Interval Type-2 TSK+ Fuzzy Inference System

Jie Li, Longzhi Yang
Department of Computer and Information Sciences
Faculty of Engineering and Environment
Northumbria University
Newcastle upon Tyne, NE1 8XT, U.K.
Email: {jie.li-2, longzhi.yang}@northumbria.ac.uk

Xin Fu†, Fei Chao‡
†School of Management
‡Department of Cognitive Science
Xiamen University
Xiamen, P.R. China
Email: {xfu, fchao}@xmu.edu.cn

Yanpeng Qu
Information Science and
Technology College
Dalian Maritime University
Dalian, P.R. China
Email: yanpengqu@dlmu.edu.cn

***Abstract*—Type-2 fuzzy sets and systems can better handle uncertainties compared to its type-1 counterpart, and the widely applied Mamdani and TSK fuzzy inference approaches have been both extended to support interval type-2 fuzzy sets. Fuzzy interpolation enhances the conventional Mamdani and TKS fuzzy inference systems, which not only enables inferences when inputs are not covered by an incomplete or sparse rule base but also helps in system simplification for very complex problems. This paper extends the recently proposed fuzzy interpolation approach TSK+ to allow the utilization of interval type-2 TSK fuzzy rule bases. One illustrative case based on an example problem from the literature demonstrates the working of the proposed system, and the application on the cart centering problem reveals the power of the proposed system. The experimental investigation confirmed that the proposed approach is able to perform fuzzy inferences using either dense or sparse interval type-2 TSK rule bases with promising results generated.**

***Keywords*—Interval type-2 TSK+, TSK fuzzy inference system, sparse rule base, imbalanced data set, fuzzy interpolation**

I. Introduction

Fuzzy inference systems are mechanisms that use fuzzy logic and fuzzy set theory to map inputs and outputs, which has been successfully applied in many application areas, such as decision making, robotic control, intrusion detection, and computer vision. A typical fuzzy inference system consists of a rule base and an inference engine. A number of inference engines have been developed, with the Mamdani inference approach [1] and TSK inference approach [2] being most intensively studied and widely applied. In particular, Mamdani fuzzy inference approach is more intuitive and suitable for handling human linguistic inputs, which usually leads to fuzzy outputs and thus a defuzzification process is typically required to convert the fuzzy outputs to crisp values for general system use. In contrast, crisp outputs are directly produced by the TSK approach, as polynomials (often 0-order or 1-order) are used as the rule consequences in TSK fuzzy model.

These conventional fuzzy inference systems have both been extended to support interval type-2 (IT2) fuzzy sets for better uncertainty management and processing. Briefly, type-2 fuzzy sets generalize the standard type-1 fuzzy sets by representing the membership of each element in a fuzzy set as a standard type-1 fuzzy set, while IT2 fuzzy sets representing the membership as a crisp interval bounded by [0, 1]. Such extension can handle rule uncertainties in a more flexible and effective way, but generally requires more computational power at the same time. Regardless of the type of fuzzy sets used, a complete dense rule base which covers the entire input domains, is always required by either Mamdani or TSK fuzzy inference approaches; otherwise, no rule can be fired and consequently no result can be generated when a given input does not overlap with any rule antecedent in the rule base.

Fuzzy interpolation, firstly proposed in [3], relaxes the requirement of dense rule bases, and thus improves the applicability of fuzzy models [4]. Given an input or observation, which does not overlap with any rule antecedent, fuzzy interpolation can still approximate the conclusion by considering the neighboring rules in the rule base by means of fuzzified polynomial (often linear) interpolation. Fuzzy interpolation has also been used to reduce the complexity of fuzzy models by removing the rules that can be approximated by their neighbors. For instance, a curvature-based rule base simplification method has been proposed to support FRI in [5], and a sparse TSK rule base generation approach was developed in [6] to support TSK fuzzy inference.

Various fuzzy interpolation approaches have been developed and applied in real world applications [7], [8]. For instance, fuzzy models reported in [3], [4], [9], [10], [11], [12], [13], [14], [15], [16], [17] were developed based on Mamdani fuzzy rule bases, and [6] and [18] were proposed to perform fuzzy inferences based on sparse TSK rule bases. Fuzzy interpolation has been employed to deal with real world applications in the field of home heating management, computer network, cyber security, and manufacturing amongst others [19], [20], [21], [22] and [23]. However, all the existing fuzzy interpolation approaches were originally proposed based on the type-1 fuzzy sets, although some have been further developed to deal with type-2 fuzzy sets and fuzzy rules.

This paper extends the recently proposed TKS+ [6] fuzzy inference to allow the utilization of IT2 TSK fuzzy rule bases. The extended fuzzy inference system TSK+ is able to work with: 1) sparse rule bases, 2) dense rule bases, 3) Type-1 fuzzy sets, and 4) interval type 2 fuzzy sets. The proposed system has been applied to two problems that have been considered in the literature. The experimental results confirmed that the proposed system not only is workable with both dense and sparse IT2 TSK fuzzy model, but also

This work is supported by the National Natural Science Foundation of China (Nos. 91746103 and 71301133).

enhances the TSK inference method when the knowledge represented in the rule base is not complete.

The rest of this paper is structured as follows. Section II introduces (interval) type-2 fuzzy sets, and the fuzzy inference approach TSK+ as well as its underpinning measurement of similarity degrees. Section III details the proposed IT2 TSK+. Section IV reports the experimentation and analyzes the experimental results. Section V concludes the paper and suggests probable future developments.

II. Background

Type-2 fuzzy sets and the TSK+ fuzzy inference approach as well as its underpinning similarity measure and are briefly introduced in this section.

A. Type-2 Fuzzy Sets

The membership grades of each element in type-1 fuzzy systems are crisp values, which minimizes the effect of uncertainty handling. Type-2 fuzzy systems can handle more uncertainty using type-2 fuzzy sets. An type-2 fuzzy set, denoted as $\tilde{A}$, can be represented as:

$$\begin{aligned}\tilde{A} =& \{((x,u),\mu_{\tilde{A}}(x,u))|\forall x \in X, \forall u \in J_x \subseteq [0,1], \\ & \mu_{\tilde{A}}(x,u) \in [0,1]\} \ , \\ =& \int_{x\in X}\int_{u \in J_x} \mu_{\tilde{A}}(x,u)/(x,u) \ ,\end{aligned} \tag{1}$$

where X is the primary domain, J_x is the primary membership for a given element x, and $\mu_{\tilde{A}}(x,u)$ denotes the secondary membership. When all $\mu_{\tilde{A}}(x,u) = 1$, $\tilde{A}$ is deteriorated as an IT2 fuzzy set, which can be expressed as:

$$\tilde{A} = \int_{x\in X}\int_{u\in J_x} 1/(x,u), \quad J_x \subseteq [0,1]. \tag{2}$$

