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RESEARCH ARTICLE

Properties of Sequences Generated by Summing the Digits of Cubed Positive Integers

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ABSTRACT

Having established some properties of sequences generated by summing the digits of squared positive integers (Okagbue et al, 2015), we go a step further to explore the properties and characteristics of sequences generated by summing the digits of cubed positive integers. The results are different from summing the digits of squared positive integers. Two distinct sequences were obtained: one generated by summing the digits of cubed positive integers and the other sequence as the complement of the first but the domain remains the positive integers. The properties of these two sequences are discussed. The properties include their decompositions, subsequences, algebraic, additive, multiplicative, divisibility, uniqueness and ratios.

Keywords: Sequence, cube, digits, subsequence, positive integers.

INTRODUCTION

Mathematically, a cube of a number is its third power or when the number is raised to power of positive 3. It can also be defined as when a number is multiplied by its square. The positive perfect cube formed a sequence which can be found in A000578 –OEIS. 1,8,27,64,125,216,343,512,729,1000,… (A)

Hardy and Wright (1980) book on number theory contained some details on cube integers or numbers. Sum of digits of some sequence of integers have appeared in scientific literature such as Cilleruelo et al, (2013) and Allouche and Shallit (2000). Some cubes are also squares; examples are 64, 729, 4096……. (B)

Some relationships between sums of consecutive cubed integers and squared integers were given by Pletser (2015). See Madachy (1979) for relationship between sum of digits of numbers and Narcissistic numbers. Broughan (2003) showed that every integer has a smallest multiple which is a sum of two cubes. Luca (2006) looked at various arithmetic properties of positive integers with fixed digit sum.

RESEARCH MOTIVATION AND OBJECTIVES

The research is to give another direction to the earlier research by Lietzmann (1948). In his book, he discovered that a cube number is the third power of its digit sum and a number is the sum of the third power of its digits. Both Lietzmann and Hardy (1993) researched on the sums of the cubes of digits while this research is the sum of the digits of cube of positive integers. This research is to show that the sum of the digits of cube positive integers generate sequence. The sequence and its properties are subject of interest in this research. Okagbue et al (2015) have written on the properties of sequences generated by summing the digits of square positive integers.

METHODOLOGY

The first 16,000 positive integers or natural numbers were cubed and their respective individual digits were summed to obtain a sequence with varying frequencies. The sequence is:
1,8,9,10,17,18,19,26,27,28,35,36,37,44,45,46,53,54,55,62,63,64,71,72,… (C)

However, when a positive number is cubed and the digits summed, the following numbers cannot be obtained which also formed a sequence. The sequence is:
2,3,4,5,6,7,11,12,13,14,15,16,20,21,22,23,24,25,29,30,31,32,33,34,… (D)

RESULTS AND DISCUSSION

PROPERTIES OF SEQUENCES (C) AND (D) 1.Decomposition **of sequence C**

Apart from the first term of sequence C, the other terms appear in the form

8	9	10	
17	18	19	
26	27	28	
35	36	37	and so on.

All the three columns formed some unique sequences.
8,17,26,35,44,53,62,71,80,89,98,… (E)
9,18,27,36,45,54,63,72,81,90,99 (F)
10,19,28,37,46,55,64,73,82,91,100,… (G)
All the three sequences have 9 as their common difference; Sequence F is the multiples of 9.

Decomposition of sequence D

The terms of sequence D appears in the form;

2	3	4	5	6	7	
11	12	13	14	15	16	
20	21	22	23	24	25	
29	30	31	32	33	34	and so on.

The six columns formed unique sequences.

2, 11, 20, 29, 38, 47, 56, 65, 74, 83, 92, … (H)
3, 12, 21, 30, 39, 48, 57, 66, 75, 84, 93, … (I)
4, 13, 22, 31, 40, 49, 58, 67, 76, 85, 94, … (J)
5, 14, 23, 32, 41, 50, 59, 68, 77, 86, 95, … (K)
6, 15, 24, 33, 42, 51, 60, 69, 78, 87, 96, … (L)
7, 16, 25, 34, 43, 52, 61, 70, 79, 88, 97, … (M)

All the six sequences have 9 as their common difference; Sequence I and L are multiples of 3.

