases of the sub-induction, assume that the present case holds true for all the cases where the total number of additive inference rules that apply at ${}^\star\tau_2$ are l or less, and prove that it still holds true for a total number of $l + 1$. Consider which additive left inference rule **Inf'** is:

1. $\wedge L$: Then the principal for **Inf'** is in the form: $H_1 \wedge H_2$ in the antecedent part of D_e. But we have inversion lemma that replaces $H_1 \wedge H_2$ with $H_1; H_2$ in $\texttt{LBIZ}_1$. Now because we have ${}^\star\tau_1 \prec {}^\star\tau_2 \preceq \Gamma$ in D, ${}^\star\tau_2$ must be in the form: $H_1 \wedge H_2; \Gamma'$ such that ${}^\star\tau_1 \prec \Gamma'$. Then it is immediate that the inversion can apply prior to **Inf**: we achieve the effect through $\wedge L$. There now remain only l additive inference rules to apply at ${}^\star\tau_2$ in $D_f \rightsquigarrow^*_{\textbf{Infs}} D'$.

2. $\vee L$: Similar.

3. $\supset L_{\{p,\top,\wedge,\vee\}}$: Similar.

4. $\supset L_{\{\supset,*\}}$: if D_f is the right premise sequent of $\supset L_{\{\supset,*\}}$ with D_e as its conclusion sequent, then reasoning is similar to the $\wedge L$ case. Otherwise, if D_f is the left premise sequent of $\supset L_{\{\supset,*\}}$, then firstly notice that **Inf** cannot be $\supset L_{\{\supset,*\}}$, since the derivation step would bring ${}^\star\tau_1$ at the outermost structural layer in the antecedent part, *i.e.* there could no longer be any additive structural layer u in the antecedent part such that ${}^\star\tau_1 \prec u$. Though fairly obvious with an informal

means, we nevertheless complete the present sub-proof by sub-induction on the length k of the transition $D_d \rightsquigarrow^k_{\mathbf{Infs}} D_e$. If it is 0, then the original transition looks like: $D \rightsquigarrow^*_{\mathbf{Infs}} D_c \rightsquigarrow_{\mathbf{Inf}} D_d \rightsquigarrow_{\mathbf{Inf'}} D_f \rightsquigarrow^*_{\mathbf{Infs}} D'$. The order of **Inf** and **Inf'** can be then swapped: $D \rightsquigarrow^*_{\mathbf{Infs}} D_c \rightsquigarrow_{\mathbf{Inf'}} D_x \rightsquigarrow_{\mathbf{Inf}} D_f \rightsquigarrow^*_{\mathbf{Infs}} D'$ such that there remain only l additive inference rules to apply at ${}^\star\tau_2$ in $D_f \rightsquigarrow^*_{\mathbf{Infs}} D'$.

For inductive cases of the sub-induction, assume that $D \rightsquigarrow^*_{\mathbf{Infs}} D_c \rightsquigarrow_{\mathbf{Inf}} D_d \rightsquigarrow^k_{\mathbf{Infs}} D_e \rightsquigarrow_{\mathbf{Inf'}} D_f \rightsquigarrow^*_{\mathbf{Infs}} D'$ can be permuted into $D \rightsquigarrow^*_{\mathbf{Infs}} D_1 \rightsquigarrow_{\mathbf{Inf'}} D_2 \rightsquigarrow^*_{\mathbf{Infs}} D_3 \rightsquigarrow_{\mathbf{Inf}} D_4 \rightsquigarrow^*_{\mathbf{Infs}} D'$ such that there remain only l additive inference rules to apply at ${}^\star\tau_2$ in $D_4 \rightsquigarrow^*_{\mathbf{Infs}} D'$, and show that such permutation is bound to be successful also for $D \rightsquigarrow^*_{\mathbf{Infs}} D_c \rightsquigarrow_{\mathbf{Inf}} D_d \rightsquigarrow^k_{\mathbf{Infs}} D_{d'} \rightsquigarrow_{(\mathbf{Inf}_a \in \mathbf{Infs})} D_e \rightsquigarrow_{\mathbf{Inf'}} D_f \rightsquigarrow^*_{\mathbf{Infs}} D'$. Consider what $\mathbf{Inf}_a$ is in the transition $D_{d'} \rightsquigarrow_{\mathbf{Inf}_a} D_e$. If $\mathbf{Inf}_a$ applies at some constituent of a multiplicative structural layer with two or more constituents, then almost vacuous. Otherwise:

(a) $\wedge L$: then, at whichever additive structural layer $u(\prec {}^\star\tau_2)$ it applies, we have the following permuted transition: $D \rightsquigarrow^*_{\mathbf{Infs}} D_c \rightsquigarrow_{\mathbf{Inf}} D_d \rightsquigarrow^k_{\mathbf{Infs}} D_{d'} \rightsquigarrow_{\mathbf{Inf'}} D_x \rightsquigarrow_{\mathbf{Inf}_a} D_f \rightsquigarrow^*_{\mathbf{Infs}} D'$.

(b) $\vee L$: similar.

(c) $\supset L_{\{\top,\wedge,\vee\}}$: similar.

(d) $\supset L_{\{\supset,*\}}$: then, D_f cannot be the left premise sequent of the inference rule since it applies at $u \prec {}^\star\tau_2$. But then similar.

(e) $*L$: similar.

Hence, in each of the cases above, induction hypothesis of the sub-induction concludes.

Then the present case concludes by induction hypothesis of the main induction.

For a general case where ${}^\star\tau_2$ is some additive structural layer which is not necessarily the outermost, the proof approach is similar to the above specific case (which acts as the base case). □

Permutability of multiplicative inference rules is proved in a similar but by far simpler manner.

Lemma 29 (Permutation of multiplicative inference rules)
Let $D : \Gamma \vdash F$ *be a named* $\texttt{LBIZ}_1$ *sequent. Then, for any transition* $D \rightsquigarrow^{+}_{\textbf{\textit{Infs}}} D'$ *with a set of inference rules* ***Infs*** *that includes all the left inference rules but no right inference rules, and for any two multiplicative structural layers* $^{\star}\tau_1$ *and* $^{\star}\tau_2$ *such that* $^{\star}\tau_1 \prec {^{\star}\tau_2} \preceq \Gamma$*, it holds that if* $*L$ *applies at* $^{\star}\tau_1$*, there exists a transition* $D \rightsquigarrow^{*}_{\textbf{\textit{Infs}}} D_a \rightsquigarrow_{*L} D_b \rightsquigarrow^{*} D'$ *such that* $*L$ *does not apply at* $^{\star}\tau_2$ *in the transition* $D_b \rightsquigarrow^{*}_{\textbf{\textit{Infs}}} D'$*.*

Proof. Assume, without loss of generality, that a name, once introduced, will never be introduced anew later in a transition. Because multiplicative unit is not present in $\texttt{BI}_{\texttt{base}}$, any multiplicative structural layer in the form: "Γ_a, Γ_b" can neither be Γ_a nor be Γ_b. And because no multiplicative implication is present in $\texttt{BI}_{\texttt{base}}$, a multiplicative structural layer, once generated, cannot reduce in the number of constituents without a $*R$ which, by the way, is not in **Infs**. It holds for all $\texttt{BI}_{\texttt{base}}$ formulas in the form $G_1 * G_2 (\preceq \Gamma)$ that $^{\star}\tau_1 \not\prec G_1 * G_2$. Then due to the assumption that $^{\star}\tau_1 \prec {^{\star}\tau_2} \preceq \Gamma$, we simply bring all the $*L$ to apply at $^{\star}\tau_2$ before any $*L$ at $^{\star}\tau_1$ by $\texttt{LBIZ}_1$ inversion. □

Definition 98 (Permutation-inversion normal $\texttt{LBIZ}_1$ derivations)
A permutation-inversion normal $\texttt{LBIZ}_1$ *derivation is a* $\texttt{LBIZ}_1$ *derivation such that*

1. *for all transitions* $D \rightsquigarrow^{*}_{\textbf{\textit{Infs}}} D_1$ *for a set of inference rules which includes all the* $\texttt{LBIZ}_1$ *left inference rules but no* $\texttt{LBIZ}_1$ *right inference rules, (assuming without loss of generality that (1)* D *is named and that (2) a name, once introduced, will never be introduced anew later), it holds that for any two additive structural layers* $^{\star}\tau_1$ *and* $^{\star}\tau_2$ *such that* $^{\star}\tau_1 \prec {^{\star}\tau_2}$*, any additive left inference rules to apply at* $^{\star}\tau_2$ *applies before any left inference rules to apply at* $^{\star}\tau_1$*.*

2. *for all transitions* $D \rightsquigarrow^{*}_{\textbf{\textit{Infs}}} D_1$ *for a set of inference rules which includes all the* $\texttt{LBIZ}_1$ *left inference rules but no* $\texttt{LBIZ}_1$ *right inference rules, (assuming without loss of generality that (1)* D *is named and that (2) a name, once introduced, will never introduced anew later), it holds that for any two multiplicative structural layers* $^{\star}\tau_1$ *and* $^{\star}\tau_2$ *such that* $^{\star}\tau_1 \prec {^{\star}\tau_2}$*, any* $*L$ *to apply at* $^{\star}\tau_2$ *applies before any* $*L$ *to apply at* $^{\star}\tau_1$*.*

3. *$\wedge R, \supset R$ apply as soon as any one of them becomes applicable. This rule has a higher priority than the following two.*

4. *$\vee R, *R$ apply as soon as either of them becomes applicable preserving derivability upwards. This rule has a higher priority than the last rule.*

5. *$*L, \wedge L, \vee L, \supset L_{\{\wedge,\vee,p,\top\}}$ apply as soon as any one of them becomes applicable.*

5.4.2.3 Main results

The second part of the equivalence proof now follows.

Proposition 24 *If $D : \Gamma \vdash F \in \mathfrak{D}_{\texttt{BI}_{\texttt{base}}}$ is* $\texttt{LBIZ}_1$*-derivable, then ${}^k\Omega_i \vdash_i F (\in \mathfrak{D}_{\texttt{PBI}})$ is* `PBI`*-derivable for any $k \geq 0$ (such that there exists, in the antecedent of D, at least one focusable structure with the k-th context degree) and ${}^k\Omega_i = \Gamma$.*[1]

Proof. The proof is most general with $k = 0$, due to the availability of `Revert` $\uparrow$ and `Revert` $\downarrow$. Hence, it suffices to prove only the case. Without loss of generality, assume $\Pi(D)$ to be permutation-inversion normal. Also we do not consider unnecessarily longer derivations than necessary where an axiom is involved: if some axiom(s) and some non-axiom are both applicable to a sequent D' preserving derivability upwards, it is (one of the) axiom(s) that applies. Proof is by induction on the derivation depth of $\Pi(D)$ and a sub-induction on sequent weight (*Cf.* Definition 61). For base cases, consider which axiom has applied.

1. id: If the antecedent part is a formula, namely a propositional variable, apply $SL_1 id$ or $SL_2 id$ depending on the value of i. Otherwise, apply $SL_2 id$ possibly with the help of `Transfer` $\downarrow$.

2. $\mathbb{1}L$: Then there exists at least one $\mathbb{1}$ in the antecedent part of D which becomes active for $\mathbb{1}L$ (in $\texttt{LBIZ}_1$). Then one such $\mathbb{1}$ can be revealed by (a sequence of) $\mathbb{1}$`Peel` $\uparrow$, $\mathbb{1}$`Peel` $\downarrow$. Then possibly with the aid of `Transfer` $\downarrow$, $\mathbb{1}L$ (in `PBI`) concludes.

3. $\top R$: trivial with $\top R$ (in `PBI`) possibly with the aid of `Transfer` $\downarrow$.

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[1] Readers are kindly reminded that, for any $k > 0$, the value of i is always determined.

For inductive cases, assume that the current proposition holds true for all the permutation-normal derivations of D with the derivation depth not greater than j and show that it still holds true for permutation-normal derivations of derivation depth $j+1$. Consider what the last $\texttt{LBIZ}_1$ inference rule applied is:

1. $\wedge R$: apply $\wedge R$ (in `PBI`; similarly for the rest) possibly with the help of `Transfer` $\downarrow$.
2. $\supset R$: apply $\supset R$ possibly with the help of `Transfer` $\downarrow$
3. $\vee R$: apply $\vee R$ possibly with the help of `Transfer` $\downarrow$.
4. $*R$: $\texttt{LBIZ}_1$ incremental weakening is achieved via $*$`Lock` $\uparrow$ and $*$`Lock` $\downarrow$ (possibly with the help of `Transfer` $\uparrow$). Then apply $*R$.
5. $\wedge L$: say that the principal is $F_a \wedge F_b$. If the $\texttt{LBIZ}_1$ inference rule applies at the outermost additive structural layer (in the antecedent part of the $\texttt{LBIZ}_1$ sequent), then $\wedge L$ concludes, possibly with the help of `Transfer` $\downarrow$ to focus the particular occurrence of the principal. If it does not apply at the outermost additive structural layer, then consider what form the consequent formula of the sequent is in:
    (a) p: by the stated assumption at the beginning of the present proposition, neither id nor $\mathbb{1}L$ is applicable to the sequent. Then there exists at least one multiplicative structural layer. Suppose that we have, in Γ, o (≥ 1) outermost multiplicative structural layers: $u_1, \cdots, u_o$. Since $\Pi(D)$ is permutation-inversion normalised, D cannot be derived unless there is some $u_i = \mathcal{A}_1, \cdots, \mathcal{A}_l$ for $1 \leq i \leq o$ and $2 \leq l$ such that $\mathcal{A}_m \vdash \mathbb{1}$, $1 \leq m \leq l$, is $\texttt{LBIZ}_1$-derivable. If the principal does not occur in $\mathcal{A}_m$, then vacuous; induction hypothesis of the sub-induction concludes, otherwise.
    (b) $\mathbb{1}$: similar.
    (c) $H_1 \vee H_2$: by the definition of a permutation-inversion normal derivation, $\vee R$ does not apply either because the antecedent part is inconsistent,[1] or because some $\vee L$(s) must be first handled. For the former, a similar approach to the previous sub-cases holds. For the latter, we have `Transfer` $\downarrow$, `Revert` $\downarrow$,

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[1] By an antecedent structure Γ being inconsistent, I mean that $\Gamma \vdash \mathbb{1}$ is derivable.

$\vee$`Lock` $\downarrow$, `Transfer` $\uparrow$, `Revert` $\uparrow$, $\vee$`Lock` $\uparrow$ to focus relevant structural layers. Then apply $\wedge L$ on the principal $F_a \wedge F_b$.

   (d) $H_1 * H_2$: by the definition of a permutation-inversion normal derivation, $*R$ does not apply either because the antecedent part is inconsistent or because inner structural layers must be first processed and incremental weakening must be then applied to conjoin relevant inner multiplicative structural layers at the outermost multiplicative structural layer. Straightforward in case it is inconsistent. Otherwise, if $\wedge L$ must apply on $F_a \wedge F_b$ in order that D be $\texttt{LBIZ}_1$-derivable, then the outermost additive structural layer of D must - because by the stated assumption $F_a \wedge F_b$ is not a constituent of the outermost additive structural layer - look like $\mathcal{M}_1; \cdots ; \mathcal{M}_m$ for $2 \leq m$ such that there exist at least two of the constituents, say $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ for which it holds either that $[(F_a \wedge F_b) \prec \mathcal{M}_\alpha]$ or that $[(F_a \wedge F_b) \prec \mathcal{M}_\beta]$. But then the constituent which does not hold $F_a \wedge F_b$ as a sub-structure is irrelevant to the $\texttt{LBIZ}_1$-derivability of D due to Lemma 13 and Corollary 1. By admissibility of weakening in $\texttt{LBIZ}_1$, then, we simply weaken the irrelevant structure, thereupon applies induction hypothesis of the sub-induction.

6. $\vee L$: similar.

7. All variations of $\supset L$: similar.

8. $*L$: similar with one difference that we be looking at multiplicative structural layers instead.

Induction hypothesis of the main induction concludes. $\square$

Theorem 12 (Equivalence of `PBI` with $\texttt{LBIZ}_1$) `PBI` *is as expressive as* $\texttt{LBIZ}_1$.

Proof. Follows from Proposition 23 and Proposition 24. $\square$

5.5 A $\texttt{BI}_{\texttt{base}}$ Decision Procedure with `PBI`

In this section I present one decision procedure, exhibiting a basic proof search methodology in phased sequent calculus. The emphasis is not on efficiency.

The termination argument is made rather intuitive and simple thanks to the physical separation between the base components rendering feasible a compositional approach. In fact, there is little difficulty involved in the proof of a finite derivation growth in each individual phase, since SL_1 derives from propositional intuitionistic logic and SL_2 from a fragment of propositional multiplicative intuitionistic linear logic, each of which is decidable (and is hence terminating). Therefore, if we succeed in finding a proof search tactic that eliminates infinite transitions between states in Figure 5.2, then the desired result follows due to (1) the finiteness of a derivation within every phase and (2) the finiteness of interaction transitions.

5.5.1 Preparations

For convenience, a phase in a `PBI` derivation tree is differentiated from later (when the derivation tree is looked backwards) phases with, for instance, a monotonously increasing super-script of a positive integer n: $SL_1^n \to SL_2^{n+1} \to SL_1^{n+2} \to SL_2^{n+3} \cdots$ or $SL_2^n \to SL_1^{n+1} \to SL_2^{n+2} \to SL_1^{n+3} \cdots$ depending on the initial phase of the given conclusion sequent $D \in \mathfrak{D}_{\texttt{PBI}}$. I first state an inversion lemma.

Definition 99 (Phase depth)
A phase depth of a `PBI` *sequent* $D \in \mathfrak{D}_{\texttt{PBI}}$*,* i.e. $\texttt{phase\_depth}(D)$*, is defined to be a relative derivation depth with respect to the particular phase in which it is, with the following inductive definition.*

- *it is 1 if (1)* D *is the conclusion sequent of a* SL_1 *or* SL_2 *axiom or if (2) there exists a transition* $D \rightsquigarrow D'$ *such that* $[D \in \mathfrak{D}_{SL_1}] \wedge^{\dagger} [D' \in \mathfrak{D}'_{SL_2}]$ *or* $[D \in \mathfrak{D}_{SL_2}] \wedge^{\dagger} [D' \in \mathfrak{D}'_{SL_1}]$.
- *it is* $\texttt{phase\_depth}(D') + 1$ *if there exits a transition* $D \rightsquigarrow_{\textbf{\textit{Inf}}} D'$ *for some one-premised* SL_1 *(resp.* SL_2*) inference rule* ***Inf*** *and some* $D' \in \mathfrak{D}_{SL_1}$ *(resp.* $D' \in \mathfrak{D}_{SL_2}$*).*

- *it is* $1 + max(\texttt{phase\_depth}(D'), \texttt{phase\_depth}(D''))$ *if there exist transitions* $D \rightsquigarrow_{\textbf{\textit{Inf}}} D'$ *and* $D \rightsquigarrow_{\textbf{\textit{Inf}}} D''$ *such that* ***Inf*** *is a two-premised* SL_1 *(resp.* SL_2*) inference rule and that* $D', D'' \in \mathfrak{D}_{SL_1}$ *(resp.* $D', D'' \in \mathfrak{D}_{SL_2}$*) are the two premise sequents.*

Lemma 30 (`PBI` inversion lemma) *In the following pairs of* `PBI` *sequents, if the sequent shown on the left is* `PBI`*-derivable within the phase depth* i*, then so is (are) the sequent(s) shown on the right.* $k \geq 0$ *and* $j \in \{0, 1\}$.

$$ {}^k\Omega_1(\langle\Gamma\rangle, F * G) \vdash_1 H \qquad {}^k\Omega_1(\langle\Gamma\rangle, F, G) \vdash_1 H \tag{5.1} $$

$$ {}^k\Omega_2(\Gamma; F \wedge G) \vdash_2 H \qquad {}^k\Omega_2(\Gamma; F; G) \vdash_2 H \tag{5.2} $$

$$ {}^k\Omega_2(\Gamma; F \vee G) \vdash_2 H \qquad \textit{both } {}^k\Omega_2(\Gamma; F) \vdash_2 H \textit{ and } {}^k\Omega_2(\Gamma; G) \vdash_2 H \tag{5.3} $$

$$ {}^k\Omega_2(\Gamma; p \supset F; p) \vdash_2 H \qquad {}^k\Omega_2(\Gamma; p; F) \vdash_2 H \tag{5.4} $$

$$ {}^k\Omega_2(\Gamma; \top \supset F) \vdash_2 H \qquad {}^k\Omega_2(\Gamma; F) \vdash_2 H \tag{5.5} $$

$$ {}^k\Omega_2(\Gamma; (F_a \wedge F_b) \supset F_c) \vdash_2 H \qquad {}^k\Omega_2(\Gamma; F_a \supset (F_b \supset F_c)) \vdash_2 H \tag{5.6} $$

$$ {}^k\Omega_2(\Gamma; (F_a \vee F_b) \supset F_c) \vdash_2 H \qquad {}^k\Omega_2(\Gamma; (F_a \supset F_c); (F_b \supset F_c)) \vdash_2 H \tag{5.7} $$

$$ {}^k\Omega_2(\Gamma; (F_a \supset F_b) \supset F_c) \vdash_2 H \qquad {}^k\Omega_2(\Gamma; F_c) \vdash_2 H \tag{5.8} $$

$$ {}^k\Omega_2(\Gamma; (F_a * F_b) \supset F_c) \vdash_2 H \qquad {}^k\Omega_2(\Gamma; F_c) \vdash_2 H \tag{5.9} $$

$$ {}^j\Omega_2 \vdash_2 F \wedge G \qquad \textit{both } {}^j\Omega \vdash_2 F \textit{and } {}^j\Omega \vdash_2 G \tag{5.10} $$

$$ {}^j\Omega_2 \vdash_2 F \supset G \qquad {}^0\Omega_2(\Gamma; F) \vdash_2 G \quad ([{}^j\Omega_2 = \Gamma]) \tag{5.11} $$

Proof. Induction hypothesis on the total number of phases to appear in the derivation of the `PBI` sequent and a sub-induction on the phase depth. Standard, otherwise. *Cf.* Lemma 1, Lemma 7, Lemma 12 and Lemma 14. □

As usual, all the axioms and fully invertible inference rules are safe to apply at any point during a derivation.

Definition 100 (Safe/unsafe rules) *We define safe and unsafe inference rules for all* `PBI` *inference rules* $\textbf{\textit{Inf}} \notin \textbf{\textit{Inf}}_I$.

Safe rules: $SL_1id, SL_2id, \mathbb{1}L, *L, \top R, \wedge L, \vee L, \supset L_{\{p,\top,\wedge,\vee\}}, \wedge R, \supset R.$

Unsafe rules: $*R, \supset L_{\{\supset,*\}}, \vee R$.

5.5.2 Detailing the intention of the mediator

I now detail the intention of the mediator in order to finitely restrict the use of interaction rules. To simplify the discussion, we may assume some 'initial' $\texttt{PBI}$ sequent (though not at all any necessity other than for simplification) of 0-th context degree (*Cf.* Proposition 24). Without loss of generality, we may also assume that every inference rule applicable in a phase while preserving derivability upwards applies before an interaction rule is called.

Thus setting up premises for reasoning, we recall that each interaction rule reflects certain intention of the mediator. For example, $\mathbb{1}\texttt{Peel} \uparrow$ and $\mathbb{1}\texttt{Peel} \downarrow$ both suppose that the antecedent is inconsistent[1] at inner structural layer(s) and (upward) discard, on the supposition, the outermost structural layer but one that holds the suspected structural layer. By this, we learn that the intention of the mediator relays along the derivation upwards, possibly resulting in a chain of $\mathbb{1}\texttt{Peel} \downarrow$ and $\mathbb{1}\texttt{Peel} \uparrow$. The particular derivation of the premise sequent closes only if what the mediator supposed turns out to be correct: that the antecedent is inconsistent and that structures to be peeled away are irrelevant to identification of the inconsistency. If it is inaccurate, then the derivation can fail and another guess may be made. There then would be no need for $\texttt{Transfer} \uparrow, \texttt{Transfer} \downarrow, \texttt{Revert} \uparrow, \texttt{Revert} \downarrow$ once a guess is made. Then, there cannot be an infinite phase switching to be initiated via $\mathbb{1}\texttt{Peel} \downarrow$ or $\mathbb{1}\texttt{Peel} \uparrow$.

A similar observation holds true also for $*\texttt{Lock} \uparrow\downarrow$ and $\vee\texttt{Lock} \uparrow\downarrow$. First, in the case of $*\texttt{Lock}$, the intention of the mediator is such that a consequent formula in the form: $H_1 * H_2$ be only locked via $*\texttt{Lock} \downarrow$ in a SL_2 phase, which is released via $*\texttt{Lock} \uparrow$: there is a handshake between $*\texttt{Lock} \downarrow$ and $*\texttt{Lock} \uparrow$. Second, in the case of $\vee\texttt{Lock}$, it is $\vee\texttt{Lock} \uparrow$ that locks a consequent formula in the form: $H_1 \vee H_2$ into a SL_1 phase. The intention of the mediator continues to be effective in the SL_1 phase, which shifts phases, upon completion of applications of required $*L$s, into the next SL_2 phase, such that all the $\vee L$s that must apply before $\vee R$ on the particular $H_1 \vee H_2$ be processed. While the intention is effective, there is no need for $\texttt{Transfer} \uparrow, \texttt{Transfer} \downarrow, *\texttt{Lock} \uparrow, *\texttt{Lock} \downarrow$ ($\texttt{Revert} \uparrow$ may still apply, as it is possible that some additive structural layer is or

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[1] By the antecedent Γ of a sequent $\Gamma \vdash F$ being inconsistent, I mean that $\Gamma \vdash \mathbb{1}$ is derivable.

becomes, for example, a formula in the form: $G_1 * G_2$).[1] Once the mediator judges (which may be accurate or inaccurate) that all the $\vee L$ that must be treated have been treated, $\vee R$ applies in a SL_2 with the help of `Revert` ↑, `Revert` ↓ to bring the degree of the context to 0 or 1 (depending on whether the outermost structural layer is additive or multiplicative).

Finally for `Transfer` ↑ and `Transfer` ↓, their intention is to switch phases so that what are unknown in the current phase be passed to the other phase which recognises them, and that a progress in derivation be consequently made.

5.5.3 Main results

Observations were made about the intention of the mediator in the previous subsection, which is reflected within the algorithm $\oint$ below. All the safe rules may apply unconditionally and in any order, whereas unsafe rules and transfer rules are generally non-deterministic because of a choice involved in:

- whether the rule should apply at all (derivability may not be upward preserved).
- how to project when $*$`Lock` ↑ applies.
- how to divide constituents of the outermost multiplicative structural layer (as the outermost structural layer) in two upon $*R$.
- which applicable inference rule should apply.

These need taken into account during a proof search by means of an information record, allowing a potential backtrack (all the relevant information such as the values of pointers and so on are to be recorded).

Algorithm $\oint$:

Input: a sequent ${}^0\Omega_i \vdash_i F \in \mathfrak{D}_{\texttt{PBI}}$ with i set either 1 or 2 appropriately.

Output: `true` or `false`.

