PIVOT DUALITY OF UNIVERSAL INTERPOLATION AND EXTRAPOLATION SPACES

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ABSTRACT. It is a widely used method, for instance in perturbation theory, to associate with a given C_0-semigroup its so-called interpolation and extrapolation spaces. In the model case of the shift semigroup acting on $L^2(\mathbb{R})$, the resulting chain of spaces recovers the classical Sobolev scale. In 2014, the second named author defined the universal interpolation space as the projective limit of the interpolation spaces and the universal extrapolation space as the completion of the inductive limit of the extrapolation spaces, provided that the latter is Hausdorff.

In this note we use the notion of the dual with respect to a pivot space in order to show that the aforementioned inductive limit is Hausdorff, already complete, and can be represented as the dual of the projective limit whenever a power of the generator of the initial semigroup is a self-adjoint operator. In the case of the classical Sobolev scale we show that the duality holds, and that the two universal spaces were already studied by Laurent Schwartz in the 1950s.

Our results and examples complement the approach of Haase, who in 2006 gave a different definition of universal extrapolation spaces in the context of functional calculi. Haase avoids the inductive limit topology precisely for the reason that it a priori cannot be guaranteed that the latter is always Hausdorff. We show that this is indeed the case provided that we start with a semigroup defined on a reflexive Banach space.

1. THE CLASSICAL SOBOLEV SCALE

We start by considering the following generic example. Let $(T(t))_{t \geqslant 0}$ denote the left shift semigroup on the Hilbert space $L^2(\mathbb{R})$ generated by the first derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ defined on the domain $D(\frac{\mathrm{d}}{\mathrm{d}x}) = \{f \in L^2(\mathbb{R}) \,;\, \frac{\mathrm{d}}{\mathrm{d}x}f \in L^2(\mathbb{R})\}$. Writing down the abstract interpolation and extrapolation spaces [5, Chapter II.5] gives the classical scale of Sobolev spaces

$$\cdots \longrightarrow \mathcal{H}^3(\mathbb{R}) \xrightarrow{i_3^2} \mathcal{H}^2(\mathbb{R}) \xrightarrow{i_2^1} \mathcal{H}^1(\mathbb{R}) \xrightarrow{i_1^0} L^2(\mathbb{R}) \xrightarrow{i_0^{-1}} \mathcal{H}^{-1}(\mathbb{R}) \xrightarrow{i_{-1}^{-2}} \mathcal{H}^{-2}(\mathbb{R}) \xrightarrow{i_{-2}^{-3}} \mathcal{H}^{-3}(\mathbb{R}) \to \cdots$$

where the maps are all continuous. Taking the projective limit of this chain of spaces, i.e., endowing the intersection $\cap_{n\in\mathbb{N}}\mathcal{H}^n(\mathbb{R})$ with the coarsest linear topology which makes the inclusions $\cap_{n\in\mathbb{N}}\mathcal{H}^n(\mathbb{R}) \to \mathcal{H}^k(\mathbb{R})$ for all $k \in \mathbb{N}$ continuous, yields the classical function space

$$\mathcal{D}_{L^2}(\mathbb{R}) = \mathrm{proj}_{n\in\mathbb{N}} \mathcal{H}^n(\mathbb{R})$$

studied by Schwartz [10, §8, p. 199]. Taking the inductive limit, i.e., endowing the union $\cup_{n\in\mathbb{N}}\mathcal{H}^{-n}(\mathbb{R})$ with the finest linear topology which makes the inclusions $\mathcal{H}^k(\mathbb{R}) \to \cup_{n\in\mathbb{N}}\mathcal{H}^{-n}(\mathbb{R})$ for all $k \in \mathbb{N}$ continuous, yields a subspace of the space of distributions which turns out to be isomorphic to the strong dual of $\mathcal{D}_{L^2}(\mathbb{R})$, i.e.,

$$\mathcal{D}'_{L^2}(\mathbb{R}) \cong \mathrm{ind}_{n\in\mathbb{N}} \mathcal{H}^{-n}(\mathbb{R})$$

in a natural way. Also this space was investigated by Schwartz [10, §8, p. 200]. Indeed, we have the following commutative diagram

$$
\begin{array}{ccccccccc}
L^2(\mathbb{R}) & \xrightarrow{i_0^{-1}} & \mathcal{H}^{-1}(\mathbb{R}) & \xrightarrow{i_{-1}^{-2}} & \mathcal{H}^{-2}(\mathbb{R}) & \xrightarrow{i_{-2}^{-3}} & \mathcal{H}^{-3}(\mathbb{R}) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \mathrm{id}_{L^2(\mathbb{R})}} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \\
L^2(\mathbb{R}) & \xrightarrow{(i_1^0)'} & \mathcal{H}^1(\mathbb{R})' & \xrightarrow{(i_2^1)'} & \mathcal{H}^2(\mathbb{R})' & \xrightarrow{(i_3^2)'} & \mathcal{H}^3(\mathbb{R})' & \longrightarrow & \cdots
\end{array}
\tag{1}
$$

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where the maps ϕ_n for $n \in \mathbb{N}$ are isomorphisms. Our first aim is to see that the corresponding inductive limits are isomorphic. We emphasize that this is not trivial just by having "step-wise" isomorphisms. Indeed, we have for instance $\mathcal{H}^{-n}(\mathbb{R}) \cong L^2(\mathbb{R})$ for each $n \in \mathbb{N}$ but $L^2(\mathbb{R}) \ncong \cup_{n \in \mathbb{N}} \mathcal{H}^{-n}(\mathbb{R})$, which shows that we have to be extremely careful when we "identify" isomorphic spaces.