An example trapezoidal IT2 fuzzy set $\tilde{A}$ is illustrated in Figure 1, which can be represented by a lower membership function (LMF), $\underline{\tilde{A}} = (\underline{a_1}, \underline{a_2}, \underline{a_3}, \underline{a_4}, \underline{w})$, and a upper membership function (UMF), $\overline{\tilde{A}} = (\overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{w})$. In this case, $\tilde{A} =< \underline{\tilde{A}}, \overline{\tilde{A}} >$, where $(\underline{a_1}, \underline{a_2}, \underline{a_3}, \underline{a_4})$ and $(\overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4})$ are respectively the four odd points of the LMF and UMF, and $\underline{w}$ and $\overline{w}$ denote respectively the degrees of confidence for $\underline{\tilde{A}}$ and $\overline{\tilde{A}}$, with $0 < \underline{w} \leqslant \overline{w} = 1$. The area between LMF and UMF, illustrated in grey in Figure 1, thus denotes the footprint of uncertainty (FOU), which represents the uncertainty of the fuzzy set $\tilde{A}$. Obviously, a larger FOU area implies a higher level of uncertainty; and the IT2 fuzzy set degenerates to a type-1 fuzzy set when $\underline{\tilde{A}}$ coincides with $\overline{\tilde{A}}$ (i.e., the area of $FOU(\tilde{A})$ is 0).

Using the concept of FOU, Equation 2 can also be rewritten as [24]:

$$\tilde{A} = 1/FOU(\tilde{A}). \tag{3}$$

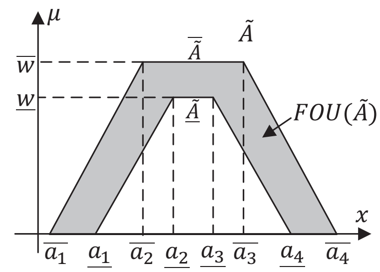


Fig. 1. LMF $\underline{\tilde{A}}$ and UMF $\overline{\tilde{A}}$ of a trapezoidal IT2 fuzzy set $\tilde{A}$

B. TSK+

The original TSK inference system generates crisp inference results by weighted averaging the sub-consequences of all fired rules using their firing strengths [2]. Obviously, no rule will be fired if a given input does not overlap with any rule antecedent, and consequently the TSK inference cannot be performed. TSK+ inference approach addressed such issue by redefining the fire strength and its implementation of similarity degree [6]. Assume that two weighted convex trapezoidal fuzzy sets in a normalized variable domain are given as $A = (a_1, a_2, a_3, a_4, w)$ and $A^{'} = (a_1^{'}, a_2^{'}, a_3^{'}, a_4^{'}, w^{'})$, where $0 < w \leqslant 1$ and $0 < w^{'} \leqslant 1$ represent the degrees of confidence for fuzzy sets A and $A^{'}$, respectively. Note that the weighted fuzzy set A will deteriorate to a normal fuzzy set when $w = 1$, which is usually simply denoted as $A = (a_1, a_2, a_3, a_4)$. The similarity degree $s(A, A^{'})$ between A and $A^{'}$ can be calculated by the following Equation:

$$s(A, A') = \left(1 - \frac{\sum_{i=1}^{4} |a_i - a_i'|}{4}\right) \cdot d \cdot \frac{min(w, w^{'})}{max(w, w^{'})}, \tag{4}$$

where d represents $distance\ factor$. This factor is a function defined below:

$$d = \begin{cases} 1 & ; \quad a_1 = a_2 = a_3 = a_4 \\ & \quad \& \ a_1^{'} = a_2^{'} = a_3^{'} = a_4^{'} \\ 1 - \frac{1}{1+e^{(-f\cdot\|A,A'\|+5)}} & ; \quad \text{otherwise,} \end{cases} \tag{5}$$

where $\|A, A'\|$ represents the distance between two fuzzy sets usually defined as the Euclidean distance of their representative values [9], and f ($f > 0$) is an adjustable sensitivity factor. A smaller value of s usually leads to a similarity degree which is more sensitive to the distance between the two fuzzy sets. According to Equation 5, the *distance factor* is not considered when fuzzy sets A and A' are both crisp. This is because the shapes of the fuzzy sets need to considered by the representative values as contributing elements of the *distance factor* when the objects are fuzzy sets, but there is no point to consider this if the objects are simply crisp numbers [25].

Suppose that a type-1 TSK rule base, sparse or dense, is comprised of n rules:

$$\begin{aligned} R_1 : & \textbf{ IF } x_1 \text{ is } A_1^1 \text{ and } \cdots x_j \text{ is } A_j^1 \cdots \text{ and } x_k \text{ is } A_k^1 \\ & \textbf{ THEN } y = f_1(x_1^1, \cdots, x_k^1), \\ & \ldots \ldots \\ R_n : & \textbf{ IF } x_1 \text{ is } A_1^n \text{ and } \cdots x_j \text{ is } A_j^n \cdots \text{ and } x_k \text{ is } A_k^n \\ & \textbf{ THEN } y = f_n(x_1^n, \cdots, x_k^n), \end{aligned} \tag{6}$$

where $A_j^i, (i \in \{1, 2, \cdots, n\}$ and $j \in \{1, 2, \cdots, k\})$ represents a convex trapezoidal fuzzy set that can be denoted as $(a_{j1}^i, a_{j2}^i, a_{j3}^i, a_{j4}^i, w^i)$. Given an input $I = (A_1, A_2, \cdots, A_k)$, either crisp or fuzzy, in the input domain, a crisp inference result can be generated by the following steps:

Step 1: Calculate the matching degrees between the given input $(A_1, A_2, \cdots, A_k)$ and rule antecedents $(A_1^i, A_2^i, \cdots, A_k^i)$ for each rule R_i using Equation 4.

Step 2: Determine the firing degree of each rule by aggregating the matching degrees between its antecedent terms and the given input values by:

$$\alpha_i = s(A_1, A_1^i) \wedge s(A_2, A_2^i) \wedge \cdots \wedge s(A_k, A_k^i) , \tag{7}$$

where $\wedge$ is a t-norm operator usually implemented as a minimum operator.

