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On A New Weighted Exponential Distribution: Theory and Application

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ABSTRACT

A new two-parameter weighted exponential distribution which has more mild algebraic properties than the existing weighted exponential distribution was studied. Explicit expressions for some of its basic statistical properties including moments, reliability analysis, quantile function and order statistics were derived. Its parameters were estimated using the method of maximum likelihood estimation. The new probability model was applied to four real data sets to assess its flexibility over the existing weighted exponential distribution.

Key words: Maximum likelihood estimation, order statistics, parameters, properties, alkaike information criteria, weighted weibull distribution

INTRODUCTION

The weighted exponential distribution being a competitor to the Weibull, Gamma and Generalized exponential distributions has received appreciable usage in the fields of engineering and medicine. It was proposed by Gupta and Kundu (2009) as a generalization of the exponential distribution and its probability density function (pdf) is defined by Eq. 1:

$$f(x) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left[1-e^{-\lambda\alpha x}\right] \quad (1)$$

For x>0, α>0, λ>0

The corresponding cumulative density function (cdf) is given by Eq. 2:

$$F(x) = \frac{\alpha+1}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}\left(1-e^{-\lambda x(1+\alpha)}\right)\right] \quad (2)$$

For x>0, α>0, λ>0

where, α is the shape parameter, λ is the scale parameter.

Its hazard function is given by Eq. 3:

$$h(x) = (\alpha+1)\beta\frac{1-e^{-\alpha\beta x}}{\alpha+1-e^{-\alpha\beta x}} \quad (3)$$

For x>0, α>0, λ>0

In recent time, the weighted exponential distribution has been rigorously explored in the area of probability distribution theory. For instance, Alqallaf *et al*. (2015) used five different estimation methods (maximum likelihood, moments, L-moments, ordinary least squares and weighted least squares) to estimate the parameters of the weighted exponential distribution. Oguntunde (2015) also generalized the weighted exponential distribution using the Exponentiated family of distributions to propose the exponentiated weighted exponential distribution.

Some other weighted distributions have also been defined in the literature, for example, the weighted inverted exponential distribution, Hussian (2013), the weighted weibull distribution, Mahdy (2013) and Shahbaz *et al*. (2010), the weighted multivariate normal distribution, Kim (2008), the weighted inverse weibull distribution, Kersey (2010), a weighted three parameter weibull distribution (Essam and Mohamed, 2013).

Meanwhile, the interest of this research is to define another version of the weighted exponential distribution following the content of Nasiru (2015) who defined another version of the weighted Weibull distribution. The rest of this article is organized as follows, the new weighted exponential distribution is defined in section 2, the basic statistical properties of the new model are derived in section 3 including the estimation of model parameters. In section 4, the model is applied to four real life data sets to assess its flexibility followed by a concluding remark.

MATERIALS AND METHODS

New weighted exponential distribution: Following Nasiru (2015), a modified weighted version of Azzalini (1985) approach was used as follows:

Let g(x) be a pdf and $\bar{G}$ be the corresponding survival (or reliability) function such that the cdf, G(x) exist. The new weighted family of distribution is given by Eq. 4:

$$f(x) = Kg(x)\bar{G}(\lambda x) \tag{4}$$

where, K is a normalizing constant.

Now, make g(x) and $\bar{G}$ to represent the pdf and survival function of the exponential distribution, respectively. Then Eq. 5 and 6 will be as:

$$g(x) = \alpha e^{-\alpha x} \quad : \quad x > 0, \alpha > 0 \tag{5}$$

and

$$\bar{G}(x) = e^{-\alpha x} \quad : \quad x > 0, \alpha > 0 \tag{6}$$

Hence, the pdf of the new weighted exponential distribution is derived by substituting Eq. 5 and 6 into Eq. 4 to give Eq. 7:

$$\begin{aligned} f(x) &= k\alpha e^{-\alpha x}\left[e^{-\alpha\lambda x}\right] \\ f(x) &= k\alpha e^{-(\alpha x + \alpha\lambda x)} \\ f(x) &= k\alpha e^{-(1+\lambda)\alpha x} \quad : \quad x > 0, \alpha > 0 \end{aligned} \tag{7}$$

Equation 7 can further be written as Eq. 8:

$$f(x) = (1+\lambda)\alpha e^{-(1+\lambda)\alpha x} \quad : \quad x > 0, \alpha > 0 \tag{8}$$

where, the normalizing constant, k = (1+λ).

