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A Note on the Minimax Distribution

Oguntunde, P. E[1], Adejumo, A. O[2]

[1]Department of Mathematics,
Covenant University, Ota, Ogun State, Nigeria

[2]Department of Statistics,
University of Ilorin, Kwara State, Nigeria

Abstract: We introduce a one parameter probability model bounded on (0, 1) support called *One Parameter Minimax distribution* which is a special case of both the Kumaraswamy distribution and Beta distribution. Its statistical properties are systematically explored; we provide explicit expressions for its moments, quantile function, reliability function and failure rate. The method of maximum likelihood estimation was used in estimating its parameter. The proposed model can be used to model data sets with increasing failure rates.

Keywords: Beta distribution, Increasing failure rate, Kumaraswamy distribution, Minimax distribution.

1.0 Introduction

According to [4] and [6], "Despite the many alternatives and generalizations, it remains fair to say that the beta distribution provides the premier family of continuous distributions on bounded support (which is taken to be (0, 1))". Let X denote a non-negative continuous random variable that follows a beta distribution with parameters 'a' and 'b', we write; $X \sim Beta(a,b)$, the probability density function is given by;

$$f(x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}; \quad x\in(0,1) \qquad (1)$$

Where $a,b>0$ are shape parameters

Besides lot of other attractive properties, the shape of beta densities could be unimodal, uniantimodal, constant, increasing or decreasing depending on the values of parameters 'a' and 'b' relative to 1 ; See [1] for details. It is good to note that the cumulative density function of the beta distribution involves an incomplete beta function; this makes the distribution not to be very tractable.

[5] introduced the Kumaraswamy distribution and many authors have explored some of the properties of this distribution. Noticeably is the work of [2] who described the distribution as a Minimax distribution sharing many desirable properties with the beta distribution. The Minimax distribution has been

described as a viable alternative to the beta distribution and its basic properties reflects that it is more tractable compared to the beta distribution.

For a random variable X having Minimax distribution with parameters ‘a’ and ‘b’, the probability density function (pdf) is given by;

$$f(x) = abx^{a-1}\left(1-x^{a}\right)^{b-1} \; ; \; x \in (0,1) \quad (2)$$

The corresponding cumulative density function (cdf) is given by;

$$F(x) = 1-\left(1-x^{a}\right)^{b} \; ; \; x \in (0,1) \quad (3)$$

Where $a,b > 0$ are shape parameters

We refer readers to [7] and [8] for some details on the Minimax distribution.

This article seeks to extend the notable work of [3] by exploring a special case of both the beta distribution and the Minimax distribution when we assume one of the shape parameters $'a'$ to equal 1, we also investigate some basic statistical properties of the resulting model.

The rest of this article is organized as follows; section two discusses how the new model was derived including derivations of some of its basic properties, section three discusses the estimation of the model parameter using the method of maximum likelihood estimation, followed by a concluding remark.

2.0 The Proposed Model

Substituting $a = 1$ in both Equation (1) and (2) gives the pdf of the proposed model to be;

$$f(x) = b(1-x)^{b-1} \; ; \; x \in (0,1) \quad (4)$$

Thus, we easily derive its cdf to give;

$$F(x) = 1-(1-x)^{b} \; ; \; x \in (0,1) \quad (5)$$

Where $b > 0$ is a shape parameter

2.1 The new model is a valid probability density function

To proof that the model in Equation (4) is a valid pdf, we show that integrating Equation (4) with respect to x from 0 to 1 equals 1. That is,

$$\int_0^1 b(1-x)^{b-1}dx = 1$$

Let $u = (1-x)$; then, $dx = -du$

Therefore,

$$\int_0^1 b(1-x)^{b-1}dx = -b\int_0^1 u^{b-1}du$$

After simple calculations,

$$\int_0^1 b(1-x)^{b-1}dx = 1$$

This completes the proof.