Step 3: Generate the final output o by integrating the sub-consequences from all rules:

$$c = \sum_{i=1}^{n} \alpha_i \cdot f_n(x_1, \cdots, x_k) \; / \; \sum_{i=1}^{n} \alpha_i \; . \tag{8}$$

III. Interval Type-2 TSK+

Many type-1 fuzzy systems have been extended to support IT2 fuzzy systems, including the TSK approach [26], [27]. Generally speaking, the inputs and all the fuzzy sets in the rule antecedents can but not necessarily be IT2 fuzzy sets in a IT2 fuzzy systems; and the consequence of IT2 TSK rules are zero or first order of polynomial functions, where the parameters can be either crisp values or a crisp interval. Assume that an IT2 sparse TSK rule base is comprised of n rules as:

$$\begin{aligned} R_1 : & \textbf{ IF } x_1 \text{ is } \tilde{A}_1^1 \text{ and } \ldots \text{ and } x_k \text{ is } \tilde{A}_k^1 \\ & \textbf{ THEN } y = \tilde{p}_0^1 + \tilde{p}_1^1 x_1^1 + \cdots + \tilde{p}_k^1 x_k^1 \\ & \cdots \\ R_i : & \textbf{ IF } x_1 \text{ is } \tilde{A}_1^i \text{ and } \ldots \text{ and } x_k \text{ is } \tilde{A}_k^i \\ & \textbf{ THEN } y = \tilde{p}_0^i + \tilde{p}_1^i x_1^i + \cdots + \tilde{p}_k^i x_k^i \\ & \cdots \\ R_n : & \textbf{ IF } x_1 \text{ is } \tilde{A}_1^n \text{ and } \ldots \text{ and } x_k \text{ is } \tilde{A}_k^n \\ & \textbf{ THEN } y = \tilde{p}_0^n + \tilde{p}_1^i x_1^n + \cdots + \tilde{p}_k^n x_k^n, \end{aligned} \tag{9}$$

where $\tilde{A}_j^i, (j \in \{1, \ldots, k\}, i \in \{1, \ldots, n\})$ is an IT2 fuzzy set regarding input variable x_j in the i^{th} rule. The consequence is a crisp polynomial function $y = f_i(x_1, \ldots, x_k) = \tilde{p}_0^i + \tilde{p}_1^i x_1^i + \cdots + \tilde{p}_k^i x_k^i$, where $\tilde{p}_j^i$ are parameters usually being crisp intervals with crisp singleton numbers being the special case. For a given input $O(\tilde{A}_1^*, \cdots, \tilde{A}_k^*)$, the steps introduced in Section II-B can be generally used for the generation of the output, but all the operations on type-1 fuzzy set involved in these steps need to be upgraded to those based on IT2 fuzzy sets. Fortunately, such upgrading has been extensively studied in the literature [24]; and all the operations on IT2 fuzzy sets used in this work can be simplified by the calculation of the FOU of the resulted IT2 fuzzy sets based on Equation 3, which in turn can be simplified by the calculation of the LMF and UMF. Note that crisp numbers, sets and type-1 fuzzy sets are special cases of IT2 fuzzy sets and thus the discussion below all based on the general case of IT2 fuzzy sets unless only a simplified version can be the case.

A. Firing Strength

The firing strength and similarity measure based on type-1 fuzzy sets, as reviewed in Section II-B, is extended in this subsection to support the development of IT2-TSK+. As each IT2 fuzzy set can be repressed by a set of embedded type-1 fuzzy sets, the matching degree between a crisp value and an IT2 fuzzy sets can be calculated as the complete set of matching degrees between the crisp value and every embedded type-1 fuzzy set within the calculated IT2 fuzzy set using Equation 4; and the calculated matching degree is thus a crisp interval.

Without losing generalisation, given a crisp singleton IT2 observation item $\tilde{A}_j$ and the corresponding antecedent value $\tilde{A}_j^i$ of rule R_i regarding the same antecedent variable x_j, their matching degree S is calculated as:

$$\tilde{s}(\tilde{A}_j^i, \tilde{A}_j) = [s(\underline{\tilde{A}_j^i}, \tilde{A}_j), s(\overline{\tilde{A}_j^i}, \tilde{A}_j)], \tag{10}$$

where $\overline{\tilde{A}_j^i}$ and $\underline{\tilde{A}_j^i}$ respectively indicate the UMF and LMF of $\tilde{A}_j^i$, and $s(\cdot, \cdot)$ represents the similarity degree between two type-1 fuzzy sets (or specifically one crisp singleton and one type-1 fuzzy set) as defined in Equation 4.

Once the similarity degrees between the observed values and the rule antecedent values of rule R_i regarding every antecedent variable $x_1, ..., x_n$ are obtained, which are then integrated as the firing strength $\tilde{\alpha}_i$ of rule R_i. The integration is implemented by a t-norm operator when the similarity degrees are crisp values as in type-1 fuzzy systems, but this needs to be extended to the meet $\sqcap$ operation when intervals are used [28]:

$$\begin{aligned} \tilde{\alpha}_i =& [\underline{\alpha_i}, \overline{\alpha_i}] \\ =& \sqcap_{j=1}^{k} \tilde{s}(\tilde{A}_j^i, \tilde{A}_j) \\ =& [\min(A_1), \max(A_1)] \sqcap ... \sqcap [\min(A_n), \max(A_n)] \\ =& [\min(\min(A_1), ..., \min(A_n)), \\ & \min(\max(A_1), ..., \max(A_n))] \\ =& [\tilde{s}(\tilde{A}_1^i, \tilde{A}_1) \sqcap \cdots \sqcap \tilde{s}(\tilde{A}_k^i, \tilde{A}_k)] \\ =& [[s(\underline{\tilde{A}_1^i}, \underline{\tilde{A}_1}), s(\overline{\tilde{A}_1^i}, \overline{\tilde{A}_1})], \cdots, \\ & [s(\underline{\tilde{A}_k^i}, \underline{\tilde{A}_k}), s(\overline{\tilde{A}_k^i}, \overline{\tilde{A}_k})]] \\ =& [s(\underline{\tilde{A}_1^i}, \underline{\tilde{A}_1}) \wedge \cdots \wedge s(\underline{\tilde{A}_k^i}, \underline{\tilde{A}_k}), \\ & s(\overline{\tilde{A}_1^i}, \overline{\tilde{A}_1}) \wedge \cdots \wedge s(\overline{\tilde{A}_k^i}, \overline{\tilde{A}_k})], \end{aligned} \tag{11}$$

where $\wedge$ represents a t-norm operation implemented as a minimum operator in this work.