The suitable notion to address our first aim is that of *equivalent inductive sequences*. Each row in the diagram (1) is a so-called inductive sequence, i.e., a sequence $(X_n, i_n^{n+1})_{n \in \mathbb{N}}$ of Banach spaces X_n and linear and continuous maps $i_n^{n+1} \colon X_n \to X_{n+1}$ for $n \in \mathbb{N}$. Two such inductive sequences $(X_n, i_n^{n+1})_{n \in \mathbb{N}}$ and $(Y_n, j_n^{n+1})_{n \in \mathbb{N}}$ are said to be equivalent, if there are increasing sequences $(k(n))_{n \in \mathbb{N}}$ and $(\ell(n))_{n \in \mathbb{N}}$ of natural numbers with $n \leqslant \ell(n) \leqslant k(n) \leqslant \ell(n+1)$ and linear and continuous maps $\alpha_n \colon Y_{\ell(n)} \to X_{k(n)}$, $\beta_n \colon X_{k(n)} \to Y_{\ell(n+1)}$ such that

$$
\begin{array}{ccccc}
\cdots \longrightarrow & X_{k(n)} & \xrightarrow{\;i_{k(n)}^{k(n+1)}\;} & X_{k(n+1)} & \longrightarrow \cdots \\
& \alpha_n \nearrow \quad \beta_n \searrow & \alpha_{n+1} \nearrow & \beta_{n+1} \searrow & \\
\cdots \longrightarrow Y_{\ell(n)} & \xrightarrow{\;j_{\ell(n)}^{\ell(n+1)}\;} & Y_{\ell(n+1)} & \xrightarrow{\;j_{\ell(n+1)}^{\ell(n+2)}\;} & Y_{\ell(n+2)} \longrightarrow \cdots
\end{array}
$$

commutes. As a matter of fact, two equivalent inductive sequences have isomorphic inductive limits.

We now see that the two inductive sequences in (1) are equivalent and we thus get that

$$
\operatorname{ind}_{n \in \mathbb{N}} \mathcal{H}^{-n}(\mathbb{R}) \cong \operatorname{ind}_{n \in \mathbb{N}} \mathcal{H}^{n}(\mathbb{R})'
$$

holds. Now we would like to conclude that the dual of a projective limit (=intersection) is equal to the inductive limit (=union) of the duals of the spaces in the sequence. This is indeed true but requires an open mapping theorem for topological vector spaces. Note that the important conclusion of the following argument is that we finally get the isomorphism (2) below. We observe that $\mathcal{D}_{L^2}(\mathbb{R})$ is a reflexive Fréchet space by Jarchow [8, Proposition 11.5.5 and Corollary 11.4.3] and thus distinguished by [8, Remark after 13.4.5 on p. 280]. Then by [8, Corollary 13.4.4] it follows that $\mathcal{D}'_{L^2}(\mathbb{R})$ is ultrabornological. As $\mathcal{H}^{-n}(\mathbb{R}) \subseteq \mathcal{H}^{-(n+1)}(\mathbb{R})$ is dense for all $n \in \mathbb{N}$, we have that the corresponding sequence is reduced and [8, Proposition 8.8.7] implies that there is a natural isomorphism

$$
\operatorname{ind}_{n \in \mathbb{N}} \mathcal{H}^{n}(\mathbb{R})' \xrightarrow{\;\varphi\;} (\operatorname{proj}_{n \in \mathbb{N}} \mathcal{H}^{n}(\mathbb{R}))'_b = \mathcal{D}'_{L^2}(\mathbb{R}) \tag{2}
$$

of linear spaces. As $\varphi|_{\mathcal{H}^{n}(\mathbb{R})'} \colon \mathcal{H}^{n}(\mathbb{R})' \to (\operatorname{proj}_{n \in \mathbb{N}} \mathcal{H}^{n}(\mathbb{R}))'_b$ is continuous, it follows that φ is continuous and hence an isomorphism by the open mapping theorem [8, Theorem 5.5.2] as its domain is—as an LB-space—in particular webbed [8, Corollary 5.3.3] and its codomain is ultrabornological.

We point out again, that the crucial fact for the above arguments is not just the existence of isomorphisms ϕ_n. In the current example this indeed might appear to be trivial by a well-known characterization of the Sobolev spaces, e.g., that $\mathcal{H}^{-n}(\mathbb{R})$ can be identified with the dual of $\mathcal{H}^{n}(\mathbb{R})$. However, exactly these identifications have to be compatible with the maps in the inductive sequences in order to induce an equivalence of the latter. Below, in Definition 2, we employ a generalization of the notion of the "dual with respect to a pivot space", see Tucsnak, Weiss [11, Chapter 2.10], in order to construct families of isomorphisms that are compatible with the linking maps of the corresponding inductive sequences. The gap that we left in the example above will be closed by Corollary 9.