Algebraic properties of sequences (C) and (D)

(a). $(C) \cap (D) = \phi$
(b). $(C) \cup (D) - \mathbb{N} - \mathbb{Z}^{+}$
$(C) \cup (D) \cup 0 \cup (\ C) \cup (\ D) = \mathbb{Z}$

Subsequences of sequence (C)

Each of the elements of sequence C also form a unique sequence. For an illustration, the first 100 positive integers can be grouped based on the numbers in sequence C. This is represented in in **figure 1.** It is also like a histogram. The values are clustered in the center. .

Additive properties of sequence C

Addition oftwo numbers ofsequence C will not necessarily yield a number inthe sequence.This is illustrated in **table 1**

Additive properties of sequence D

Addition of two numbers of sequence D will not necessarily yield a number in the sequence. This is illustrated in **table 2.**

Addition of elements of both sequences C and D

Addition of one element of sequence C and another element of sequence D will most likely yield an element of sequence D as illustrated in **table 3.**

Absolute value of the differences of any two elements of sequence C

Finding the difference of any two elements of sequence C and then obtain the corresponding absolute value will not necessarily yield an element of sequence C. This is illustrated in **table 4.**

Absolute value of the differences of any two elements of sequence D

Finding the difference of any two elements of sequence D and then obtain the corresponding absolute value will not necessarily yield an element of sequence D. This is illustrated in **table 5.**

Absolute value of the difference of elements of both sequences C and D

Find the difference of any one element of sequence C and another from sequence D and then obtain the corresponding absolute value will not necessarily yield an irregular pattern of numbers as illustrated in **table 6.** This is evidence that the sequences have no element common to them.

Multiplicative properties of sequence C

Multiplication of two numbers of sequence C yields a number in the sequence. This is illustrated in **table 7.**

Multiplicative properties of sequence D

Multiplication of two numbers of sequence D will not necessarily yield a number in the sequence. This is illustrated in **table 8.**

Multiplication of elements of both sequences C and D

Multiply of one element of sequence C and another element of sequence D will most likely yield an element of sequence D as illustrated in **table 9.**Elements of sequence C can be obtained only when the multiples of 9 in the sequence are multiply with the any element of sequence D.

Divisibility properties

1. All multiples of 9 are contained in sequence C.
2. Squares of the multiples of 3 are also contained in sequence C. The sequence is:
1,9,36,81,144,225,324,441,576,… (N)
3. All two consecutive elements of sequence C are coprime.
4. No clear multiples of numbers can be obtained from sequence D.

The observed uniqueness of sequences C and D

Any form of general additions to or subtractions from all the elements of both sequences yield different sequences.

The line graph of sequences C and D

A line graph was used to graphically display the behavior of both sequences. The first 25 terms of both sequences were used and can be seen in **figure 2.** Both sequences increase steadily in a step-like way as the number of terms increases. The behaviors are similar but sequences C have higher values than sequence D

The ratio of sequence C

A new sequence can be obtained from the ratio of the two consecutive terms of sequence

$$C.\frac{8}{1},\frac{9}{8},\frac{10}{9},\frac{17}{10},\frac{18}{17},\frac{19}{18},\frac{26}{19},\frac{27}{26},\ldots \quad (O)$$

The behavior of the sequence can be seen in **figure 3.** The sequence obtained from the ratio of sequence C converges to almost one.The closed form solution of the ratio can be written as:

$$\varphi = \frac{4}{5}\left[\frac{1+\sqrt{5}}{2}\right] \approx 1.298387$$

To see a clearer view of the graph, 8 which is first term of the sequence is removed and the corresponding line graph can be seen in **figure 4.**The first term is an extreme value. It can be seen that it is a decreasing as the number of terms increases. The ratio converges to 1.