B. Intermediate Result from Individual Rule

As reviewed in Section II-B, the TSK+ inference approach integrates the intermediate results from every individual rules in the rule base to form the final output. Given an observation $O(\tilde{A}_1, \cdots, \tilde{A}_k)$, the intermediate result $\tilde{c}^i$ led by rule R_i needs to be calculated first. As intervals are usually used as the parameters of the polynomial function in the consequence of IT2 TSK rules and the domains of input variables are normalized, each sub-consequence $\tilde{c}^i$ from rule R_i in the IT2 TSK+ system is therefore a crisp interval [26], [27]. The minimum and maximum values of $\tilde{c}^i$ can be obtained based on the given observation and the corresponding IT2 polynomial function of the rule consequence:

$$\begin{aligned} \tilde{c}^i =& \tilde{p}_0^i + \tilde{p}_1^i x_1^i + \cdots + \tilde{p}_k^i x_k^i \\ =& [\underline{\tilde{p}_0^i} + \underline{\tilde{p}_1^i} x_1 + \cdots + \underline{\tilde{p}_k^i} x_k \ , \\ & \overline{\tilde{p}_0^i} + \overline{\tilde{p}_1^i} x_1 + \cdots + \overline{\tilde{p}_k^i} x_k], \end{aligned} \tag{12}$$

where $\underline{\tilde{p}_j^i}$ and $\overline{\tilde{p}_j^i}, (j \in \{0, 1, \cdots, k\})$, denote the minimum and maximum values of crisp interval $\tilde{p}_j^i$, respectively.

C. Final Output Generation

The final output of the TSK+ system is a fuzzified weighted average of the sub-consequences from all rules. In particular, based on the obtained firing strength $\tilde{\alpha}^i$ and corresponding sub-consequence $\tilde{c}^i$, the final interval output $\tilde{c}$ can be calculated by:

$$\begin{aligned} \tilde{c} =& [\underline{\tilde{c}}, \overline{\tilde{c}}] \\ =& \int_{\tilde{c}^1 \in [\underline{\tilde{c}^1}, \overline{\tilde{c}^1}]} \cdots \int_{\tilde{c}^n \in [\underline{\tilde{c}^n}, \overline{\tilde{c}^n}]} \int_{\tilde{\alpha}^1 \in [\underline{\tilde{\alpha}^1}, \overline{\tilde{\alpha}^1}]} \cdots \int_{\tilde{\alpha}^n \in [\underline{\tilde{\alpha}^n}, \overline{\tilde{\alpha}^n}]} \\ & 1 \Bigg/ \frac{\sum_{i=1}^n \tilde{\alpha}^i \cdot \tilde{c}^i}{\sum_{i=1}^n \tilde{\alpha}^i} \ . \end{aligned} \tag{13}$$

This equation can be practically implemented by computing the two extreme values of the crisp interval, minimum $\underline{\tilde{c}}$ and maximum $\overline{\tilde{c}}$, separately:

$$\begin{cases} \underline{\tilde{c}} = \dfrac{\sum\limits_{i=1}^{L} \overline{\tilde{\alpha}^i} \underline{\tilde{c}^i} + \sum\limits_{j=L+1}^{n} \underline{\tilde{\alpha}^i} \underline{\tilde{c}^i}}{\sum\limits_{i=1}^{L} \overline{\alpha}_i + \sum\limits_{j=L+1}^{n} \underline{\alpha}_j} \\ \overline{\tilde{c}} = \dfrac{\sum\limits_{i=1}^{R} \overline{\tilde{\alpha}^i} \overline{\tilde{c}^i} + \sum\limits_{j=R+1}^{n} \underline{\tilde{\alpha}^i} \overline{\tilde{c}^i}}{\sum\limits_{i=1}^{R} \overline{\alpha}_i + \sum\limits_{j=R+1}^{n} \underline{\alpha}_j}, \end{cases} \tag{14}$$

where L and R are the *switch points* that used to make sure $\underline{\tilde{c}}$ is minimized and $\overline{\tilde{c}}$ is maximized, which can be obtained by an iterative procedure. A number of implementations on such problem have been proposed in the literature and widely used in the real world, such as Karnik-Mendel (KM) algorithms, enhanced Karnik-Mendel algorithms (EKMA), an iterative algorithm with stop condition (ISAC), and enhanced ISAC [29]. In particular, the Karnik-Mendel (KM) algorithm is adapted in this work due to its efficiency, and the details of this approach is omitted here as this is beyond the focus of this paper.

Once the output interval or special IT2 fuzzy set is generated, type reduction or defuzzification needs to be applied. This can be readily implemented by applying a simple average operation:

$$c = \frac{\underline{\tilde{c}} + \overline{\tilde{c}}}{2}. \tag{15}$$

IT2 fuzzy sets are extensions of type-1 fuzzy sets. The above proposed IT2 fuzzy TSK+ approach deteriorates to type-1 TKS+ when all the IT2 TSK fuzzy rules degenerate to type-1 ones. There are variations of IT2 rule bases. For instances, the parameters in the rule consequences can be all crisp numbers, instead of crisp intervals. In this special case, the intermediate result led by Equation 12 is a singleton number. In addition, if the $\tilde{p}_1^i = \tilde{p}_2^i = \cdots = \tilde{p}_k^i = 0$ for all rules in the rule base in Equation 9, the TSK+ model will become to a 0-order one, and Equation 12 can then be rewritten as $\tilde{c}^i = [\underline{\tilde{p}_0^i}, \overline{\tilde{p}_0^i}]$. Therefore, the proposed approach is able to be applied to perform the inference with multiple varied TSK rule bases.

Note that rough-fuzzy set based interpolation approaches may also provide similar approximation functionality practically, such as [30], [31]. However, there is an obvious theoretical difference between the two as different underpinning approximation approaches are utilized. In addition, the rough-fuzzy set based interpolation approaches are analogy-based interpolation that considers the representative values and the shapes of upper and lower membership functions separately during the reasoning processes, while the proposed approach considers the IT2 fuzzy set as signal component. The proposed approach utilizes the similarity degree value between the given inputs and rule antecedents as the rule firing strength in aggregating the final result, which follows the conventional TSK inference principle.

IV. Experimentation

The proposed IT2 TSK+ approach is validated and evaluated in this section by applying it to two cases used in the literature.

A. Illustrative Example

A two inputs and one output fuzzy inference problem, which has been used as an illustration example in the work of [32], is re-considered here. In particular, the domains of the two inputs are each fuzzy partitioned and represented by two trapezoidal IT2 fuzzy sets, and thus the rule base consists of 4 rules which covers the entire problem domain. The consequence of each rule is a crisp interval. The complete rule base is listed in Table I, where each IT2 fuzzy set is represented as $\tilde{A} = (\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4, \overline{w}; \underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{w})$, and $(\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4, \overline{w})$ denotes the UMF of the IT2 fuzzy set with degree of confidence $\overline{w}$, $(\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{w})$ indicates the LMF with $\underline{w}$ being the confidence level. Given an input vector $O = (-0.3, 0.6)$, the calculation of the final crisp inference output is detailed below.

