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An Endgame Procedure for Generating Integer Sequence

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ABSTRACT
In playing Ayo game, both opening and endgames are often stylized. The opening is very interesting with both players showing skills by the speed of their movements. However, there exists an endgame strategy in Ayo game called Completely Determined Game (CDG) such that its usefulness for ending a game should be apparent. In this paper, we present the CDG as a class of endgame strategy and describe its configuration and detailed analysis of its winning positions that generates integer sequence, and some self-replicating patterns.

Keywords- Integer sequence, Ayo game, CDG, endgame, matching group

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1. INTRODUCTION

Ayo is a game that requires rigorous calculations and strategies, with the aim of capturing as many seeds as possible [1]. Such a game is called combinatorial game, and has captured the attention of many Artificial Intelligence (AI) researchers, mathematicians and computer scientists. A rich theory on how to evaluate game positions has been developed in recent years, and it has been successfully applied to analyze certain endgame positions of Chess, [2] and Go, [3]. In playing *Ayo* game, both opening and endgames are often stylized. The opening is very interesting with both players showing skills by the speed of their movements. However, playing the game will require remembering the number of seeds in each of the twelve pits, as well as planning several moves in advance, [4].

Consequent upon this, there exist an endgame strategy in *Ayo* game called Completely Determined Game (CDG) such that its usefulness for ending a game should be apparent, and can be configured (or arranged) as the number of seeds on the board reduces to 21 [5]. The endgame strategy, in which the CDG is described in this paper for generating integer sequence, is of course very exciting with one player dominating.
The rest of this paper is arranged as follows: section 2 discusses *Ayo* game. Section 3 focuses on detail description and analysis of generating integer sequence from an endgame strategy called Completely Determined Game (CDG). A concluding remark is given in section 4.

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2. AYO GAME

Ayo game is the most popular board game among the Yorubas who occupy roughly the south-western states of Nigeria and parts of Republic of Benin. It belongs to a member of a family of board games called Mancala. The term Mancala is used to indicate a large group of related games that are played almost all over the world [6, 7]. The game of *Ayo* is one of the oldest known strategy games. It is a game of perfect information known as combinatorial games. Two persons play *Ayo* at a time with the board ("Opon") put in between the players. The board is a hollow out plank of wood consisting of two rows of six pits belonging to either row and each pit contains four seeds of the plant "*Caeselpinia crista*" [8] such that a total of forty-eight seeds are contained in a board at the start of the game. The objective of the players is to capture their opponent's seeds (as many as possible). A move consists of a player choosing a non–empty pit on his side of the board and removing all of the seeds contained in that pit. The seeds are redistributed (*sown*) one seed per pit. The pits are sown in counter–clockwise direction from the pit that has just been emptied. A capture is made, when the last pit sown is on the opponent's side, and contains after the addition of the sowing seed either two or three seeds. Thus, the seeds in the pit are captured and removed from the game. Also captured are the immediately preceding pits which meet the same conditions. One important feature of this game is that each player has to make a move such that his opponent has a legal move to play. If this does not happen, then the opponent is rewarded with all the remaining seeds on the board. This rule is referred to as the golden rule, [9]. If during the game, it is found that there are not enough seeds to make a capture, but both players can always proceed with a legal move, the game is stopped and the players are awarded seeds that reside on their respective side of the board. The initial game is rapid and much more interesting, where both the players capture seeds in quick succession. To determine the optimal strategy during the initial play is hard, and thus has not yet been studied. It involves planning at least 2–3 moves in advance, and remembering the number of seeds in every pit, [4, 8].

In playing *Ayo* game, both opening and endgames are often stylized. The opening is very interesting with both players showing skills by the speed of their movements. However, playing the game will require remembering the number of seeds in each of the twelve pits, as well as planning several moves in advance, [4]. The endgame, in which the CDG is described in this paper for generating integer sequence, is of course less exciting with one player dominating. The following section explicitly describes the CDG.