In the notation established by the second-named author in [13], the space $\operatorname{proj}_{n \in \mathbb{N}} \mathcal{H}^{n}(\mathbb{R})$ is the *universal interpolation space* associated with the shift semigroup on $L^2(\mathbb{R})$. According to the latter article, the *universal extrapolation space* is defined to be the completion of $\operatorname{ind}_{n \in \mathbb{N}} \mathcal{H}^{-n}(\mathbb{R})$. By the representation above and as the strong dual of a Fréchet space is complete, see e.g. Meise, Vogt [9, p. 296], we see however that the inductive limit is already complete and thus forming the completion is dispensable.

Haase [6, p. 143ff] and [7, p. 221ff] proposed a different notion of *universal extrapolation space*: In the situation of our example, his extrapolation space is the union $U = \cup_{n \in \mathbb{N}} \mathcal{H}^{-n}(\mathbb{R})$ in which a net $(x_\alpha)_{\alpha \in \mathcal{A}} \subseteq U$ converges by definition to $x \in U$ if

$$
\exists \, n \in \mathbb{N}, \; \alpha_0 \in \mathcal{A} \; \forall \, \alpha \geqslant \alpha_0 \colon x, \, x_\alpha \in \mathcal{H}^{-n}(\mathbb{R}) \text{ and } x_\alpha \to x \text{ in } \mathcal{H}^{-n}(\mathbb{R})
$$

holds. Haase [7, Remark 8.1.2] motivates his definition by the fact that the inductive topology even in the case of a countable sequence of Banach spaces needs not to be Hausdorff.

In [13] several sequence space cases were studied in which the inductive limit of the extrapolation spaces turned out to be Hausdorff and to be complete. In this note we go one better and provide a general theorem that establishes that the inductive limit is Hausdorff and even complete whenever our initial Banach space is reflexive. Further, we identify conditions under which the duality of universal inter- and extrapolation space holds. Above, we gave already one example in which the corresponding limit space is even well-known since a long time ago.

2. Duals with respect to a pivot space

For the whole section let X be a reflexive Banach space with norm $\|\cdot\|_X$ and

$$A\colon D(A) \to X$$

be a closed and densely defined operator on X such that $A\colon D(A) \to X$ is invertible with $A^{-1} \in L(X)$. In particular we may think of A as the generator of a C_0-semigroup on X. Note that in the latter case one can assume w.l.o.g. that A^{-1} exists and belongs to $L(X)$, see [5, p. 124].

Following [5, 13], we introduce the interpolation and extrapolation spaces with respect to the operator A. The *n-th interpolation space* is defined as

$$X_n := (D(A^n), \|\cdot\|_n), \quad \text{where} \quad \|x\|_n := \|x\|_X + \|A^n x\|_X, \quad \text{for} \quad x \in X_n.$$

Note that the norm $\|\cdot\|_n$ is the graph norm with respect to A^n. For the definition of the extrapolation spaces, we define the norm

$$\|x\|_{-n} = \|A^{-n}x\|_X, \quad \text{where} \quad A^{-n} := \left(A^{-1}\right)^n,$$

on X. We define the *n-the extrapolation space*

$$X_{-n} := (X, \|\cdot\|_{-n})^{\wedge}$$

as the completion of X with respect to the norm $\|\cdot\|_{-n}$.

As in the motivating example in the first section, we want to find a representation of the inductive limit of the spaces X_{-n} as the dual space of a projective limit of a suitable sequence of Banach spaces. Before we start this construction, we first summarize some facts which follow easily from [12, Chapter 1] and [5, Appendix B]. Given a Banach space X with dual space X', we denote by $\langle \cdot, \cdot \rangle$ the duality mapping between X and X', i.e., $\langle x, \varphi \rangle := \varphi(x)$, for $x \in X$ and $\varphi \in X'$. We put

$$D(A') = \left\{ x' \in X'\,;\, \exists\, y' \in X' \,\forall\, x \in D(A)\colon \langle Ax, x' \rangle = \langle x, y' \rangle \right\}$$

and denote by $A'\colon D(A') \to X'$, $A'x' = y'$ the dual operator of A. When we consider its powers we abbreviate $A'^{\,n} := (A')^n$.

Lemma 1. *Let X be a reflexive Banach space.*
  (i) *The operator $A'\colon D(A') \to X'$ is closed and densely defined.*
  (ii) *For $n \geqslant 1$ the operator A^n is invertible and we have $(A^n)^{-1} = A^{-n} \in L(X)$.*
  (iii) *For $n \geqslant 1$ we have $(A^{-n})' = (A')^{-n}$.* $\qquad\square$

Observe that it is well-known, that (i) fails without reflexivity and that the proof of (iii) relies on (i). Remember that we only consider reflexive spaces, although some single arguments might hold true without this assumption, as, e.g., part (ii) in the lemma above.

Using that X is reflexive, the main idea of the following construction is to find suitable subspaces of X' which, up to isomorphisms, may act as preduals of the extrapolation spaces X_{-n}.