(Remark: `curr` keeps track of the sequent to be processed. A Boolean variable `modified` indicates that in the current phase there applied at least one logical

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[1] $G_1 * G_2$ as the only one constituent of a focusable additive structural layer with the context degree of k is, in a SL_1 phase, a constituent of a multiplicative structural layer focusable *not* with the $(k+1)$th-degree context but with the $(k-1)$th-degree context.

inference rule. Its initial value is `true`. A string variable `Interact` records a label of certain interaction rule. The last two are used to prevent an infinite switching of phases.)

init: set a pointer `curr` to refer to the root `PBI` sequent ${}^0\Omega_i \vdash_i F$. Set `modified` to `true`. Set `Interact` to “irrelevant”.

$\mathbf{L_1}$: if $^\star\texttt{curr} \in \mathfrak{D}_{SL_2}$, go to $\mathbf{L_{SL_2}}$.

$\mathbf{L_{SL_1}}$: // In a $\mathbf{SL_1}$ phase

1. If a safe rule **Inf** is applicable to $^\star\texttt{curr}$, take the following derivation step: $^\star\texttt{curr} \rightsquigarrow_{\textbf{Inf}} D'$. Set `modified` to `true`. Set `curr` to refer to D'. Go to $\mathbf{L_{SL_1}}$.
2. If the consequent part of $^\star\texttt{curr}$ is a $\mathbb{1}$ and if $\mathbb{1}\texttt{Peel}\downarrow$ is applicable at all, then record a backtrack point and take the following derivation step: $^\star\texttt{curr} \rightsquigarrow_{\mathbb{1}\texttt{Peel}\downarrow} D'$. Set `Interact` to “irrelevant”. Set `modified` to `false`. Set `curr` to refer to D'. Go to $\mathbf{L_1}$.
3. If the consequent part of $^\star\texttt{curr}$ is in the form: $H_1 * H_2$ and if **Inf** $\in \{*R, \mathbb{1}\texttt{Peel}\downarrow, *\texttt{Lock}\downarrow\}$ is applicable at all, then record a backtrack point and take the following derivation step: $^\star\texttt{curr} \rightsquigarrow_{\textbf{Inf}} D'$ where D' is the only premise or the right premise in case **Inf** is $*R$. If **Inf** is $*R$, set `modified` to `true`, and store the left premise sequent and the value of `Interact` for a future processing. If **Inf** is $*\texttt{Lock}\downarrow$, set `Interact` to “$*\texttt{Lock}\downarrow$”. If **Inf** is $\mathbb{1}\texttt{Peel}\downarrow$, set `Interact` to “irrelevant”. Set `modified` to `false`. Set `curr` to refer to D'. Go to $\mathbf{L_1}$.
4. If the consequent part of $^\star\texttt{curr}$ is in the form: $H_1 \vee H_2$, and if `Interact` is “$\vee\texttt{Lock}\uparrow$”, then apply $\vee\texttt{Lock}\downarrow$ if possible at all: $^\star\texttt{curr} \rightsquigarrow_{\vee\texttt{Lock}\downarrow} D'$. Set `curr` to refer to D'. Set `modified` to `false`. Go to $\mathbf{L_1}$.
5. If `Interact` is not “$\vee\texttt{Lock}\uparrow$” and if the context-degree is 1, then take the following derivation step: $^\star\texttt{curr} \rightsquigarrow_{\texttt{Revert}\downarrow} D'$. Set `modified` to `false`. Set `curr` to refer to D'. Set `Interact` to “irrelevant”. Go to $\mathbf{L_1}$.
6. If `modified` and if **Inf** $\in \{\texttt{Transfer}\downarrow, \mathbb{1}\texttt{Peel}\downarrow\}$ is applicable at all, then record a backtrack point and apply **Inf**: $^\star\texttt{curr} \rightsquigarrow_{\textbf{Inf}} D'$. Set `Interact` to “irrelevant”. Set `modified` to `false`. Set `curr` to refer to D'. Go to $\mathbf{L_1}$.
7. Go to $\mathbf{L_E}$.

$\mathbf{L_{SL_2}}$: // In a SL_2 phase

8. If a safe rule **Inf** is applicable to *`curr`, take the following derivation step: *`curr` $\leadsto_{\textbf{Inf}} D'$ where D' is the only or the right premise sequent of **Inf**. In case **Inf** generates two premise sequents, store the left premise sequent and the value of `Interact` for a future processing. Set `modified` to `true`. Set `curr` to refer to D'. Go to $\mathbf{L_{SL_2}}$.
9. If there exists a formula in the form: $(H_1 \supset H_2) \supset H_3$ or $(H_1 * H_2) \supset H_3$ as a constituent of a focusable structural layer, then first record a backtrack point and make a judgement as to whether, upon application of $\supset L_\supset$ or $\supset L_*$ on the formula, the left premise sequent is `PBI`-derivable. If it is judged not to be the case, then delete the backtrack point just recorded and do nothing; otherwise, take the following derivation step on the particular formula: *`curr` $\leadsto_{\supset L_\supset} D'$ or *`curr` $\leadsto_{\supset L_*} D'$ where D' is the right premise sequent of the inference rule. Store the left premise sequent and the value "irrelevant" for a future processing. Set `modified` to `true`. Set `curr` to refer to D'. Go to $\mathbf{L_{SL_2}}$.
10. If the consequent part of *`curr` is a $\mathbb{1}$, and if $\mathbb{1}$`Peel` $\uparrow$ is applicable at all, then record a backtrack point and take the following derivation step: *`curr` $\leadsto_{\mathbb{1}\texttt{Peel}\uparrow} D'$. Set `Interact` to "irrelevant". Set `modified` to `false`. Set `curr` to refer to D'. Go to $\mathbf{L_1}$.
11. If the consequent part of *`curr` is in the form $H_1 * H_2$ and if `Interact` is "$*$`Lock` $\downarrow$", and if $*$`Lock` $\uparrow$ is applicable, then record a backtrack point and take the following derivation step: *`curr` $\leadsto_{*\texttt{Lock}\uparrow} D'$. Set `modified` to `false`. Set `Interact` to "irrelevant". Set `curr` to refer to D'. Go to $\mathbf{L_1}$.
12. If there exists at least one formula that is in the form $H_1 * H_2$ as the only constituent of some focusable (necessarily antecedent) structural layer, then apply **Inf** $\in \{$`Transfer` $\uparrow$, `Revert` $\uparrow\}$ (depending on the current context degree): *`curr` $\leadsto_{\textbf{Inf}} D'$.[1] Set `modified` to `false`. Set `curr` to refer to D'. If **Inf** is `Transfer` $\uparrow$, then set `Interact` to "irrelevant". Go to $\mathbf{L_1}$.

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[1] What this does is to decrement the context degree by 1 or transfer from 0 to 0.

13. If $\texttt{Interact}$ is “$\vee\texttt{Lock}\uparrow$” and if either $\vee\texttt{Lock}\uparrow$, $\vee R$ or $\texttt{Revert}\uparrow$ is applicable, then record a backtrack point and then
    (a) either $\vee\texttt{Lock}\uparrow$,
    (b) or, possibly with a sequence of $\texttt{Revert}\uparrow, \texttt{Revert}\downarrow$ to first decrease the context degree down either to 0 or 1 depending on the outermost structural layer, $\vee R$,

    to have the derivation step: $^{\star}\texttt{curr} \rightsquigarrow_{\textbf{Inf}} D'$. Set $\texttt{curr}$ to refer to D'. If the first option, set $\texttt{modified}$ to $\texttt{false}$. If the second option, set $\texttt{modified}$ to $\texttt{true}$, and set $\texttt{Interact}$ to “irrelevant”. Go to $\mathbf{L_1}$.
14. If the consequent part of $^{\star}\texttt{curr}$ is in the form $H_1 \vee H_2$, and if $\texttt{Interact}$ is not “$\vee\texttt{Lock}\uparrow$”, and if $\textbf{Inf} \in \{\mathbb{1}\texttt{Peel}\uparrow, \vee\texttt{Lock}\uparrow, \vee R\}$ is applicable, then record a backtrack point, and apply $^{\star}\texttt{curr} \rightsquigarrow_{\textbf{Inf}} D'$. If **Inf** is $\vee R$, then set $\texttt{modified}$ to $\texttt{true}$; otherwise set it to $\texttt{false}$. If **Inf** is $\vee\texttt{Lock}\uparrow$, then set $\texttt{Interact}$ to “$\vee\texttt{Lock}\uparrow$”; otherwise set it to “irrelevant”. Set $\texttt{curr}$ to refer to D'. Go to $\mathbf{L_1}$.
15. If $\texttt{modified}$ then apply $\textbf{Inf} \in \{\texttt{Transfer}\uparrow, \texttt{Revert}\uparrow\}$ if possible at all: $^{\star}\texttt{curr} \rightsquigarrow_{\textbf{Inf}} D'$. Set $\texttt{modified}$ to $\texttt{false}$. Set $\texttt{Interact}$ to “irrelevant”. Set $\texttt{curr}$ to refer to D'. Go to $\mathbf{L_1}$.

$\mathbf{L_E}$: //No more rules to apply to $\texttt{curr}$: backtrack or move to an unattempted branch.

16. If $^{\star}\texttt{curr}$ is the conclusion sequent of an axiom, then if there is a stored unexamined left premise sequent, then set $\texttt{curr}$ to refer to the sequent, set $\texttt{modified}$ to $\texttt{true}$, set $\texttt{Interact}$ to the stored string value, and go to $\mathbf{L_1}$; otherwise, return $\texttt{true}$.
17. If $^{\star}\texttt{curr}$ is not the conclusion sequent of an axiom, then backtrack to the nearest backtracking point recorded. If there is another unexamined option at the backtrack point, take some decision that has not yet been made, revert the values of $\texttt{Interact}$ and $\texttt{modified}$ to those at the backtrack point, set $\texttt{curr}$ appropriately, and go to $\mathbf{L_1}$; otherwise, return $\texttt{false}$.

The termination proof of the above algorithm is given in a compositional manner.

Lemma 31 *Each SL_1/SL_2 phase terminates.*

Proof. For each SL_1/SL_2 inference rule, the weight of the premise sequent(s) is strictly smaller than that of the conclusion sequent. There can be at most a finite number of non-deterministic choices involved in the unsafe SL_1/SL_2 inference rule. $\Box$

Lemma 32 *There are no infinite switchings between the two phases via interaction rules.*

Proof. If any logical inference rule applies backward, the sequent weight always decreases (*Cf.* Lemma 31). It hence suffices to show that we cannot have an infinite transition $D \rightsquigarrow^*_{\mathbf{Inf}_I} D'$. But if either $\mathbb{1}\texttt{Peel} \uparrow$ or $\mathbb{1}\texttt{Peel} \downarrow$ applies backward, then the sequent weight always decreases. So we only need to show that $D \rightsquigarrow^*_{\mathbf{Inf}_I \backslash \{\mathbb{1}\texttt{Peel}\uparrow, \mathbb{1}\texttt{Peel}\downarrow\}} D'$ cannot be infinite. But if $*\texttt{Lock} \downarrow$ applies backward, then the next transition is, since we attempt to have such an infinite transition, always induced by $*\texttt{Lock} \uparrow$ (*Cf.* $\mathbf{L_{SL_2}}$ of the algorithm). By the definition of projection in the rule, however, some antecedent structure is weakened away, *i.e.* the sequent weight decreases, or there is at least one formula in the form $H_1 * H_2$ in the antecedent which becomes accessible in the next SL_1 phase. Meanwhile, $*\texttt{Lock} \uparrow$ can never apply unless $\texttt{Interact}$ is "$*\texttt{Lock} \downarrow$". Therefore we only need to show that $D \rightsquigarrow^*_{\mathbf{Inf}_I \backslash \{\mathbb{1}\texttt{Peel}\uparrow, \mathbb{1}\texttt{Peel}\downarrow, *\texttt{Lock}\downarrow, *\texttt{Lock}\uparrow\}} D'$ cannot be infinite.

Now, if $\vee\texttt{Lock} \uparrow$ applies backward, since we attempt to have such an infinite transition, the next derivation step is determined to be $\vee\texttt{Lock} \downarrow$ into the next SL_2 phase. Possible steps are 12 ($\texttt{Transfer} \uparrow$ cannot apply as the context degree is by now at least 2) and 13 of the algorithm. But if 12 is taken, then there is at least one formula in the form: $H_1 * H_2$ accessible in the antecedent in the next SL_1 phase; and if 13 is taken, the context degree increases and this sequence simply repeats, which must imply, since any $\texttt{PBI}$ sequent is finite and there is at most a finite number of structural layers, that, at some point, context-degree can no longer increase via $\vee\texttt{Lock} \uparrow$, $\vee\texttt{Lock} \downarrow$. Hence we in fact only need to show that $D \rightsquigarrow^*_{\mathbf{Inf}_I \backslash \{\mathbb{1}\texttt{Peel}\uparrow, \mathbb{1}\texttt{Peel}\downarrow, *\texttt{Lock}\downarrow, *\texttt{Lock}\uparrow, \vee\texttt{Lock}\uparrow, \vee\texttt{Lock}\downarrow\}} D'$ cannot be infinite.

Next, if $\texttt{Transfer} \uparrow$ applies, then, since we attempt to have such an infinite transition, we find that neither $\texttt{Transfer} \downarrow$ nor $\texttt{Revert} \downarrow$ is possible in the shifted SL_1 phase. If $\texttt{Transfer} \downarrow$ applies, since we attempt to have such an infinite transition, the only possible step in the shifted SL_2 phase is 12. But if 12 is taken, then there exists at least one accessible formula in the form $H_1 * H_2$ in the next SL_1 phase. Therefore we now only need to show that $D \rightsquigarrow^{*}_{\{\texttt{Revert}\uparrow,\texttt{Revert}\downarrow\}} D'$ cannot be infinite.

To conclude the present proof, we simply observe that context-degree cannot keep decreasing since every `PBI` sequent is finite. $\square$

Theorem 13 **(`BI`$_{\texttt{base}}$ decision procedure)** *The algorithm* $\oint$ *terminates. It is a* `BI`$_{\texttt{base}}$ *decision procedure.*

Proof. Termination follows from Lemma 31 and Lemma 32. Proof of the second obligation follows as below. We go through each step in $\oint$ for the proof of the second obligation.

1. If a phase is of SL_1, safe SL_1 inference rules, *i.e.* SL_1 axioms and $*L$ can apply. This step ensures recognition of axioms and normalisation (up to inversion) of a given SL_1 sequent into a normalised SL_1 sequent.

2. For a normalised SL_1 sequent, if the consequent part is a $\mathbb{1}$, the sequent is not `PBI`-derivable unless its antecedent is inconsistent. This step covers all the possibilities of a structure to keep in the antecedent. To show that this step omits nothing, we need to consider the cases where the context degree is greater than or equal to 1, in which $\mathbb{1}$`Peel` $\downarrow$ does not apply due to the mismatch on the context-degree. If it is 1, however, it is impossible, given the algorithm steps, that `modified` be `false`. Then `Transfer` $\downarrow$ is bound to apply at the step 6 to pass the computation to the next SL_2 phase of 0-th context degree, as required. For other greater values of the context degree $k \geq 2$, $\oint$ never allows, in the course of a proof search, that the context-degree be k if the consequent were a $\mathbb{1}$. Therefore we must show that such sequents are redundant in derivations of `PBI` from which $\oint$ derived. However, we know that the context-degree can only increase above 2 via $\vee$`Lock` $\uparrow$. According to the intention of the mediator as detailed in the previous sub-section, the interaction rule is called because antecedent formula(s) in the form: $H_1 \vee H_2$ must first be processed such that derivability be preserved upwards. However, according to the intention, it is the case that, once those such

antecedent formula(s) have been successfully processed, then the context-degree must revert to either 0 or 1 for an application of $\vee R$, to offset the potency of the intention.

3. For a normalised SL_1 sequent, if the consequent part is in the form: $H_1 * H_2$:
    (a) if the context-degree is 0, then possibilies that could be thought of are (1) $*R$ is applicable straight ahead, preserving derivability, (2) the antecedent is inconsistent, and (3) incremental weakening must take place in order that $*R$ become applicable. (And of course an additional possibility that none of these are applicable.) This step covers all.
    (b) if it is 1, then possibilities that could be thought of is that incremental weakening must take place in order that $*R$ become applicable. This step covers all.
    (c) there is no need for consideration to arise for all the other greater context-degrees.
4. For a normalised SL_1 sequent, if the consequent part is in the form: $H_1 \vee H_2$:
    (a) the context-degree is 0: then it was not locked in the previous SL_2 phase into the current phase. So it is either that such previous SL_2 phase does not exist, *i.e.* the current phase is the first phase, or that it was revealed via a $*R$ application. Either of the cases, it is impossible, given $\oint$, that `modified` be `false`. Step 6, if possible at all, is taken, skipping this step and also step 5, as required.
    (b) it is 1: if the previous SL_2 phase has locked the consequent formula with $\vee$`Lock` $\uparrow$ into the current SL_1 phase, then by the intention of the mediator effective in the current phase, $\vee$`Lock` $\downarrow$ is the only one that may apply, which is taken care of in this step. Otherwise, the consequent formula must be in this form because the current phase is the first phase and/or because $*R$ applied in the current phase to reveal it on the consequent. Either way, if it is possible at all, `Revert` $\downarrow$ must apply to swith phases into the next SL_2 phase, and the question is whether, in such a case where `Revert` $\downarrow$ must apply, $\oint$ in fact allows the rule to apply. But in both of the

possibilities, `Interact` is assumed not to be "$\vee$`Lock` $\uparrow$", which then satisfies the conditions of the step 5, as required.

(c) it is greater: then it must be the case that there exists at least one preceding phase to the current one and that the previous phase switch was induced by $\vee$`Lock` $\uparrow$. Then by the intention of the interaction rule, only $\vee$`Lock` $\downarrow$ needs considered if possible at all, which is taken care of in this step.

5. For a normalised SL_1 sequent whose consequent is not in the form: $\mathbb{1}$, $H_1 * H_2$ or $H_1 \vee H_2$, and for which the context-degree is 1, if any `PBI` inference rule should apply, it is `Revert` $\downarrow$ only. This step always applies the interaction rule where applicable.

6. For a normalised SL_1 sequent whose consequent is not in the form: $\mathbb{1}$, $H_1 * H_2$ or $H_1 \vee H_2$, and for which the context-degree is not 1:

   (a) the context-degree is 0: then, if any `PBI` inference rule is applicable at all, it is some interaction rule. The antecedent may be inconsistent in which case $\mathbb{1}$`Peel` $\downarrow$ applies, or `Transfer` $\downarrow$ applies if, just as it is the intention of the interaction rule, any progress should be possible by the phase switch. Now the question is whether $\oint$ facilitates such. The first case is taken care of in this step. For `Transfer` $\downarrow$, $\oint$ permits the rule only if `modified` is `true`. Therefore it does not apply if the current sequent is the premise sequent of `Transfer` $\uparrow$ that switched the previous phase into the current phase. (The other interaction rules cannot be applicable under the current set of conditions.) Now consider if such should detract from expressiveness. Suppose that `Transfer` $\downarrow$ were necessary on the sequent, then in the next SL_2 phase, the context-degree must be 0, since that for the conclusion sequent of the last `Transfer` $\uparrow$ is 0, which means that there is only one formula in the antecedent of the sequent. But then, since the `Transfer` $\uparrow$ must have - due to the intention of the interaction rule - applied because no more progress was possible in the previous SL_2 phase, and since no progress was made in the switched SL_1 phase either, it is known that there would be no progress in the next SL_2 phase. Therefore it is rather immediate that the sequent cannot be `PBI`-derivable. This step is skipped into step 7 that treats the situation.

   (b) it is greater than or equal to 2: by the earlier reasoning, this case is redundant.

7. For a normalised SL_1 sequent whose consequent is not in the form: $\mathbb{1}, H_1 * H_2$ or $H_1 \vee H_2$, and for which the context-degree is not 1, and if `𝟙Peel` $\downarrow$ should apply at all, then derivability would not be preserved upwards, and further, that no progress would be possible if `Transfer` $\downarrow$ should apply, then there can be applicable no `PBI` inference rule - since the use of `Revert` $\downarrow$ is constrained - on the sequent. Such a situation is taken care of in this step. This concludes our reasoning for a SL_1 phase.

8. In a SL_2 phase, all the safe SL_2 `PBI` inference rules can apply in any order to accessible structures, as can be achieved in this step.

9. If a formula in the form: $(H_1 \supset H_2) \supset H_3$ or $(H_1 * H_2) \supset H_3$ is accessible, then $\supset L_{\supset}$ or, respectively, $\supset L_*$ applies, provided that the derivability preserves upwards for the left premise sequent of the inference rule. Note that in such a case the context degree of the left premise sequent goes below 2 (0 or 1, depending on the appearance of the antecedent part of the sequent) with certain intention of the mediator that might have been carried over from the previous phase(s) also getting offset. The decision as to whether to apply those inference rules needs taken on a non-deterministic basis, which is taken care of in this step.

10. For a SL_2 sequent which is normalised and for which, if either a $\supset L_{\supset}$ or a $\supset L_*$ (if possible at all) applies, derivability upwards will not preserve, and for which a $\mathbb{1}$ is the consequent formula:

    (a) the context-degree is 0: if the antecedent part is inconsistent, then `𝟙Peel` $\uparrow$ applies, as taken care of in this step. If not, then if any progress is possible at all in the next SL_1 phase, `Transfer` $\uparrow$ applies to induce a phase switch; otherwise, the sequent is `PBI`-underivable. No conditions of any step satisfy but steps 15 and 17 in $\oint$, which fulfills what it must.[1]

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[1] In case it is `PBI`-derivable, the step 15 instead of the step 17 may be taken. This, however, only leads to that, in the next SL_1 phase, no progress would be possible, and consequently that `modified` would remain `false` to go to step 7, as required.

   (b) it is 1: then the antecedent is inconsistent, or there is at least one formula in the form $H_1 * H_2$ which is accessible in the 0th-degree context if in a SL_1 phase, in which case `Revert` $\uparrow$ should be called such that $\mathbb{1}$`Peel` $\downarrow$ or $*L$ be subsequently applied. The latter is taken care of at step 12 which is the only step with satisfiable conditions given the current set of assumptions. For the former, if `modified`, then the step 15 is the only step with satisfiable conditions. $\oint$ does not permit any step to be taken if `modified` is `false`. The question is whether such restriction should not detract from completeness. Since the current context degree is supposed to be 1, the current sequent must be the premise sequent of `Transfer` $\downarrow$ (with the conclusion sequent of the context-degree 0).[1] But if `Transfer` $\downarrow$ had applied, $\mathbb{1}$`Peel` $\downarrow$ could have applied, and so the previous decision was wrong, which, however, would be corrected eventually through backtracking.

   (c) it is greater: redundant, since the context formula is not in the form: $H_1 \vee H_2$.

11. For a normalised SL_2 sequent such that, if either a $\supset L_{\supset}$ or a $\supset L_*$ should apply (if applicable at all), then derivability does not preserve upwards for the left premise sequent, and for which it holds that a formula in the form $H_1 * H_2$ is the consequent formula and also that it was locked in the previous phase into the current phase via $*$`Lock` $\downarrow$, then, by the intention of the interaction rule, there is - if possible at all - only one interaction rule, namely $*$`Lock` $\uparrow$ which may apply on the sequent. The process is taken care of in this step.

12. For a normalised SL_2 sequent such that, if either a $\supset L_{\supset}$ or a $\supset_*$ should apply (if applicable at all), then derivability upwards does not preserve for the left premise sequent, that its consequent formula is not a $\mathbb{1}$, and that the intention of the mediator to probe sufficient information for a consequent formula $H_a * H_b$ is not in effect, then if there exists at least one formula in the form $H_a * H_b$ which is accessible in SL_1 with the context-degree one smaller (if the current context degree is greater than or equal to 1) or 0 (if it is 0 already), then `Revert` $\uparrow$ or `Transfer` $\uparrow$ can meaningfully apply, which is taken care of in this step.

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[1] `Revert` $\downarrow$ is not adequate since the consequent formula is not in the form: $H_1 \vee H_2$. *Cf.* the sub-case 2 of the current proof to see why it is not adequate.

13. For a normalised SL_2 sequent such that, if either a $\supset L_{\supset}$ or a $\supset L_*$ should apply (if applicable at all), then derivability does not preserve upwards for the left premise sequent, if it is $\vee$`Lock` $\downarrow$ that induced the previous phase switch with the intention of the mediator of probing some antecedent formula(s) in the form $H_a \vee H_b$ presently in effect, then the context-degree in the current phase is at least 2, or greater. If there is no antecedent formulas in the form $H_c * H_d$ which are accessible in SL_1 with the context-degree one smaller (if the current context-degree is greater than or equal to 1) or 0 (if it is 0 already), then we need consider if those antecedent formula(s) in the form $H_a \vee H_b$ had been already processed. If so, the intention of the mediator in effect may be offset through $*R$ (preceding which is a sequence of `Revert` $\uparrow$ and `Revert` $\downarrow$); otherwise, the intention is carried over (if possible at all) to the next SL_1 phase via $\vee$`Lock` $\uparrow$. Both are taken care of in this step.

14. For a normalised SL_2 sequent such that, if either a $\supset L_{\supset}$ or a $\supset L_*$ should apply, then derivability does not preserve upwards for the left premise of the inference rule(s), if there are no antecedent formulas in the form $H_c * H_d$ accessible in SL_1 with the context-degree one smaller or 0, and if the consequent formula is in the form $H_1 \vee H_2$, and also if the intention of the mediator to probe some antecedent formula(s) in the form $H_a \vee H_b$ is not in effect, then there are a few possibilities.

    (a) the context-degree is 0: if the antecedent is inconsistent, then $\mathbb{1}$`Peel` $\uparrow$ applies, as possible in this step. If the consequent formula can be outright processed via $\vee R$, this step can again execute the process. If it needs locked into the next SL_1 phase, then $\vee$`Lock` $\uparrow$ should apply, which this step allows. The last two eliminate a need for `Transfer` $\uparrow$ with the particular consequent formula, to conclude that nothing is omitted in this sub-case.

    (b) it is 1: by the set of conditions here, if any `PBI` inference rule should be applicable, it is $\vee R$ or $\vee$`Lock` $\uparrow$, which is possible in this step.

    (c) it is greater: there is no need to consider this case since the intention of the mediator to search for antecedent formula(s) in the form: $H_a \vee H_b$ is not in effect.

15. For a normalised SL_2 sequent such that (1) if either a $\supset L_{\supset}$ or a $\supset L_*$ should

apply, then derivability does not preserve upwards for the left premise of the inference rule(s), that (2) the consequent formula is not in the form $\mathbb{1}, H_1 * H_2$ or $H_1 \vee H_2$, and that (3) there are no antecedent formulas in the form $H_c * H_d$ accessible in SL_1 with the context-degree one smaller or 0, then if any `PBI` inference rule should apply, it is either `Transfer` $\uparrow$ or `Revert` $\uparrow$. In case there preceeded some logical inference rules, or in case the current phase is the first phase, then this step allows either of them to apply (if applicable). The question that remains is whether nothing is omitted by the decision to not apply either in the other cases. But, due to the thrid condition set at the beginning of this sub-case, there is no formula in the form $H_a * H_b$ that may become accessible by the phase switch, which prevents any progress in the next SL_1 phase.