The corresponding cdf of the new weighted exponential distribution is given by Eq. 9:

$$F(x) = 1 - e^{-(\alpha x + \alpha\lambda x)} \quad : \quad x > 0, \alpha > 0 \tag{9}$$

where, α is a scale parameter, λ is a shape parameter.

It can be deduced immediately from Eq. 9 that, $\lim_{x\to\infty} F(x) = 1$

The pdf and cdf of the new weighted exponential distribution defined in Eq. 8 and 9, respectively, are simpler and mathematically easier to handle as compared to the existing weighted exponential pdf and cdf as defined in Eq. 1 and 2, respectively.

Possible plots for the pdf of the new weighted exponential distribution at some chosen parameter values are shown in Fig. 1 and 2.

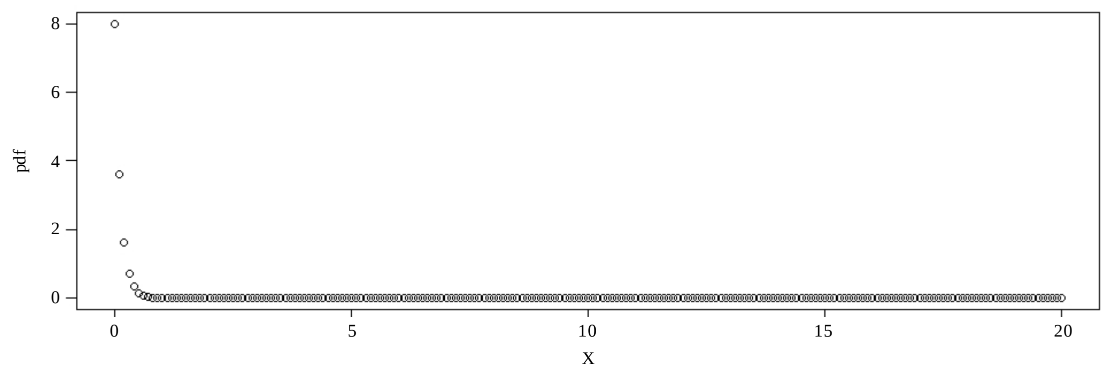

Fig. 1: Plot for the pdf at α = 2, λ = 3

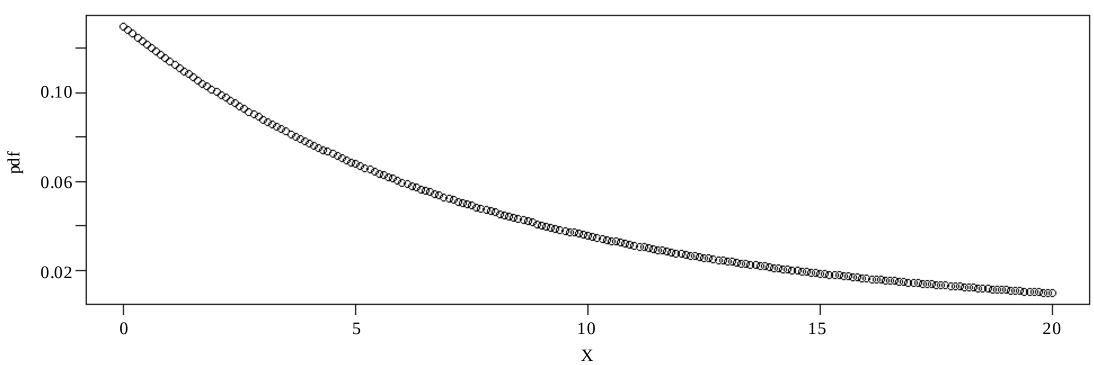

Fig. 2: Plot for the pdf at α = 0.1, λ = 0.3

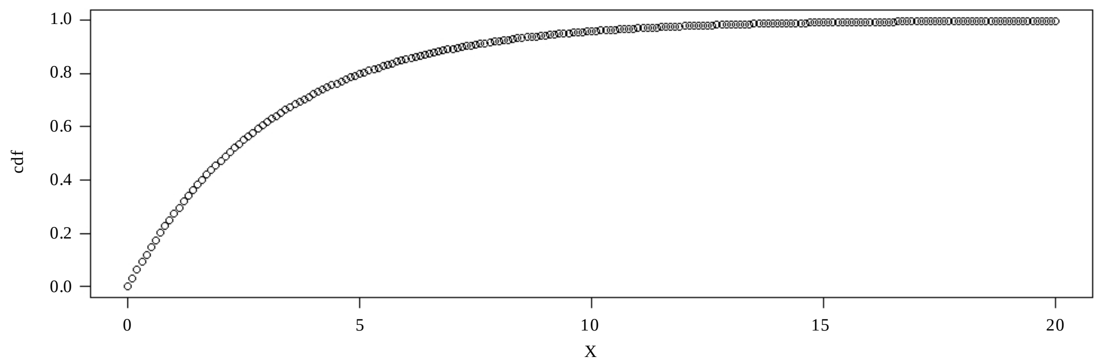

Fig. 3: Plot for the pdf at $\alpha = 0.2$, $\lambda = 0.6$

The plots in Fig. 1 and 2 indicate that as the value of 'x' increases, the curve decreases. It can be confidently concluded that the shape of proposed model is reversed J-shape or decreasing. The corresponding plot for the cdf of the new weighted exponential distribution is shown in Fig. 3.

Special case: When $\lambda = 0$, the new weighted exponential distribution reduces to give the exponential distribution.

RESULTS