TABLE I. RULE BASE FOR THE ILLUSTRATIVE EXAMPLE

No.	Inputs x_1	Inputs x_2	Output y
R_1	(-1.5,-1.5,-0.5,1.5,1;-1.5,-1.5,-1.5,0.5,1)	(-1.5,-1.5,-0.5,1.5,1;-1.5,-1.5,-1.5,0.5,1)	$\tilde{p}_0^1 = [-1, -0.9]$
R_2	(-1.5,-1.5,-0.5,1.5,1;-1.5,-1.5,-1.5,0.5,1)	(-1.5, 0.5, 1.5, 1.5,1;-0.5, 1.5, 1.5, 1.5,1)	$\tilde{p}_0^2 = [-0.6, -0.4]$
R_3	(-1.5, 0.5, 1.5, 1.5,1;-0.5, 1.5, 1.5, 1.5,1)	(-1.5,-1.5,-0.5,1.5,1;-1.5,-1.5,-1.5,0.5,1)	$\tilde{p}_0^3 = [0.4, 0.6]$
R_4	(-1.5, 0.5, 1.5, 1.5,1;-0.5, 1.5, 1.5, 1.5,1)	(-1.5, 0.5, 1.5, 1.5,1;-0.5, 1.5, 1.5, 1.5,1)	$\tilde{p}_0^4 = [0.9, 1]$

The matching degree between each given input item and each antecedent item of every rule is calculated using Equation 10, with the results shown in the 2^{nd} and 3^{rd} columns in Table II. Note that Equation 10 is an extension of Equation 4, and the sensitive factor is set to 8, which is determined empirically. Having computed the matching degrees, the firing strength of each rule is obtained using Equation 11, and the results are listed in the 4^{th} column in Table II.

TABLE II. FIRING STRENGTH FOR EXPERIMENTATION 1

i	x_1 $\tilde{s}(\tilde{A_1^i}, O)$	x_2 $\tilde{s}(\tilde{A_2^i}, O)$	Firing Strength $\tilde{\alpha}^i$
1	[0.3975, 0.6259]	[0.0443, 0.4195]	[0.0443, 0.4195]
2	[0.3975, 0.6259]	[0.5063, 0.6608]	[0.3975, 0.6259]
3	[0.1115, 0.5008]	[0.0443, 0.4195]	[0.0443, 0.4195]
4	[0.1115, 0.5008]	[0.5063, 0.6608]	[0.1115, 0.5008]

The intermediate result led by each rule is the corresponding rule consequence as the given rule base only consists of 0-order TSK fuzzy rules. By applying the KM algorithm, the switching point $L = 1$ and $R = 3$ can be calculated. From this, the final inference output interval $\tilde{c} = [\underline{\tilde{c}}, \overline{\tilde{c}}]$ can be computed as:

$$\begin{aligned}\underline{\tilde{c}} &= \frac{\overline{\tilde{\alpha}^1}\underline{\tilde{p}_0^1} + \underline{\tilde{\alpha}^2}\underline{\tilde{p}_0^2} + \underline{\tilde{\alpha}^3}\underline{\tilde{p}_0^3} + \underline{\tilde{\alpha}^4}\underline{\tilde{p}_0^4}}{\overline{\tilde{\alpha}^1} + \underline{\tilde{\alpha}^2} + \underline{\tilde{\alpha}^3} + \underline{\tilde{\alpha}^4}} \\ &= \frac{0.42 \cdot (-1) + 0.40 \cdot (-0.6) + 0.04 \cdot 0.4 + 0.11 \cdot 0.9}{0.42 + 0.40 + 0.04 + 0.11} \\ &= -0.5636\end{aligned} \quad (16)$$

$$\begin{aligned}\overline{\tilde{c}} &= \frac{\underline{\tilde{\alpha}^1}\overline{\tilde{p}_0^1} + \underline{\tilde{\alpha}^2}\overline{\tilde{p}_0^2} + \underline{\tilde{\alpha}^3}\overline{\tilde{p}_0^3} + \overline{\tilde{\alpha}^4}\overline{\tilde{p}_0^4}}{\underline{\tilde{\alpha}^1} + \underline{\tilde{\alpha}^2} + \underline{\tilde{\alpha}^3} + \overline{\tilde{\alpha}^4}} \\ &= \frac{0.04 \cdot (-0.9) + 0.40 \cdot (-0.4) + 0.04 \cdot 0.6 + 0.50 \cdot 1}{0.04 + 0.40 + 0.04 + 0.50} \\ &= 0.4064\end{aligned} \quad (17)$$

The final system output can be derived by applying a fuzzy type reduction method such as Equation 15:

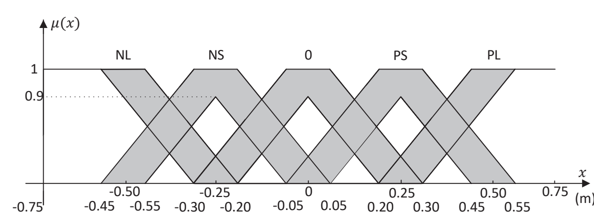
$$\begin{aligned}c &= \frac{\underline{\tilde{c}} + \overline{\tilde{c}}}{2} \\ &= \frac{-0.5636 + 0.4064}{2} = -0.0786\end{aligned} \quad (18)$$

This example demonstrates the working of the proposed TSK+ system with an IT2 fuzzy rule base. Note that the result generated by the conventional IT2 TSK inference approach as reported in the work of [32] is $\underline{\tilde{c}} = -0.6316$, $\overline{\tilde{c}} = 0.4897$ and $c = -0.0710$. The final output led by the proposed IT2 TSK+ approach is very similar to the final result reported in [32], but the generated output interval is only a subset of the one reported in [32].

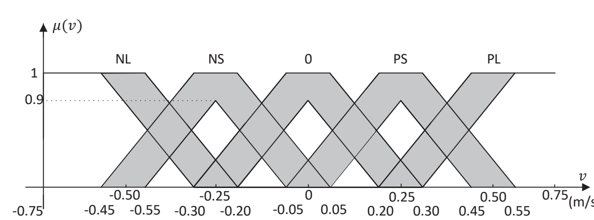
B. Cart Centering Application

In this experiment, the well-known cart centering problem, which has been considered in [26], was used for system evaluation. In this particular problem, a cart can only move along a line on a frictionless plane, and the goal is to drive the cart to the center position of the line from a given initial position on the line, which forms a typical control problem. The inputs of the controller for this problem are the current position coordinates of the cart x and the current velocity of the cart v; and the output of this fuzzy model is the force F that should be applied on the cart. In [26], the domain of cart position x was restricted from $-0.75m$ to $0.75m$; the range of cart velocity v was restricted from $-0.75m/s$ to $0.75m/s$; the output force F was defined between $-0.18m/s$ and $0.18m/s$; and the sampling time used was $t = 0.1s$. This set of parameters and constraints were also utilized in this experiment reported herein.