3. INTEGER SEQUENCE FROM AN ENDGAME STRATEGY

3.1. Completely Determined Game (CDG)

CDG is a game configuration in which there exists a series of move and capture strategies for a player X on the action of player Y such that the move process continues for the player X until only one seed remains on the board and the seed belongs to player Y. As the concept of the CDG is defined, its usefulness for ending a game should be apparent, and can be configured (or arranged) as the number of seeds on the board reduces to twenty-one.

In *Ayo* game, a CDG play exists if the configuration of seeds on the board satisfies the two conditions given below;

i. all the holes on the side of player Y (the first player) are empty except the first hole that contains only one seed
ii. as player Y moves, player X takes a move that results in a capture of two seeds. This strategy of move and capture continues until player X has captured all the seeds except one seed that is left for player Y to ensure that the golden rule is obeyed.
iii.

Figure 1 below depicts the initial configuration of player X and Y for the CDG

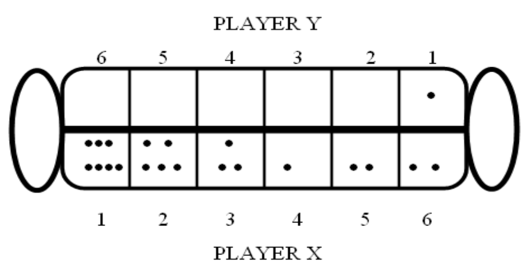

Figure 1: CDG *initial configuration in Ayo game*

The CDG configuration in figure 1 can be represented as the pattern space shown in table 1 such that when player Y moves from his pit 1, player X has two possible moves that can capture the seed in house Y, that is, moving from pit 1 or pit 6. But in order to allow the reducibility of the CDG so as to be able to capture all except one of the seeds, player X has to move from house 6 and when player Y further makes another move, player X can continue capturing in this manner till the end of the game.

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Table 1: Pattern Spaces for Player X in CDG

	Houses					
	X_1	X_2	X_3	X_4	X_5	X_6
1	2	0	0	0	0	0
2	1	3	0	0	0	0
3	0	2	4	0	0	0
4	2	2	4	0	0	0
5	1	1	3	5	0	0
6	0	0	2	4	6	0
7	2	0	2	4	6	0
8	1	3	2	4	6	0
9	0	2	1	3	5	7
10	2	2	1	3	5	7

The process whereby a CDG configuration is reduced to another such that its continuity is not hampered in any way till the last CDG possible is referred to as CDG-vector reducibility. Maxima and minima CDG-vectors enable us to define a transformation function between vectors. While it is relatively easy to determine minimal CDG-vector for any game of size n, the problem of computing maximal CDG-vector is relatively difficult. Nevertheless, we were able to efficiently compute the maximal CDG-vectors by implementing algorithm I below.

Algorithm I

Input n the size of the board

Output $x = (x_n, x_{n-1}, \ldots, x_2, x_1)$

Step 0: $x_n = n+1, d_n = 2, j = n-1$

If $j < 1$ stop otherwise go to step 1

Step 1: $x_j = (j+1)\left\lfloor \frac{(j+1)-(x_{j+1}-d_{j+1})}{j+1} \right\rfloor + (x_{j+1} - d_{j+1}$

If $j = 1$, stop otherwise, go to step 2

Step 2: $d_j = d_{j+1} + 2\left\lfloor \frac{(j+1)-(x_{j+1}-d_{j+1})}{(j+1)} \right\rfloor$

$j = j-1$

Go to step 1

End Algorithm I

A sample implementation of the above algorithm, gives the maximal CDGs in table 2 below which hold for any board size of 2n pits. The reducibility of the maximal CGD for n = 6 is shown in Table 1.