The ratio of sequence D

A new sequence can be obtained from the ratio of the two consecutive terms of sequence C

$$\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{11}{7}, \frac{12}{11}, \frac{13}{12}, \ldots \quad \text{(P)}$$

The behavior of the sequence can be seen in **figure 5** The sequence obtained from the ratio of sequence D converges to almost one. The closed form solution of the ratio can be written as:

$$\varphi = \left[\frac{7}{10}\right]^3 (1 + \sqrt{5}) \approx 1.10936$$

The subsequences obtained from the various factors or multiples of sequences C and D

Some subsequences are obtained by the various factors or multiples of both sequences such as 2n, 3n, 4n... For all cases in this section, the first sequence is for C and the second is for sequence D.

Multiples of two

A subsequence is obtained for both sequences. That is the 2nd, 4th, 6th….terms of both sequences

$$8, 10, 18, 26, 28, 36, 44, 46, 54, 62, 64, 72, 80, 82, 90, \ldots \quad \text{(Q1)}$$

$$3, 5, 7, 12, 14, 16, 21, 23, 25, 30, 32, 34, 39, 41, 43, \ldots \quad \text{(Q2)}$$

Multiples of three

A subsequence is obtained for both sequences. That is the 3rd, 6th, 9th, ….terms of both sequences

$$9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, \ldots \quad \text{(Q3)}$$

$$4, 7, 13, 16, 22, 25, 31, 34, 40, 43, 49, \ldots \quad \text{(Q4)}$$

Multiples of four

A subsequence is obtained for both sequences. That is the 4th, 8th, 12th,…terms of both sequences

$$10, 26, 36, 46, 62, 72, 82, 98, 108, 118, \ldots \quad \text{(Q5)}$$

$$5, 12, 16, 23, 30, 34, 41, 48, 52, 59, 66, \ldots \quad \text{(Q6)}$$

Multiples of five

A subsequence is obtained for both sequences. That is the 5th, 10th, 15th,..terms of both sequences

$$17, 28, 45, 62, 73, 90, 107, 118, 135, 152, \ldots \quad \text{(Q7)}$$

$$6, 14, 22, 30, 38, 43, 51, 59, 67, 75, 83, 84 \ldots \quad \text{(Q8)}$$

Prime multiples

A subsequence is obtained for both sequences. That is the 2nd, 3rd, 5th,..terms of both sequences.

$$8, 9, 17, 19, 35, 37, 53, 55, 71, 89, 91, 109, \ldots \quad \text{(Q9)}$$

$$3, 4, 6, 11, 15, 20, 24, 29, 33, 42, 47, 56, \ldots \quad \text{(Q10)}$$

The sequence C / D

A new sequence can be obtained from the ratio of sequence C to sequence D.

$$\frac{1}{2}, \frac{8}{3}, \frac{9}{4}, \frac{10}{5}, \frac{17}{6}, \frac{18}{7}, \frac{19}{11}, \frac{26}{12}, \ldots \qquad \text{(R)}$$

The sequence converges to 2 and the line graph is shown in **figure 6.** The closed form solution of the sequence C/D can be written as::

$$\varphi = \frac{639}{500}\left[\frac{1+\sqrt{5}}{2}\right] \approx 2.068329$$

The sequence D / C

A new sequence can be obtained from the ratio of sequence D to sequence C.

$$\frac{2}{1}, \frac{3}{8}, \frac{4}{9}, \frac{5}{10}, \frac{6}{17}, \frac{7}{18}, \frac{11}{19}, \frac{12}{26}, \ldots \qquad \text{(S)}$$

The sequence converges to 0.5 and the line graph is shown in **figure 6.** The closed form solution of the sequence D/C can be written as:

$$\varphi = \frac{81}{250}\left[\frac{1+\sqrt{5}}{2}\right] \approx 0.524275$$

The sequence Q1/Q2A

sequence can be generated from the ratio of sequence Q1 to Q2, since both are subsequences formed from the multiples of two of the sequences C and D respectively.