We also provide graphical representations of the pdf of the model and at various values of parameter $'b'$. For brevity purpose, we present plots for the pdf at $'b = 1'$, $'b = 4'$ and at $'b = 20'$ in Figure 1, 2 and 3 respectively.

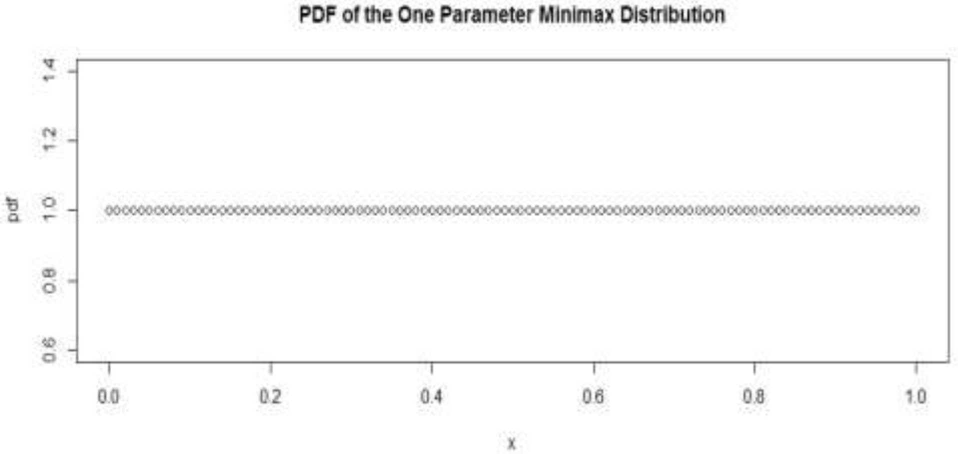

Fig. 1: The pdf of the proposed model when parameter $b = 1$

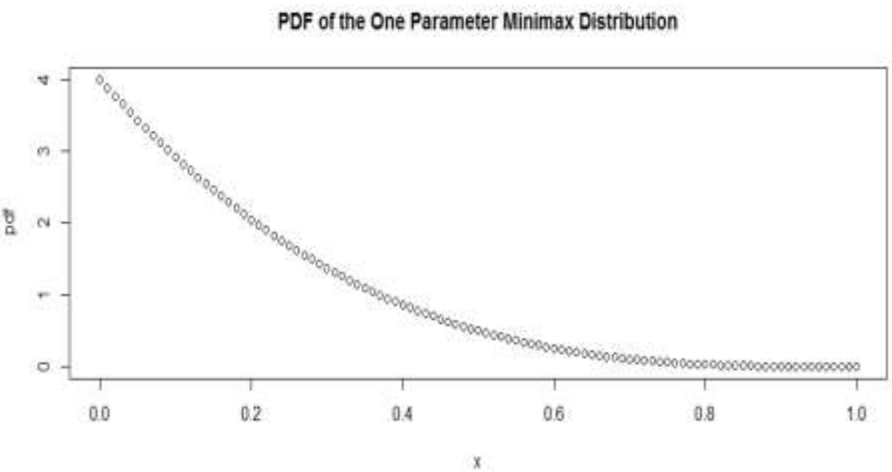

Fig. 2: The pdf of the proposed model when parameter $b = 4$

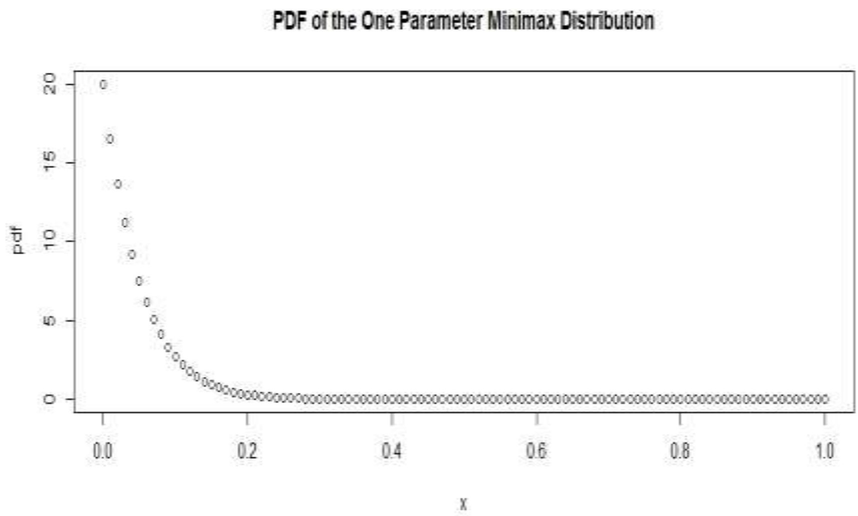

Fig. 3: The pdf of the proposed model when parameter $b = 20$

We observe in Figure 1 that the shape of the model could be "constant" and both plots in Figure 2 and 3 indicate that as the value of 'x' increases, the curve decreases. Hence, we can also say the shape of the model could be "decreasing". A plot for the cdf of the model is as shown in Figure 4;

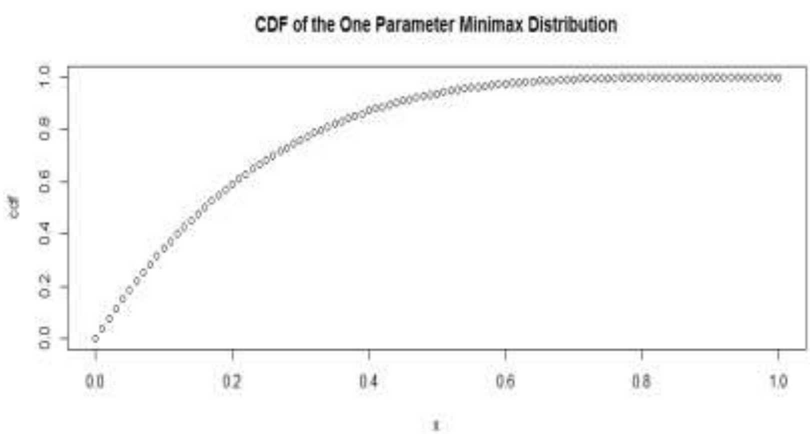

Fig. 4: The cdf of the model when parameter $'b = 4'$

2.2 Some properties of the new model

In this section, we provide some desirable properties of the new model, beginning with its moments.

2.2.1 Moments

The rth moment for a random variable X denoted by μ_r is given by;

$$\mu_r = E\left[X^r\right]$$

Hence, the rth moment for the new model in Equation (4) is given by;

$$\mu_r = b\int_0^1 x^r (1-x)^{b-1}\,dx$$

After simple calculations, we obtain the rth moment as;

$$\mu_r = bB(r+1,b) \qquad (6)$$

We can simply re-write Equation (6) as;

$$\mu_r = \frac{\Gamma(r+1)\Gamma(b+1)}{\Gamma(b+r+1)} \qquad (7)$$

Note that, $\Gamma(k) = (k-1)!$

where; $k > 0$ is an arbitrary constant.

The mean, μ_1 is easily gotten as;

$$Mean = \frac{1}{b+1} \qquad (8)$$

The variance is given by;

$$Var(X) = bB(3,b) - \left[bB(2,b)\right]^2$$

$$Var(X) = \frac{b}{(b+1)^2(b+2)} \qquad (9)$$

With the expression given in Equation (6), other higher-order moments can be obtained.