Table 2: Maximal CDGs for board size of 2n pits

	Maximal CDG-vectors																									
n	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	...																	x_n
1	2																									
2	1	3																								
3	2	2	4																							
4	1	1	3	5																						
5	1	3	2	4	6																					
6	2	2	1	3	5	7																				
7	1	3	4	2	4	6	8																			
8	2	2	3	1	3	5	7	9																		
9	1	3	1	5	2	4	6	8	10																	
10	1	1	4	4	1	3	5	7	9	11																
11	2	2	2	2	6	2	4	6	8	10	12															
12	1	1	1	1	5	1	3	5	7	9	11	13														
13	1	3	2	4	3	7	2	4	6	8	10	12	14													
14	2	2	1	3	2	6	1	3	5	7	9	11	13	15												
15	1	3	1	5	6	4	8	2	4	6	8	10	12	14	16											
16	1	1	4	4	5	3	7	1	3	5	7	9	11	13	15	17										
17	2	2	2	2	3	1	5	9	2	4	6	8	10	12	14	16	18									
18	1	1	3	5	1	7	4	8	1	3	5	7	9	11	13	15	17	19								
19	1	3	4	2	5	5	2	6	10	2	4	6	8	10	12	14	16	18	20							
20	2	2	3	1	4	4	1	5	9	1	3	5	7	9	11	13	15	17	19	21						
21	1	3	2	2	1	1	7	3	7	11	2	4	6	8	10	12	14	16	18	20	22					
22	1	1	3	5	5	7	6	2	6	10	1	3	5	7	9	11	13	15	17	19	21	23				
23	1	3	4	2	2	4	3	9	4	8	12	2	4	6	8	10	12	14	16	18	20	22	24			
24	2	2	3	1	1	3	2	8	3	7	11	1	3	5	7	9	11	13	15	17	19	21	23	25		
25	1	1	1	1	3	7	8	6	1	5	9	13	2	4	6	8	10	12	14	16	18	20	22	24	26	
26	1	3	2	4	1	5	6	4	10	4	8	12	1	3	5	7	9	11	13	15	17	19	21	23	25	27
..	..																									
..	..																									
..	..																									

3.1. Seed Capturing and Periodic Patterns

According to [4], a number of seeds could be captured (or harvested) in an unlimited pits and as well showed that as the number pits *n* increases, the sum of seeds on the board s(*n*) also increases as $n^2/pi + O(n)$, as shown in table 4 above. In their work, they sum an arithmetic sequence whose common difference is 2 and whose largest term is n (the number of houses). This sum counts the number of seeds in houses n/2 to n. Next there is an arithmetic sequence whose largest term is approximately n/2 and whose common difference is 4. This accounts for (approximately) the seeds in houses 3n/8 to n. Next there is an arithmetic sequence whose largest term is approximately 3n/8 and whose common difference is 6. This argument is continued to get the asymptotic results. But we noted that this asymptotic result varies widely from precise results for specific numbers.

Now, let us revisit the play and seed capturing strategy of the CDG earlier discussed in section 3, where the initial game distribution is shown in figure 2 above such that after the move by player Y (the first player to move), player X captures two seeds by playing from house 6. Since there is only one positional move is appropriate for a valid CDG move, otherwise the anticipated total seeds to be captured will not be possible, if there is more than one pit that has enough seeds to capture seeds for which the CDG strategy must hold, then it is the pit at the rightmost that must be selected as the best move. For example, the play and capturing strategy is given in Table 3 in bottom-up manner. That is, at *n* = 21 (total seeds on board), there are 20 seeds available for player X and only 1 for player Y that has only one option of play. When player Y moves form his own pit 1 to pit 2 (that is Y1 and Y2 respectively) the game configuration gives the *n* = 20 arrangement such that when player Y plays, the arrangement of *n* = 19 is obtained. This process is continued till the end as shown in Table 3.