In this section, some basic statistical properties of the new weighted exponential distribution are derived as follows:

Quantile function and median: The quantile function Q(u) of the new weighted exponential distribution is given by Eq. 10:

$$Q(u) = F^{-1}(u)$$

$$Q(u) = \left[\frac{\ln\left(\frac{1}{1-u}\right)}{\alpha(1+\lambda)}\right] \tag{10}$$

where, U has the uniform U(0, 1) distribution.

The median is obtained directly by substituting u = 0.5 in Eq. 10. Therefore, the median is given by Eq. 11:

$$\text{Median} = \left[\frac{\ln 2}{\alpha(1+\lambda)}\right] \tag{11}$$

Simulating from the new weighted exponential distribution is quite straight forward. Using the aids of inverse transformation method, the random variable X is given by Eq. 12:

$$X = \left[\frac{\ln\left(\frac{1}{1-u}\right)}{\alpha(1+\lambda)} \right] \tag{12}$$

Moments: The rth non-central moment μ_r is mathematically given by:

$$\mu_r = E\left(X^r\right) = \int_0^\infty x^r f(x) dx$$

In particular, the rth moment of the new weighted exponential distribution is derived by:

$$\mu_r = \int_0^\infty x^r (1+\lambda)\alpha e^{-(\alpha x + \alpha\lambda x)} dx$$

Let:

$$\theta = \alpha x + \alpha\lambda x \quad : \quad x = \frac{\theta}{\alpha(1+\lambda)}$$

$$\frac{d\theta}{dx} = \alpha(1+\lambda) \quad : \quad d\theta = \alpha(1+\lambda)dx \quad : \quad dx = \frac{d\theta}{\alpha(1+\lambda)}$$