Based on the problem described above, a 0-order IT2 TSK fuzzy model has been designed and created in [26]. In particular, five linguistic values, represented as IT2 fuzzy sets, were used to cover every domain of input variables x and v, which are negative large (NL), negative small (NS), zero (0), positive small (PS) and positive large (PL), as shown in Figure 2. Five crisp interval values were used as the output, which are labeled as NL, NS, 0, PS and PL, as listed in Table III. A dense rule base for this cart centering problem was generated in [26], as shown in Table IV.



(a) IT2 fuzzy partition for the domain of Position



(b) IT2 membership function for the domain of Velocity

Fig. 2. Fuzzy partition on input domain

In order to evaluate the proposed IT2 TSK+ approach

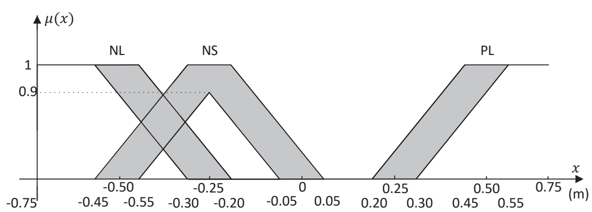
TABLE III. FUZZY PARTITION OF OUTPUT DOMAIN

Output label	Value	Linguistic value
NL	[-0.18 -0.14]	NL
NS	[-0.10 -0.06]	NS
0	[-0.02 0.02]	0
PS	[0.06 0.10]	PS
PL	[0.14 0.18]	PL

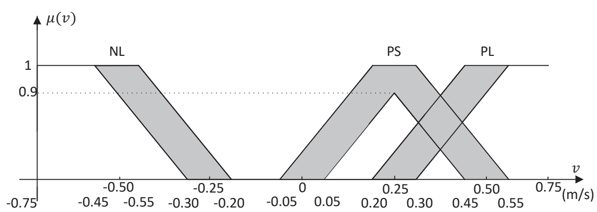
TABLE IV. DENSE RULE BASE WITH 25 RULES USED IN [26]

		Position (x)				
		NL	**NS**	**0**	**PS**	**PL**
	NL	PL	PL	PL	PS	0
	NS	PL	PL	PS	0	NS
Velocity (v)	**0**	PL	PS	0	NS	NL
	PS	PS	0	NS	NL	NL
	PL	0	NS	NL	NL	NL

working with a sparse rule base, two fuzzy sets from each input domain as shown in Figure 2 were manually removed to simulate a lack of information, and the result is shown in Figure 3. Consequently, from the incomplete information, a sparse rule base with only 9 rules was generated, as listed in Table V.



(a) Reduced number of IT2 linguistic values for Position



(b) Reduced number of IT2 linguistic values for Velocity

Fig. 3. Reduced number of IT2 linguistic values for input domain

TABLE V. MODIFIED SPARSE RULE BASE WITH 9 RULES

		Position (x)		
		NL	**NS**	**PL**
	NL	PL	PL	0
Velocity (v)	**PS**	PS	0	NL
	PL	0	NS	NL

Given an initial state of the cart $x = 0.5m$ and $v = 0.5m/s$, the conventional IT2 TSK approach and the proposed IT2 TKS+ were both applied in this experiment using the dense and sparse rule bases, if applicable, for system performance comparison, with the results demonstrated in Figure 4. In particular, the results led by the conventional IT2 TSK, of course based on the dense rule base, are shown in Figures 4(a) and 4(b); the results led by the proposed IT2 TSK+ based on the dense rule base are shown in Figures 4(c) and 4(d); and the results generated by the proposed IT2 TSK+ approach using the sparse rule base are illustrated in Figures 4(e) and 4(f).

This experiment reveals that the proposed IT2 TSK+ approach is able to generate reasonable results using either a dense or sparse rule base. From Figure 4, it is clear that the proposed IT2 TSK+ with the dense rule base took less time to drive the cart from the initial position to the goal position with relatively smooth control, compared to the performance from the conventional TSK approach based on of course the dense rule base. This indicates that the proposed IT2 TSK+ system outperforms the conventional IT2 TSK method when the dense rule base is used. Also interestingly, the proposed approach took longer to change the moving direction, which might be useful in real-world control for better dynamic stability.

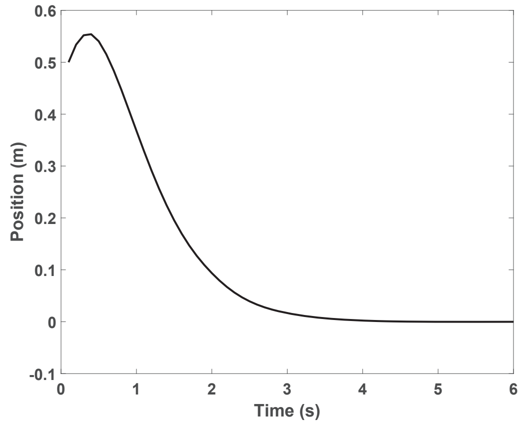
The IT2 TSK+ also successfully drive the cart to the goal position with a relatively smooth curve, although the convergence time taken by the proposed IT2 TSK+ with sparse rule base was longer than those with the dense rule base by either approaches. However, if the size of utilized rule bases are taken into account, the proposed approach can solve the same control problem with only 9 rules, while a dense rule with 25 rules is required by the conventional approach. This clearly demonstrates the power of the proposed system in system complexity reduction.

C. Discussions

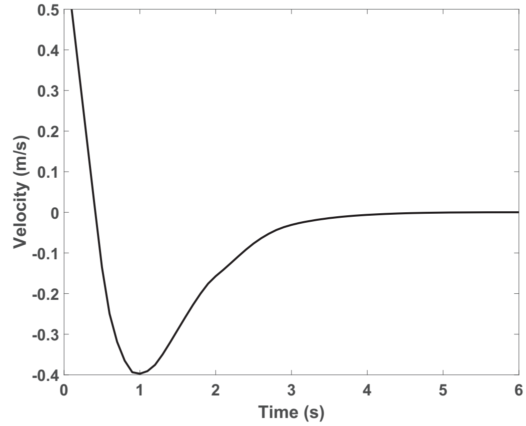
Although many fuzzy interpolation approaches have been proposed to enable fuzzy inference with sparse rule base, the majority of them were developed based on Mamdani rule bases with some being extended to support IT2 fuzzy sets. The proposed system is the first attempt to extend the TSK fuzzy system with wider applicability supporting either type-1 or IT2 fuzzy rule bases which are either dense or sparse. This will significantly improve the performance of the widely applied TSK fuzzy inference systems in real-world applications with better uncertainty management. The system also at the same time providing an effective way in system complexity reduction, especially in the ear of big data.