Definition 2. *Let X be reflexive and $n \geqslant 1$. For $x \in X$ we put*

$$\|x\|_{\star,n} = \sup \left\{ |\langle x, \psi \rangle|\,;\, \|\psi\|_{D(A'^{\,n})} \leqslant 1 \right\}$$

which defines a family of new norms on X. Using these norms, we denote by

$$X_n^d := (D(A'^{\,n}), \|\cdot\|_{D(A'^{\,n})})$$

3

the n-th interpolation space with respect to the dual operator A'. Generalizing the Hilbert space notation from [11, p. 60], we put

$$(X_n^d)^\star := (X, \|\cdot\|_{\star,n})^\wedge$$

and call this space the "dual of X_n^d with respect to the pivot space X".

The spaces X_n^d defined above will, up to isomorphisms, act as the aforementioned preduals of the extrapolation spaces X_{-n}. In order to see this relation, we need the spaces $(X_n^d)^\star$. The following lemma is used to establish isomorphisms between X_{-n} and $(X_n^d)^\star$. In the case where X is Hilbert and $n = 1$, it is proved in [11, Proposition 2.10.2].

Lemma 3. *Let X be reflexive and $n \geqslant 1$. Then, the norms $\|\cdot\|_{\star,n}$ and $\|\cdot\|_{-n}$ are equivalent.*

Proof. Let $x \in X$ be given. From Lemma 1 we get $\langle A^{-n}x, \varphi \rangle = \langle x, (A^{-n})'\varphi \rangle = \langle x, (A')^{-n}\varphi \rangle$ and thus we obtain

$$\|x\|_{-n} = \|A^{-n}x\|_X = \sup\{|\langle A^{-n}x, \varphi \rangle| \,;\, \|\varphi\|_{X'} \leqslant 1\} = \sup\{|\langle x, (A')^{-n}\varphi \rangle| \,;\, \|\varphi\|_{X'} \leqslant 1\}.$$

Since $A'^n \colon D(A'^n) \to X'$ is bijective with inverse $(A')^{-n}$ we can substitute $\psi = (A')^{-n}\varphi$ and get from the above

$$\|x\|_{-n} = \sup\{|\langle x, \psi \rangle| \,;\, \|A'^n\psi\|_{X'} \leqslant 1\}.$$

If we endow $D(A'^n)$ with the homogeneous norm $\|\cdot\|_{A',n}$, i.e., $\|x\|_{A',n} = \|A'^n x\|_X$, then the map $A'^n \colon D(A'^n) \to X'$ is an isometric isomorphism. It follows

$$\|x\|_{-n} = \sup\{|\langle x, \psi \rangle| \,;\, \|A'^n\psi\|_{X'} \leqslant 1\} = \sup\{|\langle x, \psi \rangle| \,;\, \|\psi\|_{A',n} \leqslant 1\}$$

which shows that $\|\cdot\|_{-n}$ and $\|\cdot\|_{\star,n}$ are equivalent, as the homogeneous norm is equivalent to the graph norm. $\qquad\square$

Since $A'^n \colon X_n^d \to X'$ is an isomorphism and X is reflexive, X_n^d is again reflexive. By Lemma 3 we can define

$$\phi_n \colon X_{-n} \to (X_n^d)^\star$$

to be the unique extension of id_X. Moreover, we define

$$\psi_n \colon (X_n^d)^\star \to (X_n^d)'$$

by

$$[\psi_n(z)](\varphi) = \lim_{k\to\infty} \langle z_k, \varphi \rangle_{X,X'} \quad \text{for } z \in (X_n^d)^\star, \varphi \in X_n^d$$

where $(z_k)_{k\in\mathbb{N}} \subseteq X$ with $z_k \to z$ in $(X_n^d)^\star$. Finally, we denote by $i_n \colon X \to (X_n^d)^\star$ the inclusion and by $(j_n)' \colon X \to (X_n^d)'$ the dual of the inclusion $j_n \colon X_n^d \to X'$.

Lemma 4. *Let X be reflexive and $n \geqslant 1$. Then, ψ_n is an isomorphism and $\psi_n \circ i_n = (j_n)'$ holds.*

Proof. It is straightforward to see that $(\psi_n(z))(\varphi)$ is well-defined, i.e., that the limit exists and is independent of the choice of the sequence. The linearity of $\psi_n(z)$ and ψ_n follows from the definition of ψ_n and the continuity of the vector space operations. From $\|z\|_{\star,n} = \sup\{|\langle z, \varphi \rangle| \,;\, \|\varphi\|_{D(A'^n)} \leqslant 1\}$ we may deduce $|\langle z, \varphi \rangle| \leqslant \|\varphi\|_{D(A'^n)} \|z\|_{\star,n}$ for $z \in X$ and $\varphi \in X_n^d$ and by density for $z \in (X_n^d)^\star$. Hence

$$\psi_n(z) \in (X_n^d)' \quad \text{and} \quad |(\psi_n(z))(\varphi)| \leqslant \|\varphi\|_{X_n^d} \|z\|_{\star,n}.$$

Therefore $\|\psi_n(z)\|_{(X_n^d)'} \leqslant \|z\|_{\star,n}$ and $\psi_n \colon (X_n^d)^\star \to (X_n^d)'$ is continuous.