Therefore:

$$\mu_r = \int_0^\infty \left[\frac{\theta}{\alpha(1+\lambda)} \right]^r (1+\lambda)\alpha e^{-\theta} \frac{d\theta}{\alpha(1+\lambda)}$$

$$\mu_r = \int_0^\infty \left[\frac{\theta}{\alpha(1+\lambda)} \right]^r e^{-\theta} d\theta$$

$$\mu_r = \left[\frac{1}{\alpha(1+\lambda)} \right]^r \int_0^\infty \theta^r e^{-\theta} d\theta$$

$$\mu_r = \left[\frac{1}{\alpha(1+\lambda)}\right]^r \int_0^\infty \theta^{(r+1)-1} e^{-\theta} d\theta$$

Thus, the rth moment of the new weighted exponential distribution is expressed as Eq. 13:

$$\mu_r = \left[\frac{1}{\alpha(1+\lambda)}\right]^r \Gamma(r+1) \tag{13}$$

The mean and other higher order moments can be obtained from Eq. 13. For instance, When r = 1, the mean of the proposed model is derived to give Eq. 14:

$$E(X) = \left[\frac{1}{\alpha(1+\lambda)}\right] \tag{14}$$

When, r = 2 then Eq. 5 will be as Eq. 15:

$$E(X^2) = 2\left[\frac{1}{\alpha(1+\lambda)}\right]^2 \tag{15}$$

The variance of the new weighted exponential distribution can be derived through the relation:

$$Var\ (x)\ E(X^2)\text{-}[E(x)]^2$$

$$Var(X) = 2\left[\frac{1}{\alpha(1+\lambda)}\right]^2 - \left[\frac{1}{\alpha(1+\lambda)}\right]^2$$

Hence, Eq. 15 will be as Eq. 16:

$$Var(X) = \left[\frac{1}{\alpha(1+\lambda)}\right]^2 \tag{16}$$

Reliability analysis: The survival (or reliability) function is mathematically given by $\bar{F}(x) = 1\text{-}F(x)$. Therefore, the survival function of the new weighted exponential distribution is given by Eq. 17:

$$\bar{F}(x) = e^{-(\alpha x+\alpha\lambda x)} \tag{17}$$

Equation 17 can be re-written as Eq. 18:

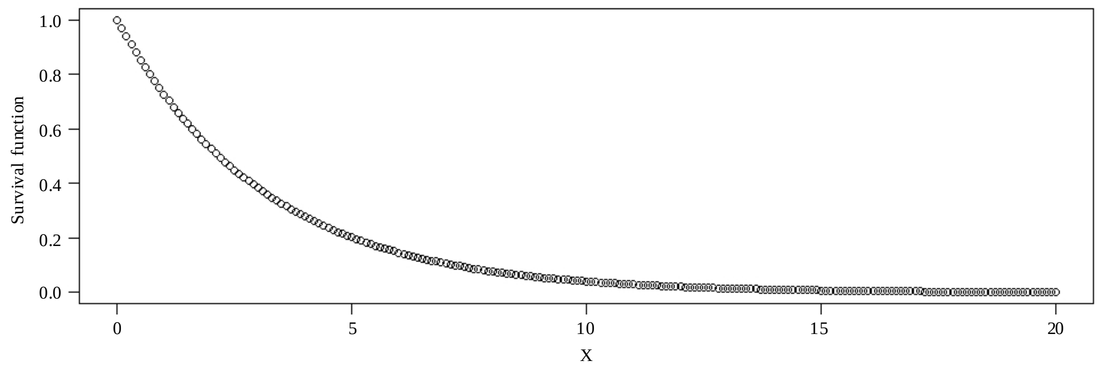

Fig. 4: Plot for the survival function at α = 0.2, λ = 0.6

$$\bar{F}(x) = e^{-(1+\lambda)\alpha x} \tag{18}$$

A possible plot for the survival function of the new weighted exponential distribution at some specified parameter values is shown in Fig. 4.

The hazard function is mathematically given by:

$$h(x) = \frac{f(x)}{1-F(x)}$$

Therefore, the hazard function of the new weighted exponential distribution is given by:

$$h(x) = \frac{(1+\lambda)\alpha e^{-(\alpha x+\alpha\lambda x)}}{e^{-(\alpha x+\alpha\lambda x)}}$$

$$h(x) = (1+\lambda)\alpha \tag{19}$$

It was observed from Eq. 19 that the new weighted exponential distribution has a constant failure rate.