The sparse rule base used in the second experiment was generated by manually removing some linguistic values from each variable domain rather arbitrarily, and thus the sparse rule base and correspondingly the inferred results may not be optimal. Therefore, better performance is expected from an optimal sparse rule base. Note that developments on sparse rule base generation have been reported in the literature [5], [6]. Although these approaches only targeted type-1 fuzzy models, the underpinning principle can be readily extended to generate sparse IT2 TSK rule base, which remains an active future work. Also, the cart is limited its movement along a straight line only in this experiment. Note that fuzzy controllers have been applied to mobile robot control with no restriction on the cart movement [33]. Such complex control problem may better reveal the capability of the proposed approach.

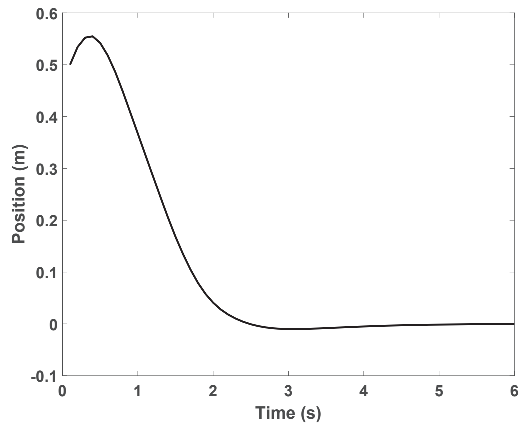
The proposed IT2 TSK+ approach can be employed to address some real-world problems. For instance, a wall-following mobile-robot controller has been proposed in [34]. In this system, an IT2 TSK fuzzy model is designed for



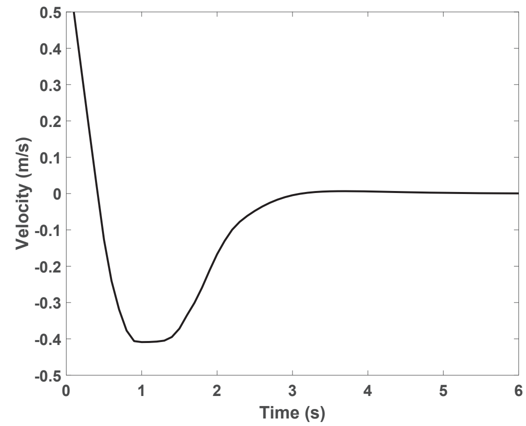
(a) Cart position by conventional TSK with dense rule base [26]



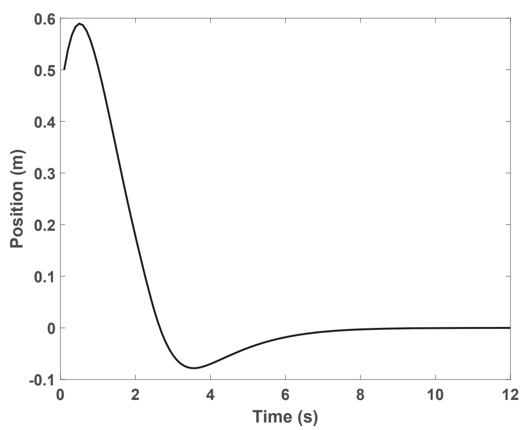
(b) Cart velocity by conventional TSK with dense rule base [26]



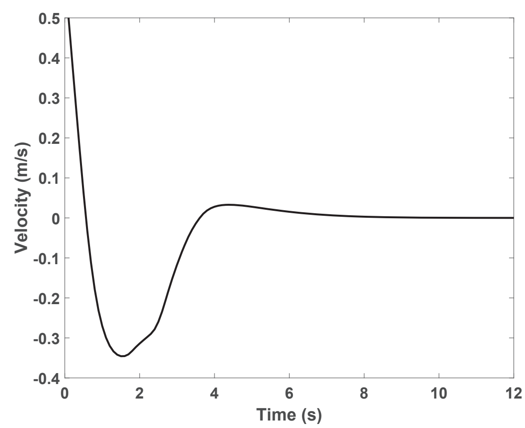
(c) Cart position by IT2 TSK+ with dense rule base



(d) Cart velocity by IT2 TSK+ with dense rule base



(e) Cart position by IT2 TSK+ with sparse rule base



(f) Cart velocity by ITS TSK+ with sparse rule base

Fig. 4. Performance comparison

mobile robot control; and the reinforcement method Q-learning is adopted to learn the IT2 TSK fuzzy rule base. The proposed IT2 TSK+ approach can be readily applied to the mobile-robot system to make the best guess of the next action rather than simply a random one, when the rule base is extremely sparse. Consequently, the total number of trials in the learning process is expected to be reduced. The proposed approach may also be used to other real-world applications which were developed based on type-1 fuzzy sets, such as [35], in an effort to boost the system performance. The implementation and the evaluation of such applications remain as a piece of future work.

V. Conclusion

This paper extended the recently proposed TSK+ fuzzy inference approach by allowing the utilization of sparse IT2 TSK rule bases as well as dense ones. Thanks to the extensive research carried out in the field of IT2 fuzzy sets and the corresponding computing approaches, this work also presented a practically feasible computing approach for real-world applications. IT2 TSK+ is therefore able to perform inferences with dense, sparse, type-1, or IT2 fuzzy rule bases. Two experiments adapted from the literature have been used for system validation and evaluation, with the first one illustrating the working of the system and the second one demonstrating the power of the proposed fuzzy inference system in mobile cart control.

Although promising, this work can be further improved in the following areas. Noting that the value of sensitivity factor (f) in the similarity measure needs to be empirically determined; it would be worthwhile to investigate how this parameter can be intelligently auto-determined. Also, it is interesting to study how the proposed approach can be further extended to work with general type-2 TSK fuzzy sets theoretically and practically. In addition, more evaluation on scaled-up real-world applications are required to fully discover the potential of the proposed IT2 TSK+ fuzzy inference system.

REFERENCES

[1] E. H. Mamdani. Application of fuzzy logic to approximate reasoning using linguistic synthesis. *Computers, IEEE Transactions on*, C-26(12):1182–1191, 1977.

[2] T. Takagi and M. Sugeno. Fuzzy identification of systems and its applications to modeling and control. *Systems, Man and Cybernetics, IEEE Transactions on*, SMC-15(1):116–132, 1985.

[3] L. T. Kóczy and K. Hirota. Approximate reasoning by linear rule interpolation and general approximation. *International Journal of Approximate Reasoning*, 9(3):197–225, 1993.