16. Derivations continue for each open branch. This step concludes the proof in case all the non-deterministic guesses were made in such a way not to blemish upward derivability.

17. $\oint$ allows backtracking to cover all the possibile non-deterministic choices.

□

For the complexity of the algorithm, I only conjecture that the brute-force approach with backtracks makes $\oint$ exponentially complex. More detailed studies into computational complexity are better opportuned once the decidability of `BI` is concluded. It is not immediate whether `BI`$_{\texttt{base}}$ has any applications.

5.6 Conclusion

The present chapter exhibited the following:

1. Illustration of the concept of phased sequent calculus as one that can be used to engineer a particular sense of logical combination.

2. A working example of phased sequent calculus `PBI` for `BI`$_{\texttt{base}}$ with the abstract state diagram capturing base-logic interactions in effect.

3. A decision procedure for the fragment.

As was purported, the concept of the phased sequent calculus was introduced as one that proof-theoretically asks what it means by a logical combination. The effectiveness of such inquisition hinges solely on a delicate effort to not combine logics, since the moment we combine them, a particular sense of logical combination is already in effect, which then wipes off any hope of studying the particular logical combination altogether (because it is then a definition, an axiom, which cannot be questioned), a rather unwanted event. It is hence a critical requirement that a platform to study logical combinations be able to express physical separation of the base logics. Phased sequent calculus, to my knowledge, is one that has come to the 'zero' closest within sequent calculus, in so doing without rendering it a *mere adjunction* of two base logics since the second principle of phased sequent calculus (*Cf.* 5.1) must still hold even for the most basic phased sequent calculus of given base logics. For a demonstration of the basic mechanisms of the platform, $\texttt{BI}_{\texttt{base}}$ as found decidable in Chapter 3 was formulated. As one practical merit of the physical separation, it becomes easier to study the manner in which base logics interact within a combined logic, which led to the delivery of a $\texttt{BI}_{\texttt{base}}$ decision procedure in a compositional manner.

Several work are related. A modern logical combination methodology was pioneered by Gabbay [1996], known as Fibring, which continues to reign in the field of combined logics as a vital concept. Phased sequent calculus shares the fundamental idea with his work, of defining a combined logic by defining the sense of logical combination, which is expressed primarily in his fifth agenda: "Program 5. Study possible natural interactions between the logics . . . which are meaningful . . . conditional logics . . . and so on." Even though the fifth agenda of Gabbay's appears to have so far gained comparatively low popularity in the post-Gabbay Fibring work, in considering one extreme vision that computer science advances: *what actually work, whatever they are, are virtuous,* it is this fifth agenda that I here believe will become a central focus in future work in combined logics.

The more refined modulated Fibring by Caleiro et al. [2005] which extends Gabbay's Fibring to a wide range of logics also relates to phased sequent calculus. Via a categorical treatment, it addresses the following collapsing problem: combining two logics, the one semantically collapses into the other. With the modulated Fibring, one can obtain a non-collapsing result if (s)he so wishes to avoid the phenomenon. For more empirical working non-collapsing examples, there are some earlier works, including `BI`, such as found in Moortgat [1997]; O'Hearn and Pym [1999] where components of a logic are strongly differentiated from those of the other(s) via indexing[1] for a tighter structural control, a familiar idea within sub-structural logics Dosěn [1993]. In the context of combining the strongly differentiated logics (such as implicit in the "most basic logical combination" in the phased sequent calculus sense), if they are to exhibit any interactions, then there must be certain entities that encompass them, resulting in the concept of Bridges in Caleiro et al. [2005] and the mediators in phased sequent calculus.

Concerning the collapsing of logics, which forms one theme also in Schechter [2011], the principle of phased sequent calculus implies that, insofar as it is possible for us to imagine two smaller disjoint worlds (of a world) in one of which a logic A and in the other of which a logic B remain potent, and insofar as it is possible for us to perceive the two smaller worlds - that is, the concurrent logical processes - at once, the moment we indeed conceive them, there would already occur a combination of the base logics within the reflection of ours in which neither is collapsing. As to whether logics ought or ought not collapse upon combination therefore, the question can be answered in the affirmative (that it ought to collapse) only if the sense of a logical combination of the base logics one entertains in his/her mind should dictate that it be so, and in the negative only if it be otherwise.

Phased sequent calculus relates to the Schechter's (where he exhibits a logical combination methodology of Juxtaposition) in that phased sequent calculus and Juxtaposition are both non-Hilbert platforms. His juxtaposed consequence relations present certain similarity to the intuition behind phased sequent calculus. But there are at the same time differences. Semantically, for example, he seeks after a philosophically natural model, the class of coherent juxtaposed models as he terms. Suppose a combined logic with two base logics for instance, then in a coherent juxtaposed model (c.j.m hereafter) for the combined logic essentially reside two models, one for each small world representative of A and respectively B. In the c.j.m it holds that, for every possible formula constructable in the combined logic from the available propositional variables and the logical connectives, it has a designated semantic value in the small world representative of A iff it does in the other small world representative of B. As the semantics then dictates, there should exist no unrecognised formula in either of the

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[1] `BI`, as we saw, uses ";" and "," which can be alternatively represented as "$,_i$" and "$,_j$" with indexes.

small worlds unless it is unrecognised in the both, in turn dictating that combined logics of the following sort: $\mathfrak{L}(\mathcal{P}, \mathcal{C}, \textbf{Inf}_1, \textbf{Inf}_2, \emptyset)$ with some connectives $\mathcal{C}$ is not generally expressible under Juxtaposed semantics.[1] By contrast, phased sequent calculus does not preclude the possibility that what is perceptible may form a part of knowledge in a small world even though it is not (yet) registered as such in the others.

Finally, there are a few work related to Fibring in sequent calculus such as by Coniglio [2007]; Cruz-fillipe and Sernadas [2005]. Coniglio [2007] considers a problem of recovering a logic by fibring its fragments in sequent calculus. It turned out that it was not generally possible to achieve the recovery since fibring may not preserve meta-property such as (if we consider intuitionistic logic) "$A; B \vdash C$ implies $A \vdash B \supset C$". Coniglio [2007] then considers meta-fibring which is a particular combination paradigm that preserves those meta-properties, achieving the recovery. Collapsing of logics, however, occur more prominently with meta-fibring than with fibring. Cruz-fillipe and Sernadas [2005] describes a general idea to achieve Fibring in sequent calculus using categorical notations. Just as in phased sequent calculus, their approach keeps logics separate. Unlike phased sequent calculus, however, phase switches are not described in terms of derivations but in terms of transference functions that map formulas that belong to a base logic in terms of formulas that belong to another base logic. In the `PBI` case above, phase switches by transfer are easily achievable in their paradigm. Any phase switches which do not carry intention to the next phase can be also captured easily. However, it is not clear how phase switches by peeling which propagate the intention of the mediator through can be concisely expressed in their methodology or if it is expressible, much less the more intricate phase switches by locking. Phased sequent calculus can simulate the idea of Cruz-fillipe and Sernadas [2005] within, but the converse is not very obvious.

To conclude, compared with the so far mentioned work, I focused more on the engineering aspect of logical combinations, having application of combined logics in mind. Sequent calculus was then a natural and timely choice. To engineer a suitable sense of combination, the abstract methodology of state diagrams was proposed for an efficient development and analysis of logical combinations themselves. It is my hope that phased sequent calculus and Fibring will mutually forge ahead the program of

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[1] Let us call back into our mind the anonymous logic as we saw earlier in this chapter and the unknown predicates.

producing adequate reasoning platforms tailored to specific applications.

Chapter 6

Thesis Conclusion

Under the theme of studying base-logic interactions of combined logics within sequent calculus, both specialisation with `BI` (Chapter 3) and `BBI` (Chapter 4), and generalisation with phased sequent calculus (Chapter 5) were covered. Critical assessments of earlier work were also presented for `BI` (Chapter 2).

Achievements as seen in Chapter 3 and Chapter 4 illustrate a general need for dedicated theoretical frameworks to adequately reason about what makes a combined logic distinct from either of the base logics. With the specific combined logics of `BI` and `BBI`, we observed that there could exist parts (namely the mutually extended parts) which should be better expressed not within the philosophy of base logics but within that of a combined logic of which they are a constituent. The importance of a close study of semantics was stressed particularly in Chapter 4 for `BBI`. This line of research to pursue a tight syntax-semantics correspondence should be a worthwhile for an efficient theorem-proving.

It is also important that we establish a solid proof-theoretical framework in which a particular sense of logical combination can be engineered. With the delivery of phased sequent calculus, I presently believe that we at last gained a platform which is application-oriented, and which, also in consultation with the accumulated knowledge of combined logics within philosophy, will promote derivations of appropriate combined logics befitting applications with a calculated logical combination in place.

6.1 Summaries of Contributions in Earlier Chapters

I now state what have been covered.

6.1.1 BI proof theory (Chapter 2, Chapter 3 and Chapter 5)

1. Critical reviews of earlier results in `BI` proof theory, identifying apparent issues in the earlier proofs of `BI` decidability and of cut elimination in `BI` sequent calculus.

2. An official proof of admissibility of `Cut` in `LBI`.

3. Delivery of α`LBI` and `LBIZ` without any structural rules which hitherto hindered scalable proof searches within `BI` sequent calculus.

4. Cut admissibility proof within [α`LBI` + `Cut`], which also vacuously extends to [`LBIZ` + `Cut`].

5. Proof of decidability of $\texttt{BI}_{\texttt{base}}$, which is a `BI` fragment without the multiplicative implication and the multiplicative unit, and a decision procedure for the fragment.

6. Emphasis of the importance of regarding `BI` as `BI` than as an extension of either intuitionistic logic or of multiplicative intuitionistic linear logic, *i.e.* the emphasis that the mutually extended parts in a combined logic may no longer possess a logical characteristic as exemplified in its base logics.

6.1.2 `BBI` proof theory (Chapter 4)

1. Delivery of `BBI` sequent calculi that exhibit a closer syntax-semantic correspondence than previously envisaged.

2. Identification of a cut-eliminable class of `BBI` sequent calculi.

6.1.3 General studies into logical combinations (Chapter 5)

1. Development of the concept of phased sequent calculus in which a sense of logical combination of base logics as defined by mediator(s) can be developed and analysed.
2. Proposal of abstract state diagrams to keep track of the intention of mediators to engineer logical combinations constructively.

6.2 Future Work

There are several work around Logic `BI` and Logic `BBI` that extend the results of this thesis.

1. Purely syntactic proof of decidability/undecidability of the full `BI`.
2. Proof/refutation of an earlier conjecture by Brotherston [2012] and Park et al. [2013] that there does not exist an analytic `BBI` sequent calculus.
3. Development of a semantically natural sound separation logic sequent calculus with user-defined inductive predicates to be competent against the currently prominent separation logic theorem provers.

Also, there are a few work that may be of interest to `BI` and `BBI` communities.

1. Certification of proofs.
2. Development of `BBI` semantics in which the multiplicative unit behaves intuitionistically, so that dedicated inference rules around the multiplicative unit for the current `BBI` semantics become unsound.
3. Investigation into decidable `BBI` fragments.

For phased sequent calculus, there are a number of future work conceivable. Here I only mention one of them that has a moderate degree of importance. Study of `Cut` in phased sequent calculus will be important. In the case of $\mathtt{BI}_{\mathtt{base}}$, it sufficed to have `PBI` as presented in Chapter 5, as it was shown equivalent to a `Cut`-free $\mathtt{BI}_{\mathtt{base}}$ sequent calculus $\mathtt{LBIZ}_1$. There would be only a superficial merit if we considered `Cut` in `PBI`, since we, speaking on a reasonable ground, know the answer. In a more general context, however, it is important that we consider the following: whether `Cut`, a rule of transitivity should only belong to base components or also to mediators.

Appendices

Appendix A - Proof of Lemma 20

By induction on $\texttt{str\_dist}(\mathbb{\Gamma}(-))$. If $\mathbb{\Gamma}(-) = -$, then $\forall W \in \texttt{ND}\, \forall m \in W$:

1. if $\neg^{\dagger}[m \models \underline{\Gamma_1^{\Delta_1}}]$, then vacuous.
2. otherwise, $[m \models (F_1 \wedge F_2) * (G_1 \vee G_2)] \leftrightarrow^{\dagger}$ $\exists m_1, m_2 \in W.[m \in m_1 \circ m_2] \wedge^{\dagger} [m_1 \models F_1 \wedge F_2] \wedge^{\dagger} [m_2 \models G_1 \vee G_2] \leftrightarrow^{\dagger}$ $\exists m_1, m_2 \in W.[m \in m_1 \circ m_2] \wedge^{\dagger} [m_1 \models F_1] \wedge^{\dagger} [m_1 \models F_2] \wedge^{\dagger} ([m_2 \models G_1] \vee^{\dagger} [m_2 \models G_2])$ $\rightarrow^{\dagger}$ $\exists m_1, m_2 \in W.[m \in m_1 \circ m_2] \wedge^{\dagger} (([m_1 \models F_1] \wedge^{\dagger} [m_2 \models G_1]) \vee^{\dagger} ([m_1 \models F_2] \wedge^{\dagger} [m_2 \models G_2])) \rightarrow^{\dagger}$ $(\exists m_1, m_2 \in W.[m \in m_1 \circ m_2] \wedge^{\dagger} [m_1 \models F_1] \wedge^{\dagger} [m_2 \models G_1])) \vee^{\dagger} (\exists m_1, m_2 \in W.[m \in m_1 \circ m_2] \wedge^{\dagger} [m_1 \models F_2] \wedge^{\dagger} [m_2 \models G_2])$, as required.

For inductive cases, assume the current lemma holds true for all $0 \leq \texttt{str\_dist}(\mathbb{\Gamma}(-)) \leq l$, and show that it still holds true for $\texttt{str\_dist}(\mathbb{\Gamma}(-)) = l + 1$.

1. $\mathbb{\Gamma}(-)$ is in the form: $(\Gamma'(-); \Gamma'_1)^{\Delta}$ such that $\texttt{str\_dist}(\Gamma'(-)) = l$ and that $\texttt{str\_dist}(\mathbb{\Gamma}(-)) = l + 1$. Then $\forall W \in \texttt{ND}\, \forall m \in W$:
   (a) if $\neg^{\dagger}[m \models \underline{\Gamma'_1} \wedge (\overline{\Delta} \supset \mathbb{1})]$, then vacuous.
   (b) otherwise, it holds that:
   $\forall W \in \texttt{ND}\, \forall m \in W.[m \models \underline{\Gamma'(\Gamma_1^{(\Delta_1;(F_1 \wedge F_2)*(G_1 \vee G_2))})} \supset \mathbb{1}] \rightarrow^{\dagger}$
   $[m \models \underline{\Gamma'(\Gamma_1^{(\Delta_1;(F_1 \wedge F_2)*(G_1 \vee G_2))})} \supset \mathbb{1}]$, by induction hypothesis.
2. $\mathbb{\Gamma}(-)$ is in the form: $(\mathbb{\Gamma}'(-), \mathbb{\Gamma}'_1)^{\Delta}$ such that $\texttt{str\_dist}(\mathbb{\Gamma}'(-)) = l$ and that $\texttt{str\_dist}(\mathbb{\Gamma}(-)) = l + 1$. Then $\forall W \in \texttt{ND}\, \forall m \in W$:
   (a) if $\neg^{\dagger}[m \models \overline{\Delta} \supset \mathbb{1}]$, then vacuous.

(b) otherwise, we show that $\neg^{\dagger}[m \models \underline{(\mathbb{\Gamma}'(\Gamma_1^{(\Delta_1;(F_1 \wedge F_2)*(G_1 \vee G_2))}), \mathbb{\Gamma}_1')}] \rightarrow^{\dagger}$ $\neg^{\dagger}[m \models \underline{(\mathbb{\Gamma}'(\Gamma_1^{(\Delta_1;(F_1 * G_1);(F_2 * G_2))}), \mathbb{\Gamma}_1')}]$. By case studies. For simplification, we denote $\mathbb{\Gamma}'(\Gamma_1^{(\Delta_1;(F_1 \wedge F_2)*(G_1 \vee G_2))})$ by $\mathbb{\Gamma}'_{pre}$ and $\mathbb{\Gamma}'(\Gamma_1^{(\Delta_1;(F_1 * G_1);(F_2 * G_2))})$ by $\mathbb{\Gamma}'_{conc}$. Now $\forall m' \in W.$:

   i. if $[m' \models \underline{\mathbb{\Gamma}'_{pre}, \mathbb{\Gamma}'_1}]$, then vacuous.

   ii. otherwise, if $\neg^{\dagger}[m' \models \underline{\mathbb{\Gamma}'_{pre}, \mathbb{\Gamma}_1}](\leftrightarrow^{\dagger}$ $\forall m_1, m_2 \in W.[m_1 \models \underline{\mathbb{\Gamma}_1}] \rightarrow^{\dagger} ([m' \in m_1 \circ m_2] \rightarrow^{\dagger} \neg^{\dagger}[m_2 \models \mathbb{\Gamma}'_{pre}])$. If $\neg^{\dagger}[m' \models \underline{\mathbb{\Gamma}'_{pre}}] \wedge^{\dagger} [m' \models \underline{\mathbb{\Gamma}'_1}]$, then by induction hypothesis we have $\forall m_1, m_2 \in W.[m_1 \models \underline{\mathbb{\Gamma}_1}] \rightarrow^{\dagger} ([m' \in m_1 \circ m_2] \rightarrow^{\dagger} \neg^{\dagger}[m_2 \models \mathbb{\Gamma}'_{cons}]) \leftrightarrow^{\dagger}$ $\neg^{\dagger}[m' \models \underline{\mathbb{\Gamma}'_{cons}, \mathbb{\Gamma}'_1}]$, as required. $\square$

Appendix B - Proof of Lemma 21

Proof is by induction on $\texttt{str\_dist}(\mathbb{\Gamma}(-))$.

1. $\mathbb{\Gamma}(-) = -$: immediate.

2. For inductive cases, assume that the current lemma holds true for all $\mathbb{\Gamma}(\mathbb{\Gamma}_1)$ for which $0 \leq \texttt{str\_dist}(\mathbb{\Gamma}(-)) \leq l$. Then we must show that it still holds true for $\texttt{str\_dist}(\mathbb{\Gamma}(-)) = l + 1$.

    (a) $\mathbb{\Gamma}(-)$ is in the form: $(\Gamma'(-); \Gamma'_1)^{\Delta}$ such that $\texttt{str\_dist}(\Gamma'(-)) = l$ and that $\texttt{str\_dist}(\mathbb{\Gamma}(-)) = l + 1$. By induction hypothesis, it holds that $\forall W \in \texttt{ND}\, \forall m \in W.\neg^{\dagger}[m \models \underline{\Gamma'(\mathbb{\Gamma}_1)}]$. Then it is immediate, by the logical equivalence around $\mathbb{1}$, that $\forall W \in \texttt{ND}\, \forall m \in W.\neg^{\dagger}[m \models \underline{(\Gamma'(\mathbb{\Gamma}_1); \Gamma'_1)^{\Delta}}] \leftrightarrow^{\dagger} [m \models \underline{\mathbb{\Gamma}(\mathbb{\Gamma}_1)} \supset \mathbb{1}]$, as required.

    (b) $\mathbb{\Gamma}(-)$ is in the form: $(\mathbb{\Gamma}'(-), \mathbb{\Gamma}'_1)^{\Delta}$ such that $\texttt{str\_dist}(\mathbb{\Gamma}'(-)) = l$ and that $\texttt{str\_dist}(\mathbb{\Gamma}(-)) = l + 1$. By induction hypothesis, it holds that $\forall W \in \texttt{ND}\, \forall m \in W.\neg^{\dagger}[m \models \underline{\mathbb{\Gamma}'(\mathbb{\Gamma}_1)}]$. Then, again by the logical equivalence around $\mathbb{1}$, it follows that $\forall W \in \texttt{ND} \forall m \in W.\neg^{\dagger}[m \models \underline{(\mathbb{\Gamma}'(\mathbb{\Gamma}_1), \mathbb{\Gamma}'_1)^{\Delta}}] \leftrightarrow^{\dagger} [m \models \underline{\mathbb{\Gamma}(\mathbb{\Gamma}_1)} \supset \mathbb{1}]$, as required. $\square$

Appendix C - Proof of Lemma 23

Proof is by induction on the relative structural distance $\texttt{str\_dist}(\mathbb{\Gamma}(-))$.

1. if $\texttt{str\_dist}(\mathbb{\Gamma}(-)) = -$, then vacuous by the semantics of $\vee$ in the $\texttt{BBI}$ Kripke non-deterministic semantics.

2. otherwise, if $\mathbb{\Gamma}(-) \leftrightharpoons_{ant} (\Gamma'(-);\Gamma'_1)^{\Delta}$ such that $\texttt{str\_dist}(\Gamma'(-)) = k$ and that $\texttt{str\_dist}(\mathbb{\Gamma}(-)) = k+1$, then $\forall W \in \texttt{ND}.(\forall m \in W.[m \models \underline{\Gamma'(F_1 \vee F_2)}]) \leftrightarrow^{\dagger} (\forall m \in W.[m \models \underline{\Gamma'(F_1)}] \vee^{\dagger} [m \models \underline{\Gamma'(F_2)}])$ by induction hypothesis. But then $\forall W \in \texttt{ND}$:

    (a) for the obligation of $(\forall m' \in W.[m' \models \underline{(\Gamma'(F_1);\Gamma'_1)^{\Delta}}] \vee^{\dagger} [m' \models \underline{(\Gamma'(F_2);\Gamma'_1)^{\Delta}}]) \rightarrow^{\dagger} (\forall m'' \in W.[m'' \models \underline{(\Gamma'(F_1 \vee F_2);\Gamma'_1)^{\Delta}}]$:

        i. if $[m' \models \underline{\Gamma'_1}] \rightarrow^{\dagger} [m' \models \overline{\Delta}]$, then vacuous.

        ii. otherwise, induction hypothesis concludes.

    (b) Similarly for the other direction.

3. otherwise, if $\mathbb{\Gamma}(-) \leftrightharpoons_{ant} (\mathbb{\Gamma}'(-),\mathbb{\Gamma}'_1)^{\Delta}$ such that $\texttt{str\_dist}(\mathbb{\Gamma}'(-)) = k$ and that $\texttt{str\_dist}(\mathbb{\Gamma}(-)) = k+1$, then $\forall W \in \texttt{ND}.(\forall m \in W.[m \models \underline{\mathbb{\Gamma}'(F_1 \vee F_2)}]) \leftrightarrow^{\dagger} (\forall m \in W.[m \models \underline{\mathbb{\Gamma}'(F_1)}] \vee^{\dagger} [m \models \underline{\mathbb{\Gamma}'(F_2)}])$ by induction hypothesis. But then $\forall W \in \texttt{ND}$:

    (a) for the obligation of $(\forall m' \in W.[m' \models \underline{(\mathbb{\Gamma}'(F_1),\mathbb{\Gamma}'_1)^{\Delta}}] \vee^{\dagger} [m' \models \underline{(\mathbb{\Gamma}'(F_2),\mathbb{\Gamma}'_1)^{\Delta}}]) \rightarrow^{\dagger} (\forall m'' \in W.[m'' \models \underline{(\mathbb{\Gamma}'(F_1 \vee F_2),\mathbb{\Gamma}'_1)^{\Delta}}])$:

        i. if $[m' \models \overline{\Delta} \supset \mathbb{1}]$, then vacuous.

        ii. otherwise, note first that we have:
        $[m' \models (\mathbb{\Gamma}'(F_1),\mathbb{\Gamma}'_1)^{\mathbb{1}}] \vee^{\dagger} [m' \models (\mathbb{\Gamma}'(F_2),\mathbb{\Gamma}'_1)^{\mathbb{1}}] \leftrightarrow^{\dagger}$
        $\exists m_1, m_2.[m' \in m_1 \circ m_2] \wedge^{\dagger} (([m_1 \models \underline{\mathbb{\Gamma}'(F_1)}] \wedge^{\dagger} [m_2 \models \underline{\mathbb{\Gamma}'_1}])$
        $\vee^{\dagger}([m_1 \models \underline{\mathbb{\Gamma}'(F_2)}] \wedge^{\dagger} [m_2 \models \underline{\mathbb{\Gamma}'_1}])) \leftrightarrow^{\dagger}$

$\exists m_1, m_2.[m' \in m_1 \circ m_2] \wedge^{\dagger} [m_1 \models \underline{\mathbb{\Gamma}'(F_1 \vee F_2)}] \wedge^{\dagger} [m_2 \models \underline{\mathbb{\Gamma}'_1}] \leftrightarrow^{\dagger}$ $[m' \models \underline{(\mathbb{\Gamma}'(F_1 \vee F_2), \mathbb{\Gamma}'_1)^{\mathbb{1}}}]$. Hence, as far as such m' (those that satisfy $\overline{\Delta}$) as m'' are concerned, we are done. But there cannot be any other m_x as m'' which does not satisfy $\underline{(\mathbb{\Gamma}'(F_1 \vee F_2), \mathbb{\Gamma}'_1)^{\Delta}}$, and which, even despite that, satisfies either $\underline{(\mathbb{\Gamma}'(F_1), \mathbb{\Gamma}'_1)^{\Delta}}$ or $\underline{(\mathbb{\Gamma}'(F_2), \mathbb{\Gamma}'_1)^{\Delta}}$, as, otherwise, such would contradict the previous (vacuous) sub-proof.

(b) Similarly for the other direction. $\square$

Appendix D - Proof of Proposition 16

By induction on derivation depth of $\Pi(D)$. If it is 1, then D is the conclusion sequent of an axiom. id and $^*\top R$ both absorb $^*\top WkL$. Neither $\mathbb{1}L$ nor $\top R$ requires any but $\mathbb{1}$ or respectively $\top$. For inductive cases, assume that the current proposition holds true for all the $\alpha\texttt{LBBI}_p$-derivations of derivation depth up to k and prove that it still holds true for $\alpha\texttt{LBBI}_p$-derivations of derivation depth $k+1$. Consider what the last inference rule applied is in $\Pi(D)$.

1. $\wedge L$: $\Pi(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots\\ D_1 : \mathbb{\Gamma}\{\Gamma_1; F_1; F_2\} \vdash \{\Delta_1\}\end{matrix}}{D : \mathbb{\Gamma}\{\Gamma_1; F_1 \wedge F_2\} \vdash \{\Delta_1\}}\ \wedge L$$

   By induction hypothesis,
   $D_1' : \mathbb{\Gamma}\{\mathbb{E}((\Gamma_1; F_1; F_2)^{\Delta_1})\} \vdash \{\mathbb{1}\}$ is $\alpha\texttt{LBBI}_p$-derivable. But then the result follows via (a forward application of) $\wedge L$. Note that we are only interested in reaching the proof of the current proposition; specifically, we are not applying $^*\top WkL$ to split the focused $F_1; F_2$ such as into $(F_1, (^*\top; \Gamma_1)); (F_2, (^*\top; \Gamma_2))$ to make it unable to apply $\wedge L$ (that is, to make it unable to have $F_1 \wedge F_2$).