$$\frac{8}{3}, \frac{10}{5}, \frac{18}{7}, \frac{26}{12}, \frac{28}{14}, \frac{36}{16}, \frac{44}{21}, \frac{46}{23}, \frac{54}{25}, \ldots \qquad \text{(T)}$$

The sequence converges to 2 and the line graph is shown in **figure 8.** The closed form solution of the sequence Q1/Q2 can be written as:

$$\varphi = \frac{323}{250}\left[\frac{1+\sqrt{5}}{2}\right] \approx 2.091256$$

These results are similar to the results of the sequence C/D, an indication that subsequences exhibit similar characteristics with their parent sequences.

The sequence Q2/Q1

A sequence can be generated from the ratio of sequence Q2 to Q1, since both are subsequences formed from the multiples of two of the sequences D and C respectively.

$$\frac{3}{8}, \frac{5}{10}, \frac{7}{18}, \frac{12}{26}, \frac{14}{23}, \frac{16}{36}, \frac{21}{44}, \frac{23}{46}, \frac{25}{54}, \ldots \qquad \text{(U)}$$

The sequence converges to 0.5 and the line graph is shown in **figure 8.**
The closed form solution of the sequence Q2/Q1 can be written as:

$$\varphi = \frac{37}{125}\left[\frac{1+\sqrt{5}}{2}\right] \approx 0.480417$$

These results are similar to the results of the sequence C/D, an indication that subsequences exhibit similar characteristics with their parent sequences but with slight variability as seen from this research.

CONCLUSION

The various observed properties of sequences generated by summing the digits of cube positive integers have been considered. More research is needed to discover more properties of the sequences. Also the subsequences exhibit similar characteristics with their parent sequences.

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1	1 10 100
8	2 5 8 11 20 50 80
9	3 6 30 60
10	4 7 40 70
17	14 17 23 47 68 74
18	9 12 15 18 21 24 45 48 51 63 81 90
19	13 16 22 25 28 34 37 52 58 64 67 85
26	26 29 32 35 38 41 44 56 59 62 65 71 77 86 98
27	27 33 36 39 42 54 57 69 72 75 78 84 87 93
28	19 31 43 46 49 55 61 73 79 82 88 91 94 97
35	53 83 89 95
36	66 96 99
37	76
44	92
45	

Figure 1 The subsequences of C for the first 100 positive integers.

Table 1 Addition of numbers of sequence C

$a+b$	1	8	9	10	17	18	19	26	…
1	2	9	10	11	18	19	20	27	…
8	9	16	17	18	25	26	27	34	…
9	10	17	18	19	26	27	28	35	…
10	11	18	19	20	27	28	29	36	…
17	18	25	26	27	34	35	36	43	…
18	19	26	27	28	35	36	37	44	…
19	20	27	28	29	36	37	38	45	…
.	.	.	.	.	.	.	.	.	…

Table 2. Addition of numbers of sequence D.

$a+b$	2	3	4	5	6	7	11	12	…
2	4	5	6	7	8	9	13	14	…
3	5	6	7	8	9	10	14	15	…
4	6	7	8	9	10	11	15	16	…
5	7	8	9	10	11	12	16	17	…
6	8	9	10	11	12	13	17	18	…
7	9	10	11	12	13	14	18	19	…
11	13	14	15	16	17	18	22	23	…
.	.	.	.	.	.	.	.	.	…

Table 3. Addition of numbers of both sequence C and D.