For $z \in X$ we have $(\psi_n(z))(\varphi) = \langle z, \varphi \rangle$ since the constant sequence $(z)_{k\in\mathbb{N}}$ converges to z. Therefore we get for $z \in X$

$$\|\psi_n(z)\|_{(X_n^d)'} = \sup\{|\langle \psi_n(z), \varphi \rangle| \,;\, \|\varphi\|_{X_n^d} \leqslant 1\} = \sup\{|\langle z, \varphi \rangle| \,;\, \|\varphi\|_{X_n^d} \leqslant 1\} = \|z\|_{\star,n}.$$

Since $X \subseteq (X_n^d)^\star$ is dense and ψ_n is continuous, we may conclude that $\|\psi_n(z)\|_{(X_n^d)'} = \|z\|_{\star,n}$ holds for all $z \in (X_n^d)^\star$. In other words ψ_n is an isometric embedding and hence has a closed range. In

order to finish the proof, we show that the range is also dense. Assume for a contradiction that $\psi_n((X_n^d)^\star) \subseteq (X_n^d)'$ is not a dense subset. Then the Hahn-Banach Theorem implies the existence of a functional $0 \neq \varphi \in (X_n^d)'' = X_n^d$ where $\langle \psi_n(z), \varphi \rangle = 0$ for all $z \in (X_n^d)^\star$. In particular, we get $\langle z, \varphi \rangle = 0$ for all $z \in X$ which is a contradiction since $X_n^d \subseteq X'$ and therefore $0 \neq \varphi \in X'$ with $\langle z, \varphi \rangle = 0$ for all $z \in X$. Finally, the equality

$$(\psi_n(i_n(z)))(\varphi) = \langle \psi_n(z), \varphi \rangle = \langle z, \varphi \rangle = ((j_n)'(z))(\varphi)$$

shows $\psi_n \circ i_n = (j_n)'$. $\qquad\qquad\square$

We mention, that the above proof was inspired by [11, Proposition 2.9.2], where a similar statement is shown when X is a Hilbert space and $n = 1$.

3. Universal inter- and extrapolation spaces

In this section, we now form inductive and projective limits of the sequences considered before. The main result of this section is the following theorem which provides a representation of the inductive limit of the extrapolation spaces for a densely defined, closed operator having a bounded inverse. In the situation of [13], where A is the generator of a C_0-semigroup, this inductive limit was called the universal extrapolation space. Note that in the latter case one can assume w.l.o.g. that A^{-1} exists and belongs to $L(X)$, see [5, p. 124].

Theorem 5. *Let X be a reflexive Banach space and $A \colon D(A) \to X$ be a densely defined, closed operator such that A^{-1} exists and belongs to $L(X)$. Then we have*

$$(\operatorname{proj}_{n\in\mathbb{N}} X_n^d)'_b \cong \operatorname{ind}_{n\in\mathbb{N}} X_{-n}. \tag{3}$$

In particular, the inductive limit $\operatorname{ind}_{n\in\mathbb{N}} X_{-n}$ is complete.

In order to prove this theorem, we first have to specify the linking maps between the spaces defined in Section 2 in more detail. We start with the linking maps between the extrapolation spaces, i.e. the maps

$$i_n^{n+1} \colon X_{-n} \to X_{-(n+1)}.$$

In [5, Remark after II.5.6], where A is always assumed to be a generator of a C_0-semigroup, it is mentioned, that

$$X_{-(n+m)} = (X_{-(n)})_{-m} \tag{4}$$

holds for all n, $m \geqslant 1$, where X_{-n} for $n \geqslant 1$ is the n-th extrapolation space as defined above. Indeed, the above equality holds in the sense of isometric isomorphisms that constitute an equivalence of inductive sequences. We first observe that, given a closed and densely defined operator A with $0 \in \rho(A)$, then A_{-n} has the same properties. Therefore we can define

$$Y_{-n} := (\cdots(X)_{-1}\cdots)_{-1}$$

by forming n-times the first extrapolation space. We denote by $j_n^{n+1} \colon Y_{-n} \to Y_{-(n+1)}$ the inclusion maps. As $X \subseteq Y_{-n}$ and $X \subseteq X_{-n}$ are dense and $\|x\|_{-n} = \|x\|_{Y_{-n}}$ holds for $x \in X$, the identity id_X extends to an isometric isomorphism $\Theta_n \colon Y_{-n} \to X_{-n}$. We define the maps $i_n^{n+1} \colon X_{-n} \to X_{-(n+1)}$ via $i_n^{n+1} := \Theta_{n+1} \circ j_n^{n+1} \circ \Theta_n^{-1}$, i.e., the diagram

$$
\begin{array}{ccccccccc}
X & \longrightarrow & \cdots & \longrightarrow & Y_{-n} & \xrightarrow{j_n^{n+1}} & Y_{-(n+1)} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle\operatorname{id}_X} & & & & \downarrow{\scriptstyle\Theta_n} & & \downarrow{\scriptstyle\Theta_{n+1}} & & \\
X & \longrightarrow & \cdots & \longrightarrow & X_{-n} & \xrightarrow[i_n^{n+1}]{} & X_{-(n+1)} & \longrightarrow & \cdots
\end{array}
$$

is commutative. Note that the mappings Θ_n are only used to define the linking maps i_n^{n+1} and will not be used the the sequel.