2. $\supset L$: $\Pi(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots\\ D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1\}\end{matrix} \qquad \begin{matrix}\vdots\\ D_2 : \mathbb{\Gamma}\{\Gamma_1; F_2\} \vdash \{\Delta_1\}\end{matrix}}{D : \mathbb{\Gamma}\{\Gamma_1; F_1 \supset F_2\} \vdash \{\Delta_1\}}\ \supset L$$

   Apply a sub-induction on the number of (new) $(^*\top; \Gamma_i')^{\Delta_i}$s that $\mathbb{E}(\Gamma_1^{(\Delta_1;F_1)})$ has introduced. If zero, then vacuous. Now suppose that the current case holds for up to l new introductions of $(^*\top; \Gamma_i')^{\Delta_i'}$ structures ($i = 1, \cdots, l$), then we have:

$$\frac{D_1'' : \mathbb{\Gamma}(\mathbb{E}(\Gamma_1^{(\Delta_1;F_1)})) \vdash \mathbb{1} \qquad D_2'' : \mathbb{\Gamma}(\mathbb{E}((\Gamma_1;F_2)^{\Delta_1})) \vdash \mathbb{1}}{D'' : \mathbb{\Gamma}(\mathbb{E}((\Gamma_1;F_1 \supset F_2)^{\Delta_1})) \vdash \mathbb{1}} \supset L$$

We must now show that it still holds when we introduce one more structure $({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}$.

(a) For any $D_1^\star : \mathbb{\Gamma}(({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}(\Gamma_1^{(\Delta_1;F_1)})) \vdash \mathbb{1}$ such that $D_1^\star \rightsquigarrow_{{}^{*}\top WkL} D_1''$, the same introduction of $({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}$ into D_2'' results in $D_2^\star : \mathbb{\Gamma}(({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}((\Gamma_1;F_2)^{\Delta_1})) \vdash \mathbb{1}$. By induction hypothesis of the sub-induction, we then have
$D^\star : \mathbb{\Gamma}(({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}((\Gamma_1;F_1 \supset F_2)^{\Delta_1})) \vdash \mathbb{1}$.
But by induction hypothesis of the main induction, both $D_1^\star$ and $D_2^\star$ are $\alpha\mathtt{LBBI}_p$-derivable. So is $D^\star$.

(b) For any $D_1^\star : \mathbb{\Gamma}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}(\Gamma_1^{\Delta_2;F_1}))^{\Delta_3}) \vdash \mathbb{1}$ such that $\Delta_2;\Delta_3 \equiv \Delta_1$ (up to assoc. and commut. of ";"), the same introduction of the new structure into D_2'' results in:
$D_2^\star : \mathbb{\Gamma}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}((\Gamma_1;F_2)^{\Delta_2}))^{\Delta_3}) \vdash \mathbb{1}$. By induction hypothesis of the sub-induction, we then have:
$D^\star : \mathbb{\Gamma}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}((\Gamma_1;F_1 \supset F_2)^{\Delta_2}))^{\Delta_3}) \vdash \mathbb{1}$.
But by induction hypothesis of the main induction, both $D_1^\star$ and $D_2^\star$, and consequently also $D^\star$ are $\alpha\mathtt{LBBI}_p$-derivable.

(c) For any $D_1^\star : \mathbb{\Gamma}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}(\Gamma_1^{\Delta_2}))^{(\Delta_3;F_1)}) \vdash \mathbb{1}$ such that $\Delta_2;\Delta_3 \equiv \Delta_1$, the same introduction into D_2'' leads to:
$D_2^\star : \mathbb{\Gamma}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}(\Gamma_1;F_2)^{\Delta_2})^{\Delta_3}) \vdash \mathbb{1}$.
By induction hypothesis of the sub-induction, we then have;
$D^\star : \mathbb{\Gamma}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}(\Gamma_1;F_1 \supset F_2)^{\Delta_2})^{\Delta_3}) \vdash \mathbb{1}$.
Induction hypothesis of the main induction then concludes.

(d) For any $D_1^\star : \mathbb{\Gamma}(\mathbb{E}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \Gamma_2^{(\Delta_2;F_1)}); \mathbb{E}'(\Gamma_3^{\Delta_3}))) \vdash \mathbb{1}$ for $\Delta_2;\Delta_3 \equiv \Delta_1$ and $\Gamma_2;\Gamma_3 \equiv \Gamma_1$, the same introduction into D_2'' leads to:
$D_2^\star : \mathbb{\Gamma}(\mathbb{E}((({}^{*}\top;\Gamma'_{k+1})^{\Delta'_{k+1}}, (\Gamma_2;F_2)^{\Delta_2}); \mathbb{E}'(\Gamma_3^{\Delta_3}))) \vdash \mathbb{1}$.
Induction hypothesis of the sub-induction and of the main-induction then conclude.

(e) The other variations: similar.

3. The other additive inference rules: simpler.

4. All the multiplicative inference rules except for $\mathrel{-\!\!*}L_{{}^{*}\!\top}$ and $\mathrel{-\!\!*}R_{{}^{*}\!\top}$: the effect is absorbed. For $\mathrel{-\!\!*}L_I$, because the essence is multiplicatively connected to the surrounding negative structures ($\Gamma_a^{\Delta_a}$ and/or $\Gamma_b^{\Delta_b}$), whether those are made an essence (or essences) does not affect the inference rule. *Cf.* the internalised weakening process within $\mathrel{-\!\!*}L_I$ as defined earlier.

5. $\mathrel{-\!\!*}R_{{}^{*}\!\top}$: $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; G\} \quad (F \in \Xi)\end{array}}{D : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F \mathrel{-\!\!*} G\}}\ \mathrel{-\!\!*}R_{{}^{*}\!\top}$$

Trivial via induction hypothesis as nothing changes in this derivation step but the principal.

6. $\mathrel{-\!\!*}L_{{}^{*}\!\top}$: similar.

7. ${}^{*}\!\top CtrL$: $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : \mathbb{\Gamma}\{({}^{*}\!\top; \Gamma_1)^{\Delta_1}, ({}^{*}\!\top; \Gamma_1)^{\Delta_1}\} \vdash \{\Delta_1\}\end{array}}{D : \mathbb{\Gamma}\{{}^{*}\!\top; \Gamma_1\} \vdash \{\Delta_1\}}\ {}^{*}\!\top CtrL$$

Vacuous to prove that D is derivable from D_1, which the current assumption (*i.e.* the above derivation) precisely shows. To prove a general case where $D' : \mathbb{\Gamma}\{\mathbb{E}(({}^{*}\!\top; \Gamma_1)^{\Delta_1})\} \vdash \{\mathbb{1}\}$ where $\mathbb{E}(({}^{*}\!\top; \Gamma_1)^{\Delta_1}) \leftrightharpoons_{ant} (\mathbb{E}'({}^{*}\!\top), ({}^{*}\!\top; \mathsf{E}(\Gamma_1))^{\Delta_1})$, by induction hypothesis we have $D_1' : \mathbb{\Gamma}(\mathbb{E}'({}^{*}\!\top), \{({}^{*}\!\top; \mathsf{E}(\Gamma_1))^{\Delta_1}, ({}^{*}\!\top; \mathsf{E}(\Gamma_1))^{\Delta_1}\}) \vdash \{\Delta_1\}$ $\alpha\mathtt{LBBI}_p$-derivable. Then ${}^{*}\!\top CtrL$ concludes. $\Box$

Appendix E - Proof of Proposition 17

By induction on derivation depth of $\Pi(D)$. If it is 1, then D is the conclusion sequent of an axiom. id and ${}^*\top R$ both absorb ${}^*\top WkL$. Neither $\mathbb{1}L$ nor $\top R$ requires any but $\mathbb{1}$ or respectively $\top$. For inductive cases, assume that the current proposition holds true for all the $\alpha\mathtt{LBBI}_p$-derivations of derivation depth up to k and prove that it still holds true for $\alpha\mathtt{LBBI}_p$-derivations of derivation depth $k+1$. Consider what the last inference rule applied is in $\Pi(D)$.

1. $\wedge L$: $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : \mathbb{\Gamma}\{\Gamma_1; F_1; F_2\} \vdash \{\Delta_1\}\end{array}}{D : \mathbb{\Gamma}\{\Gamma_1; F_1 \wedge F_2\} \vdash \{\Delta_1\}}\ \wedge L$$

   By induction hypothesis,
   $D_1' : \mathbb{\Gamma}\{\mathbb{E}((\Gamma_1; F_1; F_2)^{\Delta_1})\} \vdash \{\mathbb{1}\}$ is $\alpha\mathtt{LBBI}_p$-derivable. But then the result follows via (a forward application of) $\wedge L$. Note that we are only interested in reaching the proof of the current proposition; specifically, we are not applying ${}^*\top WkL$ to split the focused $F_1; F_2$ such as into $(F_1, ({}^*\top; \Gamma_1)); (F_2, ({}^*\top; \Gamma_2))$ to make it unable to apply $\wedge L$ (that is, to make it unable to have $F_1 \wedge F_2$).

2. $\supset L$: $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1\}\end{array} \qquad \begin{array}{c}\vdots\\ D_2 : \mathbb{\Gamma}\{\Gamma_1; F_2\} \vdash \{\Delta_1\}\end{array}}{D : \mathbb{\Gamma}\{\Gamma_1; F_1 \supset F_2\} \vdash \{\Delta_1\}}\ \supset L$$

   Apply a sub-induction on the number of (new) $({}^*\top; \Gamma_i')^{\Delta_i}$s that $\mathbb{E}(\Gamma_1^{(\Delta_1; F_1)})$ has introduced. If zero, then vacuous. Now suppose that the current case holds for up to l new introductions of $({}^*\top; \Gamma_i')^{\Delta_i'}$ structures ($i = 1, \cdots, l$), then we have:

$$\dfrac{D_1'' : \mathbb{\Gamma}(\mathbb{E}(\Gamma_1^{(\Delta_1;F_1)})) \vdash \mathbb{1} \qquad D_2'' : \mathbb{\Gamma}(\mathbb{E}((\Gamma_1;F_2)^{\Delta_1})) \vdash \mathbb{1}}{D'' : \mathbb{\Gamma}(\mathbb{E}((\Gamma_1;F_1 \supset F_2)^{\Delta_1})) \vdash \mathbb{1}} \supset L$$

We must now show that it still holds when we introduce one more structure $({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}$.

(a) For any $D_1^\star : \mathbb{\Gamma}(({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}(\Gamma_1^{(\Delta_1;F_1)})) \vdash \mathbb{1}$ such that $D_1^\star \rightsquigarrow_{{}^{*}\top WkL} D_1''$, the same introduction of $({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}$ into D_2'' results in $D_2^\star : \mathbb{\Gamma}(({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}((\Gamma_1;F_2)^{\Delta_1})) \vdash \mathbb{1}$. By induction hypothesis of the sub-induction, we then have
$D^\star : \mathbb{\Gamma}(({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}((\Gamma_1;F_1 \supset F_2)^{\Delta_1})) \vdash \mathbb{1}$.
But by induction hypothesis of the main induction, both $D_1^\star$ and $D_2^\star$ are $\alpha\mathtt{LBBI}_p$-derivable. So is $D^\star$.

(b) For any $D_1^\star : \mathbb{\Gamma}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}(\Gamma_1^{\Delta_2;F_1}))^{\Delta_3}) \vdash \mathbb{1}$ such that $\Delta_2;\Delta_3 \equiv \Delta_1$ (up to assoc. and commut. of ";"), the same introduction of the new structure into D_2'' results in:
$D_2^\star : \mathbb{\Gamma}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}((\Gamma_1;F_2)^{\Delta_2}))^{\Delta_3}) \vdash \mathbb{1}$. By induction hypothesis of the sub-induction, we then have:
$D^\star : \mathbb{\Gamma}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}((\Gamma_1;F_1 \supset F_2)^{\Delta_2}))^{\Delta_3}) \vdash \mathbb{1}$.
But by induction hypothesis of the main induction, both $D_1^\star$ and $D_2^\star$, and consequently also $D^\star$ are $\alpha\mathtt{LBBI}_p$-derivable.

(c) For any $D_1^\star : \mathbb{\Gamma}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}(\Gamma_1^{\Delta_2}))^{(\Delta_3;F_1)}) \vdash \mathbb{1}$ such that $\Delta_2;\Delta_3 \equiv \Delta_1$, the same introduction into D_2'' leads to:
$D_2^\star : \mathbb{\Gamma}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}(\Gamma_1;F_2)^{\Delta_2})^{\Delta_3}) \vdash \mathbb{1}$.
By induction hypothesis of the sub-induction, we then have;
$D^\star : \mathbb{\Gamma}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \mathbb{E}(\Gamma_1;F_1 \supset F_2)^{\Delta_2})^{\Delta_3}) \vdash \mathbb{1}$.
Induction hypothesis of the main induction then concludes.

(d) For any $D_1^\star : \mathbb{\Gamma}(\mathbb{E}((({}^{*}\top;\Gamma'_{l+1})^{\Delta'_{l+1}}, \Gamma_2^{(\Delta_2;F_1)}); \mathbb{E}'(\Gamma_3^{\Delta_3}))) \vdash \mathbb{1}$ for $\Delta_2;\Delta_3 \equiv \Delta_1$ and $\Gamma_2;\Gamma_3 \equiv \Gamma_1$, the same introduction into D_2'' leads to:
$D_2^\star : \mathbb{\Gamma}(\mathbb{E}((({}^{*}\top;\Gamma'_{k+1})^{\Delta'_{k+1}}, (\Gamma_2;F_2)^{\Delta_2}); \mathbb{E}'(\Gamma_3^{\Delta_3}))) \vdash \mathbb{1}$.
Induction hypothesis of the sub-induction and of the main-induction then conclude.

(e) The other variations: similar.

3. The other additive inference rules: simpler.

4. All the multiplicative inference rules except for $-\!\!*L_{*\top}$ and $-\!\!*R_{*\top}$: the effect is absorbed. For $-\!\!*L_I$, because the essence is multiplicatively connected to the surrounding negative structures ($\Gamma_a^{\Delta_a}$ and/or $\Gamma_b^{\Delta_b}$), whether those are made an essence (or essences) does not affect the inference rule. *Cf.* the internalised weakening process within $-\!\!*L_I$ as defined earlier.

5. $-\!\!*R_{*\top}$: $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; G\} \quad (F \in \Xi)\end{array}}{D : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F -\!\!* G\}}\ -\!\!*R_{*\top}$$

   Trivial via induction hypothesis as nothing changes in this derivation step but the principal.

6. $-\!\!*L_{*\top}$: similar.

7. ${}^{*}\top CtrL$: $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : \mathbb{\Gamma}\{({}^{*}\top; \Gamma_1)^{\Delta_1}, ({}^{*}\top; \Gamma_1)^{\Delta_1}\} \vdash \{\Delta_1\}\end{array}}{D : \mathbb{\Gamma}\{{}^{*}\top; \Gamma_1\} \vdash \{\Delta_1\}}\ {}^{*}\top CtrL$$

   Vacuous to prove that D is derivable from D_1, which the current assumption (*i.e.* the above derivation) precisely shows. To prove a general case where $D' : \mathbb{\Gamma}\{\mathbb{E}(({}^{*}\top; \Gamma_1)^{\Delta_1})\} \vdash \{\mathbb{1}\}$ where $\mathbb{E}(({}^{*}\top; \Gamma_1)^{\Delta_1}) \leftrightharpoons_{ant} (\mathbb{E}'({}^{*}\top), ({}^{*}\top; \mathsf{E}(\Gamma_1))^{\Delta_1})$, by induction hypothesis we have $D_1' : \mathbb{\Gamma}(\mathbb{E}'({}^{*}\top), \{({}^{*}\top; \mathsf{E}(\Gamma_1))^{\Delta_1}, ({}^{*}\top; \mathsf{E}(\Gamma_1))^{\Delta_1}\}) \vdash \{\Delta_1\}$ $\alpha\mathtt{LBBI}_p$-derivable. Then ${}^{*}\top CtrL$ concludes. □

Appendix F - Proof of Lemma 24

By induction on the derivation depth of a sequent. First consider easier ones (4.6) - (4.8). (4.1) - (4.4) are slightly more difficult around $\ast R_I$ (and $\mathrel{-\!\!\ast} L_I$). (4.5) and (4.9) are trivial.

$\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F \wedge G\}$: base cases are when it is the conclusion sequent of an axiom. Trivially both $\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F\}$ and $\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; G\}$ are axioms. For inductive cases, assume that this case holds true for all the derivations of derivation depth up to k, and prove that it still holds true at derivation depth $k+1$. Consider what the last inference rule applied is. Note that there is almost no relation between symbols across distinct derivations except what matter (Γ_1, Δ_1 and $F \wedge G$). Same symbols may be re-used, lest we should witness a flooding of sub/super-scripts.[1]

1. $\wedge L$: the derivation then looks either like:

$$\dfrac{\begin{matrix}\vdots \\ D_1 : \mathbb{\Gamma}\{\Gamma_1'; H_1; H_2\} \vdash \{\Delta_1; F \wedge G\}\end{matrix}}{\mathbb{\Gamma}\{\Gamma_1'; H_1 \wedge H_2\} \vdash \{\Delta_1; F \wedge G\}} \wedge L$$

or like:

$$\dfrac{\begin{matrix}\vdots \\ D_2 : \mathbb{\Gamma}\{\Gamma'(\mathbb{\Gamma}_1); H_1; H_2\} \vdash \{\Delta'\}\end{matrix}}{\mathbb{\Gamma}\{\Gamma'(\mathbb{\Gamma}_1); H_1 \wedge H_2\} \vdash \{\Delta'\}} \wedge L$$

where $\log \mathbb{\Gamma}_1 = \Delta_1; F \wedge G$, or like:

$$\dfrac{D_3 : \mathbb{\Gamma}(\mathbb{\Gamma}_1)\{\Gamma'; H_1; H_2\} \vdash \{\Delta'\}}{\mathbb{\Gamma}(\mathbb{\Gamma}_1)\{\Gamma'; H_1 \wedge H_2\} \vdash \{\Delta'\}} \wedge L$$

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[1] I do not reiterate this notice in the rest.

where $\log \mathbb{\Gamma}_1 = \Delta_1; F \wedge G$.

For the first, by induction hypothesis on D_1, both $\mathbb{\Gamma}\{\Gamma'; H_1; H_2\} \vdash \{\Delta_1; F\}$ and $\mathbb{\Gamma}\{\Gamma'; H_1; H_2\} \vdash \{\Delta_1; G\}$ are $\alpha\mathtt{LBBI}_p$-derivable. Then both $\mathbb{\Gamma}\{\Gamma'; H_1 \wedge H_2\} \vdash \{\Delta_1; F\}$ and $\mathbb{\Gamma}\{\Gamma'; H_1 \wedge H_2\} \vdash \{\Delta_1; G\}$ are $\alpha\mathtt{LBBI}_p$-derivable via $\wedge L$ (forward; I do not reiterate in the rest).

For the second, by induction hypothesis on D_2, both $\mathbb{\Gamma}\{\Gamma'(\mathbb{\Gamma}_1); H_1; H_2\} \vdash \{\Delta'\}$ with $\log \mathbb{\Gamma}_1 = \Delta_1; F$ and $\mathbb{\Gamma}\{\Gamma'(\mathbb{\Gamma}_1); H_1; H_2\} \vdash \{\Delta'\}$ with $\log \mathbb{\Gamma}_1 = \Delta_1; G$ are $\alpha\mathtt{LBBI}_p$-derivable. $\wedge L$ concludes.

The last is similar to the second case.

2. $\wedge R$: trivial in case the principal of the inference rule coincides with the "$F \wedge G$". Otherwise the derivation looks either like:

$$\dfrac{\overset{\vdots}{D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta'_1; F \wedge G; H_1\}} \qquad \overset{\vdots}{D_2 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta'_1; F \wedge G; H_2\}}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta'_1; F \wedge G; H_1 \wedge H_2\}} \wedge R$$

or like:

$$\dfrac{\overset{\vdots}{D_3 : \mathbb{\Gamma}(\mathbb{\Gamma}')\{\Gamma_1\} \vdash \{\Delta'; H_1\}} \qquad \overset{\vdots}{D_4 : \mathbb{\Gamma}(\mathbb{\Gamma}')\{\Gamma_1\} \vdash \{\Delta'; H_2\}}}{\mathbb{\Gamma}(\mathbb{\Gamma}')\{\Gamma_1\} \vdash \{\Delta'; H_1 \wedge H_2\}} \wedge R$$

where $\log \mathbb{\Gamma}' = \Delta_1; F \wedge G$.

For the first, by induction hypothesis on D_1, both $D'_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta'_1; F; H_1\}$ and $D''_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta'_1; G; H_1\}$ are $\alpha\mathtt{LBBI}_p$-derivable. Meanwhile by induction hypothesis on D_2, both $D'_2 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta'_1; F; H_2\}$ and $D''_2 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta'_1; G; H_2\}$ are $\alpha\mathtt{LBBI}_p$-derivable. Then $D' : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta'_1; F; H_1 \wedge H_2\}$ and $D'' : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta'_1; G; H_1 \wedge H_2\}$ are $\alpha\mathtt{LBBI}_p$-derivable via $\wedge R$.

Similar for the second.

3. $\vee L$: similar, straightforward.

4. $\vee R$: similar, straightforward.

5. $\supset L$: the derivation then looks like:

$$\dfrac{\overset{\vdots}{D_1 : \mathbb{\Gamma}\{\Gamma'_1\} \vdash \{\Delta'_1; F \wedge G; H_1\}} \qquad \overset{\vdots}{D_2 : \mathbb{\Gamma}\{\Gamma'_1; H_2\} \vdash \{\Delta'_1; F \wedge G\}}}{\mathbb{\Gamma}\{\Gamma'_1; H_1 \supset H_2\} \vdash \{\Delta'_1; F \wedge G\}} \supset L$$

Hence similar to $\wedge R$ case, *i.e.* induction hypothesis on both premises and a forward application of $\supset L$ for both. The other cases are trivial.

6. $\supset R$: straightforward.

7. $*L$: straightforward.

8. $*R_I$: derivation then looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : Re_1^{\Delta_a} \vdash \mathsf{S}^{+}(F_1 * G_1; \cdots ; F_l * G_l)\end{array} \qquad \begin{array}{c}\vdots\\ D_2 : Re_2^{\Delta_b} \vdash \mathsf{S}^{-}(F_1 * G_1; \cdots ; F_l * G_l)\end{array}}{\mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(\Delta';F_1*G_1;\cdots;F_l*G_l;F\wedge G)})\} \vdash \{\mathbb{1}\}}\, *R_I$$

Then in forward derivation with D_1 and D_2, if neither $Re_1^{\Delta_a}$ nor $Re_2^{\Delta_b}$ retains $F\wedge G$, do a forward weakening with F or G instead of $F\wedge G$; otherwise, inversion lemma and a forward application of $*R_I$.

9. $*R_{{}^{*}\!\top}$: Let F_x denote $F_1{*}G_1; \cdots ; F_l{*}G_l$ and let Δ_y denote $\log \mathbb{E}(\Gamma_1^{(F\wedge G;\Delta_1;F_x)})$. The derivation looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : {}^{*}\!\top \vdash \mathsf{S}^{+}(F_x)\end{array} \qquad \begin{array}{c}\vdots\\ D_2 : \mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(F\wedge G;\Delta_1;F_x)})\} \vdash \{\mathsf{S}^{-}(F_x)\}\}\end{array}}{\mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(F\wedge G;\Delta_1;F_x)})\} \vdash \mathbb{1}}\, *R_{{}^{*}\!\top}$$

Induction hypothesis on D_2 and a forward application of $*R_{{}^{*}\!\top}$ conclude.

10. $-\!\!*L_I$: straightforward. *Cf.* $*R_I$.

11. $-\!\!*R_I$: derivation then looks like:

$$\dfrac{\Gamma_1^{(F\wedge G;\Delta_1;F_1-\!\!*F_2)}, F_1 \vdash F_2}{\mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(F\wedge G;\Delta_1;F_1-\!\!*F_2)})\} \vdash \{\mathbb{1}\}}\, -\!\!*R_I$$

Induction hypothesis on the premise and then a forward application of $-\!\!*R_I$ conclude.

12. $-\!\!*R_{{}^{*}\!\top}$: derivation then looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F \wedge G; H_2\}\end{array} \quad (H_1 \in \Xi)}{D : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F \wedge G; H_1-\!\!*H_2\}}\, -\!\!*R_{{}^{*}\!\top}$$

Inversion on $F \wedge G$ and then a forward application of $-\!\!*R_{{}^{*}\!\top}$.

13. $-\!\!*L_{{}^{*}\!\top}$: straightforward.

14. ${}^{*}\!\top CtrL$: straightforward.

$\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F \vee G\}$: similar.

$\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F \supset G\}$: similar.

$\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F \mathrel{-\!\!*} G\} \quad (F \in \Xi)$: similar. The focused $F \mathrel{-\!\!*} G$ does not become the principal for $\mathrel{-\!\!*} R_I$ since F is in the collector.

Now consider the rest.

$\mathbb{\Gamma}\{\Gamma_1; F \wedge G\} \vdash \{\Delta_1\}$: trivial for the base cases. For inductive cases, assume that this case holds true for all the derivations of derivation depth up to k and show that it still holds true at derivation depth $k + 1$. Consider what the last inference rule applied is.

1. $\wedge L$: trivial if the principal of the inference rule coincides with the "$F \wedge G$".

   Otherwise the derivation looks like:

   $$\dfrac{\begin{array}{c}\vdots\\ D_1 : \mathbb{\Gamma}\{\Gamma_1'; H_1; H_2; F \wedge G\} \vdash \{\Delta_1\}\end{array}}{\mathbb{\Gamma}\{\Gamma_1'; H_1 \wedge H_2; F \wedge G\} \vdash \{\Delta_1\}}\ \wedge L$$

   By induction hypothesis on D_1, $D_1' : \mathbb{\Gamma}\{\Gamma_1'; H_1; H_2; F; G\} \vdash \{\Delta_1\}$ is $\alpha\mathtt{LBBI}_p$-derivable; a forward application of $\wedge L$ (on "$H_1; H_2$") then concludes. The other cases are trivial.

2. $\wedge R$: the derivation then looks like:

   $$\dfrac{\begin{array}{c}\vdots\\ D_1 : \mathbb{\Gamma}\{\Gamma_1'; F \wedge G\} \vdash \{\Delta_1'; H_1\}\end{array} \qquad \begin{array}{c}\vdots\\ D_2 : \mathbb{\Gamma}\{\Gamma_1'; F \wedge G\} \vdash \{\Delta_1'; H_2\}\end{array}}{\mathbb{\Gamma}\{\Gamma_1'; F \wedge G\} \vdash \{\Delta_1'; H_1 \wedge H_2\}}\ \wedge R$$

   By induction hypothesis both on D_1 and D_2,
   $D_3 : \mathbb{\Gamma}\{\Gamma_1'; F; G\} \vdash \{\Delta_1'; H_1\}$ and
   $D_4 : \mathbb{\Gamma}\{\Gamma_1'; F; G\} \vdash \{\Delta_1'; H_2\}$ are both $\alpha\mathtt{LBBI}_p$-derivable. Then a forward application of $\wedge R$ concludes. The other cases are trivial.