$a+b$	2	3	4	5	6	7	11	12	…
1	3	4	5	6	7	8	12	13	…
8	10	11	12	13	14	15	19	20	…
9	11	12	13	14	15	16	20	21	…
10	12	13	14	15	16	17	21	22	…
17	19	20	21	22	23	24	28	29	…
18	20	21	22	23	24	25	29	30	…
19	21	22	23	24	25	26	30	31	…
.	.	.	.	.	.	.	.	.	…

Table 4. Absolute value of the differences of any two elements of sequence C

$\|a-b\|$	1	8	9	10	17	18	19	26	…
1	0	7	8	9	16	17	18	25	…
8	7	0	1	2	9	10	11	18	…
9	8	1	0	1	8	9	10	17	…
10	9	2	1	0	7	8	9	16	…
17	16	9	8	7	0	1	2	9	…
18	17	10	9	8	1	0	1	8	…
19	18	11	10	9	2	1	0	7	…
.	.	.	.	.	.	.	.	.	…

Table 5Absolute value of the differences of any two elements of sequence D

$\|a-b\|$	**2**	**3**	**4**	**5**	**6**	**7**	**11**	**12**	**…**
2	0	1	2	3	4	5	9	10	…
3	1	0	1	2	3	4	8	9	…
4	2	1	0	1	2	3	7	8	…
5	3	2	1	0	1	2	6	7	…
6	4	3	2	1	0	1	5	6	…
7	5	4	3	2	1	0	4	5	…
11	9	8	7	6	5	4	0	1	…
.	.	.	.	.	.	.	.	.	…

Table 6 Absolute value of the difference of elements of both sequences C and D

$\|a-b\|$	**2**	**3**	**4**	**5**	**6**	**7**	**11**	**12**	**…**
1	1	2	3	4	5	6	10	11	…
8	6	5	4	3	2	1	3	4	…
9	7	6	5	4	3	2	2	3	…
10	8	7	6	5	4	3	1	2	…
17	15	14	13	12	11	10	6	5	…
18	16	15	14	13	12	11	7	6	…
19	17	16	15	14	13	12	8	7	…
.	.	.	.	.	.	.	.	.	…

Table 7 Multiplication of any 2 numbers of sequence C

$a \times b$	**1**	**8**	**9**	**10**	**17**	**…**
1	1	8	9	10	17	…
8	8	64	72	80	136	…
9	9	72	81	90	153	…
10	10	80	90	100	170	…
17	17	136	153	170	289	…
18	18	144	162	180	306	…
.	.	.	.	.	.	…

Table 8 Multiplication of any 2 numbers of sequence D

$\|a \times b\|$	**2**	**3**	**4**	**5**	**6**	**7**	**11**	**12**	**…**
2	4	6	8	10	12	14	22	24	…
3	6	9	12	15	18	21	33	36	…
4	8	12	16	20	24	28	44	48	…
5	10	15	20	25	30	35	55	60	…
6	12	18	24	30	36	42	66	72	…
7	14	21	28	35	42	49	77	84	…
11	22	33	44	55	66	77	121	132	…
.	.	.	.	.	.	.	.	.	…

Table 9 Multiplication of elements of both sequences C and D

$a \times b$	**2**	**3**	**4**	**5**	**6**	**7**	**11**	**12**	**…**
1	2	3	4	5	6	7	11	12	…
8	16	24	32	40	48	56	88	96	…
9	18	27	36	45	54	63	99	108	…
10	20	30	40	50	60	70	110	120	…
17	34	51	68	85	85	119	187	204	…
18	36	54	72	90	90	126	198	216	…
.	.	.	.	.	.	.	.	.	…

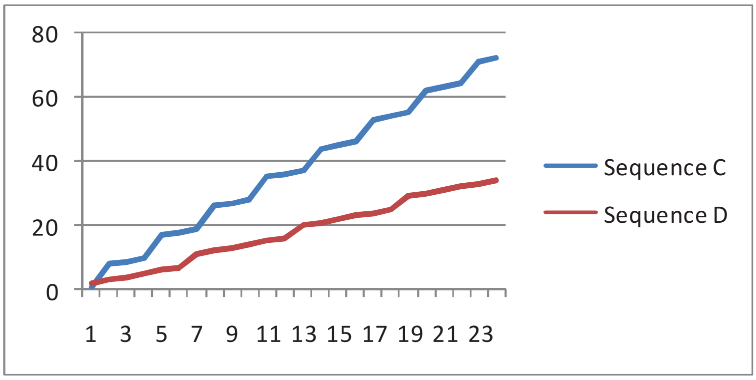

Figure 2 The line graph of sequences C and D.
x axis = the number of terms y axis = the values of the terms in both sequences