[4] L. T. Kóczy and K. Hirota. Interpolative reasoning with insufficient evidence in sparse fuzzy rule bases. *Information Sciences*, 71(1):169–201, 1993.

[5] Y. Tan, J. Li, M. Wonders, F. Chao, H. P. H. Shum, and L. Yang. Towards sparse rule base generation for fuzzy rule interpolation. In *IEEE World Congress on Computation Intelligence Internation Conference*, 2016.

[6] J. Li, L. Yang, Y. Qu, and G. Sexton. An extended takagi–sugeno–kang inference system (tsk+) with fuzzy interpolation and its rule base generation. *Soft Computing*, Nov 2017.

[7] L. Yang, Z. Zuo, F. Chao, and Y. Qu. Fuzzy interpolation systems and applications. In S. Ramakrishnan, editor, *Modern Fuzzy Control Systems and Its Applications*, chapter 03. InTech, Rijeka, 2017.

[8] L. Yang. *Fuzzy Rule Interpolation and Its Applications*. PhD dissertation, Aberystwyth University, 2011.

[9] Z. Huang and Q. Shen. Fuzzy interpolative reasoning via scale and move transformations. *Fuzzy Systems, IEEE Transactions on*, 14(2):340–359, 2006.

[10] Z. Huang and Q. Shen. Fuzzy interpolation and extrapolation: A practical approach. *Fuzzy Systems, IEEE Transactions on*, 16(1):13–28, 2008.

[11] L. Yang and Q. Shen. Towards adaptive interpolative reasoning. In *2009 IEEE International Conference on Fuzzy Systems*, pages 542–549, Aug 2009.

[12] L. Yang and Q. Shen. Adaptive fuzzy interpolation. *Fuzzy Systems, IEEE Transactions on*, 19(6):1107–1126, Dec 2011.

[13] L. Yang and Q. Shen. Closed form fuzzy interpolation. *Fuzzy Sets and Systems*, 225:1–22, 2013. Theme: Fuzzy Systems.

[14] S. Jin, R. Diao, C. Quek, and Q. Shen. Backward fuzzy rule interpolation. *IEEE Transactions on Fuzzy Systems*, 22(6):1682–1698, Dec 2014.

[15] S. Chen and W. Hsin. Weighted fuzzy interpolative reasoning based on the slopes of fuzzy sets and particle swarm optimization techniques. *Cybernetics, IEEE Transactions on*, 45(7):1250–1261, 2015.

[16] L. Yang, F. Chao, and Q. Shen. Generalised adaptive fuzzy rule interpolation. *IEEE Transactions on Fuzzy Systems*, 2017.

[17] N. Naik, R. Diao, and Q. Shen. Dynamic fuzzy rule interpolation and its application to intrusion detection. *IEEE Transactions on Fuzzy Systems*, PP(99):1–1, 2017.

[18] J. Li, Y. Qu, H. P. H. Shum, and L. Yang. Tsk inference with sparse rule bases. In *Advances in Computational Intelligence Systems*, pages 107–123. Springer, 2017.

[19] J. Li, L. Yang, H. P. H. Shum, G. Sexton, and Y. Tan. Intelligent home heating controller using fuzzy rule interpolation. 2015.

[20] L. Yang, J. Li, G. Fehringer, P. Barraclough, G. Sexton, and Y. Cao. Intrusion detection system by fuzzy interpolation. In *2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pages 1–6, July 2017.

[21] J. Li, L. Yang, X. Fu, F. Chao, and Y. Qu. Dynamic qos solution for enterprise networks using tsk fuzzy interpolation. In *2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pages 1–6, July 2017.

[22] L Yang, J Li, P Hackney, F Chao, and M Flanagan. Manual task completion time estimation for job shop scheduling using a fuzzy inference system. In *10th IEEE International Conference on Cyber, Physical and Social Computing*, 2017.

[23] Longzhi Yang, Jie Li, Fei Chao, Phil Hackney, and Mark Flanagan. Job shop planning and scheduling for manufacturers with manual operations. *Expert Systems*, 2018.

[24] J.M. Mendel, R.I. John, and F. Liu. Interval type-2 fuzzy logic systems made simple. *IEEE Transactions on Fuzzy Systems*, 14(6):808–821, 2006.

[25] Zsolt Csaba Johanyák and Szilvester Kovács. Distance based similarity measures of fuzzy sets. *Proceedings of SAMI*, 2005, 2005.

[26] N. Enyinna, A. Karimoddini, D. Opoku, A. Homaifar, and S. Arnold. Developing an interval type-2 tsk fuzzy logic controller. In *2015 Annual Conference of the North American Fuzzy Information Processing Society (NAFIPS) held jointly with 2015 5th World Conference on Soft Computing (WConSC)*, pages 1–6, Aug 2015.

[27] N. Boumella, K. Djouani, and M. Boulemden. A robust interval type-2 tsk fuzzy logic system design based on chebyshev fitting. *International Journal of Control, Automation and Systems*, 10(4):727–736, Aug 2012.

[28] L. Yang, C. Chen, N. Jin, X. Fu, and Q. Shen. Closed form fuzzy interpolation with interval type-2 fuzzy sets. In *2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pages 2184–2191, July 2014.

[29] D. Wu and M. Nie. Comparison and practical implementation of type-reduction algorithms for type-2 fuzzy sets and systems. In *2011 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2011)*, pages 2131–2138, June 2011.

[30] Chengyuan Chen, Neil Mac Parthalin, Ying Li, Chris Price, Chai Quek, and Qiang Shen. Rough-fuzzy rule interpolation. *Information Sciences*, 351:1 – 17, 2016.

[31] Shyi-Ming Chen, Shou-Hsiung Cheng, and Ze-Jin Chen. Fuzzy interpolative reasoning based on the ratio of fuzziness of rough-fuzzy sets. *Information Sciences*, 299:394 – 411, 2015.

[32] D. Wu. A brief tutorial on interval type-2 fuzzy sets and systems. 2010.

[33] D. Zhou, M. Shi, F. Chao, C. Lin, L. Yang, C. Shang, and C. Zhou. Use of human gestures for controlling a mobile robot via adaptive cmac network and fuzzy logic controller. *Neurocomputing*, 2017.

[34] C. F. Juang and C. H. Hsu. Reinforcement ant optimized fuzzy controller for mobile-robot wall-following control. *IEEE Transactions on Industrial Electronics*, 56(10):3931–3940, Oct 2009.

[35] D. Zhou, M. Shi, F. Chao, CM Lin, L. Yang, C. Shang, and C. Zhou. Use of human gestures for controlling a mobile robot via adaptive cmac network and fuzzy logic controller. *Neurocomputing*, 282:218 – 231, 2018.