3. $\vee L$: the derivation then looks like:

   $$\dfrac{\begin{array}{c}\vdots\\ D_1 : \mathbb{\Gamma}\{\Gamma_1'; F \wedge G; H_1\} \vdash \{\Delta_1\}\end{array} \qquad \begin{array}{c}\vdots\\ D_2 : \mathbb{\Gamma}\{\Gamma_1'; F \wedge G; H_2\} \vdash \{\Delta_1\}\end{array}}{\mathbb{\Gamma}\{\Gamma_1'; F \wedge G; H_1 \vee H_2\} \vdash \{\Delta_1\}}\ \vee L$$

   By induction hypothesis on both D_1 and D_2,
   $D_3 : \mathbb{\Gamma}\{\Gamma_1'; F; G; H_1\} \vdash \{\Delta_1\}$ and

$D_4 : \mathbb{\Gamma}\{\Gamma_1'; F; G; H_2\} \vdash \{\Delta_1\}$ are both $\alpha\mathtt{LBBI}_p$-derivable. Then a forward application of $\vee L$ concludes. The other cases are trivial.

4. $\vee R$: straightforward.
5. $\supset L$: straightforward.
6. $\supset R$: straightforward.
7. $\ast L$: straightforward.
8. $\ast R_I$: Let H_x denote $F_1 \ast G_1; \cdots ; F_l \ast G_l$. Then the derivation looks either like:

$$\dfrac{\begin{matrix}\vdots & \qquad & \vdots \\ D_1 : Re_1^{\Delta_a} \vdash \mathsf{S}^+(H_x) & \qquad & D_2 : Re_2^{\Delta_b} \vdash \mathsf{S}^-(H_x)\end{matrix}}{\mathbb{\Gamma}((\Gamma_1; F \wedge G)^{\Delta_1})\{\mathbb{E}(\Gamma'^{(\Delta'; H_x)})\} \vdash \{\mathbb{1}\}}\ast R_I$$

or like:

$$\dfrac{\begin{matrix}\vdots & \qquad & \vdots \\ D_3 : (Re_1(F \wedge G))^{\Delta_a} \vdash \mathsf{S}^+(H_x) & \qquad & D_4 : Re_2^{\Delta_b} \vdash \mathsf{S}^-(H_x)\end{matrix}}{\mathbb{\Gamma}\{\mathbb{E}((\Gamma'((\Gamma_1; F \wedge G)^{\Delta_1}))^{(\Delta'; H_x)})\} \vdash \{\mathbb{1}\}}\ast R_I$$

or like:

$$\dfrac{\begin{matrix}\vdots & \qquad & \vdots \\ D_5 : Re_1^{\Delta_a} \vdash \mathsf{S}^+(H_x) & \qquad & D_6 : (Re_2(F \wedge G))^{\Delta_b} \vdash \mathsf{S}^-(H_x)\end{matrix}}{\mathbb{\Gamma}\{\mathbb{E}((\Gamma'((\Gamma_1; F \wedge G)^{\Delta_1}))^{(\Delta'; H_x)})\} \vdash \{\mathbb{1}\}}\ast R_I$$

or like:

$$\dfrac{\begin{matrix}\vdots & \qquad & \vdots \\ D_7 : Re_1^{\Delta_a} \vdash \mathsf{S}^+(H_x) & \qquad & D_8 : Re_2^{\Delta_b} \vdash \mathsf{S}^-(H_x)\end{matrix}}{\mathbb{\Gamma}\{\mathbb{E}((\Gamma'((\Gamma_1; F \wedge G)^{\Delta_1}))^{(\Delta'; H_x)})\} \vdash \{\mathbb{1}\}}\ast R_I$$

The focused "$F \wedge G$" in a premise sequent is assumed coincident with the focused "$F \wedge G$" in the conclusion sequent.

For the first case, through (forward) internalised weakening with "$F; G$" instead of $F \wedge G$.

For the second and the third cases, induction hypothesis on the premise sequent in which $F \wedge G$ occurs, and then a forward application of $\ast R_I$. The last case is similar to the first case.

9. $\ast R_{\ast\top}$: similar.

10. $-\!\!*L_I$: similar.
11. $-\!\!*L_{{}^*\!\top}$: vacuous.
12. $-\!\!*R_I$: similar.
13. $-\!\!*R_{{}^*\!\top}$: vacuous.
14. ${}^*\!\top CtrL$: straightforward.

$\mathbb{\Gamma}\{\Gamma_1; F_1 \vee F_2\} \vdash \{\Delta_1\}$**:** straightforward.

$\mathbb{\Gamma}\{\Gamma_1; F{\supset}G\} \vdash \{\Delta_1\}$**:** mostly straightforward, but we consider one case.

1. $\supset L$: if the principal of the inference rule coincides with the "$F{\supset}G$", then trivial. Otherwise, the derivation looks either like:

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ D_1 : \mathbb{\Gamma}\{\Gamma'_1; F{\supset}G\} \vdash \{\Delta_1; H_1\} & D_2 : \mathbb{\Gamma}\{\Gamma'_1; F \supset G; H_2\} \vdash \{\Delta_1\}\end{array}}{\mathbb{\Gamma}\{\Gamma'_1; F{\supset}G; H_1{\supset}H_2\} \vdash \{\Delta_1\}} \supset L$$

or like:

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ D_3 : \mathbb{\Gamma}((\Gamma_1; F \supset G)^{\Delta_1})\{\Gamma'\} \vdash \{\Delta'; H_1\} & D_4 : \mathbb{\Gamma}((\Gamma_1; F \supset G)^{\Delta_1})\{\Gamma'; H_2\} \vdash \{\Delta'\}\end{array}}{\mathbb{\Gamma}((\Gamma_1; F{\supset}G)^{\Delta_1})\{\Gamma'; H_1{\supset}H_2\} \vdash \{\Delta'\}} \supset L$$

or like:

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ D_5 : \mathbb{\Gamma}\{\Gamma'((\Gamma_1; F \supset G)^{\Delta_1})\} \vdash \{\Delta'; H_1\} & D_6 : \mathbb{\Gamma}\{\Gamma'((\Gamma_1; F \supset G)^{\Delta_1}); H_2\} \vdash \{\Delta'\}\end{array}}{\mathbb{\Gamma}\{\Gamma'((\Gamma_1; F{\supset}G)^{\Delta_1}); H_1{\supset}H_2\} \vdash \{\Delta'\}} \supset L$$

For the first, by induction hypothesis on D_1 and D_2, both
$D'_1 : \mathbb{\Gamma}\{\Gamma'_1\} \vdash \{\Delta_1; F; H_1\}$ and
$D'_2 : \mathbb{\Gamma}\{\Gamma'_1; H_2\} \vdash \{\Delta_1; F\}$
are $\alpha\mathtt{LBBI}_p$-derivable. Furthermore, both $D''_1 : \mathbb{\Gamma}\{\Gamma'_1; G\} \vdash \{\Delta_1; H_1\}$ and $D''_2 : \mathbb{\Gamma}\{\Gamma'_1; G; H_2\} \vdash \{\Delta_1\}$ are $\alpha\mathtt{LBBI}_p$-derivable. $\supset L$ then concludes. Similar for the rest.

$\mathbb{\Gamma}\{\Gamma_1; F * G\} \vdash \{\Delta_1\}$**:** straightforward.

$\mathbb{\Gamma}\{\Gamma_1; F \mathrel{-\!\!*} G\} \vdash \{\Delta_1\} \quad (F \in \Xi)$ **:** straightforward. The focused $F \mathrel{-\!\!*} G$ does not become the principal for $\mathrel{-\!\!*} L_I$ since F is in the collector. $\square$

Appendix G - Proof of Lemma 25

I prove that the incremental weakening is sufficient. Then the rest follows as a trivial corollary.

$*R_I$: Under the assumption made, there exists a $\alpha\texttt{LBBI}_p$-derivable pair of $D_1 : Re_1^{\Delta_1} \vdash \mathsf{S}^+(F_x)$ and $D_2 : Re_2^{\Delta_2} \vdash \mathsf{S}^-(F_x)$ from the conclusion sequent $D : \mathbb{\Gamma}\{\mathbb{E}(\Gamma_1^{(\Delta';F_x)})\} \vdash \{\mathbb{1}\}$ such that $D \leadsto_{*R_I} D_1$ and $D \leadsto_{*R_I} D_2$. Internally $Re_1^{\Delta_1}/Re_2^{\Delta_2}$ results from a finite number of $WkL_{\texttt{LBBI}_p}$ and $WkR_{\texttt{LBBI}_p}$ applications on D as follows: $D \leadsto^*_{\{\mathbb{1}_{ps\ \texttt{LBBI}_p}, {}^*\!\top WkL_{\texttt{LBBI}_p}\}} [D' : \Gamma_1^{\Delta'} \vdash F_x] \leadsto_{syn} [D'' : \Gamma_1^{\Delta'} \vdash \mathsf{S}^+(F_x) * \mathsf{S}^-(F_x)] \leadsto^*_{\{WkL_{\texttt{LBBI}_p}, WkR_{\texttt{LBBI}_p}\}} [D''' : Re_1^{\Delta_1}, Re_2^{\Delta_2} \vdash \mathsf{S}^+(F_x) * \mathsf{S}^-(F_x)]$. In D''', in case neither $Re_1^{\Delta_1}$ nor $Re_2^{\Delta_2}$ is empty, then the outermost structural layer of the antecedent structure is multiplicative. If $\mathsf{\Gamma}_1^{\Delta'}$ in D'' was an additive structural layer, *i.e.* $\Gamma_1^{\Delta'}$ denotes $(F_1; \ldots; F_m; \mathcal{M}_1^{\mathbb{1}}; \ldots; \mathcal{M}_n^{\mathbb{1}})^{\Delta'}$ for $m + n \geq 2$, $m \geq 0$ and $n \geq 1$, then a finite number of $WkL_{\texttt{LBBI}_p}$ and $WkR_{\texttt{LBBI}_p}$ applications must have taken place at this additive structural layer (which is the outermost structural layer in $\Gamma_1^{\Delta'}$) such that (in backward derivation) all but one multiplicative structural layer $\mathcal{M}_k^{\mathbb{1}}$, $1 \leq k \leq n$ were discarded in the transition. But this process is also achieved via Wk, leading to a sequent $\mathcal{M}_k^{\mathbb{1}} \vdash \mathsf{S}^+(F_x) * \mathsf{S}^-(F_x)$. Once the outermost structural layer is multiplicative, it is either the case that some $Re_{1'}^{\Delta'_1}/Re_{2'}^{\Delta'_2}$ pair can be formed on the antecedent part for D'_1 and D'_2 such that $Re_{1'}^{\Delta'_1} \vdash \mathsf{S}^+(F_x)$ and $Re_{2'}^{\Delta'_2} \vdash \mathsf{S}^-(F_x)$ are both $\alpha\texttt{LBBI}_p$-derivable, or not. We are done if it can be formed. Otherwise, the current outermost multiplicative structural layer holds $\mathcal{A}$(s) as its constituent(s) whose $\mathcal{M}$ constituent (again only one of them) must be connected at the current outermost multiplicative structural layer, which is achieved also through Wk. This incremental process eventually produces the $Re_{1'}^{\Delta'_1}$ $/Re_{2'}^{\Delta'_2}$ pair on the antecedent part, provided that a situation that satisfies all the below conditions does not arise.

- there exists $D^* : Re_{1^*}^{\Delta_1^*}, Re_{2^*}^{\Delta_2^*} \vdash \mathsf{S}^+(F_x) * \mathsf{S}^-(F_x)$ such that $D'' \rightsquigarrow_{Wk}^* D^*$ as the internal weakening process within $*R_I$.
- not both $D_1^* : Re_{1^*}^{\Delta_1^*} \vdash \mathsf{S}^+(F_x)$ and $D_2^* : Re_{2^*}^{\Delta_2^*} \vdash \mathsf{S}^-(F_x)$ are $\alpha\mathtt{LBBI}_p$-derivable.
- there exists $D^{**} : Re_{1^{**}}^{\Delta_1^{**}}, Re_{2^{**}}^{\Delta_2^{**}} \vdash \mathsf{S}^+(F_x) * \mathsf{S}^-(F_x)$ such that $D^* \rightsquigarrow_{\{WkL_{\mathtt{LBBI}_p}, WkR_{\mathtt{LBBI}_p}\}}^* D^{**}$ (as the internal weakening process within $*R_I$).
- both $D_1^{**} : Re_{1^{**}}^{\Delta_1^{**}} \vdash \mathsf{S}^+(F_x)$ and $D_2^{**} : Re_{2^{**}}^{\Delta_2^{**}} \vdash \mathsf{S}^-(F_x)$ are $\alpha\mathtt{LBBI}_p$-derivable.
- D_1^* (resp. D_2^*) is a sequent that results from additive weakening admissibility on D_1^{**} (resp. D_2^{**}), *i.e.* there exists in $\mathtt{LBBI}_p$-space the following derivation ($i \in \{1, 2\}$):

$$\frac{D_i^{**}}{D_i^*}\ \{\text{A finite number of } WkL_{\mathtt{LBBI}_p},\ WkR_{\mathtt{LBBI}_p} \text{ applications}\}$$

Suppose, by way of showing contradiction, that there exists a $\alpha\mathtt{LBBI}_p$-derivation in which all the five conditions above satisfy. Then additive weakening admissibility dictates that $\alpha\mathtt{LBBI}_p$-derivability of D_1^{**} (resp. D_2^{**}) implies $\alpha\mathtt{LBBI}_p$-derivability of D_1^* (resp. D_2^*), a direct contradiction to the supposition.

$-\!\!*L$: Similar. The starting point is $\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, (\Gamma_2; F -\!\!* G)^{\Delta_3}$ in the conclusion sequent $D : \mathbb{\Gamma}\{(\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, (\Gamma_2; F -\!\!* G)^{\Delta_3}); \Gamma'\} \vdash \{\Delta'\}$. An application of Wk_1' is mandatory in case either Δ_3 or Γ_2 is not empty. □

Appendix H - Proof of Proposition 18

By induction on the derivation depth of $\Pi(D)$. We assume maximal pairs for $*R_I/{-\!\!*}L_I$ (*Cf.* Lemma 25). Base cases when D is the conclusion sequent of an axiom are trivial. For inductive cases, assume that the current proposition holds true for all the derivations of derivation depth up to k and show that it still holds true at derivation depth $k+1$. Consider what the $\alpha\mathtt{LBBI}_p$ inference rule last applied is by cases on where ϕ: the active negative (ϕ_n) or positive (ϕ_p) formula, is in $\mathbb{\Gamma}\{\Gamma_1;\Gamma_2;\Gamma_2\}$ or respectively in $\{\Delta_1;H;H\}$.

$\wedge L$ and ϕ_n is $F_1 \wedge F_2$: if ϕ_n does not appear in Γ_2, induction hypothesis on the premise sequent. Otherwise $\Pi(D)$ looks like:

$$\cfrac{\begin{array}{c}\vdots\\ D_1 : \mathbb{\Gamma}((\Gamma_1;\Gamma_2\{\Gamma_3;F_1;F_2\};\Gamma_2((\Gamma_3;F_1\wedge F_2)^{\Delta'}))^{(\Delta_1;H;H)}) \vdash \{\Delta'\}\end{array}}{\mathbb{\Gamma}((\Gamma_1;\Gamma_2\{\Gamma_3;F_1\wedge F_2\};\Gamma_2((\Gamma_3;F_1\wedge F_2)^{\Delta'}))^{(\Delta_1;H;H)}) \vdash \{\Delta'\}}\,\wedge L$$

$D_1' : \mathbb{\Gamma}\{\Gamma_1;\Gamma_2((\Gamma_3;F_1;F_2)^{\Delta'});\,\Gamma_2((\Gamma_3;F_1;F_2)^{\Delta'})\} \vdash \{\Delta_1;H;H\}$ is $\alpha\mathtt{LBBI}_p$ derivable (inversion lemma); then also $D_1'' : \mathbb{\Gamma}\{\Gamma_1;\Gamma_2((\Gamma_3;F_1;F_2)^{\Delta'})\} \vdash \{\Delta_1;H\}$ (induction hypothesis); then a forward application of $\wedge L$ concludes.

$\wedge R$ and ϕ_p is $F_1 \wedge F_2$: if ϕ_p is not H, induction hypothesis on both of the premises. Otherwise:

$$\cfrac{\begin{array}{c}\vdots\\ D_1 : \mathbb{\Gamma}\{\Gamma_1;\Gamma_2;\Gamma_2\} \vdash \{\Delta_1;F_1;F_1\wedge F_2\}\end{array} \qquad \begin{array}{c}\vdots\\ D_2 : \mathbb{\Gamma}\{\Gamma_1;\Gamma_2;\Gamma_2\} \vdash \{\Delta_1;F_2;F_1\wedge F_2\}\end{array}}{\mathbb{\Gamma}\{\Gamma_1;\Gamma_2;\Gamma_2\} \vdash \{\Delta_1;F_1\wedge F_2;F_1\wedge F_2\}}\,\wedge R$$

Then both $D_1' : \mathbb{\Gamma}\{\Gamma_1;\Gamma_2;\Gamma_2\} \vdash \{\Delta_1;F_1;F_1\}$ and
$D_2' : \mathbb{\Gamma}\{\Gamma_1;\Gamma_2;\Gamma_2\} \vdash \{\Delta_1;F_2;F_2\}$ are $\alpha\mathtt{LBBI}_p$-derivable;

$D_1'' : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; F_1\}$ and
$D_2'' : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; F_2\}$ are also $\alpha\texttt{LBBI}_p$-derivable (induction hypothesis); then a forward application of $\wedge R$ on D_1'' and D_2'' concludes.

$\vee L$ and ϕ_n is $F_1 \vee F_2$: similar, straightforward. I say simply straightforward to also mean a similar case in the rest.

$\vee R$ and ϕ_p is $F_1 \vee F_2$: straightforward.

$\supset L$ and ϕ_n is $F_1 \supset F_2$: if it does not appear in Γ_2, then induction hypothesis on both of the premises. Otherwise $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ D_1 & D_2 \end{array}}{\mathbb{\Gamma}((\Gamma_1; \Gamma_2\{\Gamma_3; F_1 \supset F_2\}; \Gamma_2((\Gamma_3; F_1 \supset F_2)^{\Delta'}))^{(\Delta_1;H;H)}) \vdash \{\Delta'\}} \supset L$$

where
$D_1 : \mathbb{\Gamma}((\Gamma_1; \Gamma_2\{\Gamma_3\}; \Gamma_2((\Gamma_3; F_1 \supset F_2)^{\Delta'}))^{(\Delta_1;H;H)}) \vdash \{\Delta'; F_1\}$ and
$D_2 : \mathbb{\Gamma}((\Gamma_1; \Gamma_2\{\Gamma_3; F_2\}; \Gamma_2((\Gamma_3; F_1 \supset F_2)^{\Delta'}))^{(\Delta_1;H;H)}) \vdash \{\Delta'\}$.
Then
$D_1' : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2((\Gamma_3)^{\Delta';F_1}); \Gamma_2((\Gamma_3)^{\Delta';F_1})\} \vdash \{\Delta_1; H; H\}$ and
$D_2' : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2((\Gamma_3; F_2)^{\Delta'}); \Gamma_2((\Gamma_3; F_2)^{\Delta'})\} \vdash \{\Delta_1; H; H\}$ are $\alpha\texttt{LBBI}_p$-derivable (inversion lemma);
$D_1'' : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2((\Gamma_3)^{\Delta';F_1})\} \vdash \{\Delta_1; H\}$ and
$D_2'' : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2((\Gamma_3; F_2)^{\Delta'})\} \vdash \{\Delta_1; H\}$
are also $\alpha\texttt{LBBI}_p$-derivable (induction hypothesis); a forward application of $\supset L$ then concludes.

$\supset R$ and ϕ_p is $F_1 \supset F_2$: if ϕ_p it not H, then induction hypothesis. Otherwise, $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{c} \vdots \\ D_1 : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2; \Gamma_2; F_1\} \vdash \{\Delta_1; F_2; F_1 \supset F_2\} \end{array}}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2; \Gamma_2\} \vdash \{\Delta_1; F_1 \supset F_2; F_1 \supset F_2\}} \supset R$$

Then $D_1' : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2; \Gamma_2; F_1; F_1\} \vdash \{\Delta_1; F_2; F_2\}$ is $\alpha\texttt{LBBI}_p$-derivable (inversion lemma). Then $D_1' : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2; F_1\} \vdash \{\Delta_1; F_2\}$ is also $\alpha\texttt{LBBI}_p$-derivable (induction hypothesis); a forward application of $\supset R$ then concludes.

$*L$ **and** ϕ_n **is** $F_1 * F_2$**:** If ϕ_n does not appear in Γ_2, then induction hypothesis. Otherwise $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : \mathbb{\Gamma}((\Gamma_1; \Gamma_2\{\Gamma_3; (F_1, F_2)\}; \Gamma_2((\Gamma_3; F_1 * F_2)^{\Delta'}))^{(\Delta_1;H;H)}) \vdash \{\Delta'\}\end{array}}{\mathbb{\Gamma}((\Gamma_1; \Gamma_2\{\Gamma_3; F_1 * F_2\}; \Gamma_2((\Gamma_3; F_1 * F_2)^{\Delta'}))^{(\Delta;H)}) \vdash \{\Delta'\}} *L$$

Then
$D_1' : \mathbb{\Gamma}(\Gamma_1; \Gamma_2((\Gamma_3; (F_1, F_2))^{\Delta'}); \Gamma_2((\Gamma_3; (F_1, F_2))^{\Delta'})) \vdash \{\Delta_1; H; H\}$
is $\alpha\mathtt{LBBI}_p$-derivable (inversion lemma);
$D_1'' : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2((\Gamma_3; (F_1, F_2))^{\Delta'})\} \vdash \{\Delta_1; H\}$
is also $\alpha\mathtt{LBBI}_p$-derivable (induction hypothesis); a forward application of $*L$ then concludes.

$*R_I$ **and** ϕ_p **is** $F_1 * G_1$**:** For this case and $*R_{*\top}$, synthesis operations may be taking place on several formulas in the form: $F_i * F_j$. Let us call those that are used in the process of synthesising $\mathsf{S}^+(\cdots) * \mathsf{S}^-(\cdots)$ active for the inference rule. If ϕ_p is not active, then induction hypothesis. Otherwise, supposing that G_x denotes $F_1 * G_1; \cdots ; F_l * G_l$, $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : Re_1^{\Delta_a} \vdash \mathsf{S}^+(G_x)\end{array} \qquad \begin{array}{c}\vdots\\ D_2 : Re_2^{\Delta_b} \vdash \mathsf{S}^-(G_x)\end{array}}{D : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2; \Gamma_2\} \vdash \{\Delta_1; F_1 * G_1; F_1 * G_1\}} *R_I$$

where $\mathbb{\Gamma}((\Gamma_1; \Gamma_2; \Gamma_2)^{(\Delta_1; F_1*G_1; F_1*G_1)}) \leftrightharpoons_{ant} \mathbb{\Gamma}'(\mathbb{E}(\Gamma_3^{(\Delta_3; G_x; F_1*G_1)}))$. Let us consider the internalised transitions for this $\alpha\mathtt{LBBI}_p$ inference rule:
$D \rightsquigarrow^*_{\{*\top WkL, \mathbb{1}_{ps}\}} [D' : \Gamma_3^{(\Delta_3; F_1*G_1)} \vdash G_x] \rightsquigarrow_{syn}$
$[D'' : \Gamma_3^{(\Delta_3; F_1*G_1)} \vdash \mathsf{S}^+(G_x) * \mathsf{S}^-(G_x)] \rightsquigarrow^*_{Wk} [D''' : Re_1^{\Delta_a}, Re_2^{\Delta_b} \vdash \mathsf{S}^+(G_x) * \mathsf{S}^-(G_x)]$.
At the internal state of D''', if neither $Re_1^{\Delta_a}$ nor $Re_2^{\Delta_2}$ is empty, then by the definition of a maximal $Re_1^{\Delta_a}/Re_2^{\Delta_b}$ pair, the duplicated $F_1 * G_1$ must have been (upward) weakened away in the internal transition $D'' \rightsquigarrow D'''$. For the same reason, if $\Gamma_3^{\Delta_3} \leftrightharpoons_{ant} \Gamma_2; \Gamma_2; \Gamma_4$ (for some Γ_4), then at least one of the Γ_2 must have been (upward) weakened away, to conclude.
On the other hand, if either $Re_1^{\Delta_a}$ or $Re_2^{\Delta_b}$ is empty and the other is $\Gamma_3^{(\Delta_3; F_1*G_1)}$, then we replace the derivation by $*R_I$ with that by $*R_{*\top}$, then induction hypothesis and then a forward application of $*R_{*\top}$ to conclude. Note, in this case, that

from the applicability of $*R_I$ we know that the context of the antecedent of the right premise sequent can be got rid of without blemishing the upward derivability via admissibilities of $\mathbb{1}_{ps\,\texttt{LBBI}_p}$ and ${}^{*}\!\top WkL_{\texttt{LBBI}_p}$.

$*R_{*\top}$ **and** ϕ_p **is** $F_1 * G_1$**:** Induction hypothesis and a forward application of $*R_{*\top}$.

$-\!\!*L_I$ **and** ϕ_n **is** $(\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_c; F_1 -\!\!* F_2)^{\Delta_c}))^{\Delta_4}$**:** if ϕ_n does not appear in Γ_2, then induction hypothesis. If it is in Γ_2, then $\Pi(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots\\ D_1 : Re_1^{\Delta_d} \vdash F_1\end{matrix} \quad (\Xi \cap F_1 = \emptyset) \qquad \begin{matrix}\vdots\\ D_2\end{matrix}}{D}\; -\!\!*L_I$$

where

$D_2 : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2(((Re_2^{\Delta_e}, F_2); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_c; F_1 -\!\!* F_2)^{\Delta_c})))^{\Delta''});$
$\Gamma_2((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_c; F_1 -\!\!* F_2)^{\Delta_c}))^{\Delta''})\} \vdash \{\Delta_1; H; H\}$

and

$D : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_c; F_1 -\!\!* F_2)^{\Delta_c}))^{\Delta''});$
$\Gamma_2((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_c; F_1 -\!\!* F_2)^{\Delta_c}))^{\Delta''})\} \vdash \{\Delta_1; H; H\}$.
Then
$D_2' : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2(((Re_2^{\Delta_e}, F_2); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_c; F_1 -\!\!* F_2)^{\Delta_c})))^{\Delta''});$
$\Gamma_2(((Re_2^{\Delta_e}, F_2); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_c; F_1 -\!\!* F_2)^{\Delta_c}))^{\Delta''}))\} \vdash \{\Delta_1; H; H\}$

is $\alpha\texttt{LBBI}_p$-derivable (additive weakening admissibility);

$D_2'' : \mathbb{\Gamma}\{\Gamma_1; \Gamma_2(((Re_2^{\Delta_b}, F_2); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_c; F_1 -\!\!* F_2)^{\Delta_c})))^{\Delta''})\} \vdash \{\Delta_1; H\}$

is also $\alpha\texttt{LBBI}_p$-derivable (induction hypothesis); a forward application of $-\!\!*L_I$ then concludes.
If, on the other hand, Γ_2 is in ϕ_n, then

1. if it is not in Γ_a, nor in Γ_b, nor in Γ_c after the internalised backward ${}^{*}\!\top WkL$, then the "$\Gamma_2; \Gamma_2$" must have been multiplicatively weakened away. Simpler

than the above case.