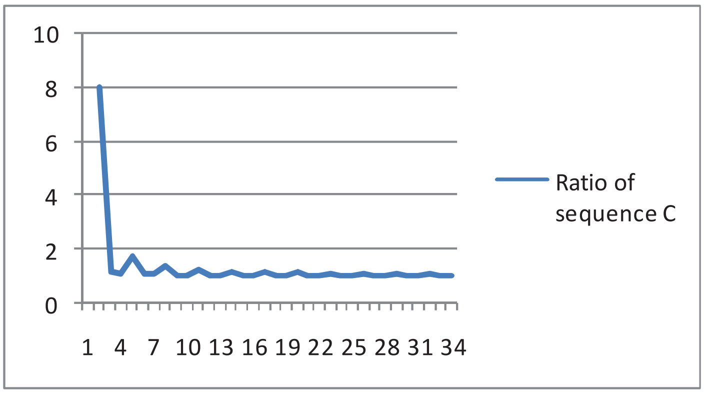

Figure 3 The ratio of sequence C
x axis = terms of the sequence y axis = ratio of 2 consecutive numbers of sequence C

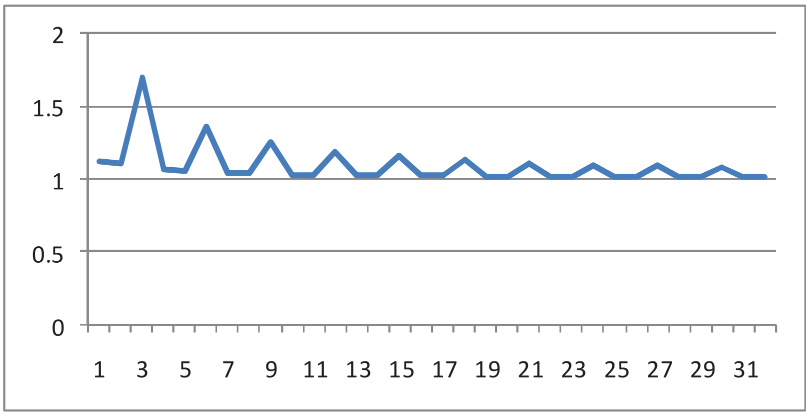

Figure 4 The ratio of sequence C with the first term removed
x axis = terms of the sequence y axis = ratio of 2 consecutive numbers of sequence C

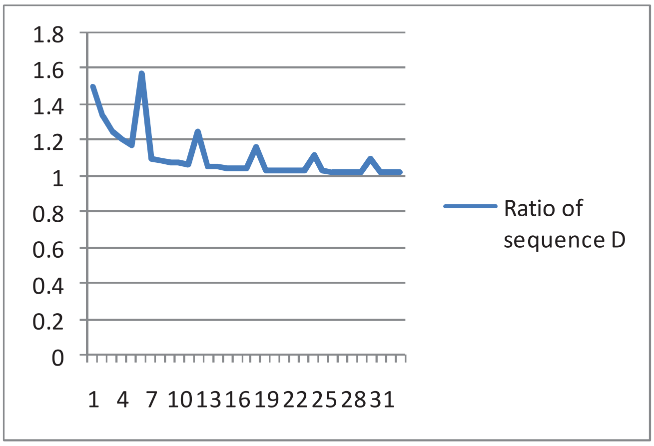

Figure 5The ratio of sequence D
x axis = terms of the sequence y axis = ratio of 2 consecutive numbers of sequence D