Since the space $(X_n^d)^\star$ is defined as the completion of X with respect to the norm $\|\cdot\|_{\star,n}$, X is dense in $(X_n^d)^\star$ for all $n \geqslant 1$. We can therefore define the mapping

$$(i_n^{n+1})^\star \colon (X_n^d)^\star \to (X_{n+1}^d)^\star$$

as the extension of the identity on X. In addition, we define the mapping

$$(i_n)^\star\colon X \to (X_n^d)^\star$$

to be the canonical embedding of X into $(X_n^d)^\star$. In other words, the above construction means that $(i_n^{n+1})^\star$ is the unique continuous linear mapping satisfying $(i_{n+1})^\star = (i_n^{n+1})^\star \circ (i_n)^\star$. Since X_n^d is defined as $D(A'^n)$ equipped with the graph norm, the mappings

$$j_n := \mathrm{id}_X|_{D(A'^n)}\colon X_n^d \to X', \text{ and } j_{n+1}^n := \mathrm{id}_{X_n^d}|_{D(A'^{n+1})}\colon X_{n+1}^d \to X_n^d$$

are continuous inclusions. By [2, Proposition 6.2] the operators also have dense images. Therefore, we obtain by duality the maps

$$(j_n)'\colon X \to (X_n^d)' \text{ and } (j_{n+1}^n)'\colon (X_n^d)' \to (X_{n+1}^d)'$$

which satisfy $(j_{n+1})' = (j_{n+1}^n)' \circ (j_n)'$.

Proof of Theorem 5. We divide the proof into three steps.

As a first step we show that the sequence $(\phi_n)_{n\in\mathbb{N}}$ defined in Section 2 is an equivalence between the inductive sequences $(X_{-n}, i_n^{n+1})_{n\in\mathbb{N}}$ and $((X_n^d)^\star, (i_n^{n+1})^\star)_{n\in\mathbb{N}}$. We have to show that the diagram

$$
\begin{array}{ccccccccc}
X & \longrightarrow & \cdots & \longrightarrow & X_{-n} & \xrightarrow{i_n^{n+1}} & X_{-(n+1)} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle\mathrm{id}_X} & & & & \downarrow{\scriptstyle\phi_n} & & \downarrow{\scriptstyle\phi_{n+1}} & & \\
X & \longrightarrow & \cdots & \longrightarrow & (X_n^d)^\star & \xrightarrow[(i_n^{n+1})^\star]{} & (X_{n+1}^d)^\star & \longrightarrow & \cdots
\end{array}
\quad,
\tag{5}
$$

is commutative. Note that by definition both $\phi_n \circ (i_n^{n+1})^\star$ and $i_n^{n+1} \circ \phi_{n+1}$, restricted to X, coincide with the identity on X, which implies the commutativity since X is dense both in X_{-n} and $(X_{n+1}^d)^\star$.

As a second step, we show that the sequence of maps $(\psi_n)_{n\in\mathbb{N}}$ is an equivalence between the inductive sequences $((X_n^d)^\star, (i_n^{n+1})^\star)_{n\in\mathbb{N}}$ and $((X_n^d)', (j_{n+1}^n)')_{n\in\mathbb{N}}$. Here, we have a commutative diagram

$$
\begin{array}{ccccccccc}
X & \longrightarrow & \cdots & \longrightarrow & (X_n^d)^\star & \xrightarrow{(i_n^{n+1})^\star} & (X_{n+1}^d)^\star & \longrightarrow & \cdots \\
\downarrow{\scriptstyle\mathrm{id}_X} & & & & \downarrow{\scriptstyle\psi_n} & & \downarrow{\scriptstyle\psi_{n+1}} & & \\
X & \longrightarrow & \cdots & \longrightarrow & (X_n^d)' & \xrightarrow[(j_{n+1}^n)']{} & (X_{n+1}^d)' & \longrightarrow & \cdots
\end{array}
\tag{6}
$$

by Lemma 4. Indeed, for $x \in X$ we have

$$\psi_{n+1}((i_n^{n+1})^\star((i_n)^\star(x))) = \psi_{n+1}((i_{n+1})^\star(x)) = (j_{n+1})'(x)$$
$$= (j_{n+1}^n)'((j_n)'(x)) = (j_{n+1}^n)'(\psi_n((i_n)^\star(x)))$$

because $(i_{n+1})^\star = (i_n^{n+1})^\star \circ (i_n)^\star$ and $(j_{n+1})' = (j_{n+1}^n)' \circ (j_n)'$. Since $(i_n)^\star(X)$ is a dense subspace of $(X_n^d)^\star$ the commutativity follows.

Using these facts, we are now able to to prove the claimed duality result. By [9, 25.12, 25.14 and 25.15], $\mathrm{proj}_{n\in\mathbb{N}} X_n^d$ is a reflexive space, being a projective limit of reflexive Banach spaces, and therefore it is distinguished; in particular its dual is isomorphic to $\mathrm{ind}_{n\in\mathbb{N}}(X_n^d)'$ by [4, Definition 9 and Proposition 10, pp. 84–85]. $\qquad\square$

Remark 6. We mention that (5) yields $\mathrm{ind}_{n\in\mathbb{N}} X_{-n} \cong \mathrm{ind}_{n\in\mathbb{N}}(X_n^d)^\star$, even if the limit spaces are not Hausdorff. In particular the universal extrapolation space $X_{-\infty} = (\mathrm{ind}_{n\in\mathbb{N}} X_{-n})^\wedge$ in the sense of [13] exists if the sequence of the pivot duals is known to have a Hausdorff inductive limit.