2. if it is in Γ_c after the multiplicative weakening, then the internalised backward additive weakening must weaken away the "$\Gamma_2; \Gamma_2$". Again simpler.
3. if it is in Γ_a (or in Γ_b) after both of the weakening processes, then similar to $*R_I$ case.

$-\!\!*R_I$ and ϕ_p is $F_1 -\!\!* F_2$: If ϕ_p is not H, then induction hypothesis. Otherwise, $CtrR$ is absorbed in this inference rule.

$-\!\!*L_{*\top}$ and ϕ_n is $F_1 -\!\!* F_2$: By the applicability of the inference rule, F_1 is in the collector Ξ. Inversion lemma, induction hypothesis and then a forward application of $-\!\!*L_{*\top}$ conclude.

$-\!\!*R_{*\top}$ and ϕ_p is $F_1 -\!\!* F_2$: straightforward.

${}^{*}\!\top CtrL$: the effect of $CtrR$ is absorbed in this inference rule. Induction hypothesis (and a forward application of ${}^{*}\!\top CtrL$) conclude.

□

Appendix I - Proof of Proposition 20

Proof is by induction on derivation depth of $\Pi(D)$ into both directions. But first we prove the following ground case: $D' : \mathbb{\Gamma} \vdash \mathbb{1}$ is $\alpha\texttt{LBBI}_p$-derivable with an empty collector Ξ' iff it is $[\texttt{LBBI}_p\text{- }\texttt{Cut}]$ derivable with Ξ'.

Into the *only if* direction, assume that D' is $\alpha\texttt{LBBI}_p$-derivable with Ξ', then show that there is a $[\texttt{LBBI}_p \text{ - } \texttt{Cut}]$-derivation for each $\alpha\texttt{LBBI}_p$ derivation with the same empty collector. Modified rules are derivable in $\texttt{LBBI}_p$, as stated in 4.4.1. All the other $\alpha\texttt{LBBI}_p$ inference rules are identical to a corresponding $\texttt{LBBI}_p$ inference rule. No derivations with Ξ' involve either $\mathrel{-\!\!*} L_{{}^{*}\top}$ or $\mathrel{-\!\!*} R_{{}^{*}\top}$ derivation steps.

Into the *if* direction, assume that D' is $[\texttt{LBBI}_p\text{- }\texttt{Cut}]$-derivable, then show that there is a $\alpha\texttt{LBBI}_p$-derivation for each, using Ξ'.

If derivation depth of $\Pi(D')$ is 1, *i.e.* if D' is the conclusion sequent of an axiom, we need to show that $F \vdash F$ is $\alpha\texttt{LBBI}_p$-derivable for $F \in \mathfrak{F}_{\texttt{BBI}}$. Hence a sub-induction on $\texttt{f\_depth}(F)$. If it is 1, *i.e.* if it is a propositional variable $p, \top, \mathbb{1}$ or ${}^{*}\top$, then $id, \top R, \mathbb{1}L$ or respectively ${}^{*}\top R$.

For inductive cases of the sub-induction, assume that it holds true for all formulas of formula depth up to k. We must now show that it still holds true for all the formulas $F \in \mathfrak{F}_{\texttt{BBI}}$ of formula depth $k+1$.

1. $F = F_1 \wedge F_2$:

$$\dfrac{\dfrac{\dfrac{D_1 : F_1 \vdash F_1}{F_1; F_2 \vdash F_1}\,\text{Wk L} \qquad \dfrac{D_2 : F_2 \vdash F_2}{F_1; F_2 \vdash F_2}\,\text{Wk L}}{F_1; F_2 \vdash F_1 \wedge F_2}\,\wedge R}{F_1 \wedge F_2 \vdash F_1 \wedge F_2}\,\wedge L$$

   Induction hypothesis on both D_1 and D_2, and then appropriate $\alpha\texttt{LBBI}_p$ forward derivation steps to reach the conclusion sequent in $\alpha\texttt{LBBI}_p$-space.

2. $F = F_1 \vee F_2$: straightforward.

3. $F = F_1 \supset F_2$: straightforward.

4. $F = F_1 * F_2$:

$$\dfrac{\dfrac{D_1 : F_1 \vdash F_1 \qquad D_2 : F_2 \vdash F_2}{F_1, F_2 \vdash F_1 * F_2} {*R_I}}{F_1 * F_2 \vdash F_1 * F_2} {*L}$$

5. $F = F_1 {-\!\!*} F_2$:

$$\dfrac{\dfrac{F_1 \vdash F_1 \qquad F_2 \vdash F_2}{F_1 {-\!\!*} F_2, F_1 \vdash F_2} {-\!\!*L_I}}{F_1 {-\!\!*} F_2 \vdash F_1 {-\!\!*} F_2} {-\!\!*R_I}$$

to conclude.

For inductive cases, assume that the current proposition into the *if* direction holds true for [$\texttt{LBBI}_p$- $\texttt{Cut}$]-derivations of derivation depth up to k, then show that it still holds true at derivation depth of $k+1$. Consider what the last [$\texttt{LBBI}_p$- $\texttt{Cut}$] rule applied is.

$\mathbb{1}_{ps}$**:** Proposition 15.

$\wedge L$**:** same inference rule in $\alpha\texttt{LBBI}_p$.

$\vee L$**:** same.

$\wedge R$**:** same.

$\vee R$**:** same.

$\supset L$**:** same.

$\supset R$**:** same.

$*L$**:** same.

$*R_{I\,\texttt{LBBI}_p}$**:** $\Pi_{\texttt{LBBI}_p}(D)$ looks like:

$$\frac{\begin{array}{cc}\vdots & \vdots \\ D_1 : \Gamma_1^{\Delta_1} \vdash F_1 & D_2 : \Gamma_2^{\Delta_2} \vdash F_2\end{array}}{\Gamma_1^{\Delta_1}, \Gamma_2^{\Delta_2} \vdash F_1 * F_2}\ *R_{I\ \mathtt{LBBI}_p}$$

By induction hypothesis, both D_1 and D_2 are also $\alpha\mathtt{LBBI}_p$-derivable. But then it is straightforward to show that D is also $\alpha\mathtt{LBBI}_p$-derivable by a forward application of $*R_{I\ \alpha\mathtt{LBBI}_p}$.

$*R_{{}^{*}\top\ \mathtt{LBBI}_p}$**:** $\Pi_{\mathtt{LBBI}_p}(D)$ looks like:

$$\frac{\begin{array}{cc}\vdots & \vdots \\ D_1 : {}^{*}\top \vdash F_1 & D_2 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_2\}\end{array}}{D : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta; F_1 * F_2\}}\ *R_{{}^{*}\top\ \mathtt{LBBI}_p}$$

Both D_1 and D_2 are $\alpha\mathtt{LBBI}_p$-derivable (induction hypothesis); $D_2' : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_2; F_1 * F_2\}$ is also $\alpha\mathtt{LBBI}_p$-derivable (induction hypothesis and weakening admissibility); then a forward application of $*R_{{}^{*}\top\ \alpha\mathtt{LBBI}_p}$ concludes.

$-\!\!*R_{I\ \mathtt{LBBI}_p}$**:** similar, straightforward.

$-\!\!*R_{{}^{*}\top\ \mathtt{LBBI}_p}$**:** same.

$-\!\!*L_{I\ \mathtt{LBBI}_p}$**:** $\Pi_{\mathtt{LBBI}_p}(D)$ looks like:

$$\frac{\begin{array}{cc}\vdots & \vdots \\ D_1 : \Gamma_1^{\Delta_1} \vdash F & D_2 : \mathbb{\Gamma}\{\Gamma_2^{\Delta_2}, G\} \vdash \{\Delta\}\end{array}}{D : \mathbb{\Gamma}\{\Gamma_1^{\Delta_1}, \Gamma_2^{\Delta_2}, F -\!\!* G\} \vdash \{\Delta\}}\ -\!\!*L_{I\ \mathtt{LBBI}_p}$$

Both D_1 and D_2 are $\alpha\mathtt{LBBI}_p$-derivable. Straightforward by Proposition 17.

$-\!\!*L_{{}^{*}\top\ \mathtt{LBBI}_p}$**:** same.

$WkL_{\mathtt{LBBI}_p}$**:** Proposition 17.

$WkR_{\mathtt{LBBI}_p}$**:** Proposition 17.

$CtrL_{\mathtt{LBBI}_p}$**:** Proposition 18.

$CtrR_{\mathtt{LBBI}_p}$**:** Proposition 18.

$^*\top WkL_{\texttt{LBBI}_p}$**:** Proposition 16.

$^*\top CtrL_{\texttt{LBBI}_p}$**:** With Proposition 17.

$dR_{\texttt{LBBI}_p}$**:** Proposition 19.

To conclude the proof, we must show that $\Xi_{\alpha\texttt{LBBI}_p} = \Xi_{\texttt{LBBI}_p}$. For this, we note that neither $[\texttt{LBBI}_p - \texttt{Cut}]$ nor $\alpha\texttt{LBBI}_p$ introduces a new distinct formula in the form $F_1 -\!\!* F_2$ in the course of a backward derivation. Meanwhile, all the sub-formulas at any given point of a backward derivation in [$\texttt{LBBI}_p$- Cut] or $\alpha\texttt{LBBI}_p$ derivation tree must comprise sub-formulas up to synthesis operations of the conclusion of the derivation tree. Also the synthesis operations do not modify formulas in the form $F_1 -\!\!* F_2$. These mean in particular that if F_1 in the form $F_1 -\!\!* F_2$ is a sub-formula of a given sequent such that F_1 is in the collector in use, then F_1 needs shown [$\texttt{LBBI}_p$- Cut] (or $\alpha\texttt{LBBI}_p$) derivable (by the definition of a collector). However, the test (to show that both $F_1 \vdash {}^*\top$ and ${}^*\top \vdash F_1$ are derivable) does not involve a question of whether $F_1 -\!\!* F_2$ is already in the collector. That is, there exists a partial pre-order on the tests themselves whose base case is the ground case as we saw earlier. $\square$

Appendix J - Proof of Proposition 21

Proof is by induction on cut rank and a sub-induction on cut level. Here, (U, V) denotes, for some $[\alpha\texttt{LBBI}_p + \texttt{Cut}]^-$ inference rules U and V, that one of the premises has been just derived with U and the other with V. In the following, the first derivation is the original derivation tree, and the second derivation a permuted derivation tree of the original derivation tree.

Throughout, I make use of the following derivable `Cut`:

$$\dfrac{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F\} \qquad \mathbb{\Gamma}\{((F; {}^{*}\top; \Gamma_2)^{\Delta_2})^{\times n}\} \vdash \{\Delta_2\}}{\mathbb{\Gamma}\{{}^{*}\top; \Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2\}}\ \texttt{BBI-MultiCut}$$

where $\mathbb{\Gamma}^{\times n}$ denotes $\underbrace{\mathbb{\Gamma}, \cdots, \mathbb{\Gamma}}_{n}$.

(id, id)**:**

1.

$$\dfrac{\dfrac{}{\mathbb{\Gamma}\{p; \Gamma_1\} \vdash \{p; \Delta_1\}}\ id \qquad \dfrac{}{\mathbb{\Gamma}\{p; \Gamma_2\} \vdash \{p; \Delta_2\}}\ id}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; p\} \vdash \{\Delta_1; \Delta_2; p\}}\ \texttt{Cut}$$

$\Rightarrow$

$$\dfrac{}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; p\} \vdash \{\Delta_1; \Delta_2; p\}}\ id$$

Incidentally we analyse in case an essence appears (where the cut formula is active for id on both of the premises). First and foremost, we recall that the significance of an essence is only notational. Hence, its presence causes a difference from the above permutation if the left premise sequent $\mathbb{\Gamma}\{\mathsf{E}((p; \Gamma_1)^{(p;\Delta_1)})\} \vdash \{\log \mathbb{E}((p; \Gamma_1)^{(p;\Delta_1)})\}$ is in the form:
$\mathbb{\Gamma}'\{\Gamma_1'\} \vdash \{(p; \Delta_1')\}$ where:

- A. $\mathbb{\Gamma}'(\Gamma_1'^{(p;\Delta_1')}) \leftrightharpoons_{ant} \mathbb{\Gamma}(\mathbb{E}((p; \Gamma_1)^{(p;\Delta_1)}))$.

- $\Gamma'_1 \not\asymp_{ant} \Gamma''_2; p$ for some Γ''_2.

The left sequent of the `Cut` is then: $\mathbb{\Gamma}'\{\Gamma'_1\} \vdash \{p; \Delta'_1\}$. But then the context for the right premise sequent is also $\mathbb{\Gamma}'(-)$. This implies that the right premise sequent is at least in the form: $\mathbb{\Gamma}'\{p; \Gamma_4\} \vdash \{\Delta_4\}$ (for some Γ_4 and Δ_4) such that [B. $\mathbb{\Gamma}'((p; \Gamma_4)^{\Delta_4}) \leftrightharpoons_{ant} \mathbb{\Gamma}(\mathbb{E}((p; \Gamma_2)^{(p;\Delta_2)}))$]. Applying `Cut`, we derive $\mathbb{\Gamma}'\{\Gamma_3; \Gamma_4\} \vdash \{\Delta_3; \Delta_4\}$. If $\Delta_4 \asymp_{ant} p; \Delta'_4$ (for some Δ'_4; similarly in the rest), then we have that $\mathbb{\Gamma}'((\Gamma_3; \Gamma_4)^{\Delta_3;\Delta'_4;p}) \leftrightharpoons_{ant} \mathbb{\Gamma}(\mathbb{E}((\Gamma_5; p)^{(\Delta_5;p)}))$ by B. Otherwise, we again have that
$\mathbb{\Gamma}'((\Gamma_3; \Gamma_4)^{(\Delta_3;\Delta_4)}) \leftrightharpoons_{ant} \mathbb{\Gamma}(\mathbb{E}(\Gamma_6; p)^{(\Delta_6;p)})$ by A and B.

2.

$$\dfrac{\dfrac{}{\mathbb{\Gamma}\{\Gamma_1; p\} \vdash \{p; \Delta_1\}}\ id \qquad \dfrac{}{\mathbb{\Gamma}\{\Gamma_2; p; q\} \vdash \{q; \Delta_2\}}\ id}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; p; q\} \vdash \{q; \Delta_1; \Delta_2\}}\ \texttt{Cut}$$

$\Rightarrow$

$$\dfrac{}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; p; q\} \vdash \{q; \Delta_1; \Delta_2\}}\ id$$

(a) If an essence is required in one of the premise sequents for the id: trivial if it is the left premise sequent since the conclusion sequent of `Cut` would then look like: $\mathbb{\Gamma}'\{\Gamma_3; q\} \vdash \{\Delta_3; q\}$. If it is the right premise sequent, then the negative structure whose exponent holds the p is in the conclusion sequent of the `Cut`, which implies that the conclusion sequent is in the form: $\mathbb{\Gamma}''(\mathbb{E}((\Gamma_4; p)^{(p;\Delta_4)})) \vdash \mathbb{1}$.

(b) If an essence is required in both of the premise sequents and if the sequents are not in the form considered so far:

  i. if the left premise sequent is in the form
  $\mathbb{\Gamma}_1(p; \Gamma''_1(\mathbb{\Gamma}_2\{\Gamma''_2\})) \vdash \{p; \Delta''\}$ such that $\log \mathbb{\Gamma}_2(\Gamma_2''^{(p;\Delta'')}) = \Delta_x$:

    A. if the right premise sequent is in the form
    $\mathbb{\Gamma}_1(p; \Gamma''_1(\mathbb{\Gamma}_2(p; q; \Gamma'''_2(\mathbb{\Gamma}_3\{\Gamma'''_3\})))) \vdash \{q; \Delta'''_3\}$: then the conclusion sequent would be:
    $\mathbb{\Gamma}_1(p; \Gamma''_1(\mathbb{\Gamma}_2(q; \Gamma''_2; \Gamma'''_2(\mathbb{\Gamma}_3\{\Gamma'''_3\})))) \vdash \{q; \Delta'''_3\}$ such that $\log \mathbb{\Gamma}_2(\cdots) = \Delta_x; \Delta''$. But then, since id applies on the right premise sequent, it also applies on the conclusion sequent.

    B. if it is in the form
    $\mathbb{\Gamma}_1\{p; \Gamma''_1(\mathbb{\Gamma}_2((p; q; \Gamma'''_2)^{\Delta'''_2}))\} \vdash \{q; \Delta'''_3\}$: then the conclusion se-

        quent would be:
        $\mathbb{\Gamma}_1\{p;\Gamma_1''(\mathbb{\Gamma}_2((q;\Gamma_2''';\Gamma_2'')^{(\Delta_2''';\Delta'')}))\} \vdash \{q;\Delta_3'''\}$. Again id applies on this sequent because it does on the right premise sequent.

        C. the rest: similar.

    ii. if the left premise sequent is in the form $\mathbb{\Gamma}_1\{\mathbb{\Gamma}_1''((p;\Gamma_2)^{\Delta_2})\} \vdash \{p;\Delta''\}$: then almost symmetrical to the previous case study.

3. The rest (; if the cut formula is not the principal in either of the premise sequents): trivial.

$(id, \mathbb{1}L)$:
trivial if the cut formula is not $\mathbb{1}$. Otherwise, similar to (id, id).

$(id, \top R)$:
trivial if the cut formula is not $\top$. Otherwise similar to (id, id) case.

$(id, {}^{*}\top R)$:
straightforward.

$(id, \vee L)$:

1.

$$\dfrac{\dfrac{}{D_1 : \mathbb{\Gamma}\{\Gamma_1;p\} \vdash \{\Delta_1;p\}}\,id \qquad \dfrac{D_2 : \mathbb{\Gamma}\{p;\Gamma_2;F_1\} \vdash \{\Delta_2\} \qquad D_3 : \mathbb{\Gamma}\{p;\Gamma_2;F_2\} \vdash \{\Delta_2\}}{\mathbb{\Gamma}\{p;\Gamma_2;F_1 \vee F_2\} \vdash \{\Delta_2\}}\,\vee L}{\mathbb{\Gamma}\{\Gamma_1;p;\Gamma_2;F_1 \vee F_2\} \vdash \{\Delta_1;\Delta_2\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{D_1 \qquad D_2}{\mathbb{\Gamma}\{\Gamma_1;p;\Gamma_2;F_1\} \vdash \{\Delta_1;\Delta_2\}}\,\texttt{Cut} \qquad \dfrac{D_1 \qquad D_3}{\mathbb{\Gamma}\{\Gamma_1;p;\Gamma_2;F_2\} \vdash \{\Delta_1;\Delta_2\}}\,\texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_1;p;\Gamma_2;F_1 \vee F_2\} \vdash \{\Delta_1;\Delta_2\}}\,\vee L$$

2.

$$\dfrac{D_2 : \mathbb{\Gamma}(p;\Gamma_1(\mathbb{\Gamma}_2\{p;\Gamma_4;F_1\})) \vdash \{\Delta_2\} \qquad D_3 : \mathbb{\Gamma}(p;\Gamma_1(\mathbb{\Gamma}_2\{p;\Gamma_4;F_2\})) \vdash \{\Delta_2\}}{D_4 : \mathbb{\Gamma}(p;\Gamma_1(\mathbb{\Gamma}_2\{p;\Gamma_4;F_1 \vee F_2\})) \vdash \{\Delta_2\}}\,\vee L$$

$$\dfrac{\dfrac{}{D_1 : \mathbb{\Gamma}(p;\Gamma_1(\mathbb{\Gamma}_2\{\Gamma_3\})) \vdash \{p;\Delta\}}\,id \qquad D_4}{\mathbb{\Gamma}(p;\Gamma_1(\mathbb{\Gamma}_2\{\Gamma_3;\Gamma_4;F_1 \vee F_2\})) \vdash \{\Delta;\Delta_2\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{D_1 \qquad D_2}{\mathbb{\Gamma}(p;\Gamma_1(\mathbb{\Gamma}_2\{\Gamma_3;\Gamma_4;F_1\}))\vdash\{\Delta;\Delta_2\}}\,\texttt{Cut} \qquad \dfrac{D_1 \qquad D_3}{\mathbb{\Gamma}(p;\Gamma_1(\mathbb{\Gamma}_2\{\Gamma_3;\Gamma_4;F_2\}))\vdash\{\Delta;\Delta_2\}}\,\texttt{Cut}}{\mathbb{\Gamma}(p;\Gamma_1(\mathbb{\Gamma}_2\{\Gamma_3;\Gamma_4;F_1\vee F_2\}))\vdash\{\Delta;\Delta_2\}}\,\vee L$$

3.

$$\dfrac{\dfrac{}{D_1:\mathbb{\Gamma}\{\mathbb{\Gamma}_1((p;\Gamma_2)^{\Delta_1})\}\vdash\{p;\Delta\}}\,id \qquad \dfrac{D_2:\mathbb{\Gamma}\{p;\Gamma_3;F_1\}\vdash\{\Delta_2\} \qquad D_3:\mathbb{\Gamma}\{p;\Gamma_3;F_2\}\vdash\{\Delta_2\}}{\mathbb{\Gamma}\{p;\Gamma_3;F_1\vee F_2\}\vdash\{\Delta_2\}}\,\vee L}{\mathbb{\Gamma}\{\Gamma_3;F_1\vee F_2;\mathbb{\Gamma}_1((p;\Gamma_2)^{\Delta_1})\}\vdash\{\Delta;\Delta_2\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{D_1 \qquad D_2}{\mathbb{\Gamma}\{\Gamma_3;F_1;\mathbb{\Gamma}_1((p;\Gamma_2)^{\Delta_1})\}\vdash\{\Delta;\Delta_2\}}\,\texttt{Cut} \qquad \dfrac{D_1 \qquad D_3}{\mathbb{\Gamma}\{\Gamma_3;F_2;\mathbb{\Gamma}_1((p;\Gamma_2)^{\Delta_2})\}\vdash\{\Delta;\Delta_2\}}\,\texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_3;F_1\vee F_2;\mathbb{\Gamma}_1((p;\Gamma_2)^{\Delta_1})\}\vdash\{\Delta;\Delta_2\}}\,\vee L$$

4.

$$\dfrac{D_2:\mathbb{\Gamma}\{\Gamma_2;F_1\}\vdash\{\Delta_2\} \qquad D_3:\mathbb{\Gamma}\{\Gamma_2;F_2\}\vdash\{\Delta_2\}}{D_4:\mathbb{\Gamma}\{\Gamma_2;F_1\vee F_2\}\vdash\{\Delta_2\}}\,\vee L$$

$$\dfrac{\dfrac{}{D_1:\mathbb{\Gamma}\{\Gamma_1;p\}\vdash\{p;F_1\vee F_2;\Delta_1\}}\,id \qquad D_4}{\mathbb{\Gamma}\{\Gamma_1;p;\Gamma_2\}\vdash\{p;\Delta_1;\Delta_2\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{}{\mathbb{\Gamma}\{\Gamma_1;p;\Gamma_2\}\vdash\{p;\Delta_1;\Delta_2\}}\,id$$

5.

Let $\mathbb{\Gamma}_x(-)$ denote $(p;{}^{*}\top;\Gamma_4((\Gamma_3;-)^{\Delta'}))^{\Delta_2}$ for this case. Assume $n\geq 2$.

$$\dfrac{\dfrac{D_2:\mathbb{\Gamma}\{\mathbb{\Gamma}_x(F_1),(\mathbb{\Gamma}_x(F_1\vee F_2))^{\times n-1}\}\vdash\{\Delta_2\} \qquad D_3:\mathbb{\Gamma}\{\mathbb{\Gamma}_x(F_2),(\mathbb{\Gamma}_x(F_1\vee F_2))^{\times n-1}\}\vdash\{\Delta_2\}}{\mathbb{\Gamma}\{(\mathbb{\Gamma}_x(F_1\vee F_2))^{\times n}\}\vdash\{\Delta_2\}}\,\vee L}{D_4:\mathbb{\Gamma}\{\Gamma_x(F_1\vee F_2)\}\vdash\{\Delta_2\}}\,{}^{*}\top CtrL$$

$$\dfrac{\dfrac{}{D_1:\mathbb{\Gamma}\{\mathbb{\Gamma}_1((p;\Gamma_2)^{\Delta_3})\}\vdash\{p;\Delta_1\}}\,id \qquad D_4}{\mathbb{\Gamma}\{\mathbb{\Gamma}_1((p;\Gamma_2)^{\Delta_3});{}^{*}\top;\Gamma_4((\Gamma_3;F_1\vee F_2)^{\Delta'})\}\vdash\{\Delta_1;\Delta_2\}}\,\texttt{Cut}$$

$\Rightarrow$

We derive the following first by inversion lemma and `BBI-MultiCut`.

$$\dfrac{D_1 \qquad D_2':\mathbb{\Gamma}\{(\mathbb{\Gamma}_x(F_1))^{\times n}\}\vdash\{\Delta_2\}}{D_5:\mathbb{\Gamma}\{\mathbb{\Gamma}_1((p;\Gamma_2)^{\Delta_3});{}^{*}\top;\Gamma_4((\Gamma_3;F_1)^{\Delta'}))\}\vdash\{\Delta_1;\Delta_2\}}\,\texttt{BBI-MultiCut}$$

$$\dfrac{D_1 \qquad D_3':\mathbb{\Gamma}\{(\mathbb{\Gamma}_x(F_2))^{\times n}\}\vdash\{\Delta_2\}}{D_6:\mathbb{\Gamma}\{(\mathbb{\Gamma}_1((p;\Gamma_2)^{\Delta_3});{}^{*}\top;\Gamma_4((\Gamma_3;F_2)^{\Delta'}))\}\vdash\{\Delta_1;\Delta_2\}}\,\texttt{BBI-MultiCut}$$

The permutation is then:

$$\dfrac{D_5 \qquad D_6}{\mathbb{\Gamma}\{\mathbb{\Gamma}_1((p;\Gamma_2)^{\Delta_3}); {*\top}; \Gamma_4((\Gamma_3; F_1 \vee F_2)^{\Delta'})\} \vdash \{\Delta_1; \Delta_2\}}\,\vee L$$

6. The rest: straightforward.

$(id, \vee R)$**:**

1.

$$\dfrac{\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1; F_2; \Delta_1; p\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1 \vee F_2; \Delta_1; p\}}\,\vee R \qquad \dfrac{}{D_2 : \mathbb{\Gamma}\{\Gamma_2; p\} \vdash \{p; \Delta_2\}}\,id}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1\} \vdash \{p; \Delta_2; F_1 \vee F_2; \Delta_1\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{D_1 \qquad D_2}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{p; \Delta_2; F_1; F_2; \Delta_1\}}\,\texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{p; \Delta_2; F_1 \vee F_2; \Delta_1\}}\,\vee R$$

2. The rest: straightforward.

$(id, \wedge L), (id, \wedge R), (id, \supset L), (id, \supset R)$**:**

straightforward.