Table 3: Play and Capturing Strategy in CDG

n	X_1	X_2	X_3	X_4	X_5	X_6	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
1	0	0	0	0	0	0	1	0	0	0	0	0
2	0	0	0	0	0	2	0	1	0	0	0	0
3	0	0	0	0	0	2	1	0	0	0	0	0
4	0	0	0	0	3	1	0	1	0	0	0	0
5	0	0	0	0	3	1	1	0	0	0	0	0
6	0	0	0	4	2	0	0	1	0	0	0	0
7	0	0	0	4	2	0	1	0	0	0	0	0
8	0	0	0	4	2	2	0	1	0	0	0	0
9	0	0	0	4	2	2	1	0	0	0	0	0
10	0	0	5	3	1	1	0	1	0	0	0	0
11	0	0	5	3	1	1	1	0	0	0	0	0
12	0	6	4	2	0	0	0	1	0	0	0	0
13	0	6	4	2	0	0	1	0	0	0	0	0
14	0	6	4	2	0	2	0	1	0	0	0	0
15	0	6	4	2	0	2	1	0	0	0	0	0
16	0	6	4	2	3	1	0	1	0	0	0	0
17	0	6	4	2	3	1	1	0	0	0	0	0
18	7	5	3	1	2	0	0	1	0	0	0	0
19	7	5	3	1	2	0	1	0	0	0	0	0
20	7	5	3	1	2	2	0	1	0	0	0	0
21	7	5	3	1	2	2	1	0	0	0	0	0

Following this description, we obtain a capturing sequence given in table 8 below. For example, if we consider a game configuration of the form 6, 4, 2, 3, 1, 1; such that one seed belongs to player Y and the remaining 16 seeds belong to player X as shown in figure 2 below.

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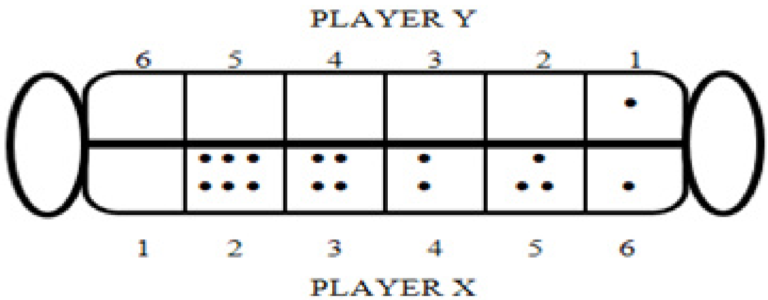

Figure 2: A Typical CDG configuration for capturing Sequence in Ayo game

From this configuration, as soon as player Y moves from his pit 1 to pit 2, player X must move from his pit 5 that has 3 seeds to capture 2 seeds from pit 2 of house Y. the capturing sequence here is 1,3. Now, there is going to be 2 seeds available in pit 6 of player X such that when player Y moves from his pit 1 to pit 2, the capturing sequence again give 1, 2 thereby making the sequence to be 1, 3, 1, 2. Continuing this way, the entire capturing sequence becomes 1,3,1,2,1,6,1,5,1,2,1,4,1,3,1,2,1. Table 4 below depicts the distribution of seeds and the capturing sequence for up to 21 seeds such that player X has 20 seeds and player Y has only 1 seed to obey the CDG rule.