In the case where X is not only a Banach space but a Hilbert space and the operator A is in addition self-adjoint, we may get the following stronger result. For a Hilbert space X, in order to avoid confusion with the duality mapping, we use the notation $(\cdot, \cdot)$ for the scalar product of X.

Theorem 7. *Let X be a Hilbert space and $A\colon D(A) \to X$ be a densely defined, self-adjoint operator such that A^{-1} exists and belongs to $L(X)$. Then we have*

$$(\mathrm{proj}_{n\in\mathbb{N}} X_n)'_b \cong \mathrm{ind}_{n\in\mathbb{N}} X_{-n}. \tag{7}$$

In particular, the inductive limit $\mathrm{ind}_{n\in\mathbb{N}} X_{-n}$ is complete.

Proof. In order to prove this theorem, we want to show that for all $n \in \mathbb{N}$ the spaces X_n and X_n^d are isomorphic in a way which is compatible with the construction in Section 2 and the proof of Theorem 5. In other words, we have to find an isomorphism between X and X' which is "canonical enough for our purposes".

Since A is self-adjoint, by the spectral theorem (see eg. [6, Theorem D.5.1]) there is an measure space (Ω, μ), a real valued continuous function f on Ω and a unitary operator $U\colon X \to L^2(\Omega, \mu)$ mapping $D(A)$ onto $D(M_f)$ and satisfying

$$A = U^\star \circ M_f \circ U, \tag{8}$$

where $M_f g := fg$ for $g \in D(M_f)$.

The conjugation mapping $L^2(\Omega, \mu) \to L^2(\Omega, \mu), f \mapsto \bar{f}$, where $\bar{f}(t) := \overline{f(t)}$, is an antilinear isometry and hence the mapping

$$J\colon X \to X, \ x \mapsto U^\star(\overline{U(x)}) \tag{9}$$

is an antilinear isometry onto. As the domain $D(M_f)$ is invariant under complex conjugation, i.e., $\overline{D(M_f)} = D(M_f)$, and since $U^\star(D(M_f)) = D(A)$, we may deduce $J(D(A)) = D(A)$ and inductively $J(X_n) = X_n$ for all $n \in \mathbb{N}$. The identites

$$(J \circ A)(x) = U^\star(\overline{UAx}) = U^\star(\overline{((U \circ U^\star \circ M_f \circ U)(x))}) = U^\star(\overline{fU(x)}) = U^\star(\bar{f}\overline{U(x)})$$
$$= (U^\star \circ M_f \circ U)(U^\star(\overline{U(x)})) = (A \circ J)(x), \tag{10}$$

which follow from (8), (9), and the fact that f is real valued, show that A and J commute.

We denote by $J_X\colon X \to X', x \mapsto (\cdot, x)$ the canonical map, which, by the Fréchet-Riesz Theorem, is an antilinear, isometric map onto. Hence the composition

$$J_X \circ J\colon X \to X', \quad x \mapsto (\cdot, J(x)) \tag{11}$$

is a linear isometry onto.

Note that a direct computation shows that J_X maps $D(A) = D(A^\star)$ onto $D(A')$ and hence inductively X_n onto X_n^d. In addition, it is easy to see that $A = J_X^{-1} \circ A' \circ J_X$. From this we may conclude that $J_X \circ J$ maps X_n onto X_n^d. Moreover the computation

$$\|(J \circ J_X^{-1})(\varphi)\|_n = \|J(J_X^{-1})(\varphi)\|_X + \|A^n(J \circ J_X^{-1})(\varphi)\|_X = \|J_X^{-1}(\varphi)\|_X + \|A^n(J_X^{-1}(\varphi))\|_X$$
$$= \|\varphi\|_{X'} + \|(J_X \circ A \circ J_X^{-1}) \circ \cdots \circ (J_X \circ A \circ J_X^{-1})\varphi\|_{X'} \tag{12}$$
$$= \|\varphi\|_{X'} + \|(A')^n\varphi\|_{X'} = \|\varphi\|_{X_n^d}$$

for $\varphi \in X_n^d$, shows that $J_X \circ J|_{X_n}$ is even an isometry onto. Since we are only using restrictions of the mapping $J_X \circ J$, the diagram

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & X_{n+1} & \longrightarrow & X_n & \longrightarrow \cdots \longrightarrow & X \\
& {\scriptstyle J_X \circ J|_{X_{n+1}}}\downarrow & & {\scriptstyle J_X \circ J|_{X_n}}\downarrow & & {\scriptstyle J_X \circ J}\downarrow \\
\cdots \longrightarrow & X_{n+1}^d & \longrightarrow & X_n^d & \longrightarrow \cdots \longrightarrow & X',
\end{array} \tag{13}
$$

where the horizontal arrow are inclusions, is commutative. From this we may deduce that

$$\mathrm{proj}_{n\in\mathbb{N}} X_n \cong \mathrm{proj}_{n\in\mathbb{N}} X_n^d,$$

which in combination with Theorem 5 finishes the proof. $\qquad\square$

We conclude this section with two corollaries. The first is the existence result for the universal extrapolation space, cf. the last paragraph of Section 1.