$(id, *L)$**:**

similar to $(id, \wedge L)$.

$(id, *R_I), (id, *R_{*\top}), (id, -\!\!*R_I)$**:**

straightforward, we will go through these ($(U, *R_I)$, $(U, *R_{*\top})$ and $(U, -\!\!*R_I)$) later for more involved cases.

$(id, -\!\!*R_{*\top})$**:**

1.

$$\dfrac{\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_2; p\}(F_1 \in \Xi)}{D_3 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 -\!\!* F_2; p\}}\,-\!\!*R_{*\top} \qquad \dfrac{}{D_2 : \mathbb{\Gamma}\{\Gamma_2; p\} \vdash \{\Delta_2\}}\,id}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2; F_1 -\!\!* F_2\}}\,\texttt{Cut}$$

where $\mathbb{\Gamma}((\Gamma_2;p)^{\Delta_2}) \leftrightharpoons_{ant} \mathbb{\Gamma}_A(\mathbb{E}((\Gamma_B;p)^{(\Delta_A;p)}))$ for some context $\mathbb{\Gamma}_A(\cdots)$, and some essence construct: $\mathbb{E}((\Gamma_B;p)^{(\Delta_A;p)})$.

$\Rightarrow$

$$\dfrac{\dfrac{D_1 \qquad D_2}{\mathbb{\Gamma}\{\Gamma_1;\Gamma_2\} \vdash \{\Delta_1;\Delta_2;F_2\}(F_1 \in \Xi)}\,\texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_1;\Gamma_2\} \vdash \{\Delta_1;\Delta_2;F_1 \mathbin{-\!\!*} F_2\}}\,{-\!\!*}R_{{}^{*}\top}$$

2. The rest: straightforward.

$(id, {-\!\!*}L_I)$**:**

there are a few sub-cases, but we will examine later with more involved cases.

$(id, {-\!\!*}L_{{}^{*}\top})$**:**

straightforward.

$(id, {}^{*}\top CtrL)$**:**

taken care of in `BBI-MultiCut`.

$(\top R, \top R)$**:**

straightforward.

$(\top R, \mathbb{1}L)$**:**

1. $$\dfrac{\dfrac{}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\top;\Delta_1;\mathbb{1}\}}\,\top R \qquad \dfrac{}{\mathbb{\Gamma}\{\Gamma_2;\mathbb{1}\} \vdash \{\Delta_2\}}\,\mathbb{1}L}{\mathbb{\Gamma}\{\Gamma_1;\Gamma_2\} \vdash \{\Delta_1;\Delta_2;\top\}}\,\texttt{Cut}$$

   $\Rightarrow$

   $$\dfrac{}{\mathbb{\Gamma}\{\Gamma_1;\Gamma_2\} \vdash \{\Delta_1;\Delta_2;\top\}}\,\top R$$

2. $$\dfrac{\dfrac{}{\mathbb{\Gamma}\{\Gamma_1;\mathbb{1}\} \vdash \{\Delta_1;\top\}}\,\mathbb{1}L \qquad \dfrac{}{\mathbb{\Gamma}\{\Gamma_2;\top\} \vdash \{\Delta_2\}}\,\top R}{\mathbb{\Gamma}\{\Gamma_1;\Gamma_2;\mathbb{1}\} \vdash \{\Delta_1;\Delta_2\}}\,\texttt{Cut}$$

   $\Rightarrow$

   $$\dfrac{}{\mathbb{\Gamma}\{\Gamma_2;\Gamma_2;\mathbb{1}\} \vdash \{\Delta_1;\Delta_2\}}\,\mathbb{1}L$$

3. The rest: straightforward.

$(\top R, {}^{*}\top R)$**:**

1.

$$\dfrac{\dfrac{}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; \top\}}\,\top R \qquad \dfrac{}{\mathbb{\Gamma}\{\top; \Gamma_2; {}^{*}\top\} \vdash \{{}^{*}\top; \Delta_2\}}\,{}^{*}\top R}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2; {}^{*}\top\} \vdash \{{}^{*}\top; \Delta_2; \Delta_1\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2; {}^{*}\top\} \vdash \{{}^{*}\top; \Delta_2; \Delta_1\}}\,{}^{*}\top R$$

2.

$$\dfrac{\dfrac{}{\mathbb{\Gamma}\{\Gamma_1; {}^{*}\top\} \vdash \{{}^{*}\top; \Delta_1\}}\,{}^{*}\top R \qquad \dfrac{}{\mathbb{\Gamma}\{\Gamma_2; {}^{*}\top\} \vdash \{\top; \Delta_2\}}\,\top R}{\mathbb{\Gamma}\{\Gamma_1; {}^{*}\top; \Gamma_2\} \vdash \{\top; \Delta_2; \Delta_1\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{}{\mathbb{\Gamma}\{\Gamma_1; {}^{*}\top; \Gamma_2\} \vdash \{\top; \Delta_2; \Delta_1\}}\,\top R$$

3. The rest: straightforward.

$(\top R, \wedge L)$**:**

1.

$$\dfrac{\dfrac{}{D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\top; \Delta_1\}}\,\top R \qquad \dfrac{D_2 : \mathbb{\Gamma}\{\Gamma_2; \top; F_1; F_2\} \vdash \{\Delta_2\}}{\mathbb{\Gamma}\{\Gamma_2; \top; F_1 \wedge F_2\} \vdash \{\Delta_2\}}\,\wedge L}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; F_1 \wedge F_2\} \vdash \{\Delta_1; \Delta_2\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{D_1 \qquad D_2}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; F_1; F_2\} \vdash \{\Delta_1; \Delta_2\}}\,\texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; F_1 \wedge F_2\} \vdash \{\Delta_1; \Delta_2\}}\,\wedge L$$

2. The rest: since the first case is almost identical to $(id, \vee L)$ (in fact simpler because $\wedge L$ is a single-premise rule), I omit reiterating the cut elimination procedure for the remaining cases.

$(\top R, \wedge R), (\top R, \vee L), (\top R, \supset L), (\top R, \supset R), (\top R, *L), (\top, *R_I)$:
straightforward.

$(\top R, *R_{{}^{*}\top}), (\top R, \mathrel{-\!\!*} R_I), (\top R, \mathrel{-\!\!*} R_I), (\top R, \mathrel{-\!\!*} R_{{}^{*}\top}), (\top R, \mathrel{-\!\!*} L_I)$:
straightforward, more involved cases later.

$(\top R, \mathrel{-\!\!*} L_{{}^{*}\top})$**:**
straightforward.

$(\top R, {}^{*}\!\top CtrL)$:

taken care of in `BBI-MultiCut`.

$(\mathbb{1}L, \mathbb{1}L)$:

$$\dfrac{\dfrac{}{\mathbb{\Gamma}\{\Gamma_1; \mathbb{1}\} \vdash \{\mathbb{1}; \Delta_1\}}\,\mathbb{1}L \qquad \dfrac{}{\mathbb{\Gamma}\{\Gamma_2; \mathbb{1}\} \vdash \{\Delta_2\}}\,\mathbb{1}L}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; \mathbb{1}\} \vdash \{\Delta_1; \Delta_2\}}\,\mathtt{Cut}$$

$\Rightarrow$

$$\dfrac{}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; \mathbb{1}\} \vdash \{\Delta_1; \Delta_2\}}\,\mathbb{1}L$$

$(\mathbb{1}L, {}^{*}\!\top R)$:

1.

$$\dfrac{\dfrac{}{\mathbb{\Gamma}\{\Gamma_1; \mathbb{1}\} \vdash \{\Delta_1; {}^{*}\!\top\}}\,\mathbb{1}L \qquad \dfrac{}{\mathbb{\Gamma}\{\Gamma_2; {}^{*}\!\top\} \vdash \{{}^{*}\!\top; \Delta_2\}}\,{}^{*}\!\top R}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; \mathbb{1}\} \vdash \{{}^{*}\!\top; \Delta_2; \Delta_1\}}\,\mathtt{Cut}$$

$\Rightarrow$

$$\dfrac{}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; \mathbb{1}\} \vdash \{{}^{*}\!\top; \Delta_2; \Delta_1\}}\,\mathbb{1}L$$

2.

$$\dfrac{\dfrac{}{\mathbb{\Gamma}\{\Gamma_1; {}^{*}\!\top\} \vdash \{{}^{*}\!\top; \Delta_1; \mathbb{1}\}}\,{}^{*}\!\top R \qquad \dfrac{}{\mathbb{\Gamma}\{\Gamma_2; {}^{*}\!\top; \mathbb{1}\} \vdash \{\Delta_2\}}\,\mathbb{1}L}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; {}^{*}\!\top\} \vdash \{{}^{*}\!\top; \Delta_1; \Delta_2\}}\,\mathtt{Cut}$$

$\Rightarrow$

$$\dfrac{}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; {}^{*}\!\top\} \vdash \{{}^{*}\!\top; \Delta_1; \Delta_2\}}\,{}^{*}\!\top R$$

$(\mathbb{1}L, \{\wedge L, \wedge R, \vee L, \vee R, \supset L, \supset R, *L, *R_I, *R_{{}^{*}\!\top}, -\!\!*R_I, -\!\!*R_{{}^{*}\!\top}, -\!\!*L_I, -\!\!*L_{{}^{*}\!\top}\})$:

straightforward.

$(\mathbb{1}L, {}^{*}\!\top CtrL)$:

taken care of in `BBI-MultiCut`.

$({}^{*}\!\top R, {}^{*}\!\top R)$**:**

straightforward.

$({}^{*}\!\top R,$ **the rest**$)$**:**

straightforward. *Cf.* id cases.

$(\wedge L, \wedge L)$**:**

straightforward.[1]

$(\wedge L, \wedge R)$**:**

1.

$$\dfrac{\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1; \Delta_1\} \qquad D_2 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_2; \Delta_1\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1 \wedge F_2; \Delta_1\}} \wedge R \qquad \dfrac{D_3 : \mathbb{\Gamma}\{\Gamma_2; F_1; F_2\} \vdash \{\Delta_2\}}{\mathbb{\Gamma}\{\Gamma_2; F_1 \wedge F_2\} \vdash \{\Delta_2\}} \wedge L}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1\} \vdash \{\Delta_1; \Delta_2\}} \texttt{Cut}$$

$\Rightarrow$

$$\begin{array}{c} \dfrac{D_2 \qquad \dfrac{D_1 \qquad D_3}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; F_2\} \vdash \{\Delta_2; \Delta_1\}} \texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; \Gamma_1\} \vdash \{\Delta_1; \Delta_2; \Delta_1\}} \texttt{Cut} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \; \{(CtrL), (CtrR)\} \\ \mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2\} \end{array}$$

Recall that additive contraction is admissible in $\alpha\texttt{LBBI}_p$. A dotted line indicates that the conclusion is derivable with not a greater derivation depth than the premise sequent is.

2. For this case, let $\mathbb{\Gamma}_x(-)$ denote $(\Gamma_2; {}^{*}\!\top; -)^{\Delta_2}$.

$$\begin{array}{c} \dfrac{D_3 : \mathbb{\Gamma}\{(\mathbb{\Gamma}_x(F_1 \wedge F_2))^{\times n-1}, \mathbb{\Gamma}_x(F_1; F_2)\} \vdash \{\Delta_2\}}{D_5 : \mathbb{\Gamma}\{(\mathbb{\Gamma}_x(F_1 \wedge F_2))^{\times n}\} \vdash \{\Delta_2\}} \wedge L \\ \hline\hline D_6 : \mathbb{\Gamma}\{\Gamma_2; {}^{*}\!\top; F_1 \wedge F_2\} \vdash \{\Delta_2\} \end{array} \; {}^{*}\!\top CtrL$$

$$\dfrac{\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1; \Delta_1\} \qquad D_2 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_2; \Delta_1\}}{D_4 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1 \wedge F_2; \Delta_1\}} \wedge R \qquad D_6}{\mathbb{\Gamma}\{\Gamma_2; {}^{*}\!\top; \Gamma_1\} \vdash \{\Delta_1; \Delta_2\}} \texttt{Cut}$$

$\Rightarrow$

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[1] In fact, additive cases are all similar to classical logic cases up to `BBI-MultiCut` and the surrounding context.

$$\dfrac{D_2 \qquad \dfrac{D_1 \qquad D_3' : \mathbb{\Gamma}\{(\mathbb{\Gamma}_x(F_1;F_2))^{\times n}\} \vdash \{\Delta_2\}}{\mathbb{\Gamma}\{\Gamma_x(\Gamma_1;F_2)\} \vdash \{\Delta_1;\Delta_2\}}\,\texttt{BBI-MultiCut}}{\dfrac{\mathbb{\Gamma}\{\Gamma_x(\Gamma_1;\Gamma_1)\} \vdash \{\Delta_1;\Delta_1;\Delta_2\}}{\mathbb{\Gamma}\{\Gamma_2;{}^{*}\!\top;\Gamma_1\} \vdash \{\Delta_1;\Delta_2\}}\,\{(CtrL),(CtrR)\}}\,\texttt{Cut}$$

where D_3' derives from D_3 via inversion lemma.

3. The rest: straightforward.

$(\wedge L, \vee L)$**:**

straightforward.

$(\wedge L, \vee R)$**:**

Let $\mathbb{\Gamma}_x(-)$ denote $(\Gamma_2;{}^{*}\!\top;-;H_1 \vee H_2)^{\Delta_2}$.

1.

$$\dfrac{\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{H_1;H_2;\Delta_1\}}{D_3 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{H_1 \vee H_2;\Delta_1\}}\,\vee R \qquad \dfrac{\dfrac{D_2 : \mathbb{\Gamma}\{(\mathbb{\Gamma}_x(F_1 \wedge F_2))^{\times n-1}, \mathbb{\Gamma}_x(F_1;F_2)\} \vdash \{\Delta_2\}}{D_4 : \mathbb{\Gamma}\{(\mathbb{\Gamma}_x(F_1 \wedge F_2))^{\times n}\} \vdash \{\Delta_2\}}\,\wedge L}{\mathbb{\Gamma}\{\Gamma_x(F_1 \wedge F_2)\} \vdash \{\Delta_2\}}\,{}^{*}\!\top CtrL}{\mathbb{\Gamma}\{\Gamma_2;{}^{*}\!\top;\Gamma_1;F_1 \wedge F_2\} \vdash \{\Delta_1;\Delta_2\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{D_3 \qquad D_2' : \mathbb{\Gamma}\{((\Gamma_2;{}^{*}\!\top;F_1;F_2;H_1 \vee H_2)^{\Delta_2})^{\times n}\} \vdash \{\Delta_2\}}{\mathbb{\Gamma}\{\Gamma_2;{}^{*}\!\top;\Gamma_1;F_1;F_2\} \vdash \{\Delta_1;\Delta_2\}}\,\texttt{BBI-MultiCut}}{\mathbb{\Gamma}\{\Gamma_2;{}^{*}\!\top;\Gamma_1;F_1 \wedge F_2\} \vdash \{\Delta_1;\Delta_2\}}\,\wedge L$$

D_2' derives from D via inversion lemma.

2. Let $\mathbb{\Gamma}_x(-)$ denote $(\Gamma_2;{}^{*}\!\top;-)^{\Delta_2}$.

$$\dfrac{\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{H_1;H_2;F_1 \wedge F_2;\Delta_1\}}{D_3 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{H_1 \vee H_2;F_1 \wedge F_2;\Delta_1\}}\,\vee R \qquad \dfrac{\dfrac{D_2 : \mathbb{\Gamma}\{(\mathbb{\Gamma}_x(F_1 \wedge F_2))^{\times n-1}, \mathbb{\Gamma}_x(F_1;F_2)\} \vdash \{\Delta_2\}}{\mathbb{\Gamma}\{(\mathbb{\Gamma}_x(F_1 \wedge F_2))^{\times n}\} \vdash \{\Delta_2\}}\,\wedge L}{D_4 : \mathbb{\Gamma}\{\Gamma_x(F_1 \wedge F_2)\} \vdash \{\Delta_2\}}\,{}^{*}\!\top CtrL}{\mathbb{\Gamma}\{\Gamma_2;{}^{*}\!\top;\Gamma_1\} \vdash \{\Delta_1;H_1 \vee H_2;\Delta_2\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{D_1 \qquad D_4}{\mathbb{\Gamma}\{\Gamma_2;{}^{*}\!\top;\Gamma_1\} \vdash \{\Delta_1;H_1;H_2;\Delta_2\}}\,\texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_2;{}^{*}\!\top;\Gamma_1\} \vdash \{\Delta_1;H_1 \vee H_2;\Delta_2\}}\,\vee R$$

3. The rest: straightforward.

$(\wedge L, \supset L)$**:**

1.

$$\dfrac{D_2 : \mathbb{\Gamma}\{\Gamma_2; G\} \vdash \{\Delta_2; H_1\} \qquad D_3 : \mathbb{\Gamma}\{\Gamma_2; G; H_2\} \vdash \{\Delta_2\}}{D_4 : \mathbb{\Gamma}\{\Gamma_2; G; H_1 {\supset} H_2\} \vdash \{\Delta_2\}} \supset L$$

$$\dfrac{\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1; F_1; F_2\} \vdash \{G; \Delta_1\}}{\mathbb{\Gamma}\{\Gamma_1; F_1 \wedge F_2\} \vdash \{G; \Delta_1\}} \wedge L \qquad D_4}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; F_1 \wedge F_2; H_1 {\supset} H_2\} \vdash \{\Delta_1; \Delta_2\}} \texttt{Cut}$$

Permutation is straightforward by $\texttt{Cut}(D_1, D_2)$ and $\texttt{Cut}(D_1, D_3)$.

2. The rest: straightforward, that is, all the answers can be found in the patterns previously studied.

$(\wedge L, \supset R)$**:**

Let $\mathbb{\Gamma}_x(-)$ denote $(\Gamma_2; {}^*\top; H_1 \supset H_2; -)^{\Delta_2}$.

1.

$$\dfrac{\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1; H_1\} \vdash \{H_2; \Delta_1\}}{D_3 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{H_1 {\supset} H_2; \Delta_1\}} \supset R \qquad \dfrac{\dfrac{D_2 : \mathbb{\Gamma}\{(\mathbb{\Gamma}_x(F_1 \wedge F_2))^{\times n-1}, \mathbb{\Gamma}_x(F_1; F_2)\} \vdash \{\Delta_2\}}{D_4 : \mathbb{\Gamma}\{(\mathbb{\Gamma}_x(F_1 \wedge F_2))^{\times n}\} \vdash \{\Delta_2\}} \wedge L}{\mathbb{\Gamma}\{\Gamma_x(F_1 \wedge F_2)\} \vdash \{\Delta_2\}} {}^*\top CtrL}{\mathbb{\Gamma}\{\Gamma_2; {}^*\top; \Gamma_1; F_1 \wedge F_2\} \vdash \{\Delta_1; \Delta_2\}} \texttt{Cut}$$

We saw a similar pattern before.

2. The rest: straightforward.

$(\wedge L, *L)$**:**

1.

$$\dfrac{\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1; F_1; F_2\} \vdash \{H_1 * H_2; \Delta_1\}}{\mathbb{\Gamma}\{\Gamma_1; F_1 \wedge F_2\} \vdash \{H_1 * H_2; \Delta_1\}} \wedge L \qquad \dfrac{D_2 : \mathbb{\Gamma}\{\Gamma_2; (H_1, H_2)\} \vdash \{\Delta_2\}}{D_3 : \mathbb{\Gamma}\{\Gamma_2; H_1 * H_2\} \vdash \{\Delta_2\}} *L}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; F_1 \wedge F_2\} \vdash \{\Delta_1; \Delta_2\}} \texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{D_1 \qquad D_3}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; F_1; F_2\} \vdash \{\Delta_1; \Delta_2\}} \texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; F_1 \wedge F_2\} \vdash \{\Delta_1; \Delta_2\}} \wedge L$$

2. Similar when ${}^*\top CtrL$ applies on the right premise sequent, up to the use of `BBI-MultiCut`.

3. The rest: straightforward.

$(\wedge L, *R_I)$**:**

Let $\mathbb{\Gamma}_x(-)$ denote $(\Gamma_2; {}^{*}\top; H_1 * H_2; -)^{\Delta_2}$.

1.

$$\dfrac{\dfrac{D_1 : Re_1^{\Delta_1} \vdash H_1 \qquad D_2 : Re_2^{\Delta_2} \vdash H_2}{D_4 : \mathbb{\Gamma}\{\Gamma_1^{\Delta_1}\} \vdash \{H_1 * H_2\}}\,{*R_I} \qquad \dfrac{\dfrac{D_3 : \mathbb{\Gamma}\{(\mathbb{\Gamma}_x(F_1 \wedge F_2))^{\times n-1}, \mathbb{\Gamma}_x(F_1; F_2)\} \vdash \{\Delta_2\}}{\mathbb{\Gamma}\{((\mathbb{\Gamma}_x(F_1 \wedge F_2))^{\times n}))\} \vdash \{\Delta_2\}}\,{\wedge L}}{\mathbb{\Gamma}\{\Gamma_x(F_1 \wedge F_2)\} \vdash \{\Delta_2\}}\,{{}^{*}\top CtrL}}{\mathbb{\Gamma}\{\Gamma_2; {}^{*}\top; \Gamma_1; F_1 \wedge F_2\} \vdash \{\Delta_1; \Delta_2\}}\,\mathtt{Cut}$$

We saw this pattern before. Note that it does not matter what the essence for the $*R_I$ is, and consequently what the $Re_1^{\Delta_a}/Re_2^{\Delta_b}$ pair is, since the permutation concludes with intuitionistic '`Cut`'s.

2. The rest: straightforward.

$(\wedge L, *R_{{}^{*}\top})$**:**

similar.

$(\wedge L, \{-\!\!*R_I, -\!\!*R_{{}^{*}\top}, -\!\!*L_I, -\!\!*L_{{}^{*}\top}\})$**:**

straightforward.

$(\wedge L, {}^{*}\top CtrL)$**:**

taken care of in `BBI-MultiCut`.

$(\wedge R, \wedge R)$**:**

1.

$$\dfrac{D_3 : \mathbb{\Gamma}\{\Gamma_2; F_1 \wedge F_2\} \vdash \{\Delta_2; H_1\} \qquad D_4 : \mathbb{\Gamma}\{\Gamma_2; F_1 \wedge F_2\} \vdash \{\Delta_2; H_2\}}{D_6 : \mathbb{\Gamma}\{\Gamma_2; F_1 \wedge F_2\} \vdash \{\Delta_2; H_1 \wedge H_2\}}\,{\wedge R}$$

$$\dfrac{\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1; \Delta_1\} \qquad D_2 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_2; \Delta_1\}}{D_5 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1 \wedge F_2; \Delta_1\}}\,{\wedge R} \qquad D_6}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2; H_1 \wedge H_2\}}\,\mathtt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{D_5 \qquad D_3}{\mathbb{\Gamma}\{\Gamma_2;\Gamma_1\} \vdash \{\Delta_2; H_1; \Delta_1\}}\,\texttt{Cut} \qquad \dfrac{D_5 \qquad D_4}{\mathbb{\Gamma}\{\Gamma_2;\Gamma_1\} \vdash \{\Delta_2; H_2; \Delta_1\}}\,\texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_2;\Gamma_1\} \vdash \{\Delta_1; \Delta_2; H_1 \wedge H_2\}}\,\wedge R$$

2. The rest: straightforward.

$(\wedge R, \{\vee L, \vee R, \supset L, \supset R\})$**:**

straightforward.

$(\wedge R, *L)$**:**

1.

$$\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1; \Delta_1\} \qquad D_2 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_2; \Delta_1\}}{D_4 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1 \wedge F_2; \Delta_1\}}\,\wedge R$$

$$\dfrac{D_4 \qquad \dfrac{D_3 : \mathbb{\Gamma}\{\Gamma_2; F_1 \wedge F_2; (H_1, H_2)\} \vdash \{\Delta_2\}}{\mathbb{\Gamma}\{\Gamma_2; F_1 \wedge F_2; H_1 * H_2\} \vdash \{\Delta_2\}}\,*L}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; H_1 * H_2\} \vdash \{\Delta_1; \Delta_2\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{D_4 \qquad D_3}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; (H_1, H_2)\} \vdash \{\Delta_1; \Delta_2\}}\,\texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; H_1 * H_2\} \vdash \{\Delta_1; \Delta_2\}}\,*L$$

2.

$$\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1; H_1 * H_2; \Delta_1\} \qquad D_2 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_2; H_1 * H_2; \Delta_1\}}{D_5 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1 \wedge F_2; H_1 * H_2; \Delta_1\}}\,\wedge R$$

$$\dfrac{D_5 \qquad \dfrac{D_3 : \mathbb{\Gamma}\{\Gamma_2; (H_1, H_2)\} \vdash \{\Delta_2\}}{D_4 : \mathbb{\Gamma}\{\Gamma_2; H_1 * H_2\} \vdash \{\Delta_2\}}\,*L}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1\} \vdash \{\Delta_1; \Delta_2; F_1 \wedge F_2\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{D_1 \qquad D_4}{\mathbb{\Gamma}\{\Gamma_2;\Gamma_1\} \vdash \{\Delta_1; F_1; \Delta_2\}}\,\texttt{Cut} \qquad \dfrac{D_2 \qquad D_4}{\mathbb{\Gamma}\{\Gamma_2;\Gamma_1\} \vdash \{\Delta_1; F_2; \Delta_2\}}\,\texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_2;\Gamma_1\} \vdash \{\Delta_1; F_1 \wedge F_2; \Delta_2\}}\,\wedge R$$

3. The rest: straightforward.

$(\wedge R, \{*R_I, *R_{*\top}, \mathrel{-\!\!*}R_I, \mathrel{-\!\!*}R_{*\top}\})$**:**

straightforward.