Order statistics: The pdf of the ith order statistic for a random sample $X_1,...,X_n$ from a pdf f(x) and an associated distribution function F(x) is given by Eq. 20:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} \left[1-F(x)\right]^{n-i} \tag{20}$$

where, f(x) and F(x) in Eq. 20 are the pdf and cdf of the new weighted exponential distribution, respectively. Therefore, the pdf of the ith order statistic for a random sample $X_1,...,X_n$ from the new weighted exponential distribution is expressed as Eq. 21:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}(1+\lambda)\alpha e^{-(\alpha x+\alpha\lambda x)} \left[1-e^{-(\alpha x+\alpha\lambda x)}\right]^{i-1} \left[e^{-(\alpha x+\alpha\lambda x)}\right]^{n-i} \tag{21}$$

The pdf of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the new weighted exponential distribution are respectively given by Eq. 22 and 23:

$$f_{(X_1)}(x) = n(1+\lambda)\alpha e^{-(\alpha x+\alpha\lambda x)} \times \left[e^{-(\alpha x+\alpha\lambda x)}\right]^{n-1} \tag{22}$$

and

$$f_{(X_n)}(x) = n(1+\lambda)\alpha e^{-(\alpha x+\alpha\lambda x)} \times \left[1-e^{-(\alpha x+\alpha\lambda x)}\right]^{n-1} \tag{23}$$

Estimation: Let $X_1,...,X_2,...X_n$ denote random samples drawn from the weighted exponential distribution with parameters α and λ as defined in Eq. 7. Using the method of Maximum Likelihood Estimation (MLE), the likelihood function L is given by:

$$L\left(\tilde{X} \mid \alpha,\lambda\right) = \prod_{i=1}^{n}\left\{(1+\lambda)\alpha e^{-(\alpha x+\alpha\lambda x)}\right\}$$

Let the log-likelihood function be denoted by l, therefore,

$$l = n\log(1+\lambda) + n\log\alpha - \alpha(1+\lambda)\sum_{i=1}^{n} x_i$$

Differentiating l with respect to parameters α and λ given as Eq. 24 and 25:

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} - (1+\lambda)\sum_{i=1}^{n} x_i \tag{24}$$

$$\frac{\partial l}{\partial\lambda} = \frac{n}{(1+\lambda)} - \alpha\sum_{i=1}^{n} x_i \tag{25}$$

Equating 24 and 25 to zero and solving simultaneously gives the maximum likelihood estimates of parameters α and λ.

Application: To assess the flexibility of the new weighted exponential distribution over the existing weighted exponential distribution, four real data sets are used. The analysis involved in this research was performed with the aid of R software.

Data set 1: This data has been previously used by Ghitany *et al.* (2008) and Alqallaf *et al.* (2015). It represents the waiting time (measured in min) of 100 bank customers before service is being rendered. The data is as follows:

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7	2.9	3.1	3.2
3.3	3.5	3.6	4.0	4.1	4.2	4.2	4.3	4.3	4.4	4.4	4.6	4.7
4.7	4.8	4.9	4.9	5.0	5.3	5.5	5.7	5.7	6.1	6.2	6.2	6.2
6.3	6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6	7.7	8.0	8.2	8.6
8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6	9.7	9.8	10.7	10.9	11.0
11.0	11.1	11.2	11.2	11.5	11.9	12.4	12.5	12.9	13.0	13.1	13.3	13.6
13.7	13.9	14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4	18.9	19.0	19.9
20.6	21.3	21.4	21.9	23.0	27.0	31.6	33.1	38.5.				

The summary of the data is given in Table 1.

The estimates of the parameters, log-likelihood and Alkaike Information Criteria (AIC) for the data on waiting time of bank customers are generated and the result is as presented in Table 2.