Corollary 8. *Let X be a reflexive Banach space and $A\colon D(A) \to X$ be the generator of a C_0-semigroup such that A^{-1} exists and belongs to $L(X)$. Then the inductive limit $\mathrm{ind}_{n\in\mathbb{N}} X_{-n}$ is complete and $X_{-\infty} = \mathrm{ind}_{n\in\mathbb{N}} X_{-n}$ holds.* $\qquad\square$

The second corollary fills the gap that we left in our initial example of the classical Sobolev scale considered in Section 1.

Corollary 9. *Let X be a Hilbert space and $A\colon D(A) \to X$ be the generator of a C_0-semigroup such that A^{-1} exists and belongs to $L(X)$. If there is a $k \in \mathbb{N}$ such that $A^k\colon D(A^k) \to X$ is self-adjoint, then*

$$(\mathrm{proj}_{n\in\mathbb{N}} X_n)'_b \cong \mathrm{ind}_{n\in\mathbb{N}} X_{-n}$$

holds.

Proof. Since A^k is self-adjoint, we can apply Theorem 7 to A^k by Lemma 1(ii). The Corollary follows since the inductive sequences (X_n, i_n^{n+1}) and $(X_{kn}, i_{kn+k-1}^{k(n+1)} \circ \cdots \circ i_{nk}^{nk+1})$ are equivalent and hence the projective limits $\mathrm{proj}_{n\in\mathbb{N}} X_n$ and $\mathrm{proj}_{n\in\mathbb{N}} X_{kn}$ coincide. $\qquad\square$

4. Examples and open problems

With the above theory we are able to extend our initial Hilbert space example form Section 1 to arbitrary Lebesgue exponent $p \in (1, \infty)$.

We consider $X = L^p(\mathbb{R})$ for $1 < p < \infty$ and $A = \frac{\mathrm{d}}{\mathrm{d}x}$ with $D(A) = \left\{ f \in L^p(\mathbb{R}) \;;\; \frac{\mathrm{d}}{\mathrm{d}x} f \in L^p(\mathbb{R}) \right\}$. We get $X_n = W^{p,n}(\mathbb{R})$ and $X_\infty = \mathcal{D}_{L^p}(\mathbb{R})$. As duals with respect to the pivot space $X = L^p(\mathbb{R})$, we get $X_n^d = W^{q,n}(\mathbb{R})$, where $1/p + 1/q = 1$ and $(X_n^d)^\star = W^{p,-n}(\mathbb{R})$. Indeed, the completion of $L^p(\mathbb{R})$ with respect to the norm $\| \cdot \|_{\star,n}$ is a classical characterisation of the space $W^{p,-n}(\mathbb{R})$, cf. Adams, Fournier [1, 3.13 on p. 64]. From Lemma 3 we now get that $X_{-n} = W^{p,-n}(\mathbb{R})$. Finally, Theorem 5 implies that

$$X_{-\infty} = \mathrm{ind}_{n\in\mathbb{N}} W^{p,-n}(\mathbb{R}) = \mathcal{D}'_{L^p}(\mathbb{R})$$

since $\mathrm{proj}_{n\in\mathbb{N}}(X_n^d)' = \mathrm{proj}_{n\in\mathbb{N}} W^{q,n}(\mathbb{R}) = \mathcal{D}_{L^q}(\mathbb{R})$ and $\mathcal{D}'_{L^p}(\mathbb{R}) = (\mathcal{D}_{L^q}(\mathbb{R}))'$. The case $p = 2$ is the classical Sobolev scale considered in the introduction.

Unfortunately, the question of how to deal with the case of non-reflexive Banach spaces remains open and therefore other typical spaces on which for instance the shift semigroup can be studied cannot be treated yet. However, also here the universal interpolation space can appear to be a well-studied space. For example in the case $X = \mathcal{C}_0(\mathbb{R})$ and $A = \frac{\mathrm{d}}{\mathrm{d}x}$ with $D(A) = \mathcal{C}_0^1(\mathbb{R})$, we get $X_n = \mathcal{C}_0^n(\mathbb{R})$ and $X_\infty = \mathcal{C}_0^\infty(\mathbb{R})$ which was again studied by Schwartz [10] under the name $\dot{\mathcal{B}}(\mathbb{R})$ as a predual of $\mathcal{D}'_{L^1}(\mathbb{R})$. The well known space of distributions vanishing at infinity is the space

$$\dot{\mathcal{B}}'(\mathbb{R}) = \overline{\mathcal{E}'(\mathbb{R})}^{\mathcal{D}'_{L^\infty}},$$

whose dual is $\mathcal{D}_{L^1}(\mathbb{R})$, see e.g. [3, Proposition 7]. Note that $\dot{\mathcal{B}}'(\mathbb{R})$ is not the dual space of $\dot{\mathcal{B}}(\mathbb{R})$. It seems to be natural to conjecture that $\dot{\mathcal{B}}'(\mathbb{R})$ coincides with the universal extrapolation space $X_{-\infty}$.

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