$(\wedge R, \mathrel{-\!\!*}L_I)$**:**

---

1. Let $\mathbb{\Gamma}'(-)$ denote $\mathbb{\Gamma}((\mathbb{\Gamma}_a(-), \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_3; H_1 \mathbin{-\!\!*} H_2)^{\Delta_3}))^{\Delta_4})$.

$$
\begin{prooftree}
\AxiomC{D_3}
\AxiomC{D_4}
\RightLabel{$\mathbin{-\!\!*} L_I$}
\BinaryInfC{$D_6 : \mathbb{\Gamma}((\mathbb{\Gamma}_a\{F_1 \wedge F_2; \Gamma_2\}, \mathbb{\Gamma}_b^{\Delta_b}, \mathbb{E}((\Gamma_3; H_1 \mathbin{-\!\!*} H_2)^{\Delta_3}))^{\Delta_4}) \vdash \{\Delta_2\}$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$D_1 : \mathbb{\Gamma}'\{\Gamma_1\} \vdash \{\Delta_1; F_1\}$}
\AxiomC{$D_2 : \mathbb{\Gamma}'\{\Gamma_1\} \vdash \{\Delta_1; F_2\}$}
\RightLabel{$\wedge R$}
\BinaryInfC{$\mathbb{\Gamma}((\mathbb{\Gamma}_a\{\Gamma_1\}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_3; H_1 \mathbin{-\!\!*} H_2)^{\Delta_3})^{\Delta_4}) \vdash \{\Delta_1; F_1 \wedge F_2\}$}
\AxiomC{D_6}
\RightLabel{$\texttt{Cut}$}
\BinaryInfC{$\mathbb{\Gamma}((\mathbb{\Gamma}_a\{\Gamma_2; \Gamma_1\}, \mathbb{\Gamma}_b^{\Delta_b}, \mathbb{E}((\Gamma_3; H_1 \mathbin{-\!\!*} H_2)^{\Delta_3}))^{\Delta_4}) \vdash \{\Delta_2; \Delta_1\}$}
\end{prooftree}
$$

$\Rightarrow$

$$
\begin{prooftree}
\AxiomC{D_2}
\AxiomC{D_1}
\AxiomC{$D_6' : \mathbb{\Gamma}'\{F_1; F_2; \Gamma_2\} \vdash \{\Delta_2\}$}
\RightLabel{$\texttt{Cut}$}
\BinaryInfC{$\mathbb{\Gamma}((\mathbb{\Gamma}_a\{\Gamma_2; F_2; \Gamma_1\}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_3; H_1 \mathbin{-\!\!*} H_2)^{\Delta_3}))^{\Delta_4}) \vdash \{\Delta_1; \Delta_2\}$}
\RightLabel{$\texttt{Cut}$}
\BinaryInfC{$\mathbb{\Gamma}((\mathbb{\Gamma}_a\{\Gamma_2; \Gamma_1; \Gamma_1\}, \Gamma_b^{\Delta_b}, \mathbb{E}((\Gamma_3; H_1 \mathbin{-\!\!*} H_2)^{\Delta_3}))^{\Delta_4}) \vdash \{\Delta_1; \Delta_1; \Delta_2\}$}
\dottedLine
\RightLabel{(Ctr L) (Ctr R)}
\UnaryInfC{$\mathbb{\Gamma}((\mathbb{\Gamma}_a\{\Gamma_2; \Gamma_1\}, \Gamma_b^{\Delta_b}, \mathbb{E}(\Gamma_3; H_1 \mathbin{-\!\!*} H_2)^{\Delta_3})^{\Delta_4}) \vdash \{\Delta_1; \Delta_2\}$}
\end{prooftree}
$$

2. The rest: straightforward.

$(\wedge R, \mathbin{-\!\!*} L_{*\top})$:

straightforward.

$(\wedge R, {}^{*}\!\top CtrL)$:

taken care of in `BBI-MultiCut`.

$(\vee L, \vee L)$:

straightforward.

$(\vee L, \vee R)$:

1.

$$
\begin{prooftree}
\AxiomC{$D_2 : \mathbb{\Gamma}\{\Gamma_2; F_1\} \vdash \{\Delta_2\}$}
\AxiomC{$D_3 : \mathbb{\Gamma}\{\Gamma_2; F_2\} \vdash \{\Delta_2\}$}
\RightLabel{$\vee L$}
\BinaryInfC{$D_4 : \mathbb{\Gamma}\{\Gamma_2; F_1 \vee F_2\} \vdash \{\Delta_2\}$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1; F_2; \Delta_1\}$}
\RightLabel{$\vee R$}
\UnaryInfC{$\mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1 \vee F_2; \Delta_1\}$}
\AxiomC{D_4}
\RightLabel{$\texttt{Cut}$}
\BinaryInfC{$\mathbb{\Gamma}\{\Gamma_2; \Gamma_1\} \vdash \{\Delta_1; \Delta_2\}$}
\end{prooftree}
$$

$\rightsquigarrow$

$$\dfrac{\dfrac{D_3 \qquad \dfrac{D_1 \qquad D_2}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1\} \vdash \{\Delta_1; F_2; \Delta_2\}}\,\texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2; \Delta_2\}}\,\texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1\} \vdash \{\Delta_1; \Delta_2\}}\,\{\text{(Ctr L) (Ctr R)}\}$$

2. The rest: straightforward.

$(\vee L, \{\supset L, \supset R, *L, *R_I, *R_{*\top}, \mathrel{-\!\!*}R_I, \mathrel{-\!\!*}R_{*\top}, \mathrel{-\!\!*}L_I, \mathrel{-\!\!*}L_{*\top}\})$:

straightforward.

$(\vee L, {}^{*}\!\top CtrL)$:

taken care of in `BBI-MultiCut`.

$(\vee R, \vee R)$:

straightforward.

$(\vee R,$ the rest$)$:

straightforward.

$(\supset L, \supset L)$:

straightforward.

$(\supset L, \supset R)$:

1.

$$\dfrac{\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1; F_1\} \vdash \{F_2; \Delta_1\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{F_1 \supset F_2; \Delta_1\}}\supset R \qquad \dfrac{D_2 : \mathbb{\Gamma}\{\Gamma_2\} \vdash \{\Delta_2; F_1\} \qquad D_3 : \mathbb{\Gamma}\{\Gamma_2; F_2\} \vdash \{\Delta_2\}}{\mathbb{\Gamma}\{\Gamma_2; F_1 \supset F_2\} \vdash \{\Delta_2\}}\supset L}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1\} \vdash \{\Delta_1; \Delta_2\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{\dfrac{D_2 \qquad D_1}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2; F_2\}}\,\texttt{Cut} \qquad D_3}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2; \Gamma_2\} \vdash \{\Delta_1; \Delta_2; \Delta_2\}}\,\texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2\}}\,\{\text{(Ctr L) (Ctr R)}\}$$

2. The rest: straightforward.

($\supset L$, the rest):

straightforward.

($\supset R, \supset R$)**:**

straightforward.

($\supset R$, **the rest**)**:**

straightforward.

($*L, *L$)**:**

straightforward.

($*L, *R_I$)**:**

1.

$$\dfrac{\dfrac{D_1 : Re_1^{\Delta_a} \vdash F_1 \qquad D_2 : Re_2^{\Delta_b} \vdash F_2}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 * F_2\}}\, *R_I \qquad \dfrac{D_3 : \mathbb{\Gamma}\{\Gamma_2; (F_1, F_2)\} \vdash \{\Delta_2\}}{\mathbb{\Gamma}\{\Gamma_2; F_1 * F_2\} \vdash \{\Delta_2\}}\, *L}{\mathbb{\Gamma}\{\Gamma_2; \Gamma_1\} \vdash \{\Delta_1; \Delta_2\}}\, \texttt{Cut}$$

$\Rightarrow$

$$\begin{array}{c} \dfrac{D_2 \qquad \dfrac{D_1 \qquad D_3}{\mathbb{\Gamma}\{\Gamma_2; (Re_1^{\Delta_a}, F_2)\} \vdash \{\Delta_2\}}\, \texttt{Cut}}{\mathbb{\Gamma}\{\Gamma_2; (Re_1^{\Delta_a}, Re_2^{\Delta_b})\} \vdash \{\Delta_2\}}\, \texttt{Cut} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \; \{\text{(Wk L) (Wk R)}\} \\ \mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2\} \end{array}$$

It does not so much matter what $Re_1^{\Delta_a}/Re_2^{\Delta_b}$ pair is or what $\mathbb{E}(\Gamma_A^{(\Delta'; F_1 * F_2)})$ such that $\mathbb{\Gamma}'(\mathbb{E}(\Gamma_A^{(\Delta'; F_1 * F_2)})) \leftrightharpoons_{ant} \mathbb{\Gamma}(\Gamma_1^{(\Delta_1; F_1 * F_2)})$ is for this proof to go through.

2. The rest: straightforward.

($*L, *R_{*\top}$)**:**

1.

$$\dfrac{\dfrac{D_1 : *\top \vdash F_1 \qquad D_2 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 * G_1; G_1\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 * G_1\}}\, *R_{*\top} \qquad \dfrac{D_3 : \mathbb{\Gamma}\{\Gamma_2; (F_1, G_1)\} \vdash \{\Delta_2\}}{D_4 : \mathbb{\Gamma}\{\Gamma_2; F_1 * G_1\} \vdash \{\Delta_2\}}\, *L}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2\}}\, \texttt{Cut}$$

$\Rightarrow$

$$\cfrac{\cfrac{D_2 \qquad D_4}{\mathbb{\Gamma}\{\Gamma_1;\Gamma_2\} \vdash \{\Delta_1;\Delta_2;G_1\}}\,\texttt{Cut} \qquad \cfrac{D_1 \qquad D_3}{\mathbb{\Gamma}\{\Gamma_2;G_1\} \vdash \{\Delta_2\}}\,\texttt{Cut}}{\cfrac{\mathbb{\Gamma}\{\Gamma_1;\Gamma_2;\Gamma_2\} \vdash \{\Delta_1;\Delta_2;\Delta_2\}}{\mathbb{\Gamma}\{\Gamma_1;\Gamma_2\} \vdash \{\Delta_1;\Delta_2\}}\,\{\text{(Ctr L) (Ctr R)}\}}\,\texttt{Cut}$$

2. The rest: straightforward.

$(*L, \{-\!\!*R_I, -\!\!*R_{*\top}, -\!\!*L_I, -\!\!*L_{*\top}\})$:

straightforward.

$(*L, {*}\!\top CtrL)$:

taken care of in `BBI-MultiCut`.

$(*R_I, *R_I)$:

1. If the cut formula does not occur in $Re_3^{\Delta_c}$ or in $Re_4^{\Delta_d}$;

$$\cfrac{\cfrac{D_1 : Re_1^{\Delta_a} \vdash F_1 \qquad D_2 : Re_2^{\Delta_b} \vdash F_2}{D_5 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 * F_2\}}\,*R_I \qquad \cfrac{D_3 : Re_3^{\Delta_c} \vdash H_1 \qquad D_4 : Re_4^{\Delta_d} \vdash H_2}{\mathbb{\Gamma}\{\Gamma_2; F_1 * F_2\} \vdash \{\Delta_2; H_1 * H_2\}}\,*R_I}{\mathbb{\Gamma}\{\Gamma_2;\Gamma_1\} \vdash \{\Delta_1;\Delta_2; H_1 * H_2\}}\,\texttt{Cut}$$

$\Rightarrow$

$$\cfrac{D_3 \qquad D_4}{\mathbb{\Gamma}\{\Gamma_2;\Gamma_1\} \vdash \{\Delta_1;\Delta_2; H_1 * H_2\}}\,*R_I$$

If the cut formula occurs either in $Re_3^{\Delta_c}$ or in $Re_4^{\Delta_d}$ then assume $Re_3^{\Delta_c}$ to be holding it with no loss of generality. The permutation will be then;

$$\cfrac{\cfrac{D_2 \qquad \cfrac{D_1 \qquad D_3' : Re_3^{\Delta_c}(F_1, F_2) \vdash H_1}{Re_3^{\Delta_c}(Re_1^{\Delta_a}, F_2) \vdash H_1}\,\texttt{Cut}}{Re_3^{\Delta_c}(Re_1^{\Delta_a}, Re_2^{\Delta_b}) \vdash H_1}\,\texttt{Cut} \qquad D_4}{\mathbb{\Gamma}\{\Gamma_1;\Gamma_2\} \vdash \{\Delta_1;\Delta_2; H_1 * H_2\}}\,*R_I$$

In the internalised weakening for the $*R_I$ application (looked from premise to conclusion), Γ_1 is recovered from "$Re_1^{\Delta_a}, Re_2^{\Delta_b}$".

2. The rest: straightforward.

$(*R_I,$ the rest$)$:

straightforward.

($*R_{^*\!\top}, *R_{^*\!\top}$):

straightforward.

($*R_{^*\!\top}$, the rest):

straightforward.

($-\!\!*R_I, -\!\!*R_I$):

straightforward.

($-\!\!*R_I, -\!\!*L_I$):

1.

$$\dfrac{D_2 : Re_1^{\Delta_c} \vdash F_1\ (F_1 \cap \Xi = \emptyset) \qquad D_3 : \mathbb{\Gamma}(((Re_2^{\Delta_d}, F_2); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \{\Gamma_c; F_1 -\!\!* F_2\}))^{\Delta_4}) \vdash \{\Delta_3\}}{D_5 : \mathbb{\Gamma}((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \{\Gamma_c; F_1 -\!\!* F_2\})^{\Delta_4}) \vdash \{\Delta_3\}}\ -\!\!*L_I$$

$$\dfrac{\dfrac{D_1 : \Gamma_1^{\Delta_1}, F_1 \vdash F_2\ (F_1 \cap \Xi = \emptyset)}{D_4 : \mathbb{\Gamma}((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \{\Gamma_2\})^{\Delta_4}) \vdash \{F_1 -\!\!* F_2; \Delta_2\}}\ -\!\!*R_I \qquad D_5}{\mathbb{\Gamma}((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \{\Gamma_c; \Gamma_2\})^{\Delta_4}) \vdash \{\Delta_2; \Delta_3\}}\ \texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{\dfrac{D_2 \qquad \dfrac{D_1 \qquad \dfrac{\dfrac{D_4}{D_4' : \mathbb{\Gamma}(((Re_2^{\Delta_d}, F_2); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \{\Gamma_2\}))^{\Delta_4}) \vdash \{F_1 -\!\!* F_2; \Delta_2\}}\ \{\text{(Wk L) (Wk R)}\} \qquad D_3}{\mathbb{\Gamma}(((Re_2^{\Delta_d}, \{F_2\}); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, (\Gamma_c; \Gamma_2)^{(\Delta_2;\Delta_3)}))^{\Delta_4}) \vdash \{\mathbb{1}\}}\ \texttt{Cut}}{\mathbb{\Gamma}(((Re_2^{\Delta_d}, \Gamma_1^{\Delta_1}, F_1); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, (\Gamma_c; \Gamma_2)^{(\Delta_2;\Delta_3)}))^{\Delta_4}) \vdash \mathbb{1}}\ \texttt{Cut}}{\mathbb{\Gamma}\{(Re_2^{\Delta_d}, \Gamma_1^{\Delta_1}, Re_1^{\Delta_c}); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, (\Gamma_c; \Gamma_2)^{(\Delta_2;\Delta_3)})\} \vdash \{\Delta_4\}}\ \texttt{Cut}}{\mathbb{\Gamma}\{(\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, (\Gamma_c; \Gamma_2)^{(\Delta_2;\Delta_3)}); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, (\Gamma_c; \Gamma_2)^{(\Delta_2;\Delta_3)})\} \vdash \{\Delta_4\}}\ \{\text{(Wk L) (Wk R) } ({}^*\!\top \text{ Wk L})\}}{\mathbb{\Gamma}\{\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, (\Gamma_c; \Gamma_2)^{(\Delta_2;\Delta_3)}\} \vdash \{\Delta_4\}}\ \{\text{(Ctr L)}\}$$

2.

$$\dfrac{\dfrac{D_2 : Re_1^{\Delta_c} \vdash F_1\ (\Xi \cap F_1 = \emptyset) \qquad D_3 : \mathbb{\Gamma}(((Re_2^{\Delta_d}, F_2); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}'(({}^*\!\top; \Gamma_4; F_1 -\!\!* F_2)^{\Delta_3})))^{\Delta_4}) \vdash \mathbb{1}}{D_5 : \mathbb{\Gamma}((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}'(({}^*\!\top; \Gamma_4; F_1 -\!\!* F_2)^{\Delta_3}))^{\Delta_4}) \vdash \mathbb{1}}\ -\!\!*L_I}{D_6 : \mathbb{\Gamma}((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{E}(({}^*\!\top; \Gamma_4; F_1 -\!\!* F_2)^{\Delta_3}))^{\Delta_4}) \vdash \mathbb{1}}\ {}^*\!\top CtrL$$

$$\dfrac{\dfrac{D_1 : \Gamma_1^{\Delta_1}, F_1 \vdash F_2\ (\Xi \cap F_1 = \emptyset)}{D_4 : \mathbb{\Gamma}((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A\{\Gamma_1\})^{\Delta_4}) \vdash \{F_1 -\!\!* F_2; \Delta_1\}}\ -\!\!*R_I \qquad D_6}{\mathbb{\Gamma}((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A\{{}^*\!\top; \Gamma_B; \Gamma_1\})^{\Delta_4}) \vdash \{\Delta_1; \Delta_A\}}\ \texttt{Cut}$$

where the following hold.

(a) $\mathbb{E}(({}^{*}\top; \Gamma_4; F_1 -\!\!* F_2)^{\Delta_3}) \leftharpoondown\!\!\!\!\!\!\rightharpoondown_{ant} \mathbb{\Gamma}_A(({}^{*}\top; \Gamma_B; F_1 -\!\!* F_2)^{\Delta_A})$.

(b) $\mathbb{E}'(({}^{*}\top; \Gamma_4; F_1 -\!\!* F_2)^{\Delta_3}) \leftharpoondown\!\!\!\!\!\!\rightharpoondown_{ant} \mathbb{\Gamma}_A(((({}^{*}\top; \Gamma_B; F_1 -\!\!* F_2)^{\Delta_A})^{\times n})^{\Delta_A})$.

$\Rightarrow$

$$\dfrac{\dfrac{\dfrac{D_2 \qquad \dfrac{D_1 \qquad \dfrac{D'_4 : \mathbb{\Gamma}(((Re_2^{\Delta_d}, F_2); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A\{\Gamma_1\}))^{\Delta_4}) \vdash \{F_1 -\!\!* F_2; \Delta_1\} \qquad D_3}{\mathbb{\Gamma}(((Re_2^{\Delta_d}, F_2); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A\{{}^{*}\top; \Gamma_B; \Gamma_1\}))^{\Delta_4}) \vdash \{\Delta_1; \Delta_A\}}\,\texttt{BBI-MultiCut}}{\mathbb{\Gamma}(((Re_2^{\Delta_d}, F_1, \Gamma_1^{\Delta_1}); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A\{{}^{*}\top; \Gamma_B; \Gamma_1\}))^{\Delta_4}) \vdash \{\Delta_1; \Delta_A\}}\,\texttt{Cut}}{\mathbb{\Gamma}(((Re_1^{\Delta_c}, Re_2^{\Delta_d}, \Gamma_1^{\Delta_1}); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A\{{}^{*}\top; \Gamma_B; \Gamma_1\}))^{\Delta_4}) \vdash \{\Delta_1; \Delta_A\}}\,\texttt{Cut}}{D^*}\,\text{(Wk L) (Wk R) (}{}^{*}\top\text{ Wk L)}}{\mathbb{\Gamma}((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A\{{}^{*}\top; \Gamma_B; \Gamma_1\})^{\Delta_4}) \vdash \{\Delta_1; \Delta_A\}}\,\text{(Ctr L)}$$

where

$D^* : \mathbb{\Gamma}((((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A(({}^{*}\top; \Gamma_B; \Gamma_1)^{(\Delta_1;\Delta_A)})); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A\{{}^{*}\top; \Gamma_B; \Gamma_1\}))^{\Delta_4}) \vdash \{\Delta_1; \Delta_A\}$.

3. Let $\mathbb{\Gamma}_x$ denote $\mathbb{\Gamma}_A(({}^{*}\top; \Gamma_c; F_1 -\!\!* F_2)^{\Delta_A})$.

$$\dfrac{\dfrac{D_2 : Re_1^{\Delta_c}, \mathbb{\Gamma}_x^{\times m} \vdash F_1 \quad (F \cap \Xi = \emptyset) \qquad D_3 : \mathbb{\Gamma}(((Re_2^{\Delta_d}, F_2, \mathbb{\Gamma}_x^{\times l}); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, (\mathbb{\Gamma}_x^{\times n})^{\Delta_A}))^{\Delta_4}) \vdash \mathbb{1}}{\mathbb{\Gamma}((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, (\mathbb{\Gamma}_x^{\times n})^{\Delta_A})^{\Delta_4}) \vdash \mathbb{1}}\,-\!\!*L_I}{D_5 : \mathbb{\Gamma}(\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_x) \vdash \mathbb{1}}\,{}^{*}\top CtrL$$

$$\dfrac{\dfrac{D_1 : \Gamma_1^{\Delta_1}, F_1 \vdash F_2 \quad (\Xi \cap F_1 = \emptyset)}{D_4 : \mathbb{\Gamma}((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A\{\Gamma_1\})^{\Delta_4}) \vdash \{F_1 -\!\!* F_2; \Delta_1\}}\,-\!\!*R_I \qquad D_5}{\mathbb{\Gamma}((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A\{{}^{*}\top; \Gamma_c; \Gamma_1\})^{\Delta_4}) \vdash \{\Delta_1; \Delta_A\}}\,\texttt{Cut}$$

where $0 \leq m + l \leq n - 1$.

$\Rightarrow$

We have:

$$\dfrac{D_1}{D'_1 : \Gamma_1^{\Delta_1} \vdash F_1 -\!\!* F_2}\,-\!\!*R_I$$

We then have:

$$\dfrac{D'_1 \qquad D_2}{D_6 : Re_1^{\Delta_c}, \mathbb{\Gamma}_A(({}^{*}\top; \Gamma_c; \Gamma_1)^{(\Delta_1;\Delta_A)}) \vdash F_1}\,\texttt{BBI-MultiCut}$$

$$\dfrac{D'_1 \qquad D_3}{D_7 : \mathbb{\Gamma}(((Re_2^{\Delta_d}, F_2, \mathbb{\Gamma}_A(({}^{*}\top; \Gamma_c; \Gamma_1)^{(\Delta_1;\Delta_A)})); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A(({}^{*}\top; \Gamma_c; \Gamma_1)^{(\Delta_1;\Delta_A)})))^{\Delta_4}) \vdash \mathbb{1}}\,\texttt{BBI-MultiCut}^*$$

by recalling the multiplicity in the intuitionistic `Cut`. Then;

$$\begin{array}{c}
\dfrac{D_6 \qquad \dfrac{D_1 \qquad D_7}{\mathbb{\Gamma}(((Re_2^{\Delta_d}, \Gamma_1^{\Delta_1}, F_1, \mathbb{\Gamma}_A(({}^{*}\top; \Gamma_c; \Gamma_1)^{(\Delta_1;\Delta_A)})); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A(({}^{*}\top; \Gamma_c; \Gamma_1)^{(\Delta_1;\Delta_A)})))^{\Delta_4}) \vdash \mathbb{1}}\,\mathtt{Cut}}{\mathbb{\Gamma}(((Re_2^{\Delta_d}, \Gamma_1^{\Delta_1}, Re_1^{\Delta_c}, (\mathbb{\Gamma}_A(({}^{*}\top; \Gamma_c; \Gamma_1)^{(\Delta_1;\Delta_A)}))^{\times 2}); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A(({}^{*}\top; \Gamma_c; \Gamma_1)^{(\Delta_1;\Delta_A)})))^{\Delta_4}) \vdash \mathbb{1}}\,\mathtt{Cut} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ \textbf{Infs} \\
\dfrac{\mathbb{\Gamma}((((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, (\mathbb{\Gamma}_A(({}^{*}\top; \Gamma_c; \Gamma_1)^{(\Delta_1;\Delta_A)}))^{\times 4}); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A(({}^{*}\top; \Gamma_c; \Gamma_1)^{(\Delta_1;\Delta_A)})))^{\Delta_4}) \vdash \mathbb{1}}{\mathbb{\Gamma}((((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A(({}^{*}\top; \Gamma_c; \Gamma_1)^{(\Delta_1;\Delta_A)})); (\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A(({}^{*}\top; \Gamma_c; \Gamma_1)^{(\Delta_1;\Delta_A)})))^{\Delta_4}) \vdash \mathbb{1}}\,{}^{*}\top CtrL \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ (CtrL) \\
\mathbb{\Gamma}((\Gamma_a^{\Delta_a}, \Gamma_b^{\Delta_b}, \mathbb{\Gamma}_A(({}^{*}\top; \Gamma_c; \Gamma_1)^{(\Delta_1;\Delta_A)}))^{\Delta_4}) \vdash \mathbb{1}
\end{array}$$

where $WkL, WkR, {}^{*}\top WkL \in$ **Infs**.

4. The rest: straightforward.

$(\mathrel{-\!\!*}R_I, \mathrel{-\!\!*}L_{{}^{*}\top})$:

straightforward. The principal for either of the premise sequents of the particular `Cut` instance cannot become the principal for the other.

$(\mathrel{-\!\!*}R_I, {}^{*}\top CtrL)$:

taken care of in `BBI-MultiCut`.

$(\mathrel{-\!\!*}R_{{}^{*}\top}, \mathrel{-\!\!*}L_{{}^{*}\top})$:

1.

$$\dfrac{\dfrac{D_1 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_2\}\ (F_1 \in \Xi)}{D_3 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 \mathrel{-\!\!*} F_2\}}\,\mathrel{-\!\!*}R_{{}^{*}\top} \qquad \dfrac{D_2 : \mathbb{\Gamma}\{\Gamma_2; F_2\} \vdash \{\Delta_2\}\ (F_1 \in \Xi)}{D_4 : \mathbb{\Gamma}\{\Gamma_2; F_1 \mathrel{-\!\!*} F_2\} \vdash \{\Delta_2\}}\,\mathrel{-\!\!*}L_{{}^{*}\top}}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2\}}\,\mathtt{Cut}$$

$\Rightarrow$

$$\dfrac{D_1 \qquad D_2}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_2\} \vdash \{\Delta_1; \Delta_2\}}\,\mathtt{Cut}$$

2. The rest: straightforward.

$(\mathrel{-\!\!*}R_{{}^{*}\top}, {}^{*}\top CtrL)$:

taken care of in `BBI- MultiCut`.

$(\mathrel{-\!\!*}L_I,$ **the rest**$)$**:**
straightforward.

$(\mathrel{-\!\!*}L_{{}^{*}\!\top},$ **the rest**$)$**:**
straightforward.

The rest: straightforward. ☐

Note that, in case the principal of either of the `Cut` premise sequents does not reside in the same additive structural layer as the cut formula, permutation is very simple (via inversion lemma and depth-preserving weakening admissibilities if needed). For example, we may have the following derivation:

$$\dfrac{\dfrac{D_1 : {}^{*}\!\top \vdash F_1 \qquad D_2 : \mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 * G_1; G_1\}}{\mathbb{\Gamma}\{\Gamma_1\} \vdash \{\Delta_1; F_1 * G_1\}}{*R_{{}^{*}\!\top}} \qquad \dfrac{D_3 : Re_1^{\Delta_a} \vdash H_1 \qquad D_4 : Re_2^{\Delta_b} \vdash H_2}{\mathbb{\Gamma}'\{\Gamma_2\} \vdash \{\Delta_2; H_1 * H_2\}}{*R_I}}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_A\} \vdash \{\Delta_1; \Delta_A\}}\,\texttt{Cut}$$

where $\mathbb{\Gamma}'((\Gamma_2)^{(\Delta_2; H_1 * H_2)}) \leftrightharpoons_{ant} \mathbb{\Gamma}((\Gamma_A; F_1 * G_1)^{\Delta_A})$ with the condition that $\mathbb{\Gamma}'(-) \not\leftrightharpoons_{ant} \mathbb{\Gamma}'(-)$, which simply permutes into;

$$\dfrac{D_3 \qquad D_4}{\mathbb{\Gamma}\{\Gamma_1; \Gamma_A\} \vdash \{\Delta_1; \Delta_A\}}\, *R_I$$

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