Data set 2: This data represents the lifetime of 20 electronic components. The data has been previously used by Teimouri and Gupta (2013) and Nasiru (2015). The data is as follows:

0.03	0.22	0.73	1.25	1.52	1.80	2.38	2.87	3.14	4.72
0.12	0.35	0.79	1.41	1.79	1.94	2.40	2.99	3.17	5.09

The summary of the data is given in Table 3.

The estimates of the parameters, log-likelihood and Alkaike Information Criteria (AIC) for the data on electric components are generated and the result is as presented in Table 4.

Data set 3: Data is from an accelerated life test of 59 conductors. It has been previously used by Nasiri *et al.* (2011). The data is as follows:

6.545	9.289	7.543	6.956	6.492	5.459	8.120	4.706	8.687	2.997
8.591	6.129	11.038	5.381	6.958	4.288	6.522	4.137	7.459	7.495
6.573	6.538	5.589	6.087	5.807	6.725	8.532	6.663	6.369	7.024
8.336	9.218	7.945	6.869	6.352	4.700	6.948	9.254	5.009	7.489
7.398	6.033	10.092	7.496	4.531	7.974	8.799	7.683	7.224	7.365
6.923	5.640	5.434	7.937	6.515	6.476	6.071	10.491	5.923	

The summary of the data is given in Table 5.

The estimates of the parameters, log-likelihood and Alkaike Information Criteria (AIC) for the data on conductors are generated and the result is as presented in Table 6.

Data set 4: Data represents the remission times (in months) of a random sample of 128 bladder cancer patients. It has previously been used by Lee and Wang (2003). The data is as follows:

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98
6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50
2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51	2.54	3.70	5.17	7.28
9.74	14.76	26.31	0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33	5.49	7.66	11.25	17.14
79.05	1.35	2.87	5.62	7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93
11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25
8.37	12.02	2.02	3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76
12.07	21.73	2.0	3.36	6.93	8.65	12.63	22.69				

The summary of the data is given in Table 7.

The estimates of the parameters, log-likelihood and Alkaike Information Criteria (AIC) for the data on remission time of blood cancer patients are generated and the result is as presented in Table 8.

Table 1: Summary of data on waiting time of bank customers

Minimum	Q1	Q2	Q3	Mean	Maximum	Variance	Skewness	Kurtosis
0.800	4.675	8.100	13.020	9.877	38.500	52.37411	1.472765	5.540292

Source: Ghitany *et al*. (2008) and Alqallaf *et al*. (2015)

Table 2: Performance of distributions (standard error in parenthesis)

Distributions	Estimates	Log-likelihood	AIC	Rank
Weighted exponential	$\hat{\alpha}$ = 0.70332	-317.2319	638.4638	1
	(1.29176)			
	$\hat{\lambda}$ = 0.16069			
	(0.04709)			
New weighted exponential	$\hat{\alpha}$ = 0.004735	-329.0209	662.0418	2
	(0.002437)			
	$\hat{\lambda}$ = 20.383945			
	(10.790418)			

Source: Software: R, AIC: Alkaike information criteria

Table 3: Summary of data on electronic components

Minimum	Q1	Q2	Q3	Mean	Maximum	Variance	Skewness	Kurtosis
0.030	0.775	1.795	2.900	1.936	5.090	2.062373	0.6025412	2.720165

Teimouri and Gupta (2013) and Nasiru (2015)

Table 4: Performance of distributions (standard error in parenthesis)

Distributions	Estimates	Log-likelihood	AIC	Rank
Weighted exponential	$\hat{\alpha}$ = 252.3707	-33.14827	70.29655	1
	(11.8638)			
	$\hat{\lambda}$ = 0.5187			
	(0.1151)			
New weighted exponential	$\hat{\alpha}$ = 21.76117	-33.20731	70.41463	2
	(8.27912)			
	$\hat{\lambda}$ = -0.97626			
	(0.01048)			