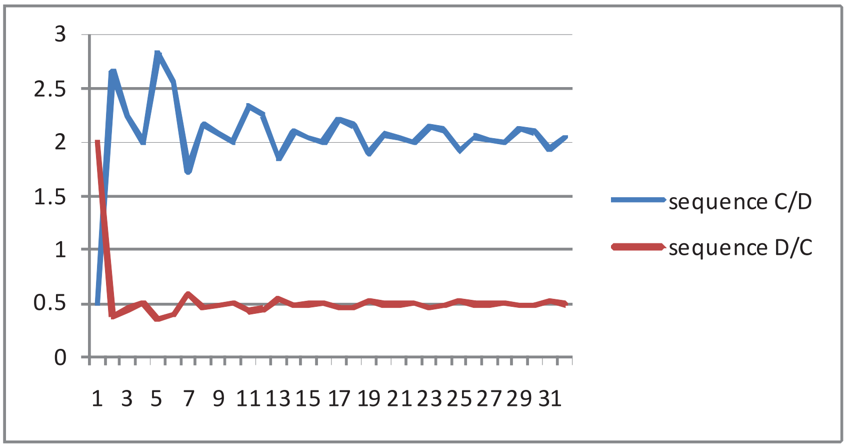

Figure 6 The sequences C/D and D/C
x axis = terms of both sequences y axis = sequences C/D and D/C.

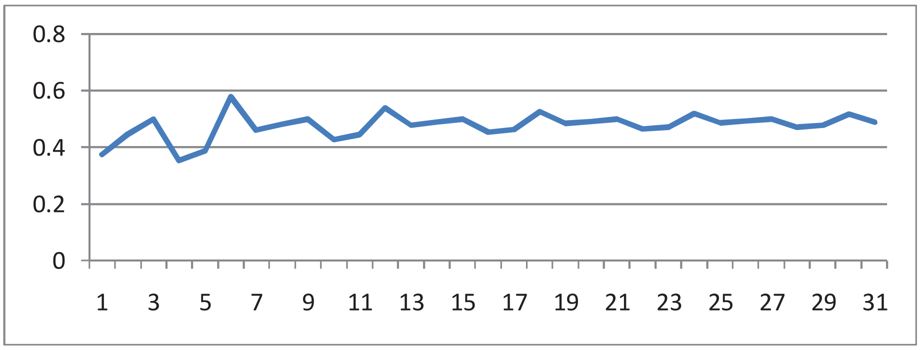

Figure 7 The sequence D/ C with the first term removed
x axis = terms of the sequence y axis = sequence D.C with first term removed.

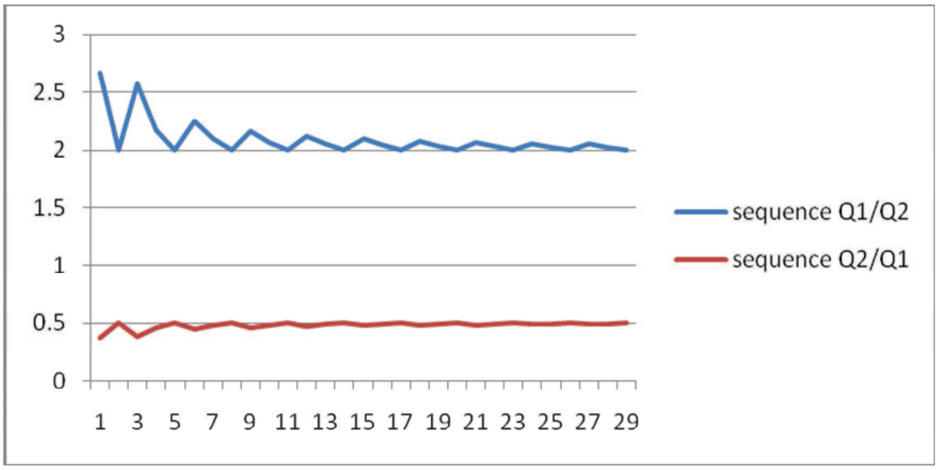

Figure 8 The sequences Q1/Q2 and Q2/Q1
x axis = terms of both sequences y axis = sequences Q1/Q2 and Q2/Q1.