2.2.2 The Quantile Function and Median

The quantile function Q is an alternative to the pdf as it is a way of prescribing a probability distribution and it can be derived as the inverse of the cdf. With this understanding, we give the expression for the quantile function of the new model as;

$$Q(u) = 1-(1-u)^{1/b} \qquad (10)$$

The median of the model can be obtained by substituting $u = 0.5$ into Equation (10). Therefore, we have;

$$Median = 1-(1-0.5)^{1/b}$$

This can be simplified to give;

$$Median = 1-(0.5)^{1/b} \qquad (11)$$

For simulation purposes, random variables from the new model can be obtained using the expression;

$$X = 1-(1-u)^{1/b} \qquad (12)$$

where $U \sim Uniform(0,1)$

2.2.3 Reliability Analysis

The reliability (survival function) for the model in Equation (4) is derived by;

$$S(x) = 1-F(x)$$

$$= 1-\left[1-(1-x)^b\right]$$

$$S(x) = (1-x)^b \qquad (13)$$

The corresponding plot for the survival function of the model is as shown in Figure 5;

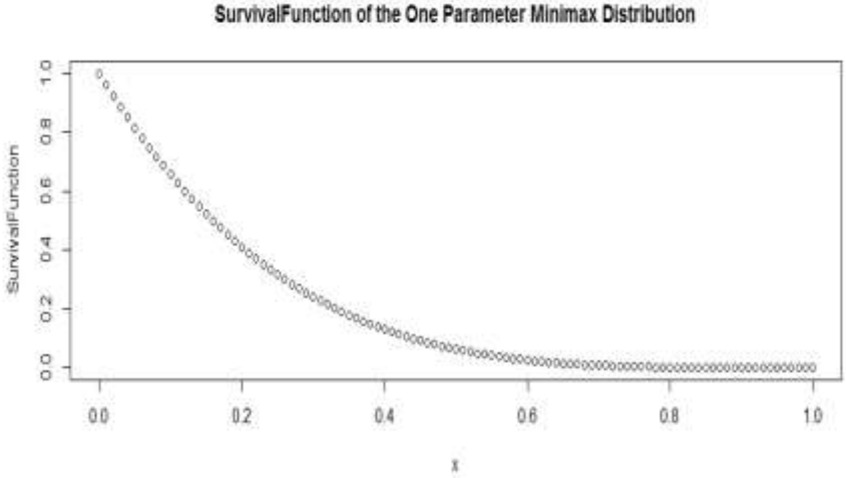

Fig. 5: The Survival Function of the model when parameter $'b = 4'$

The probability that a system having age $'x'$ units of time will survive up to $'x+t'$ units of time for $x > 0, b > 0$ is given by;

$$S(t \mid x) = \frac{S(x+t)}{S(x)}$$

$$S(t \mid x) = \frac{\left(1-\left(x+t\right)\right)^b}{\left(1-x\right)^b} \qquad (14)$$

The corresponding hazard function is thus given by;

$$h(x) = \frac{f(x)}{1-F(x)}$$

$$= \frac{b\left(1-x\right)^{b-1}}{1-\left[1-\left(1-x\right)^b\right]}$$

$$h(x) = \frac{b}{\left(1-x\right)} \qquad (15)$$

We obtain plots for the hazard function of the model at various parameter values but the plot at $b = 4$ is as shown in Figure 6;