Table 4: Distribution of Seeds and Capturing Sequence for up to 21 Seeds

n	X_1	X_2	X_3	X_4	X_5	X_6	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Capturing Sequence
1	0	0	0	0	0	0	1	0	0	0	0	0	1
2	0	0	0	0	0	2	0	1	0	0	0	0	2,1
3	0	0	0	0	0	2	1	0	0	0	0	0	1,2,1
4	0	0	0	0	3	1	0	1	0	0	0	0	3,1,2,1
5	0	0	0	0	3	1	1	0	0	0	0	0	1,3,1,2,1
6	0	0	0	4	2	0	0	1	0	0	0	0	4,1,3,1,2,1
7	0	0	0	4	2	0	1	0	0	0	0	0	1,4,1,3,1,2,1
8	0	0	0	4	2	2	0	1	0	0	0	0	2,1,4,1,3,1,2,1
9	0	0	0	4	2	2	1	0	0	0	0	0	1,2,1,4,1,3,1,2,1
10	0	0	5	3	1	1	0	1	0	0	0	0	5,1,2,1,4,1,3,1,2,1
11	0	0	5	3	1	1	1	0	0	0	0	0	1,5,1,2,1,4,1,3,1,2,1
12	0	6	4	2	0	0	0	1	0	0	0	0	6,1,5,1,2,1,4,1,3,1,2,1
13	0	6	4	2	0	0	1	0	0	0	0	0	1,6,1,5,1,2,1,4,1,3,1,2,1
14	0	6	4	2	0	2	0	1	0	0	0	0	2,1,6,1,5,1,2,1,4,1,3,1,2,1
15	0	6	4	2	0	2	1	0	0	0	0	0	1,2,1,6,1,5,1,2,1,4,1,3,1,2,1
16	0	6	4	2	3	1	0	1	0	0	0	0	3,1,2,1,6,1,5,1,2,1,4,1,3,1,2,1
17	0	6	4	2	3	1	1	0	0	0	0	0	1,3,1,2,1,6,1,5,1,2,1,4,1,3,1,2,1
18	7	5	3	1	2	0	0	1	0	0	0	0	4,1,3,1,2,1,6,1,5,1,2,1,4,1,3,1,2,1
19	7	5	3	1	2	0	1	0	0	0	0	0	1,4,1,3,1,2,1,6,1,5,1,2,1,4,1,3,1,2,1
20	7	5	3	1	2	2	0	1	0	0	0	0	5,1,4,1,3,1,2,1,6,1,5,1,2,1,4,1,3,1,2,1
21	7	5	3	1	2	2	1	0	0	0	0	0	1,5,1,4,1,3,1,2,1,6,1,5,1,2,1,4,1,3,1,2,1

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From Table 4, we noticed that when the game states for which proper maximal CDG occurs for 6 pits, 5 pit, through 1 pit is given as n = 1, 3, 5, 9, 11, 17, 21. This interestingly, correspond the sequence A007952 in the online Encyclopedia of Integer Sequence.

Again, there could be a self-replicating pattern while distributing seeds in *Ayo* game, which [10] referred to as matching groups. A matching group is made up of a successive pits in *Ayo* whose numbers of seeds are given by the sequence $n, n-1, \ldots, 2, 1$.
A typical move will result in replicating the initial pattern with a right-shift by distributing seeds from the pit having the largest number of seeds, and placing one seed in each succeeding pit as in figure 3.

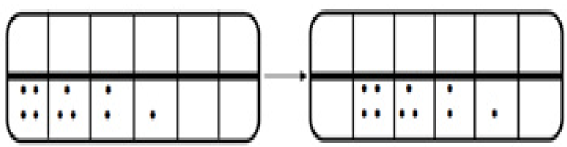

Figure 3: Matching Group in Ayo

If the distribution is left uninterrupted, repeated application will allow it to propagate in this way as far as needed. For example, if we consider a board configuration that has 10 seeds as 5, 3, and 2, a number of distributions could be made and obtain a distribution that results in a matching group as shown below.

[5, 3, 2] → [4, 3, 1, 1, 1] → [4, 2, 2, 2] → [3, 3, 3, 1] → [4, 4, 2] → [5, 3, 1, 1] → [4, 2, 2, 1, 1] → [3, 3, 2, 2] → [4, 3, 3] → [4, 4, 1, 1] → [5, 2, 2, 1] → [3, 3, 2, 1, 1] → [4, 3, 2, 1]

In the same light, there could be some configurations that will result in the initial pattern but not necessarily a matching group after some periods. In other word, the matching group rule can be used to produce periodic behaviour thereby showing that some numbers of seeds exhibits certain periodicity. Here, the initial configuration of seeds will follow the matching group rule and return back to the initial state after some iterations, which gives its period. For example, if we have 4 seeds arranged as 2, 1, 1, one will only obtain the initial configuration after 3 iterations, thereby making number 4 to be a period 3 system as shown in figure 4 below.

Sequel to the above, we then settled down and present some numbers of seeds and their corresponding behavioural pattern in Table 5 below.

Table 5: Pattern Behaviour of up to 21 Seeds in *Ayo*.