Software: R, AIC: Alkaike information criteria

Table 5: Summary of data on conductors

Minimum	Q1	Q2	Q3	Mean	Maximum	Variance	Skewness	Kurtosis
2.997	6.052	6.869	7.810	6.929	11.040	2.48013	0.2195643	3.28098

Nasiri *et al*. (2011)

Table 6: Performance of distributions (standard error in parenthesis)

Distributions	Estimates	Log-likelihood	AIC	Rank
Weighted exponential	$\hat{\alpha}$ = 7.154e-06	-152.0032	308.0063	1
	(4.75e-01)			
	$\hat{\lambda}$ = 2.886e-01			
	(7.192e-02)			
New weighted exponential	$\hat{\alpha}$ = 34.558133	-173.2091	350.4182	2
	(11.604921)			
	$\hat{\lambda}$ = -0.995824			
	(0.001505)			

Software: R, AIC: Alkaike information criteria

Table 7: Summary of data on remission time of blood cancer patients

Min.	Q1	Q2	Q3	Mean	Maximum	Variance	Skewness	Kurtosis
0.080	3.348	6.395	11.840	9.366	79.050	110.425	3.286569	18.48308

Lee and Wang (2003)

Table 8: Performance of distributions (standard error in parenthesis)

Distributions	Estimates	Log-Likelihood	AIC	Rank
Weighted Exponential	$\hat{\alpha}$ = 12.758291 (4.394094) $\hat{\lambda}$ = 0.114534 (0.009849)	-413.0449	830.0897	1
New Weighted Exponential	$\hat{\alpha}$ = 0.01885 (0.01058) $\hat{\lambda}$ = 4.66350 (3.13879)	-414.3419	832.6838	2

Software: R, AIC: Alkaike information criteria

DISCUSSION

In this article, the model with the lowest Alkaike Information Criteria (AIC) is considered to be the best fit, this is in accordance with what is obtainable in some other notable researches, Bourguignon *et al*. (2014) and Oguntunde and Adejumo (2015) for example. For all the four (4) data sets used, the AIC of the new weighted exponential distribution is higher than that of the existing weighted exponential distribution. This in turn means that the existing weighted exponential distribution proposed by Gupta and Kundu (2009) is more flexible than the new weighted exponential distribution based on the applications provided in this research. Meanwhile, the proposed new weighted exponential distribution is more tractable. This result is pointing out that, the fact that a model is mathematically friendly and simple does not mean the model is better, more robust or more flexible. This result is also similar to a result in Adepoju *et al*. (2014) where, the Kumaraswamy Nakagami distribution which is expected be more tractable than the Beta-Nakagami distribution proposed by Shittu and Adepoju (2013) did not perform better than the Beta Nakagami distribution when applied to a flood data set.

Meanwhile, the result is in contrary to some trends in the literature where newly proposed models are better than the existing ones. For instance, readers can go through Cordeiro *et al*. (2008) where the Beta Weibull distribution was shown to be better than the Weibull distribution, Idowu and Ikegwu (2013) who defined the Beta-Weighted Weibull distribution as being better than the existing Weighted Weibull distribution, Tahir *et al*. (2015), who proposed the Weibull Lomax distribution as being better than Lomax distribution and some other generalizations, Khan *et al*. (2013) who proposed the Transmuted Inverse Weibull distribution as being better than the existing inverse weibull distribution. Further research is needed to compare the flexibility of the new weighted weibull distribution defined by Nasiru (2015) with the existing weighted weibull distribution in order to validate our claim because the proposed new weighted exponential distribution is a special case of the new weighted weibull distribution.

CONCLUSION

The new two-parameter weighted exponential distribution has been successfully defined. The shape of the model is reversed J-shape. Explicit expressions are provided for its basic mathematical properties. The model shares some basic properties with the Exponential distribution as discussed in section 3, for instance, it has a constant failure rate. The model is considered to be more tractable than the existing weighted exponential distribution defined by Gupta and Kundu (2009). The model is applied to four real data sets to assess its potentiality but it was discovered that the existing weighted exponential distribution is more flexible than the new weighted exponential distribution.

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