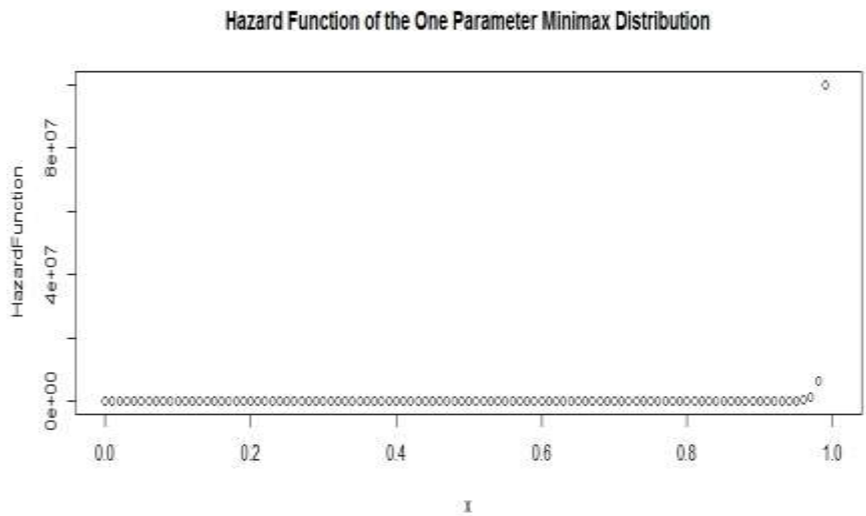

Fig. 6: The Hazard Function of the model when $b = 4$

Figure 6 shows that the proposed model has an increasing hazard rate.

3.0 Estimation

In an attempt to estimate the shape parameter of the proposed model, we employ the use of the method of the Maximum Likelihood Estimation (MLE). Let $X_1, X_2, ..., X_n$ be a random sample of 'n' independently and identically distributed random variables each distributed according to the model defined in Equation (4), the likelihood function L is given by;

$$L(x_1, x_2, ..., x_n \mid b) = \prod_{i=1}^{n}\left[b(1-x)^{b-1}\right]$$

$$= b^n \sum_{i=1}^{n}(1-x_i)^{b-1}$$

Let $l = \log L(x_1, x_2, ..., x_n \mid b)$

$$l = n\ln b + (b-1)\sum_{i=1}^{n}\ln(1-x_i)$$

Differentiating l with respect to 'b' and equating the result to zero gives;

$$\frac{\partial l}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\ln(1-x_i)$$

Hence;

$$\sum_{i=1}^{n}\ln(1-x_i) = -\frac{n}{b}$$

$$b = -\frac{n}{\sum_{i=1}^{n}\ln(1-x_i)} \qquad (16)$$

4.0 Conclusion

This article presents a model called One parameter Minimax distribution which was obtained as a special case of both the Beta distribution and the Kumaraswamy distribution. The resulting model is bounded on a (0, 1) support and its shape of the model could be "constant" or "decreasing" (depending on the value of the parameter). Expressions for the rth moment, mean, variance, quantile function, survival function and failure rate are provided. The method of maximum likelihood estimation (MLE) was proposed in estimating the only parameter in the model. The model shows an increasing hazard rate, thus, we propose the use of the distribution in modeling situations where risk is low at the beginning but increases with time.

References

Johnson, N. L., Kotz, S., and Balakrishnan, N. "Continuous Univariate Distributions", Vol. 2, 2nd Edition, New York: Wiley, 1994

Jones M. C. "The Complementary Beta Distribution", *Journal of Statistical Planning and Inference*, 104, 329-337, 2002

Jones M. C. Kumaraswamy's distribution: "A beta-type distribution with some tractability advantages". *Statistical Methodology*, Vol. 6, 70-81, 2009

Kotz, S., and Van Dorp, J. R. "Beyond Beta; Other Continuous Families of Distributions with Bounded Support and Applications", *World Scientific Press, New Jersey, U.S.A.*, pp: 289, 2004

Kumaraswamy, P. "A generalized probability density function for double-bounded random processes". *Journal of Hydrology*, Vol. 46, 79-88, 1980

Nadarajah, S., and Gupta, A. K. "Generalizations and Related Univariate Distributions," in "Hanbook of Beta Distribution and Its Applications, eds. A.K. Gupta, and S. Nadarajah", *Dekker, New York, U.S.A.*, pp. 97-163, 2004

Oguntunde, P. E., Babatunde, O. S., Ogunmola, A. O. "Theoretical Analysis of the Kumaraswamy-Inverse Exponential Distribution" *International Journal of Statistics and Applications*, Vol. 4, No. 2, 113-116, 2014

Oguntunde, P. E, Odetunmibi O. A, Okagbue H. I, Babatunde O. S, Ugwoke P. O. "The Kumaraswamy-Power Distribution: A Generalization of the Power Distribution" *International Journal of Mathematical Analysis*, Vol. 9, no. 13, 637-645, 2015.