Total Seeds	Pattern Behaviour
1	Matching Group
2	Period 2
3	Matching Group
4	Period 3
5	Period 3
6	Matching Group
7	Period 4
8	Period 4
9	Period 4
10	Matching Group
11	Period 5
12	Period 5
13	Period 5
14	Period 5
15	Matching Group
16	Period 6
17	Period 6
18	Period 6
19	Period 6
20	Period 6
21	Matching Group

Again, we can see that the numbers of seeds that lead to matching groups are 1, 3, 6, 10, 15, 21, . (see Table 5), which is represented as a triangular sequence A000217 in the online Encyclopedia of Integer Sequence as shown in figure 4, and can generally be obtained as [n(n+1)] /2 as shown in Table 6.

$$\mathbf{1}$$
$$1+2 = \mathbf{3}$$
$$(1+2) + 3 = \mathbf{6}$$
$$(1+2+3) + 4 = \mathbf{10}$$
$$(1+2+3+4) + 5 = \mathbf{15}$$
$$(1+2+3+4+5) + 6 = \mathbf{21}$$
$$\ldots\ldots$$

Figure 4: Triangular Sequence from Matching Group Numbers in *Ayo*

Table 6: Generation of n^{th} term for Matching Group Numbers in *Ayo*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
[n(n+1)]/2	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171	190	210

3.2. Generating Function

The generating function for the sequence $a_0, a_1 \ldots$ is defined to be the function

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$$f(n) = \begin{cases} \left\lfloor \frac{1}{3}(n+5) \right\rfloor + C & n < 25 \\ 10 + \left\lfloor \frac{1}{4}(n-25) \right\rfloor ; & n \geq 25 \end{cases}$$

Where

$$C = \begin{cases} 1 & ; \ n = 15 \\ -1 & ; \ n = 1, \ 2, \ or \ \ 4 \\ 0 & ; \ otherwise \end{cases}$$

Further study on the staircases pattern in table 2 showed that there is a fair correlation between *n*, *s(n)*, and *f(n)* as in table 7 below. Where *n* is the board size, *s(n)* is the sum of seeds of the maximal CDG, and *f(n)*.is the number of blocks of arithmetic sequences that appear in the maximal CDG in table 2, in line with the implementation of algorithm I.

It is worthy to note that in *Ayo* game playing, whenever a player captures 25 seeds or more the player emerges as a winner. It then means in clear term that the generating function determines a winning number of seeds.

Table 7: Computational Results

n	S(n)	f(n)		n	S(n)	f(n)		n	S(n)	f(n)
1	2	1		18	130	7		35	440	12
2	4	1		19	148	8		36	452	12
3	8	2		20	152	8		37	496	13
4	10	2		21	172	8		38	508	13
5	16	3		22	196	9		39	538	13
6	20	3		23	208	9		40	568	13
7	28	4		24	212	9		41	610	14
8	32	4		25	238	10		42	620	14
9	40	4		26	256	10		43	646	14
10	46	5		27	272	10		44	670	14
11	56	5		28	280	10		45	716	15
12	58	5		29	320	11		46	730	15
13	76	6		30	328	11		47	778	15
14	80	6		31	358	11		48	820	15
15	100	7		32	370	11		49	838	16
16	106	7		33	400	12		50	868	16
17	116	7		34	416	12				

The result of the sequence *f(n)* therefore characterized *Ayo* as a game that generates an integer sequence of the form:

1, 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 7, 7, 7, 7, 8, 8, 8, 9, 9, 9, 10, 10, 10, 10, 11, 11, 11, 11, 12, 12, 12,, 12, 13, 13, 13, 13, 14, 14, 14, 14, 15, 15, 15, 15, 16, 16, 16, 16, …

4. CONCLUDING REMARK

The work described the CDG play as an endgame strategy with some of its mathematical characteristics, which if can be achieved in any instant of play can guarantee a player winning a particular tournament. One intriguing motivation in this work is seeing Ayo game giving mathematical concepts worthy of research work in integer sequence.

Similarly, the staircase pattern in table 2 is equally worthy of investigating as this work is by no means the end of investigation on the pattern. It is our belief that the nature of the integer sequence obtained in this work would interest number theory researchers.

An open problem we wish to give in this work that calls for further research interest is, how do we derive an algorithm that can lure the opponent into the CDG play at any instant of play?

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